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► **To cite this version:**

Marie Kratz, José R. Leon. Level curves, crossings and specular points for Gaussian models.. 2007.
hal-00239290v1

HAL Id: hal-00239290

<https://hal.science/hal-00239290v1>

Preprint submitted on 5 Feb 2008 (v1), last revised 13 May 2008 (v2)

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Curve crossings and specular points, d’après Longuet-Higgins.

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Abstract

We use the Hermite expansion for the number of crossings of a differentiable curve by a stationary process to study the number of specular points of a curve and to understand its dynamical behavior, in particular asymptotically, providing some CLT.

1 Introduction

Longuet-Higgins in the fifties and the sixties (see [17] and [18]) developed a heuristic method that allows counting the number of specular points of a random surface when a ray of light hits it. His main motivation was the study of the light reflection on the sea surface aimed at determining the height of the waves. By using elements of geometrical optics he showed that the specular points of the surface $W(t, x)$ are the crossings of $\partial_x W(t, x)$ to the curve $\psi(x) = -\kappa x$, where κ is a constant depending on the distances between the light source and the surface and between the observer and the surface. He also studied the speed in which these specular points move. Finally he considered the time evolution of specular points, showing that the specular points evolve during a period of time, up to the vanishing of the surface curvature. The creation or annihilation of specular points may be called a “twinkle”: in those points the light reflection is more intense and there is a flash. Longuet-Higgins obtained also a formula that counts the number of such twinkles.

It should be noticed that all the Longuet-Higgins formulae were obtained in a heuristic form, nevertheless it is worth to mention that all of them are strictly exact.

In fact, these formulae are generalizations of the well known Rice formula ([19]) designed to count the mean number of roots of a random curve. The formula for the mean number of specular points is nothing more than an application of Cramer-Leadbetter formula ([9]) that counts the mean number of crossings of a differentiable curve. The formula for the speed of specular points is a consequence of a generalization of Rice formula to the bidimensional case, obtained by using the “co-aire” formula ([7]). The mean number of twinkles can be rigorously computed by using the formula that counts the number of solutions of a system of random equations obtained by Cabaña ([7]) and more recently by Azaïs and Wschebor ([3]). In this work we propose to go beyond, thanks to our method ([12], [14]) which allows to explain well the Longuet-Higgins discovery. We look for the representation into the Itô-Wiener Chaos (Hermite expansion) for the number of specular points and for the speed of these points. We are interested in obtaining variance results or Central Limit theorems (CLT) that allow to build confidence intervals, having in mind applications to sea modelling. We also start the study of twinkles, by providing its mean number.

Recently there has been a renewed interest in applying the generalizations of the Rice formulae to explain some difficult phenomena in optics (see [4] and [5]). The present work could be considered as a first intent to tackle mathematically such type of problems.

The paper is organized as follows. In section 2, we get the expansion into Hermite polynomials for the

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^o2000 *Mathematics Subject Classification*: Primary 60G15; Secondary 60G10, 60G60, 60G70; Thirdly 78A97

Keywords: crossings, CLT, Gaussian fields, Hermite polynomials, level curve, specular point, twinkle.

number of crossings of a differentiable curve by another method than the one proposed by Slud, who obtained the same type of results ([20]). Indeed, instead of approaching the number of crossings of a differentiable curve by the number of crossings of a polygonal Gaussian process, we approach the process then the crossings by a smoothed-by-convolution process. Then the asymptotic behavior of this expansion is described, according to the form of the level curve, namely if the curve is periodic or linear. Section 3 is devoted to the study of the specular points, when considering on one hand the speed of specular points, in particular their asymptotic distribution, on the other hand the number of twinkles.

2 Asymptotic behavior of the curve-crossings number

Hermite polynomial expansion (or Multiple Wiener-Itô Integrals) may be a powerful tool to represent and to study nonlinear functionals of stationary Gaussian processes. That is why we will first provide the Hermite expansion for the curve-crossings number, in order to obtain in an easier way its asymptotic behavior, depending on the curve's type.

2.1 Hermite expansion for the curve-crossings number

Let $X = \{X_t, t \in \mathbb{R}\}$ be a centered stationary Gaussian process, variance one, with twice differentiable correlation function r given by $r(\tau) = \int_{-\infty}^{\infty} e^{i\tau\lambda} F(d\lambda)$, where F is the spectral measure.

As in Cramér & Leadbetter ([9]), let us define the number of crossings of a differentiable function ψ by the process X , as the random variable

$$N_t^X(\psi) = \text{card}\{s \leq t : X_s = \psi_s\}.$$

$N_t^X(\psi)$ can also be seen as the number $N_t^Y(0)$ of zero crossings by the non-stationary (but stationary in the sense of the covariance) Gaussian process $Y = \{Y_s, s \in \mathbb{R}\}$ defined by $Y_s := X_s - \psi_s$, i.e. $N_t^X(\psi) = N_t^Y(0)$.

As in [15], suppose that r satisfies on $[0, \delta]$, $\delta > 0$,

$$r(\tau) = 1 + \frac{r''(0)}{2}\tau^2 + \theta(\tau), \quad \text{with } \theta(\tau) > 0, \quad \frac{\theta(\tau)}{\tau^2} \rightarrow 0, \quad \frac{\theta'(\tau)}{\tau} \rightarrow 0, \quad \theta''(\tau) \rightarrow 0, \quad \text{as } \tau \rightarrow 0, \quad (1)$$

that the nonnegative function L defined by $L(\tau) := \frac{\theta''(\tau)}{\tau} = \frac{r''(\tau) - r''(0)}{\tau}$, $\tau > 0$, satisfies the Geman condition:

$$\exists \delta > 0, \quad L \in L^1([0, \delta]) \quad (2)$$

and assume that the modulus of continuity of $\dot{\psi}$ defined by $\gamma(\tau) := \sup_{u \in [0, t]} \sup_{|s| \leq \tau} |\dot{\psi}(u+s) - \dot{\psi}(u)|$ is such

that

$$\int_0^\delta \frac{\gamma(s)}{s} ds < \infty. \quad (3)$$

These conditions imply that $N_t^X(\psi)$ has a finite variance, as it has been proved by the authors in [15].

Let $(H_n)_{n \geq 0}$ be the Hermite polynomials, defined by $H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}$, which constitute a complete orthogonal system in the Hilbert space $L^2(\mathbb{R}, \varphi(u)du)$, φ denoting the standard normal density. Let $a_l(m)$ be the coefficients in this basis of the function $|\cdot - m|$.

Proposition 1 *Under the conditions (1), (2) and (3), the number of crossings $N_t^X(\psi)$ of the function ψ by the process X has the following expansion in $L^2(\Omega)$:*

$$N_t^X(\psi) = \sqrt{-r''(0)} \sum_{q=0}^{\infty} \int_0^t \sum_{l=0}^q \frac{H_{q-l}(\psi_s) \varphi(\psi_s)}{(q-l)!} a_l \left(\frac{\dot{\psi}_s}{\sqrt{-r''(0)}} \right) H_{q-l}(X_s) H_l \left(\frac{\dot{X}_s}{\sqrt{-r''(0)}} \right) ds. \quad (4)$$

This type of results has been considered for the first time by Slud in [20]; he provided a MWI expansion of $N_t^X(\psi)$ by approximating the crossings of the process X by those of the discrete version of X .

Our approach is completely different and generalizes the one introduced in [12]; it consists in approaching the process, then the crossings, by using a smoothed-by-convolution process, for which the expansion can be readily obtained.

Our method, maybe more intuitive, revealed itself to be quite useful in different applications, as for the study on the number of maxima and the one on the length of a level curve of a random field (see [14]). Here it will be applied to other situations, as for the study of the dynamic of specular points.

Remark. Proposition 1 recovers the result under the only hypothesis that the number of crossings belongs to $L^2(\Omega)$, thus completes the results of [12] for a fixed level $\psi_t \equiv x$ and for many functions ψ .

Nevertheless, we do not know whether, under the Geman condition, $\int_0^\delta \frac{\gamma(s)}{s} ds = \infty$ implies that $N_t^X(\psi)$ does not belong to $L^2(\Omega)$.

Proof: Since the proof is based on the approach developed in [12] and on technical tools provided in [15], only the main ideas will be given. The proof can be sketched in four main steps.

- *Smooth approximation of X_t .*

Consider a twice differentiable even density function ϕ with support in $[-1, 1]$ and define the continuous twice differentiable smoothed process as

$$X_t^\varepsilon = \frac{1}{\varepsilon} \int_{-\infty}^{\infty} \phi\left(\frac{t-u}{\varepsilon}\right) X_u du.$$

Let B be a complex Brownian motion such that $E\left[B(d\lambda)\overline{B(d\lambda')}\right] = F(d\lambda)\mathbf{1}_{(\lambda=\lambda')}$.

We can write

$$X_t = \int_{-\infty}^{\infty} e^{it\lambda} B(d\lambda) \quad \text{and} \quad X_t^\varepsilon = \int_{-\infty}^{\infty} e^{it\lambda} \hat{\phi}(\varepsilon\lambda) B(d\lambda),$$

where $\hat{\phi}$ denotes the Fourier transform of ϕ .

The correlation function of the process X_t^ε given by $r_\varepsilon(\tau) = \int_{-\infty}^{\infty} e^{i\tau\lambda} |\hat{\phi}(\varepsilon\lambda)|^2 F(d\lambda)$ satisfies

$$r_\varepsilon(0) \xrightarrow{\varepsilon \rightarrow 0} r(0) = \int_{-\infty}^{\infty} F(d\lambda) = 1, \quad (5)$$

and

$$r_\varepsilon(\tau) = r_\varepsilon(0) + \frac{r_\varepsilon''(0)}{2} \tau^2 + \theta_{X^\varepsilon}(\tau),$$

with θ_{X^ε} satisfying the same conditions as θ in (1).

Let us denote by $\sigma_\varepsilon := \sqrt{r_\varepsilon(0)}$ and $\eta_\varepsilon := \sqrt{-r_\varepsilon''(0)}$, and let us define the normalized process $\tilde{X}_t^\varepsilon = \frac{X_t^\varepsilon}{\sigma_\varepsilon}$ of variance 1 and correlation function ρ_ε , such that

$$\rho_\varepsilon(\tau) = 1 + \frac{\rho_\varepsilon''(0)}{2} \tau^2 + \theta_\varepsilon(\tau),$$

where $\rho_\varepsilon''(\tau) = \frac{r_\varepsilon''(\tau)}{r_\varepsilon(0)}$ and $\theta_\varepsilon(\tau) = \frac{\theta_{X^\varepsilon}(\tau)}{r_\varepsilon(0)}$.

Note that the smoothed process X_t^ε has the fourth derivative of its correlation function r_ε finite in 0, and so does \tilde{X}_t^ε , since these processes are twice differentiable:

$$r_\varepsilon^{(iv)}(0) < \infty \quad \text{and} \quad \rho_\varepsilon^{(iv)}(0) < \infty,$$

Consider also the smoothed function $\psi_t^\varepsilon = \frac{1}{\varepsilon} \int_{-\infty}^{\infty} \phi\left(\frac{t-u}{\varepsilon}\right) \psi_u du$, and let us write $\tilde{\psi}_t^\varepsilon = \frac{\psi_t^\varepsilon}{\sigma_\varepsilon}$.

The number of $\tilde{\psi}_t^\varepsilon$ - crossings by \tilde{X}_t^ε is the number of ψ_t^ε - crossings by X_t^ε :

$$N_{\varepsilon,t}^\psi = \text{card}\{s \leq t : \tilde{X}_s^\varepsilon = \tilde{\psi}_s^\varepsilon\} = \text{card}\{s \leq t : X_s^\varepsilon = \psi_s^\varepsilon\}.$$

- *Hermite expansion for $N_{\varepsilon, t}^{\psi}$.*

As in [12], let us introduce

$$N_{t, h}^{X^\varepsilon}(\psi) := \frac{1}{h} \int_0^t \varphi \left(\frac{\tilde{X}_s^\varepsilon - \tilde{\psi}_s^\varepsilon}{h} \right) \frac{|\dot{X}_s^\varepsilon - \dot{\psi}_s^\varepsilon|}{\sigma_\varepsilon} ds = \frac{\eta_\varepsilon}{h\sigma_\varepsilon} \int_0^t \varphi \left(\frac{\tilde{X}_s^\varepsilon - \tilde{\psi}_s^\varepsilon}{h} \right) \left| \frac{\dot{X}_s^\varepsilon}{\eta_\varepsilon} - \frac{\dot{\psi}_s^\varepsilon}{\eta_\varepsilon} \right| ds, \quad (6)$$

where φ denotes the standard Gaussian density and $h > 0$.

By using the same approach as in [12], we can write in $L^2(\Omega)$,

$$N_{t, h}^{X^\varepsilon}(\psi) = \frac{\eta_\varepsilon}{\sigma_\varepsilon} \sum_{q=0}^{\infty} \int_0^t \left[\sum_{l=0}^q c_{q-l}(\tilde{\psi}_s^\varepsilon, h) a_l \left(\frac{\dot{\psi}_s^\varepsilon}{\eta_\varepsilon} \right) H_{q-l}(\tilde{X}_s^\varepsilon) H_l \left(\frac{\dot{X}_s^\varepsilon}{\eta_\varepsilon} \right) \right] ds, \quad (7)$$

where the Hermite coefficients $c_k(y, h)$ of the function $\frac{1}{h} \varphi \left(\frac{\cdot - y}{h} \right)$ are given by

$$c_k(y, h) = \frac{1}{k!} \int_{-\infty}^{\infty} \varphi(v) H_k(y - hv) \varphi(y - hv) dv \xrightarrow{h \rightarrow 0} \frac{H_k(y) \varphi(y)}{k!}, \quad (8)$$

and where the Hermite coefficients $a_l(m)$ of the function $|\cdot - m|$ are given by

$$\begin{aligned} a_0(m) &= \mathbb{E}|Z - m|, \quad \text{where } Z \text{ is a standard Gaussian r.v.,} \\ &= m [2\Phi(m) - 1] + \sqrt{\frac{2}{\pi}} e^{-\frac{m^2}{2}} = \sqrt{\frac{2}{\pi}} \left[1 + \int_0^m \int_0^u e^{-\frac{v^2}{2}} dv du \right], \end{aligned} \quad (9)$$

$$\text{and } a_l(m) = (-1)^{l+1} \sqrt{\frac{2}{\pi}} \frac{1}{l!} \int_0^1 e^{-\frac{m^2 y^2}{2}} H_l(-my) y^{l-2} dy, \quad l \geq 1. \quad (10)$$

So, as in [12], by taking $h \rightarrow 0$ in (7), we deduce the expansion of $N_{\varepsilon, t}^{\psi}$:

$$N_{\varepsilon, t}^{\psi} = \frac{\eta_\varepsilon}{\sigma_\varepsilon} \sum_{q=0}^{\infty} \int_0^t \sum_{l=0}^q \frac{H_{q-l} \left(\frac{\psi_s^\varepsilon}{\sigma_\varepsilon} \right) \varphi \left(\frac{\dot{\psi}_s^\varepsilon}{\sigma_\varepsilon} \right)}{(q-l)!} a_l \left(\frac{\dot{\psi}_s^\varepsilon}{\eta_\varepsilon} \right) H_{q-l} \left(\frac{X_s^\varepsilon}{\sigma_\varepsilon} \right) H_l \left(\frac{\dot{X}_s^\varepsilon}{\eta_\varepsilon} \right) ds. \quad (11)$$

- *Convergence in $L^2(\Omega)$ of $N_{\varepsilon, t}^{\psi}$ to $N_t^X(\psi)$ as $\varepsilon \rightarrow 0$.*

We start proving that $\mathbb{E}[N_{\varepsilon, t}^{\psi}]^2 \xrightarrow{\varepsilon \rightarrow 0} \mathbb{E}[N_t^X(\psi)]^2$.

Since we have, by using (11) and the uniform convergence,

$$\mathbb{E}[N_{\varepsilon, t}^{\psi}] = \frac{\eta_\varepsilon}{\sigma_\varepsilon} \int_0^t \varphi \left(\frac{\psi_s^\varepsilon}{\sigma_\varepsilon} \right) a_0 \left(\frac{\dot{\psi}_s^\varepsilon}{\eta_\varepsilon} \right) ds \xrightarrow{\varepsilon \rightarrow 0} \eta \int_0^t \varphi(\psi_s) a_0 \left(\frac{\dot{\psi}_s}{\eta} \right) ds = \mathbb{E}[N_t^X(\psi)],$$

we only need to prove the convergence of the second factorial moment.

The second factorial moment of the number of $\tilde{\psi}^\varepsilon$ -crossings by \tilde{X}^ε can be deduced from the one of the number of zero-crossings by Y (see Cramér & Leadbetter, [9], p.209), namely

$$M_2^{\varepsilon, \psi} = \int_0^t \int_0^t dt_1 dt_2 \int_{\mathbb{R}^2} \left| \dot{x}_1 - \dot{\tilde{\psi}}_{t_1}^\varepsilon \right| \left| \dot{x}_2 - \dot{\tilde{\psi}}_{t_2}^\varepsilon \right| p_{t_1, t_2}^\varepsilon \left(\tilde{\psi}_{t_1}^\varepsilon, \dot{x}_1, \tilde{\psi}_{t_2}^\varepsilon, \dot{x}_2 \right) d\dot{x}_1 d\dot{x}_2, \quad (12)$$

where $p_{t_1, t_2}^\varepsilon(x_1, \dot{x}_1, x_2, \dot{x}_2)$ is the joint density of the vector $(\tilde{X}_{t_1}^\varepsilon, \dot{\tilde{X}}_{t_1}^\varepsilon, \tilde{X}_{t_2}^\varepsilon, \dot{\tilde{X}}_{t_2}^\varepsilon)$.

The formula holds whether $M_2^{\varepsilon, \psi}$ is finite or not.

It can also be expressed as

$$M_2^{\varepsilon, \psi} = 2 \int_0^t \int_{t_1}^t I_\varepsilon(t_1, t_2) dt_2 dt_1 = 2 \int_0^t \int_0^{t-t_1} I_\varepsilon(t_1, t_1 + \tau) d\tau dt_1,$$

where

$$\begin{aligned} I_\varepsilon(t_1, t_2) &:= \int_{\mathbb{R}^2} |\dot{x}_1 - \dot{\tilde{\psi}}_{t_1}^\varepsilon| |\dot{x}_2 - \dot{\tilde{\psi}}_{t_2}^\varepsilon| p_{t_1, t_2}^\varepsilon(\tilde{\psi}_{t_1}^\varepsilon, \dot{x}_1, \tilde{\psi}_{t_2}^\varepsilon, \dot{x}_2) d\dot{x}_1 d\dot{x}_2 \\ &= p_{t_1, t_2}^\varepsilon(\tilde{\psi}_{t_1}^\varepsilon, \tilde{\psi}_{t_2}^\varepsilon) \mathbb{E} \left[|\dot{X}_{t_1}^\varepsilon - \dot{\tilde{\psi}}_{t_1}^\varepsilon| |\dot{X}_{t_2}^\varepsilon - \dot{\tilde{\psi}}_{t_2}^\varepsilon| \mid \tilde{X}_{t_1}^\varepsilon = \tilde{\psi}_{t_1}^\varepsilon, \tilde{X}_{t_2}^\varepsilon = \tilde{\psi}_{t_2}^\varepsilon \right], \end{aligned}$$

where $p_{t_1, t_2}^\varepsilon(x_1, x_2)$ is the joint density of the Gaussian vector $(\tilde{X}_{t_1}^\varepsilon, \tilde{X}_{t_2}^\varepsilon)$.

By using the following regression model:

$$(R) \quad \begin{cases} \dot{X}_{t_1}^\varepsilon = \zeta_\varepsilon + \alpha_1 \tilde{X}_{t_1}^\varepsilon + \alpha_2 \tilde{X}_{t_2}^\varepsilon \\ \dot{X}_{t_2}^\varepsilon = \zeta_\varepsilon^* - \beta_1 \tilde{X}_{t_1}^\varepsilon - \beta_2 \tilde{X}_{t_2}^\varepsilon \end{cases}$$

where $\alpha_1(t_2 - t_1) = \beta_2(t_2 - t_1) = \frac{\rho_\varepsilon'(t_2 - t_1)\rho_\varepsilon(t_2 - t_1)}{1 - \rho_\varepsilon^2(t_2 - t_1)}$, $\alpha_2(t_2 - t_1) = \beta_1(t_2 - t_1) = -\frac{\rho_\varepsilon'(t_2 - t_1)}{1 - \rho_\varepsilon^2(t_2 - t_1)}$,

and $(\zeta_\varepsilon, \zeta_\varepsilon^*)$ is jointly Gaussian such that $\sigma_\varepsilon^2(t_2 - t_1) = \text{Var}(\zeta_\varepsilon) = \text{Var}(\zeta_\varepsilon^*) = -\rho_\varepsilon''(0) - \frac{(\rho_\varepsilon'(t_2 - t_1))^2}{1 - \rho_\varepsilon^2(t_2 - t_1)}$

and $\text{Cov}(\zeta_\varepsilon, \zeta_\varepsilon^*) = -\rho_\varepsilon''(t_2 - t_1) - \frac{(\rho_\varepsilon'(t_2 - t_1))^2 \rho_\varepsilon(t_2 - t_1)}{1 - \rho_\varepsilon^2(t_2 - t_1)}$,

we obtain

$$I_\varepsilon(t_1, t_1 + \tau) = p_\tau^\varepsilon(\tilde{\psi}_{t_1}^\varepsilon, \tilde{\psi}_{t_1 + \tau}^\varepsilon) \mathbb{E} \left[|\zeta_\varepsilon + \alpha_1(\tau) \tilde{\psi}_{t_1}^\varepsilon + \alpha_2(\tau) \tilde{\psi}_{t_1 + \tau}^\varepsilon - \dot{\tilde{\psi}}_{t_1}^\varepsilon| |\zeta_\varepsilon^* - \alpha_2(\tau) \tilde{\psi}_{t_1}^\varepsilon - \alpha_1(\tau) \tilde{\psi}_{t_1 + \tau}^\varepsilon - \dot{\tilde{\psi}}_{t_1 + \tau}^\varepsilon| \right].$$

For each $\tau > 0$ and by using the uniform convergence, we have

$$\lim_{\varepsilon \rightarrow 0} I_\varepsilon(t_1, t_1 + \tau) = I(t_1, t_1 + \tau) := \int_{\mathbb{R}^2} |\dot{x}_1 - \dot{\psi}_{t_1}| |\dot{x}_2 - \dot{\psi}_{t_1 + \tau}| p_{t_1, t_1 + \tau}(\psi_{t_1}, \dot{x}_1, \psi_{t_1 + \tau}, \dot{x}_2) d\dot{x}_1 d\dot{x}_2,$$

and thus $\lim_{\varepsilon \rightarrow 0} \int_0^t \int_\delta^t I_\varepsilon(t_1, t_1 + \tau) d\tau dt_1 = \int_0^t \int_\delta^t I(t_1, t_1 + \tau) d\tau dt_1$.

We now have to prove that

$$\limsup_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \int_0^t \int_0^\delta I_\varepsilon(t_1, t_1 + \tau) d\tau dt_1 = 0.$$

We can write

$$\int_0^t \int_0^\delta I_\varepsilon(t_1, t_1 + \tau) d\tau dt_1 \leq \sum_{i=1}^3 \sum_{j=1}^4 \int_0^t \int_0^\delta p_\tau^\varepsilon(\tilde{\psi}_{t_1}^\varepsilon, \tilde{\psi}_{t_1 + \tau}^\varepsilon) \mathbb{E}[I_i J_j] d\tau dt_1,$$

where

$$I_1 = \left| \zeta_\varepsilon - \frac{\rho_\varepsilon'(\tau)}{1 + \rho_\varepsilon(\tau)} \tilde{\psi}_{t_1}^\varepsilon \right|, \quad I_2 = \left| \left(1 + \frac{\tau \rho_\varepsilon'(\tau)}{1 - \rho_\varepsilon^2(\tau)} \right) \dot{\tilde{\psi}}_{t_1 + \eta\tau}^\varepsilon \right|, \quad I_3 = |\dot{\tilde{\psi}}_{t_1 + \eta\tau}^\varepsilon - \dot{\tilde{\psi}}_{t_1}^\varepsilon|$$

and

$$\begin{aligned} J_1 &= \left| \zeta_\varepsilon^* + \frac{\rho_\varepsilon'(\tau)}{1 + \rho_\varepsilon(\tau)} \tilde{\psi}_{t_1}^\varepsilon \right|, & J_2 &= \left| \rho_\varepsilon(\tau) \left(1 + \frac{\tau \rho_\varepsilon'(\tau)}{1 - \rho_\varepsilon^2(\tau)} \right) \dot{\tilde{\psi}}_{t_1 + \nu\tau}^\varepsilon \right|, \\ J_3 &= |(\rho_\varepsilon(\tau) - 1) \dot{\tilde{\psi}}_{t_1 + \nu\tau}^\varepsilon|, & J_4 &= |\dot{\tilde{\psi}}_{t_1 + \nu\tau}^\varepsilon - \dot{\tilde{\psi}}_{t_1 + \tau}^\varepsilon|, \end{aligned}$$

with $0 \leq \eta, \nu \leq 1$.

Let C be some positive constant which may vary from equation to equation.

First we have

$$\int_0^t \int_0^\delta p_\tau^\varepsilon(\psi_{t_1}^\varepsilon, \psi_{t_1 + \tau}^\varepsilon) \mathbb{E}[I_1 J_1] d\tau dt_1 \leq C \int_0^\delta f_\varepsilon(\tau) \frac{1}{\tau} \left(\sigma_\varepsilon^2(\tau) + \left(\frac{\rho_\varepsilon'(\tau)}{1 + \rho_\varepsilon(\tau)} \right)^2 \right) d\tau, \quad (13)$$

where $f_\varepsilon(\tau) := \frac{\tau}{\sqrt{1 - \rho_\varepsilon(\tau)}}$.

Note that

$$f_\varepsilon(\tau) = \left(\frac{-\rho_\varepsilon''(0)}{2} - \frac{\theta_\varepsilon(\tau)}{\tau^2} \right)^{-1/2} \xrightarrow{\varepsilon \rightarrow 0} \frac{\tau}{\sqrt{1 - r(\tau)}}.$$

By using similar techniques as the ones introduced to prove the lemmas in [15], we can prove that f_ε is uniformly bounded when $\varepsilon \rightarrow 0$ in $[0, \delta]$, $\delta > 0$,

$$f_\varepsilon(\tau) \leq C, \quad (14)$$

that

$$\frac{\sigma_\varepsilon^2(\tau)}{\tau} \leq \mathbf{C} \frac{\tau \theta_\varepsilon'(\tau) - \theta_\varepsilon(\tau)}{\tau^3} \leq \mathbf{C} L(\tau), \quad (15)$$

from which we deduce under the Geman condition, by applying (14) and the Dominated Convergence Theorem, that

$$\lim_{\varepsilon \rightarrow 0} \int_0^\delta \frac{\sigma_\varepsilon^2(\tau)}{\sqrt{1 - \rho_\varepsilon(\tau)}} d\tau = \int_0^\delta \frac{\sigma^2(\tau)}{\sqrt{1 - r(\tau)}} d\tau, \quad \text{with } \sigma^2(\tau) := -r''(0) - \frac{r'{}^2(\tau)}{1 - r^2(\tau)}, \quad (16)$$

which tends to zero as $\delta \rightarrow 0$.

We can also prove that

$$\lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \int_0^\delta \frac{1}{\sqrt{1 - \rho_\varepsilon(\tau)}} \left(\frac{\rho_\varepsilon'(\tau)}{1 + \rho_\varepsilon(\tau)} \right)^2 d\tau = 0, \quad (17)$$

by using the uniform bound (14) for $f_\varepsilon(\tau)$, and the fact that $\tau \left(\frac{\rho_\varepsilon'(\tau)}{\tau} \right)^2 \leq \mathbf{C} \tau (r''(0))^2$.

We can conclude by combining (13), (14), (16) and (17).

Now let us study $\int_0^t \int_0^\delta p_\tau^\varepsilon(\tilde{\psi}_{t_1}^\varepsilon, \tilde{\psi}_{t_1+\tau}^\varepsilon) \mathbb{E}[I_1 J_2] d\tau dt_1$.

We have

$$\int_0^t \int_0^\delta p_\tau^\varepsilon(\tilde{\psi}_{t_1}^\varepsilon, \tilde{\psi}_{t_1+\tau}^\varepsilon) \mathbb{E}[I_1 J_2] d\tau dt_1 \leq \mathbf{C} \left\{ \int_0^\delta f_\varepsilon(\tau) d\tau + \int_0^\delta (f_\varepsilon(\tau))^{3/2} \left(\frac{\theta''(\tau)}{\tau} + \frac{\theta(\tau)}{\tau^3} \right) d\tau \right\}.$$

Again, the uniform bound (14) for $f_\varepsilon(\tau)$ and the Geman condition allow to conclude that

$$\lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \int_0^t \int_0^\delta p_\tau^\varepsilon(\tilde{\psi}_{t_1}^\varepsilon, \tilde{\psi}_{t_1+\tau}^\varepsilon) \mathbb{E}[I_1 J_2] d\tau dt_1 = 0.$$

The term $\int_0^t \int_0^\delta p_\tau^\varepsilon(\tilde{\psi}_{t_1}^\varepsilon, \tilde{\psi}_{t_1+\tau}^\varepsilon) \mathbb{E}[I_2 J_1] d\tau dt_1$ can be bounded in the same way.

Now, since $p_\tau^\varepsilon(\tilde{\psi}_{t_1}^\varepsilon, \tilde{\psi}_{t_1+\tau}^\varepsilon) J_3 \leq \mathbf{C}(\sqrt{1 - \rho_\varepsilon(\tau)})$, we obtain

$$\sum_{i=1}^3 \lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \int_0^t \int_0^\delta p_\tau^\varepsilon(\tilde{\psi}_{t_1}^\varepsilon, \tilde{\psi}_{t_1+\tau}^\varepsilon) \mathbb{E}[I_i J_3] d\tau dt_1 = 0.$$

Finally, the fact that $|\dot{\tilde{\psi}}_{t+\tau}^\varepsilon - \dot{\tilde{\psi}}_t^\varepsilon| \leq \mathbf{C}\gamma(\tau)$ and $p_\tau^\varepsilon(\tilde{\psi}_{t_1}^\varepsilon, \tilde{\psi}_{t_1+\tau}^\varepsilon) I_3 \leq \mathbf{C} f_\varepsilon(\tau) \frac{\gamma(\tau)}{\tau}$ implies

$$\sum_{i=1}^3 \lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \int_0^t \int_0^\delta p_\tau^\varepsilon(\tilde{\psi}_{t_1}^\varepsilon, \tilde{\psi}_{t_1+\tau}^\varepsilon) \mathbb{E}[I_i J_4] d\tau dt_1 = 0.$$

Thus we have proved that

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}[N_{\varepsilon, t}^\psi]^2 = \mathbb{E}[N_t^X(\psi)]^2. \quad (18)$$

Since it can be easily proved, in the same way as in [12] (p.245), that

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}[N_t^X(\psi)N_{\varepsilon,t}^\psi] = \mathbb{E}[N_t^X(\psi)]^2, \quad (19)$$

we deduce from (18) and (19) that $\|N_{\varepsilon,t}^\psi - N_t^X(\psi)\|_{L^2(\Omega)} \xrightarrow{\varepsilon \rightarrow 0} 0$.

• *Conclusion.*

Since, by Fatou lemma, for each Q positive,

$$\begin{aligned} & \sum_{q=0}^Q \mathbb{E} \left[\int_0^t \sum_{l=0}^q \frac{H_{q-l}(\psi_s)\varphi(\psi_s)}{(q-l)!} a_l \left(\frac{\dot{\psi}_s}{\sqrt{-r''(0)}} \right) H_{q-l}(X_s) H_l \left(\frac{\dot{X}_s}{\sqrt{-r''(0)}} \right) ds \right]^2 \\ & \leq \lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\sqrt{\frac{-r''_\varepsilon(0)}{r_\varepsilon(0)}} N_{\varepsilon,t}^\psi \right]^2 = \mathbb{E}[\sqrt{-r''(0)} N_t^X(\psi)]^2, \end{aligned}$$

the formal expansion in the right hand side of (4) defines a random variable in $L^2(\Omega)$, denoted by \mathcal{N}_t . Moreover

$$\mathbb{E} \left[\frac{1}{\eta} N_t^X(\psi) - \mathcal{N}_t \right]^2 \leq 2 \left(\mathbb{E} \left[\frac{N_t^X(\psi)}{\sqrt{-r''(0)}} - \sqrt{\frac{r_\varepsilon(0)}{-r''_\varepsilon(0)}} N_{\varepsilon,t}^\psi \right]^2 + \mathbb{E} \left[\sqrt{\frac{r_\varepsilon(0)}{-r''_\varepsilon(0)}} N_{\varepsilon,t}^\psi - \mathcal{N}_t \right]^2 \right).$$

The first term of the right hand side, tends to zero as shown in the previous step. For the second term, an argument of continuity for the projections into the chaos entails the result (see [12]). \square

Note that we considered the case where $r(0) = 1$; it can be easily generalized to a process X satisfying $\mathbb{E}[X_t^2] = r(0)$ when noticing that $N_t^X(\psi) = N_{\frac{t}{\sqrt{r(0)}}}^{Y_t}(0)$, where $Y_s = \frac{1}{\sqrt{r(0)}} \left(X(s\sqrt{r(0)}) - \psi(s\sqrt{r(0)}) \right)$; so the previous expansion becomes for such general process X

$$N_t^X(\psi) = \sqrt{\frac{-r''(0)}{r(0)}} \sum_{q=0}^{\infty} \int_0^t \sum_{l=0}^q \frac{H_{q-l} \left(\frac{\psi_s}{\sqrt{r(0)}} \right) \varphi \left(\frac{\psi_s}{\sqrt{r(0)}} \right)}{(q-l)!} a_l \left(\frac{\dot{\psi}_s}{\sqrt{-r''(0)}} \right) H_{q-l} \left(\frac{X_s}{\sqrt{r(0)}} \right) H_l \left(\frac{\dot{X}_s}{\sqrt{-r''(0)}} \right) ds.$$

2.2 Asymptotic behavior according to the level curve's type

In this section we shall describe the asymptotic behavior of the expansion of the number of curve crossings given in the previous section according to the form of the curve ψ .

On one hand, if the function ψ is periodic, then a CLT can be deduced, since we may say that we remain in an ergodic situation similar to the fixed barrier problem.

On the other hand, if ψ is defined by $\psi_s := ks$, we shall prove that the number of ψ -crossings of X belongs to $L^2(\Omega)$ when $s \rightarrow \infty$.

An example can be mentioned in each of these two situations.

In the periodic case, a cosine barrier ψ can be considered, e.g. $\psi(x) = \sqrt{2A} \cos(\omega x)$, as first studied by Rice, then fully developed by Cramer & Leadbetter (see [9]).

The second case encounters the number of specular points of a curve.

In what follows, we shall study both situations, providing a CLT for the number of crossings of a periodic curve ψ by a stationary Gaussian process, then studying the asymptotical behavior of the second moment of the number of specular points as $x \rightarrow \infty$.

2.2.1 Periodic curve

The expansion of $N_t^X(\psi)$ can be written as

$$N_t^X(\psi) = \sqrt{\frac{-r''(0)}{r(0)}} \sum_{q=0}^{\infty} \int_0^t \sum_{l=0}^q \kappa_{ql}(s) H_{q-l} \left(\frac{X_s}{\sqrt{r(0)}} \right) H_l \left(\frac{\dot{X}_s}{\sqrt{-r''(0)}} \right) ds,$$

$$\text{where } \kappa_{ql}(s) := \frac{H_{q-l}\left(\frac{\psi_s}{\sqrt{r(0)}}\right) \varphi\left(\frac{\psi_s}{\sqrt{r(0)}}\right)}{(q-l)!} a_l\left(\frac{\dot{\psi}_s}{\sqrt{-r''(0)}}\right).$$

Suppose that the function ψ is p -periodic, i.e. $\psi_{s+p} = \psi_s$.

Note that $\kappa_{ql}(s)$ is also a p -periodic function, for all q and l .

We are interested in the asymptotic behavior of the r.v. $N_t^X(\psi) - \mathbb{E}N_t^X(\psi)$ as $t \rightarrow \infty$.

We will consider the case $t = np$, from which the general case will be easily deduced.

Theorem 1 *Let X and ψ both satisfying the hypothesis of Proposition 1.*

Let $\chi(s) := \left(|r(u)| + \frac{|r'(u)|}{\sqrt{-r''(0)}}, \frac{|r'(u)|}{\sqrt{-r''(0)}} + \frac{|r''(u)|}{-r''(0)} \right)$. If

$$\int_0^\infty \chi(s) ds < \infty \tag{20}$$

then

$$\mathcal{L}\left(\frac{N_{np}^X(\psi) - \mathbb{E}N_{np}^X(\psi)}{\sqrt{n}}\right) \xrightarrow{n \rightarrow \infty} \mathcal{N}\left(0, \sum_{q=1}^\infty \sigma^2(q)\right),$$

where \implies stands for the weak convergence of measures and $\sigma^2(q)$ is defined in (36) below. The same result holds when replacing np by t .

Proof. Let us write

$$N_t^X(\psi) - \mathbb{E}N_t^X(\psi) = \sqrt{\frac{-r''(0)}{r(0)}} \sum_{q=1}^\infty \int_0^t \sum_{l=0}^q \kappa_{ql}(s) H_{q-l}\left(\frac{X_s}{\sqrt{r(0)}}\right) H_l\left(\frac{\dot{X}_s}{\sqrt{-r''(0)}}\right) ds,$$

and let define

$$Y_{Q,n}^k := \sum_{q=Q+1}^\infty \int_{(k-1)p}^{kp} \sum_{l=0}^q \kappa_{ql}(s) H_{q-l}\left(\frac{X_s}{\sqrt{r(0)}}\right) H_l\left(\frac{\dot{X}_s}{\sqrt{-r''(0)}}\right) ds,$$

which by periodicity can also be written as

$$Y_{Q,n}^k = \sum_{q=Q+1}^\infty \int_0^p \sum_{l=0}^q \kappa_{ql}(u) H_{q-l}\left(\frac{X_{u+(k-1)p}}{\sqrt{r(0)}}\right) H_l\left(\frac{\dot{X}_{u+(k-1)p}}{\sqrt{-r''(0)}}\right) du.$$

Thus we have

$$\sum_{k=1}^n Y_{Q,n}^k = \sum_{q=Q+1}^\infty \int_0^{np} \sum_{l=0}^q \kappa_{ql}(s) H_{q-l}\left(\frac{X_s}{\sqrt{r(0)}}\right) H_l\left(\frac{\dot{X}_s}{\sqrt{-r''(0)}}\right) ds.$$

First step - Let us prove that

$$\lim_{Q \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} \left[\left(\sum_{k=1}^n Y_{Q,n}^k \right)^2 \right] = 0. \tag{21}$$

Let us compute the second moment of $\sum_{k=1}^n Y_{Q,n}^k$, which can be written as

$$\mathbb{E} \left[\left(\sum_{k=1}^n Y_{Q,n}^k \right)^2 \right] = \sum_{k=1}^n \mathbb{E} [(Y_{Q,n}^k)^2] + \sum_{k \neq m} \mathbb{E} [Y_{Q,n}^k Y_{Q,n}^m]. \tag{22}$$

First we have

$$\mathbb{E}[(Y_{Q,n}^k)^2] = \sum_{q=Q+1}^{\infty} \mathbb{E} \left(\left[\int_0^p \sum_{l=0}^q \kappa_{ql}(u) H_{q-l} \left(\frac{X_u}{\sqrt{r(0)}} \right) H_l \left(\frac{\dot{X}_u}{\sqrt{-r''(0)}} \right) du \right]^2 \right) = \mathbb{E}[(Y_{Q,n}^1)^2],$$

by using the orthogonality of the chaos, the stationarity and the p -periodicity. Therefore we obtain

$$\frac{1}{n} \sum_{k=1}^n \mathbb{E}[(Y_{Q,n}^k)^2] = \mathbb{E}[(Y_{Q,n}^1)^2] \rightarrow 0 \text{ as } Q \rightarrow \infty. \quad (23)$$

Suppose now that $k > m$ and write

$$\mathbb{E}(Y_{Q,n}^k Y_{Q,n}^m) = \sum_{q=Q+1}^{\infty} \int_0^p \int_0^p \sum_{l_1=0}^q \sum_{l_2=0}^q \kappa_{ql_1}(u) \kappa_{ql_2}(v) I_{ql_1 l_2}(k-m, u, v) dudv,$$

where $I_{ql_1 l_2}(k-m, u, v)$ is defined for $k > m$ by

$$I_{ql_1 l_2}(k-m, u, v) = \mathbb{E} \left[H_{q-l_1} \left(\frac{X_{u+(k-m)p}}{\sqrt{r(0)}} \right) H_{l_1} \left(\frac{\dot{X}_{u+(k-m)p}}{\sqrt{-r''(0)}} \right) H_{q-l_2} \left(\frac{X_v}{\sqrt{r(0)}} \right) H_{l_2} \left(\frac{\dot{X}_v}{\sqrt{-r''(0)}} \right) \right]. \quad (24)$$

Thus we have

$$\begin{aligned} \sum_{k \neq m} \mathbb{E}(Y_{Q,n}^k Y_{Q,n}^m) &= 2 \sum_{k>m} \sum_{q=Q+1}^{\infty} \int_0^p \int_0^p \sum_{l_1=0}^q \sum_{l_2=0}^q \kappa_{ql_1}(u) \kappa_{ql_2}(v) I_{ql_1 l_2}(k-m, u, v) dudv \\ &= 2n \sum_{j=1}^{n-1} \left(1 - \frac{j}{n}\right) \sum_{q=Q+1}^{\infty} \int_0^p \int_0^p \sum_{l_1=0}^q \sum_{l_2=0}^q \kappa_{ql_1}(u) \kappa_{ql_2}(v) I_{ql_1 l_2}(j, u, v) dudv. \end{aligned} \quad (25)$$

Let us introduce, as in Arcones ([1]),

$$\chi(u) := \sup(\eta_1(u), \eta_2(u)), \text{ where } \eta_1(u) := |r(u)| + \frac{|r'(u)|}{\sqrt{-r''(0)}} \text{ and } \eta_2(u) := \frac{|r'(u)|}{\sqrt{-r''(0)}} + \frac{|r''(u)|}{-r''(0)}. \quad (26)$$

Notice that χ is the supremum of the sum of the absolute values of the off-diagonal terms in the column vectors belonging to the covariance matrix for the Gaussian vector $\left(X_0, \frac{\dot{X}_0}{\sqrt{-r''(0)}}, X_u, \frac{\dot{X}_u}{\sqrt{-r''(0)}} \right)$.

Then by using Arcones inequality (see [1] or [21]), we obtain

$$\begin{aligned} \sum_{l_1=0}^q \sum_{l_2=0}^q \kappa_{ql_1}(u) \kappa_{ql_2}(v) I_{ql_1 l_2}(j, u, v) &\leq \chi^q(u + jp - v) \mathbb{E}^{1/2} \left(\left[\sum_{l_1=0}^q \kappa_{ql_1}(u) H_{q-l_1} \left(\frac{X_0}{\sqrt{r(0)}} \right) H_{l_1} \left(\frac{\dot{X}_0}{\sqrt{-r''(0)}} \right) \right]^2 \right) \\ &\quad \times \mathbb{E}^{1/2} \left(\left[\sum_{l_1=0}^q \kappa_{ql_1}(v) H_{q-l_1} \left(\frac{X_0}{\sqrt{r(0)}} \right) H_{l_1} \left(\frac{\dot{X}_0}{\sqrt{-r''(0)}} \right) \right]^2 \right). \end{aligned}$$

Suppose that $\int_0^\infty \chi(u) du < \infty$ and that there exists a positive integer a such that $\chi(v) < \rho$ for $v > a$. So (25) can be rewritten as

$$\begin{aligned} \sum_{k \neq m} \mathbb{E}(Y_{Q,n}^k Y_{Q,n}^m) &= 2(I_{a1} + I_{a2}), \text{ with} \quad (27) \\ I_{a1} &:= \sum_{j=1}^a \left(1 - \frac{j}{n}\right) \sum_{q=Q+1}^{\infty} \int_0^p \int_0^p \sum_{l_1=0}^q \sum_{l_2=0}^q \kappa_{ql_1}(u) \kappa_{ql_2}(v) I_{ql_1 l_2}(j, u, v) dudv \end{aligned}$$

and

$$I_{a2} := \sum_{j=a+1}^{n-1} \left(1 - \frac{j}{n}\right) \sum_{q=Q+1}^{\infty} \int_0^p \int_0^p \sum_{l_1=0}^q \sum_{l_2=0}^q \kappa_{ql_1}(u) \kappa_{ql_2}(v) I_{ql_1 l_2}(j, u, v) dudv.$$

On one hand we have

$$I_{a1} \leq \sum_{j=1}^a \left(1 - \frac{j}{n}\right) \sum_{q=Q+1}^{\infty} \mathbb{E} \left(\left[\int_0^p \sum_{l_1=0}^q \kappa_{ql_1}(u) H_{q-l_1} \left(\frac{X_0}{\sqrt{r(0)}} \right) H_{l_1} \left(\frac{\dot{X}_0}{\sqrt{-r''(0)}} \right) du \right]^2 \right) \xrightarrow{Q \rightarrow \infty} 0. \quad (28)$$

On the other hand, I_{a2} can be bounded as

$$\begin{aligned} I_{a2} &\leq \sum_{j=a+1}^{n-1} \left(1 - \frac{j}{n}\right) \sum_{q=Q+1}^{\infty} \int_0^p \int_0^p \chi^q(u + jp - v) \mathbb{E}^{1/2} \left(\left[\sum_{l_1=0}^q \kappa_{ql_1}(u) H_{q-l_1} \left(\frac{X_0}{\sqrt{r(0)}} \right) H_{l_1} \left(\frac{\dot{X}_0}{\sqrt{-r''(0)}} \right) \right]^2 \right) \\ &\quad \times \mathbb{E}^{1/2} \left(\left[\sum_{l_1=0}^q \kappa_{ql_1}(v) H_{q-l_1} \left(\frac{X_0}{\sqrt{r(0)}} \right) H_{l_1} \left(\frac{\dot{X}_0}{\sqrt{-r''(0)}} \right) \right]^2 \right) dvdu \\ &\leq \sum_{q=Q+1}^{\infty} \int_0^p \int_{u+ap}^{u+np} \chi^q(w) \mathbb{E}^{1/2} \left(\left[\sum_{l_1=0}^q \kappa_{ql_1}(u) H_{q-l_1} \left(\frac{X_0}{\sqrt{r(0)}} \right) H_{l_1} \left(\frac{\dot{X}_0}{\sqrt{-r''(0)}} \right) \right]^2 \right) \\ &\quad \times \mathbb{E}^{1/2} \left(\left[\sum_{l_1=0}^q \kappa_{ql_1}(u - w) H_{q-l_1} \left(\frac{X_0}{\sqrt{r(0)}} \right) H_{l_1} \left(\frac{\dot{X}_0}{\sqrt{-r''(0)}} \right) \right]^2 \right) dwdu \\ &\leq \sum_{q=Q+1}^{\infty} \rho^{q-2} \int_{ap}^{\infty} \chi^2(w) \int_0^p \mathbb{E}^{1/2} \left(\left[\sum_{l_1=0}^q \kappa_{ql_1}(u) H_{q-l_1} \left(\frac{X_0}{\sqrt{r(0)}} \right) H_{l_1} \left(\frac{\dot{X}_0}{\sqrt{-r''(0)}} \right) \right]^2 \right) \\ &\quad \times \mathbb{E}^{1/2} \left(\left[\sum_{l_1=0}^q \kappa_{ql_1}(u - w) H_{q-l_1} \left(\frac{X_0}{\sqrt{r(0)}} \right) H_{l_1} \left(\frac{\dot{X}_0}{\sqrt{-r''(0)}} \right) \right]^2 \right) dudw, \end{aligned}$$

by making the change of variables $w = u + jp - v$ and by using again the periodicity of κ_{ql} .

This last upper bound tends to 0 as $Q \rightarrow \infty$, since we have

$$\begin{aligned} \mathbb{E} \left(\left[\sum_{l=0}^q \kappa_{ql}(u) H_{q-l} \left(\frac{X_0}{\sqrt{r(0)}} \right) H_l \left(\frac{\dot{X}_0}{\sqrt{-r''(0)}} \right) \right]^2 \right) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} f_q^2(u, x_1, x_2) e^{-\frac{x_1^2 + x_2^2}{2}} dx_1 dx_2 \\ &= \kappa_{ql}^2(u) (q-l)! l! \\ &= \sum_{l=0}^q \frac{H_{q-l}^2 \left(\frac{\psi_s}{\sqrt{r(0)}} \right) \varphi^2 \left(\frac{\psi_s}{\sqrt{r(0)}} \right)}{(q-l)!} a_l^2 \left(\frac{\dot{\psi}_s}{\sqrt{-r''(0)}} \right) l! \\ &\leq \mathbf{C} \mathbb{E} \left[\left(Z + \frac{\dot{\psi}_s}{\sqrt{-r''(0)}} \right)^2 \right] \\ &\leq 2\mathbf{C} \left(1 + \frac{\|\dot{\psi}\|_{\infty}^2}{-r''(0)} \right), \end{aligned}$$

with f defined by $f_q(u, x_1, x_2) = \sum_{l=0}^q \kappa_{ql}(u) H_{q-l}(x_1) H_l(x_2)$ and \mathbf{C} some constant independent of q , as a consequence of Proposition 3 in Imkeller *et al.* (see [11]). Hence

$$I_{a2} \rightarrow 0 \text{ as } Q \rightarrow \infty. \quad (29)$$

Combining (22), (23), (27), (28) and (29) conclude the proof of the first step. \square

Second step - We are now interested in the asymptotical variance $\lim_{n \rightarrow \infty} \sigma_n^2$ of

$$\mathcal{F}_n := \frac{N_{np}^X(\psi) - \mathbb{E}N_{np}^X(\psi)}{\sqrt{n}}. \quad (30)$$

We have

$$\sigma_n^2 = \frac{-r''(0)}{r(0)} \sum_{q=1}^{\infty} \sigma_n^2(q), \quad \text{with}$$

$$\sigma_n^2(q) := \frac{1}{n} \mathbb{E} \left[\left(\int_0^{np} \sum_{l=0}^q \kappa_{ql}(s) H_{q-l} \left(\frac{X_s}{\sqrt{r(0)}} \right) H_l \left(\frac{\dot{X}_s}{\sqrt{-r''(0)}} \right) ds \right)^2 \right]. \quad (31)$$

Note that Fatou lemma implies that

$$\sum_{q=2}^{\infty} \lim_{n \rightarrow \infty} \sigma_n^2(q) \leq \lim_{n \rightarrow \infty} \mathbb{E} \left[\sum_{k=1}^n Y_{1,n}^k \right]^2 < \infty.$$

We will study the asymptotical behavior as $n \rightarrow \infty$ of each component $\sigma_n^2(q)$ of the variance of \mathcal{F}_n .

We have

$$\begin{aligned} \sigma_n^2(q) &= \frac{1}{n} \mathbb{E} \left[\left(\int_0^{np} \sum_{l=0}^q \kappa_{ql}(s) H_{q-l} \left(\frac{X_s}{\sqrt{r(0)}} \right) H_l \left(\frac{\dot{X}_s}{\sqrt{-r''(0)}} \right) ds \right)^2 \right] \\ &= \frac{1}{n} \mathbb{E} \left[\left(\sum_{k=1}^n \int_0^p \sum_{l=0}^q \kappa_{ql}(u) H_{q-l} \left(\frac{X_{u+(k-1)p}}{\sqrt{r(0)}} \right) H_l \left(\frac{\dot{X}_{u+(k-1)p}}{\sqrt{-r''(0)}} \right) du \right)^2 \right] \\ &:= \frac{1}{n} \mathbb{E} \left[\left(\sum_{k=1}^n Z_k^q \right)^2 \right] = \mathbb{E}([Z_1^q]^2) + \frac{1}{n} \sum_{k_1 \neq k_2} \mathbb{E}[Z_{k_1}^q Z_{k_2}^q] \\ &= \mathbb{E}([Z_1^q]^2) + \frac{2}{n} \sum_{k_1=1}^{n-1} \sum_{k_2=k_1+1}^n \mathbb{E}[Z_{k_1}^q Z_{k_2}^q] \end{aligned} \quad (32)$$

For the first term on the RHS, we have

$$\begin{aligned} \mathbb{E}[Z_1^q]^2 &= \int_0^p \int_0^p \sum_{l_1=0}^q \sum_{l_2=0}^q \kappa_{ql_1}(u) \kappa_{ql_2}(v) \mathbb{E} \left[H_{q-l_1} \left(\frac{X_u}{\sqrt{r(0)}} \right) H_{l_1} \left(\frac{\dot{X}_u}{\sqrt{-r''(0)}} \right) H_{q-l_2} \left(\frac{X_v}{\sqrt{r(0)}} \right) H_{l_2} \left(\frac{\dot{X}_v}{\sqrt{-r''(0)}} \right) \right] dudv = \\ &2 \int_0^p \int_0^u \sum_{l_1=0}^q \sum_{l_2=0}^q \kappa_{ql_1}(u) \kappa_{ql_2}(u-z) \mathbb{E} \left[H_{q-l_1} \left(\frac{X_z}{\sqrt{r(0)}} \right) H_{l_1} \left(\frac{\dot{X}_z}{\sqrt{-r''(0)}} \right) H_{q-l_2} \left(\frac{X_0}{\sqrt{r(0)}} \right) H_{l_2} \left(\frac{\dot{X}_0}{\sqrt{-r''(0)}} \right) \right] dzdv. \end{aligned} \quad (33)$$

For the other terms on the RHS, since by stationarity we have

$$\begin{aligned} \mathbb{E}[Z_{k_1}^q Z_{k_2}^q] &= \int_0^p \int_0^p \sum_{l_1=0}^q \sum_{l_2=0}^q \kappa_{ql_1}(u) \kappa_{ql_2}(v) \times \\ &\quad \mathbb{E} \left[H_{q-l_1} \left(\frac{X_{u+(k_1-1)p}}{\sqrt{r(0)}} \right) H_{l_1} \left(\frac{\dot{X}_{u+(k_1-1)p}}{\sqrt{-r''(0)}} \right) H_{q-l_2} \left(\frac{X_{v+(k_2-1)p}}{\sqrt{r(0)}} \right) H_{l_2} \left(\frac{\dot{X}_{v+(k_2-1)p}}{\sqrt{-r''(0)}} \right) \right] dudv \\ &= \int_0^p \int_0^p \sum_{l_1=0}^q \sum_{l_2=0}^q \kappa_{ql_1}(u) \kappa_{ql_2}(v) I_{ql_1 l_2}(k_2 - k_1, u, v) dudv, \end{aligned}$$

it comes, by making the change of variables $z = u - v + jp$, and by using the periodicity and stationarity of κ_{ql} ,

$$\begin{aligned}
\frac{1}{n} \sum_{k_1=1}^{n-1} \sum_{k_2=k_1+1}^n \mathbb{E}[Z_{k_1}^q Z_{k_2}^q] &= \frac{1}{n} \sum_{k_1=1}^{n-1} \sum_{k_2=k_1+1}^n \int_0^p \int_0^p \sum_{l_1=0}^q \sum_{l_2=0}^q \kappa_{ql_1}(u) \kappa_{ql_2}(v) I_{ql_1 l_2}(k_2 - k_1, u, v) dudv \\
&= \sum_{j=1}^{n-1} \left(1 - \frac{j}{n}\right) \int_0^p \int_0^p \sum_{l_1=0}^q \sum_{l_2=0}^q \kappa_{ql_1}(u) \kappa_{ql_2}(v) I_{ql_1 l_2}(j, u - v, 0) dudv \\
&= \int_0^p \sum_{j=1}^{n-1} \left(1 - \frac{j}{n}\right) \int_{u+(j-1)p}^{u+jp} \sum_{l_1=0}^q \sum_{l_2=0}^q \kappa_{ql_1}(u) \kappa_{ql_2}(u - z) I_{ql_1 l_2}(0, z, 0) dz du \\
&= \int_0^p \int_u^{\infty} \sum_{j=1}^{n-1} \left(1 - \frac{j}{n}\right) 1_{[u+(j-1)p, u+jp]}(z) \sum_{l_1=0}^q \sum_{l_2=0}^q \kappa_{ql_1}(u) \kappa_{ql_2}(u - z) I_{ql_1 l_2}(0, z, 0) dz du.
\end{aligned}$$

But on one hand, it holds that

$$\sum_{j=1}^{n-1} \left(1 - \frac{j}{n}\right) 1_{[u+(j-1)p, u+jp]}(z) \xrightarrow{n \rightarrow \infty} 1_{[u, \infty)}(z),$$

on the other hand, we can apply the Dominated Convergence Theorem (DCT), splitting the inner integral into two parts, namely on the interval $[0, a]$, with a chosen such that $\chi(z) < \rho < 1$ for $z > a$, and on $[a, \infty)$ respectively (use the uniform continuity to interchange limit and integral in the former integral, and the DCT, by using Arcones inequality once again, in the latter integral), therefore we obtain

$$\frac{1}{n} \sum_{k_1 \neq k_2} \mathbb{E}[Z_{k_1}^q Z_{k_2}^q] \xrightarrow{n \rightarrow \infty} 2 \int_0^p \int_u^{\infty} \sum_{l_1=0}^q \sum_{l_2=0}^q \kappa_{ql_1}(u) \kappa_{ql_2}(u - z) I_{ql_1 l_2}(0, z, 0) dz du. \quad (34)$$

Combining (32), (33) and (34) provide

$$\sigma_n^2(q) \xrightarrow{n \rightarrow \infty} 2 \int_0^{\infty} \sum_{l_1=0}^q \sum_{l_2=0}^q \left(\int_0^p \kappa_{ql_1}(u) \kappa_{ql_2}(u - z) du \right) I_{ql_1 l_2}(0, z, 0) dz,$$

namely

$$\sigma^2(q) := \lim_{n \rightarrow \infty} \sigma_n^2(q) = \int_0^{\infty} \sum_{l_1=0}^q \sum_{l_2=0}^q \tilde{\kappa}_{ql_1} * \kappa_{ql_2}(z) I_{ql_1 l_2}(0, z, 0) dz, \quad (35)$$

with the notation $\tilde{h}(u) := h(-u)$ and $h * g(u) := \int_0^p h(u - v)g(v)dv$.

By using the same method as in Kratz & León (see [14], p.653), we can conclude that

$$\mathbb{E}[\mathcal{F}_n^2] := \mathbb{E} \left[\left(\frac{N_{np}^X(\psi) - \mathbb{E}N_{np}^X(\psi)}{\sqrt{n}} \right)^2 \right] \xrightarrow{n \rightarrow \infty} \frac{-r''(0)}{r(0)} \sum_{q=1}^{\infty} \sigma^2(q), \quad \sigma^2(q) \text{ satisfying (35)}.$$

Last step - CLT for \mathcal{F}_n under the hypothesis of m -dependence on X .

Under this hypothesis, $\sigma^2(q)$ becomes

$$\sigma_m^2(q) = 2 \int_0^m \sum_{l_1=0}^q \sum_{l_2=0}^q \tilde{\kappa}_{ql_1} * \kappa_{ql_2}(z) I_{ql_1 l_2}(0, z, 0) dz. \quad (36)$$

Let us define

$$\mathcal{F}_{Q,n} := \frac{1}{\sqrt{n}} \int_0^{np} g_q(s, X_s, \dot{X}_s) ds, \quad \text{with } g_q(s, X_s, \dot{X}_s) := \sqrt{\frac{-r''(0)}{r(0)}} \sum_{q=0}^Q \sum_{l=0}^q \kappa_{ql}(s) H_{q-l} \left(\frac{X_s}{\sqrt{r(0)}} \right) H_l \left(\frac{\dot{X}_s}{\sqrt{-r''(0)}} \right).$$

By using (21), it holds that

$$\lim_{Q \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbb{E} \left[(\mathcal{F}_n - \mathcal{F}_{Q,n})^2 \right] = 0. \quad (37)$$

Hence, as exposed in Kratz & León ([14]), it is enough to show that $\mathcal{F}_{Q,n}$ satisfies a CLT to obtain the same result for \mathcal{F}_n .

Since we can write

$$\mathcal{F}_{Q,n} = \frac{1}{\sqrt{n}} \sum_{k=1}^n \int_0^p g_Q(s, X_{s+(k-1)p}, \dot{X}_{s+(k-1)p}) ds := \frac{1}{\sqrt{n}} \sum_{k=1}^n Y_k = \frac{1}{\sqrt{n}} \sum_{k=1}^n \theta_{(k-1)p} Y_1^n,$$

when introducing θ_t as the shift operator of path t associated to X , under our hypothesis on X , the random variables Y_k are $(\frac{m}{p} + 1)$ -dependent and moreover are identically distributed.

Then an immediate application of the Hoeffding-Robbins CLT for m -dependent sequences ([10]) gives

$$\mathcal{L}(\mathcal{F}_{Q,n}) \xrightarrow[n \rightarrow \infty]{\Rightarrow} \mathcal{N}\left(0, \sum_{q=1}^Q \sigma_m^2(q)\right).$$

Then in the same form of [14] we can conclude the result under the more general weak dependence assumption (20). \square

2.2.2 Linear curve and specular points

We are interested in describing the behavior of the specular points in a random curve.

As described by Longuet-Higgins ([17]), specular points are the moving images of a light source reflected at different points on a wave-like surface.

Let us consider a Gaussian field W defined on $\mathbb{R}^+ \times \mathbb{R}$, \mathbb{R}^+ for the temporal variable and \mathbb{R} for the spatial one.

The first derivatives with respect to the spatial variable x and the temporal variable t will be denoted by $\partial_x W$ and $\partial_t W$ respectively; the second derivatives will be denoted by $\partial_{xx} W$, $\partial_{tt} W$, $\partial_{tx} W$ and $\partial_{xt} W$.

The spectral representation of W is given by

$$W(t, x) = \int e^{i(\lambda - |\lambda|^{1/2}t)} \sqrt{f(\lambda)} dB(\lambda), \quad (38)$$

where the real and imaginary parts of the complex-Gaussian process $B = B(\lambda, \lambda \geq 0)$ are independent-increment real Gaussian processes with $\text{var}(\text{Re}(B(\lambda))) = \text{var}(\text{Im}(B(\lambda))) = F(\lambda)/2$ and $B(-\lambda) = \bar{B}(\lambda)$ a.s. (for more details, see for instance [20] or [16]).

The correlation function r is given by

$$r(t, x) = \int e^{i(\lambda x - \omega t)} f(\lambda) d\lambda = 2 \int_0^\infty \cos(\lambda x - \lambda^{1/2}t) f(\lambda) d\lambda. \quad (39)$$

First let us consider the process at a fixed time, for instance $t = 0$.

In this case, $W(0, x)$ is a centered stationary Gaussian process with correlation function

$$r(0, x) := r(x) = 2 \int_0^\infty \cos(\lambda x) f(\lambda) d\lambda.$$

Let us define a curve in the plan (x, z) by the equation $z = W(0, x)$.

Suppose that the coordinates of a point-source of light and an observer are $(0, h_1)$ and $(0, h_2)$ respectively, situated at heights h_1 and h_2 above the mean surface level. A specular point is characterized by the equations (see [17]):

$$\partial_x W(0, x) = -\kappa x \quad \text{with} \quad \kappa = \frac{1}{2} \left(\frac{1}{h_1} + \frac{1}{h_2} \right),$$

which can be interpreted as a crossing of the curve $\psi(x) := -\kappa x$ by the process $\partial_x W(0, x)$. Let us assume that r is four times differentiable and that r'' satisfies the Geman condition:

$$\frac{r^{(iv)}(x) - r^{(iv)}(0)}{x} \in L^1([0, \delta]). \quad (40)$$

Theorem 2 Under the Geman condition (40), the number $N_{[0,x]}$ of specular points in the interval $[0, x]$ has the following expansion in $L^2(\Omega)$:

$$N_{[0,x]} = \frac{\gamma}{\eta} \sum_{q=0}^{\infty} \int_0^x F_q \left(s, \frac{\partial_x W(0, s)}{\eta}, \frac{\partial_{xx} W(0, s)}{\gamma} \right) ds, \quad (41)$$

where $\eta = \sqrt{-r''(0)}$, $\gamma = \sqrt{r^{(iv)}(0)}$, a_l defined in (10), and

$$F_q \left(s, \frac{\partial_x W(0, s)}{\eta}, \frac{\partial_{xx} W(0, s)}{\gamma} \right) := \sum_{l=0}^q \frac{H_{q-l} \left(\frac{-\kappa s}{\eta} \right) \varphi \left(\frac{-\kappa s}{\eta} \right)}{(q-l)!} a_l \left(\frac{-\kappa}{\gamma} \right) H_{q-l} \left(\frac{\partial_x W(0, s)}{\eta} \right) H_l \left(\frac{\partial_{xx} W(0, s)}{\gamma} \right).$$

Its expectation is given by

$$\mathbb{E}[N_{[0,x]}] = \sqrt{\frac{2}{\pi}} \frac{\kappa}{\eta} \left(\int_0^{\frac{\kappa}{\gamma}} e^{-\frac{u^2}{2}} du + \frac{\gamma}{\kappa} e^{-\frac{\kappa^2}{2\gamma^2}} \right) \int_0^x \varphi \left(\frac{\kappa s}{\eta} \right) ds, \quad (42)$$

and its variance by

$$\begin{aligned} \text{Var}(N_{[0,x]}) &= \\ & 2 \frac{\gamma}{\eta} \sum_{q=1}^{\infty} \int_0^x \int_0^{x-s} \mathbb{E} \left[F_q \left(s, \frac{\partial_x W(0, s)}{\eta}, \frac{\partial_{xx} W(0, s)}{\gamma} \right) F_q \left(s + \tau, \frac{\partial_x W(0, s + \tau)}{\eta}, \frac{\partial_{xx} W(0, s + \tau)}{\gamma} \right) \right] d\tau ds. \end{aligned}$$

Under m -dependence, i.e. when assuming that $-r''(\tau) = 0$ for $\tau > m$, its asymptotic variance, as $x \rightarrow \infty$, is given by

$$\begin{aligned} \text{Var}(N_{[0,\infty]}) &= \\ & 2 \frac{\gamma}{\eta} \sum_{q=1}^{\infty} \int_0^{\infty} \int_0^m \mathbb{E} \left[F_q \left(s, \frac{\partial_x W(0, s)}{\eta}, \frac{\partial_{xx} W(0, s)}{\gamma} \right) F_q \left(s + \tau, \frac{\partial_x W(0, s + \tau)}{\eta}, \frac{\partial_{xx} W(0, s + \tau)}{\gamma} \right) \right] d\tau ds. \end{aligned} \quad (43)$$

Remarks.

i) Note that the expectation (42) of $N_{[0,x]}$ was first obtained by Longuet-Higgins in [17] and [18], who considered a bilateral formula, i.e. $\mathbb{E}[N_{[-x,x]}]$, and showed that its limit as $x \rightarrow \infty$ satisfies

$$\lim_{\frac{\gamma}{\kappa} \rightarrow \infty} \frac{\kappa}{\gamma} \mathbb{E}[N_{[0,+\infty]}] = \sqrt{\frac{1}{2\pi}} \quad ;$$

it can be interpreted by saying that the number of specular points increases proportionally to the distance between the observer and the sea surface, since when $h_1 = h_2$, $\frac{1}{\kappa}$ represents this latter distance.

It is also interesting to notice that the number of specular points is a finite random variable over all the line, on the contrary to the behavior of the crossings to a fixed level or of the number of maxima.

ii) We consider the hypothesis of m -dependence to simplify the computations and with the aim of using in the next section the method settled in [14].

Proof.

Since the correlation function of $\partial_x W(0, x)$ is $-r''(x)$, Proposition 1 provides the Hermite expansion of $N_{[0,x]}$ under the Geman condition (40), from which we deduce the expectation

$$\mathbb{E}[N_{[0,x]}] = \frac{\gamma}{\eta} \int_0^x \varphi \left(\frac{\kappa s}{\eta} \right) a_0 \left(\frac{-\kappa}{\gamma} \right) ds = (42),$$

and also the variance.

Let us now study the limit of the variance of $N_{[0,x]}$ as $x \rightarrow \infty$, under the m -dependence.

★ For each $q \geq 1$, we obtain

$$\begin{aligned} & \lim_{x \rightarrow \infty} \int_0^x \int_0^{x-s} \mathbb{E} \left[F_q \left(s, \frac{\partial_x W(0,s)}{\eta}, \frac{\partial_{xx} W(0,s)}{\gamma} \right) F_q \left(s+\tau, \frac{\partial_x W(0,s+\tau)}{\eta}, \frac{\partial_{xx} W(0,s+\tau)}{\gamma} \right) \right] d\tau ds \\ &= \int_0^\infty \int_0^m \mathbb{E} \left[F_q \left(s, \frac{\partial_x W(0,s)}{\eta}, \frac{\partial_{xx} W(0,s)}{\gamma} \right) F_q \left(s+\tau, \frac{\partial_x W(0,s+\tau)}{\eta}, \frac{\partial_{xx} W(0,s+\tau)}{\gamma} \right) \right] d\tau ds. \end{aligned}$$

The next inequality is a consequence of the diagram formula (see [6]):

$$\begin{aligned} & \left| \mathbb{E} \left[F_q \left(s, \frac{\partial_x W(0,s)}{\eta}, \frac{\partial_{xx} W(0,s)}{\gamma} \right) F_q \left(s+\tau, \frac{\partial_x W(0,s+\tau)}{\eta}, \frac{\partial_{xx} W(0,s+\tau)}{\gamma} \right) \right] \right| \\ & \leq \mathbf{C} \left| \frac{\kappa s}{\eta} \right|^q \varphi \left(\frac{\kappa s}{\eta} \right) \left| \frac{\kappa(s+\tau)}{\eta} \right|^q \varphi \left(\frac{\kappa(s+\tau)}{\eta} \right) \sum_{a+b+c=q} |r''(\tau)|^a |r'''(\tau)|^b |r^{(iv)}(\tau)|^c, \end{aligned}$$

and the term equals zero when $\tau > m$. The Dominate Convergence Theorem allows then to compute the limit inside of the double integral.

★ We shall study the asymptotic behavior as $x \rightarrow \infty$ of the second moment of $N_{[0,x]}$.

First let us consider the second factorial moment, and for ease of notation we shall suppose bellow that $-r''(0) = 1$. We have

$$M_2^\psi = 2 \int_0^x \int_0^{x-s} p_\tau(-\kappa s, -\kappa(s+\tau)) G(s, \tau) d\tau ds,$$

where

$$\begin{aligned} G(s, \tau) &= \mathbb{E} \left(\left| \zeta - \frac{r'''(\tau)r''(\tau)}{1-(r''(\tau))^2} \kappa s - \frac{r'''(\tau)}{1-(r''(\tau))^2} \kappa(s+\tau) + \kappa \right| \right. \\ & \quad \left. \times \left| \zeta^* + \frac{r'''(\tau)}{1-(r''(\tau))^2} \kappa s + \frac{r'''(\tau)r''(\tau)}{1-(r''(\tau))^2} \kappa(s+\tau) + \kappa \right| \right), \end{aligned}$$

$\psi(x) = -\kappa x$, p_τ is the density of $(\partial_x W(0,s), \partial_x W(0,s+\tau))$ and (ζ, ζ^*) is defined in the same way as in (R). Then

$$M_2^\psi = 2 \int_0^x \int_\delta^{x-s} p_\tau(-\kappa s, -\kappa(s+\tau)) G(s, \tau) ds d\tau + 2 \int_0^x \int_0^\delta p_\tau(-\kappa s, -\kappa(s+\tau)) G(s, \tau) ds d\tau.$$

By using the Cauchy-Schwarz inequality, we have

$$G(s, \tau) \leq \mathbf{C} \left[\mathbb{E} \zeta^2 + \left(\frac{r'''(\tau)r''(\tau)}{1-(r''(\tau))^2} \right)^2 + \left(\frac{r'''(\tau)r''(\tau)}{1-(r''(\tau))^2} \right)^2 + \kappa^2 \right] (s^2 + (s+\tau)^2).$$

Moreover, we can write

$$p_\tau(-\kappa s, -\kappa(s+\tau)) = \frac{1}{2\pi\sqrt{1-(r''(\tau))^2}} e^{-\frac{\kappa^2}{1-r''(\tau)}(s+\frac{1}{2}\tau)^2} e^{-\frac{\kappa^2\tau^2}{4(1+r''(\tau))}}. \quad (44)$$

If $\tau \in [\delta, x-s]$, it holds that $p_\tau(-\kappa s, -\kappa(s+\tau)) \leq \mathbf{C} e^{-\frac{\kappa^2}{1-r''(\tau)}(s+\frac{1}{2}\tau)^2}$.

Thus applying the Dominate Convergence Theorem gives

$$\lim_{x \rightarrow \infty} \int_0^x \int_\delta^{x-s} p_\tau(-\kappa s, -\kappa(s+\tau)) G(s, \tau) ds d\tau = \int_0^\infty \int_\delta^m p_\tau(-\kappa s, -\kappa(s+\tau)) G(s, \tau) ds d\tau.$$

To bound $G(s, \tau)$ in the interval $[0, \delta]$, we can proceed as in the previous section when bounding the terms $\mathbb{E}[I_i J_i]$; all the terms can be bounded by functions belonging to $L^2[0, \delta]$, therefore the dominated convergence theorem yields

$$\lim_{x \rightarrow \infty} \int_0^x \int_0^{x-s} p_\tau(-\kappa s, -\kappa(s + \tau)) G(s, \tau) ds d\tau = \int_0^\infty \int_0^m p_\tau(-\kappa s, -\kappa(s + \tau)) G(s, \tau) ds d\tau < \infty.$$

On one hand, via Fatou lemma we have

$$\begin{aligned} \infty &> \lim_{x \rightarrow \infty} \mathbb{E}[N_{[0, x]}^2] \geq \lim_{x \rightarrow \infty} \text{Var} \left(\frac{\eta}{\gamma} N_{[0, x]} \right) \\ &= \lim_{x \rightarrow \infty} 2 \sum_{q=1}^\infty \int_0^x \int_0^{x-s} \mathbb{E} \left[F_q \left(s, \frac{\partial_x W(0, s)}{\eta}, \frac{\partial_{xx} W(0, s)}{\gamma} \right) F_q \left(s + \tau, \frac{\partial_x W(0, s + \tau)}{\eta}, \frac{\partial_{xx} W(0, s + \tau)}{\gamma} \right) \right] ds d\tau \\ &\geq 2 \sum_{q=1}^\infty \int_0^\infty \int_0^m \mathbb{E} \left[F_q \left(s, \frac{\partial_x W(0, s)}{\eta}, \frac{\partial_{xx} W(0, s)}{\gamma} \right) F_q \left(s + \tau, \frac{\partial_x W(0, s + \tau)}{\eta}, \frac{\partial_{xx} W(0, s + \tau)}{\gamma} \right) \right] ds d\tau. \end{aligned}$$

On the other hand, $N_{[0, x]} \uparrow N_{[0, \infty]}$ and so $\mathbb{E}[N_{[0, x]}^2] \rightarrow \mathbb{E}[N_{[0, \infty]}^2]$, as $x \rightarrow \infty$.

We can then conclude to (43). \square

Remark: the computations may be done in dimension 2, explicitly in what concerns the expectation, but we would face some difficulty for the second moment since the determinant appearing in the integrand of it would prevent from having a Hermite polynomial expansion.

3 Study of specular points

3.1 Speed of specular points

In this section we shall study the speed of specular points, in particular their asymptotic distribution and a CLT.

Let $\tilde{W}(t, x, y)$ represent the waves modeling the sea surface. We shall look at a wave in one direction, that is for fixed coordinate y , for instance when $y = 0$. Thus we consider from now on $W(t, x) = \tilde{W}(t, x, 0)$.

Our main goal is the study of the number of specular points having a given velocity. The use of the co-aire formula being the starting point of this study, let us recall this formula.

Let W be a three times continuously differentiable bivariate function ($W \in \mathcal{C}^3$), and let us define the level curve by

$$\mathcal{C}(u) = \{(x_1, x_2) \in Q(t, M) : W(x_1, x_2) = u\}, \quad \text{where } Q(t, M) = [0, t] \times [0, M].$$

If v denotes a vector field, \mathbf{n} denotes the vector normal to the level curve $\mathcal{C}(u)$ and $d\sigma$ the length measure of $\mathcal{C}(u)$, since W is \mathcal{C}^3 , then by Green formula we obtain, for some suitable function g ,

$$\int_{-\infty}^\infty g(u) \int_{\mathcal{C}(u)} \langle v, \mathbf{n} \rangle d\sigma du = \int_{Q(t, M)} g(W) \langle v, \nabla W \rangle dx_1 dx_2,$$

from which can be deduced immediately:

- i) if $v = \frac{\nabla W}{\|\nabla W\|}$, then $\int_{-\infty}^\infty g(u) \mathcal{L}_Q(\mathcal{C}(u)) du = \int_Q g(W) \|\nabla W\| dx_1 dx_2$, where $\mathcal{L}_Q(\mathcal{C}(u))$ denotes the length of the curve $\mathcal{C}(u)$;
- ii) more generally, if $v = \zeta(\alpha) \frac{\nabla W}{\|\nabla W\|}$, where ζ denotes a continuous real function defined on $[0, 2\pi]$ and α is defined by $\nabla W := \|\nabla W\| \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}$, then we have $\int_{-\infty}^\infty g(u) \int_{\mathcal{C}(u)} \zeta(\alpha(s)) d\sigma(s) du = \int_Q g(W) \zeta(\alpha) \|\nabla W\| dx_1 dx_2$.

Note this formula still holds when considering ζ as the indicator function of a measurable set, by using the monotone convergence theorem (see for instance [8] and [2]).

Let us go back to our study and let us keep the same notation as in the previous section, namely W and r satisfy (38) and (39) respectively.

Note that $\mathbf{W}(t, x) = (\partial_x W(t, x), \partial_{xx} W(t, x), \partial_{tx} W(t, x))^t$ is Gaussian, with covariance matrix $\Delta = (\Delta_{ij})_{1 \leq i, j \leq 3}$, such that

$$\begin{aligned} \Delta_{11} &= \mathbb{E}[\partial_x W^2] = M_{20} & \Delta_{22} &= \mathbb{E}[\partial_{xx} W^2] = M_{40} & \Delta_{33} &= \mathbb{E}[\partial_{xt} W^2] = M_{22} \\ \Delta_{12} &= \mathbb{E}[\partial_x W \partial_{xx} W] = \Delta_{13} = \mathbb{E}[\partial_x W \partial_{xt} W] = 0 & \text{and} & & \Delta_{23} &= \mathbb{E}[\partial_{xx} W \partial_{xt} W] = M_{31}. \end{aligned}$$

Moreover, for fixed (t, x) , $\partial_x W(t, x)$ is independent of the vector $(\partial_{xx} W(t, x), \partial_{xt} W(t, x))$.

Let us consider, as in [18], the simplified condition to have a specular point, that is, the spatial derivative of the field takes some given value u , namely

$$\partial_x W(t, x) = u.$$

Then a consequence of the implicit function theorem is:

$$\partial_{xx} W dx + \partial_{xt} W dt = 0, \quad \text{i.e.} \quad \frac{dx}{dt} = -\frac{\partial_{xt} W}{\partial_{xx} W}.$$

We are interested in the number of specular points which speed is bounded below by $-v_2$ and above by $-v_1$. Thus we define

$$\tilde{N}_{sp}(s, u, v_1, v_2) := \# \left\{ z \leq M : \partial_x W(s, z) = u; v_1 \leq \frac{\partial_{xt} W}{\partial_{xx} W} \leq v_2 \right\} \quad \text{for } 0 \leq s \leq t.$$

Moreover our interest is in mean time, hence we define

$$N_{sp}(u, v_1, v_2, t) := \frac{1}{t} \int_0^t \tilde{N}_{sp}(s, u, v_1, v_2) ds. \quad (45)$$

Let g a continuous and bounded function. By using the Banach formula we get

$$\begin{aligned} \int_{-\infty}^{\infty} g(u) N_{sp}(u, v_1, v_2, t) du &= \frac{1}{t} \int_0^t \int_{-\infty}^{\infty} g(u) \tilde{N}_{sp}(s, u, v_1, v_2) du ds \\ &= \frac{1}{t} \int_0^t \int_0^M g(\partial_x W(s, z)) \mathbb{1}_{[v_1, v_2]} \left(\frac{\partial_{xt} W(s, z)}{\partial_{xx} W(s, z)} \right) |\partial_{xx} W(s, z)| dz ds \\ &= \int_{-\infty}^{\infty} g(u) \int_{\mathcal{C}(u)} \zeta(\alpha(s)) d\sigma(s) du, \end{aligned}$$

when choosing $\zeta(\alpha(s)) = \mathbb{1}_{[v_1, v_2]}(\tan \alpha(s)) |\cos \alpha(s)|$ and by using the co-airé formula in the last equality. Note that this type of integrals has been first considered in [8].

Therefore, by independence and stationarity, we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} g(u) \mathbb{E}[N_{sp}(u, v_1, v_2, t)] du &= \frac{1}{t} \int_{-\infty}^{\infty} g(u) \mathbb{E} \left[\int_{\mathcal{C}(u)} \zeta(\alpha(s)) d\sigma(s) \right] du \\ &= M \int_{-\infty}^{\infty} g(u) \frac{e^{-\frac{u^2}{2M_{20}}}}{\sqrt{2\pi M_{20}}} du \mathbb{E} \left[\mathbb{1}_{[v_1, v_2]} \left(\frac{\partial_{xt} W(0, 0)}{\partial_{xx} W(0, 0)} \right) |\partial_{xx} W(0, 0)| \right], \end{aligned}$$

and by duality it comes

$$\mathbb{E}[N_{sp}(u, v_1, v_2, t)] = M \frac{e^{-\frac{u^2}{2M_{20}}}}{\sqrt{2\pi M_{20}}} \mathbb{E} \left[\mathbb{1}_{[v_1, v_2]} \left(\frac{\partial_{xt} W(0, 0)}{\partial_{xx} W(0, 0)} \right) |\partial_{xx} W(0, 0)| \right], \quad (46)$$

which holds for a.s. u , nevertheless it can be shown that is true for all u (see [7] and [8]).

The expression (46) can be interpreted as the expected number of specular points in $[0, t]$ with speed

belonging to $[-v_2, -v_1]$.

Let us compute the expectation in (46). We have

$$\begin{aligned} \mathbb{E} \left[\mathbb{1}_{[v_1, v_2]} \left(\frac{\partial_{xt} W(0, 0)}{\partial_{xx} W(0, 0)} \right) \left| \partial_{xx} W(0, 0) \right| \right] &= \frac{1}{2\pi\sqrt{\lambda_1\lambda_2}} \int_0^\infty \int_{\beta+\arctan v_1}^{\beta+\arctan v_2} |\cos(\theta - \beta)| r^2 e^{-\frac{r^2}{2} \left(\frac{\cos^2 \theta}{\lambda_1} + \frac{\sin^2 \theta}{\lambda_2} \right)} d\theta dr \\ &= \frac{\lambda_1\lambda_2}{\sqrt{8\pi}} \int_{\beta+\arctan v_1}^{\beta+\arctan v_2} |\cos(\theta - \beta)| \frac{1}{(\lambda_1 \sin^2 \theta + \lambda_2 \cos^2 \theta)^{3/2}} d\theta, \end{aligned}$$

where

$$\begin{cases} \lambda_1, \lambda_2 \text{ are the eigenvalues of the covariance matrix of } (\partial_{xx} W(0, 0), \partial_{xt} W(0, 0)), \\ \text{and } \beta \text{ is the rotation angle which turns diagonal this covariance matrix.} \end{cases} \quad (47)$$

Thus

$$\gamma_0 := \mathbb{E}[N_{sp}(u, v_1, v_2, t)] = M \frac{e^{-\frac{u^2}{2M_{20}}} \lambda_1 \lambda_2}{4\pi\sqrt{M_{20}}} \int_{\beta+\arctan v_1}^{\beta+\arctan v_2} |\cos(\theta - \beta)| \frac{1}{(\lambda_1 \sin^2 \theta + \lambda_2 \cos^2 \theta)^{3/2}} d\theta. \quad (48)$$

We are now looking at the Hermite expansion of the mean number $N_{sp}(u, v_1, v_2, t)$ of specular points in $[0, t]$ with speed belonging to $[-v_2, -v_1]$, defined in (45), in view of obtaining a CLT for $N_{sp}(u, v_1, v_2, t)$ as $t \rightarrow \infty$. We can prove the following:

Theorem 3 *If $W(s, z) \in \mathcal{C}^3$, then the Hermite expansion of the functional $N_{sp}(u, v_1, v_2, t)$ is given by*

$$\begin{aligned} N_{sp}(u, v_1, v_2, t) &= \sum_{l=0}^{\infty} \sum_{0 \leq n+m \leq l} \frac{H_{l-(n+m)} \left(\frac{u}{\sqrt{M_{20}}} \right) \varphi \left(\frac{u}{\sqrt{M_{20}}} \right) g_{n,m}}{\sqrt{M_{20}} [l - (n+m)]!} \\ &\quad \times \frac{1}{t} \int_0^t \int_0^M H_{l-(n+m)} \left(\frac{\partial_x W(s, z)}{\sqrt{M_{20}}} \right) H_n \left(\frac{\eta_1(s, z)}{\sqrt{\lambda_1}} \right) H_n \left(\frac{\eta_2(s, z)}{\sqrt{\lambda_2}} \right) dz ds, \end{aligned}$$

where $(\eta_1(s, z), \eta_2(s, z))$ is normally distributed $\mathcal{N} \left(0, \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \right)$ with λ_1, λ_2 defined in (47),

and where

$$g_{n,m} = \frac{1}{2\pi n! m!} \int_0^\infty \int_{\arctan a}^{\arctan b} \left| \sqrt{\lambda_1} \cos \beta \cos \theta - \sqrt{\lambda_2} \sin \beta \sin \theta \right| H_n(\rho \cos \theta) H_m(\rho \sin \theta) e^{-\frac{\rho^2}{2}} \rho^2 d\theta d\rho, \quad (49)$$

with

$$a := \sqrt{\frac{\lambda_1}{\lambda_2}} \left(\frac{v_1 - \tan \beta}{1 + v_1 \tan \beta} \right) \quad \text{and} \quad b := \sqrt{\frac{\lambda_1}{\lambda_2}} \left(\frac{v_2 - \tan \beta}{1 + v_2 \tan \beta} \right). \quad (50)$$

Moreover, under m -dependence, we have

$$\sqrt{t} (N_{sp}(u, v_1, v_2, t) - \gamma_0) \Rightarrow \mathcal{N}(0, \sigma^2(x)), \quad \text{as } t \rightarrow \infty,$$

with $\sigma^2(x)$ computed in (54) below and γ_0 defined in (48).

Proof. The proof of this result relies mainly on the application of the co-aire formula and the method used in [14]. Suppose w.l.o.g. $\mathbb{E} \partial_x^2 W(t, z) = M_{20} = 1$.

It can be showed that, in $L^2(\Omega)$,

$$\frac{1}{h} \int_{-\infty}^{\infty} \varphi \left(\frac{u-y}{h} \right) \left[\int_{\mathcal{C}(y)} \zeta(\alpha(s)) d\sigma(s) \right] dy \xrightarrow{h \rightarrow 0} \int_{\mathcal{C}(u)} \zeta(\alpha(s)) d\sigma(s) = t N_{sp}(u, v_1, v_2),$$

which will be used to provide the Hermite expansion of $N_{sp}(u, v_1, v_2)$.

By using the co-aire formula, we obtain

$$\begin{aligned} &\frac{1}{h} \int_{-\infty}^{\infty} \varphi \left(\frac{u-y}{h} \right) \left[\int_{\mathcal{C}(y)} \zeta(\alpha(s)) d\sigma(s) \right] dy \\ &= \frac{1}{h} \int_0^t \int_0^M \varphi \left(\frac{u - \partial_x W(s, z)}{h} \right) \mathbb{1}_{[v_1, v_2]} \left(\frac{\partial_{xt} W(s, z)}{\partial_{xx} W(s, z)} \right) |\partial_{xx} W(s, z)| dz ds. \end{aligned}$$

Let consider, at fixed (s, z) , the covariance matrix of the random vector $(\partial_{xx}W(s, z), \partial_{xt}W(s, z))$ which can be diagonalized with a rotation matrix of angle β , independent of the point (s, z) because of the stationarity of the process $(\partial_{xx}W(\cdot, \cdot), \partial_{xt}W(\cdot, \cdot))$.

Thus we can write, for fixed (s, z) ,

$$\begin{pmatrix} \partial_{xx}W(s, z) \\ \partial_{xt}W(s, z) \end{pmatrix} = \begin{pmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{pmatrix} \begin{pmatrix} \eta_1(s, z) \\ \eta_2(s, z) \end{pmatrix},$$

with $(\eta_1(s, z), \eta_2(s, z))$ normally distributed $\mathcal{N}\left(0, \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\right)$.

We have

$$\begin{aligned} \mathbb{1}_{[v_1, v_2]} \left(\frac{\partial_{xt}W(s, z)}{\partial_{xx}W(s, z)} \right) |\partial_{xx}W(s, z)| &= \mathbb{1}_{[v_1, v_2]}(\tan(\beta + \eta(s, z))) |\eta_1(s, z) \cos \beta - \eta_2(s, z) \sin \beta| \\ &= \mathbb{1}_{[\arctan v_1 - \beta, \arctan v_2 - \beta]}(\eta(s, z)) |\eta_1(s, z) \cos \beta - \eta_2(s, z) \sin \beta| \\ &= \mathbb{1}_{\left[\frac{v_1 - \tan \beta}{1 + v_1 \tan \beta}, \frac{v_2 - \tan \beta}{1 + v_2 \tan \beta}\right]} \left(\frac{\eta_2(s, z)}{\eta_1(s, z)} \right) |\eta_1(s, z) \cos \beta - \eta_2(s, z) \sin \beta| \\ \left(\frac{\sqrt{\lambda_1} \eta_2(s, z)}{\sqrt{\lambda_2} \eta_1(s, z)} \right) &= \mathbb{1}_{[a, b]} \left(\frac{Z_1(s, z)}{Z_2(s, z)} \right) \left| \sqrt{\lambda_1} Z_1(s, z) \cos \beta - \sqrt{\lambda_2} Z_2(s, z) \sin \beta \right|, \end{aligned}$$

where $\eta(s, z) := \arctan\left(\frac{\eta_2(s, z)}{\eta_1(s, z)}\right)$ and $Z_i(s, z) := \left(\frac{\eta_i(s, z)}{\sqrt{\lambda_i}}\right)$, $i = 1, 2$, satisfying for fixed (s, z) , $(Z_1(s, z), Z_2(s, z)) \rightsquigarrow \mathcal{N}(0, Id)$.

And the Hermite coefficients $(g_{n, m})$ of the functional

$$G(Z_1, Z_2) := \mathbb{1}_{[a, b]} \left(\frac{Z_2}{Z_1} \right) \left| \sqrt{\lambda_1} Z_1 \cos \beta - \sqrt{\lambda_2} Z_2 \sin \beta \right|, \text{ for } (Z_1, Z_2) \rightsquigarrow \mathcal{N}(0, Id),$$

can be computed as

$$\begin{aligned} g_{n, m} &= \frac{1}{n!m!} \int_{\mathbb{R}^2} G(z_1, z_2) H_n(z_1) H_m(z_2) \varphi(z_1) \varphi(z_2) dz_1 dz_2 \\ &= \frac{1}{2\pi n!m!} \int_0^\infty \int_0^{2\pi} G(\rho \cos \theta, \rho \sin \theta) H_n(\rho \cos \theta) H_m(\rho \sin \theta) e^{-\frac{\rho^2}{2}} \rho d\theta d\rho \\ &= \frac{1}{2\pi n!m!} \int_0^\infty \int_{\arctan a}^{\arctan b} \left| \sqrt{\lambda_1} \cos \beta \cos \theta - \sqrt{\lambda_2} \sin \beta \sin \theta \right| H_n(\rho \cos \theta) H_m(\rho \sin \theta) e^{-\frac{\rho^2}{2}} \rho^2 d\theta d\rho, \end{aligned}$$

i.e. (49). So we obtain

$$\begin{aligned} &\frac{1}{h} \int_{-\infty}^\infty \varphi\left(\frac{u-v}{h}\right) \left[\int_{\mathcal{C}(v)} \zeta(\alpha(s)) d\sigma(s) \right] du \\ &= \sum_{l=0}^\infty \sum_{0 \leq n+m \leq l} \frac{c_{l-(n+m)}(u, h) g_{n, m}}{[l - (n+m)]!} \int_0^t \int_0^M H_{l-(n+m)}(\partial_x W(s, z)) H_m(Z_1(s, z)) H_n(Z_2(s, z)) dz ds, \end{aligned}$$

with c_k defined in (8).

Hence the Hermite expansion of $N_{sp}(u, v_1, v_2, t)$ can be deduced in the same way as in [14], namely

$$\frac{1}{h} \int_{-\infty}^\infty \varphi\left(\frac{u-y}{h}\right) \left[\int_{\mathcal{C}(y)} \zeta(\alpha(s)) d\sigma(s) \right] dy \xrightarrow{h \rightarrow 0} N_{sp}(u, v_1, v_2), \text{ in } L^2(\Omega),$$

where $N_{sp}(u, v_1, v_2, t) =$

$$\sum_{l=0}^\infty \sum_{0 \leq n+m \leq l} \frac{H_{l-(n+m)}(u) \varphi(u) g_{n, m}}{[l - (n+m)]!} \frac{1}{t} \int_0^t \int_0^M H_{l-(n+m)}(\partial_x W(s, z)) H_m\left(\frac{\eta_1(s, z)}{\sqrt{\lambda_1}}\right) H_n\left(\frac{\eta_2(s, z)}{\sqrt{\lambda_2}}\right) dz ds.$$

To obtain the Central Limit Theorem for $N_{sp}(u, v_1, v_2, t)$ when $t \rightarrow \infty$, we suppose, for simplicity and to be in the same context as in [14] in order to apply our method, that the field $W(s, x)$ is m -dependent in time, that is, its correlation function satisfies $r(s, z) = 0$, whenever $s > m$, for all z . The proof consists in an L^2 -approximation of $N_{sp}(u, v_1, v_2, t)$ by the finite expansion

$$N_{sp}^Q(u, v_1, v_2, t) := \sum_{l=0}^Q \sum_{0 \leq n+m \leq l} \frac{H_{l-(n+m)}(u) \varphi(u) g_{n,m}}{(l-(n+m))!} \quad (51)$$

$$\times \frac{1}{t} \int_0^t \int_0^M H_{l-(n+m)}(\partial_x W(s, z)) H_m \left(\frac{\eta_1(s, z)}{\sqrt{\lambda_1}} \right) H_n \left(\frac{\eta_2(s, z)}{\sqrt{\lambda_2}} \right) dz ds.$$

Then we shall study the asymptotic behavior of occupation functionals of the form

$$S_t(x) = \int_0^t \int_0^x F(\mathbf{W}(s, z)) dz ds,$$

where $F(., ., .)$ is a polynomial function and $\mathbf{W}(t, x) = (\partial_x W(t, x), \partial_{xx} W(t, x), \partial_{tx} W(t, x))^t$ the Gaussian random field already defined, and finally use it to get the asymptotic behavior of $N_{sp}^Q(u, v_1, v_2, t)$ to conclude to the CLT.

Let us start with the study of $S_t(x)$.

Put

$$\mathbf{R}(t, x) := \mathbb{E}[\mathbf{W}(t, x) \mathbf{W}(0, 0)^t], \quad \text{and} \quad \Sigma := \mathbf{R}(0, 0) = \mathbb{E}[\mathbf{W}(0, 0) \mathbf{W}(0, 0)^t].$$

Defining

$$\mathbf{Y}(t, x) := \Sigma^{-1/2} \mathbf{W}(t, x),$$

it holds that

$$\mathbf{Cov}[Y(0, 0)] = I, \quad \mathbb{E}[\mathbf{Y}(t, x) \mathbf{Y}(0, 0)^t] = \Sigma^{-1/2} \mathbf{R}(t, x) \Sigma^{-1/2},$$

and, since we supposed $M_{20} = 1$,

$$\mathbf{Y}(t, x) = \left(\partial_x W(t, x), \frac{\eta_1(t, x)}{\sqrt{\lambda_1}}, \frac{\eta_2(t, x)}{\sqrt{\lambda_2}} \right)^t := (Y_1(t, x), Y_2(t, x), Y_3(t, x))^t. \quad (52)$$

Then we can write

$$S_t(x) = \int_0^t \int_0^x F \circ \Sigma^{1/2}(\mathbf{Y}(s, z)) dz ds. \quad (53)$$

Note that $F \circ \Sigma^{1/2}$ is still a polynomial function, and that the field $\mathbf{Y}(t, x)$, from its definition, is also m -dependent in time.

Given that $F \circ \Sigma^{1/2} \in L^2(\phi(x_1)\phi(x_2)\phi(x_3)dx_1dx_2dx_3)$, we have

$$S_t(x) = \sum_{|\mathbf{k}| \leq N} c_{\mathbf{k}} \int_0^t \int_0^x H_{k_1}(Y_1(s, z)) H_{k_2}(Y_2(s, z)) H_{k_3}(Y_3(s, z)) dz ds,$$

where $\mathbf{k} = (k_1, k_2, k_3)$ and $|\mathbf{k}| = k_1 + k_2 + k_3$.

Defining

$$I(|\mathbf{k}|, k_2, k_3, s, z) := H_{|\mathbf{k}|-(k_2+k_3)}(Y_1(s, z)) H_{k_2}(Y_2(s, z)) H_{k_3}(Y_3(s, z)),$$

we can write

$$S_t(x) = \sum_{|\mathbf{k}|=0}^N \sum_{0 \leq k_2+k_3 \leq |\mathbf{k}|} c_{(|\mathbf{k}|-(k_2+k_3), k_2, k_3)} \int_0^t \int_0^x I(|\mathbf{k}|, k_2, k_3, s, z) dz ds.$$

The random variables $(Y_i(0, 0), 1 \leq i \leq 3)$ being independent standard Gaussian, using Parseval's equality gives

$$\mathbb{E}[(F \circ \Sigma^{1/2}(Y_1(0, 0), Y_2(0, 0), Y_3(0, 0)))^2] = \sum_{|\mathbf{k}| \leq N} c_{\mathbf{k}}^2 k_1! k_2! k_3! < \infty.$$

For $t > m$, we obtain

$$\begin{aligned}
t\mathbb{E} \left[\left(\frac{S_t(x)}{t} - c_0 \right)^2 \right] &= 2 \sum_{|\mathbf{k}|=1}^N \sum_{0 \leq k_2+k_3 \leq |\mathbf{k}|} \sum_{0 \leq m_2+m_3 \leq |\mathbf{k}|} c_{(|\mathbf{k}|-(k_2+k_3), k_2, k_3)} c_{(|\mathbf{k}|-(m_2+m_3), m_2, m_3)} \\
&\quad \times \int_0^m \int_0^x (x-z) \mathbb{E}[I(|\mathbf{k}|, k_2, k_3, s, z) I(|\mathbf{k}|, m_2, m_3, 0, 0)] dz ds \\
&:= \sigma^2(x),
\end{aligned} \tag{54}$$

by using the Diagram formula and the Dominated Convergence Theorem in the second equality.

Thus we have $\frac{S_t(x)}{t} \rightarrow c_0$ in probability, when $t \rightarrow \infty$; moreover an almost sure result holds too.

Let us study the weak limit of the sequence $\sqrt{t} \left(\frac{S_t(x)}{t} - c_0 \right)$.

Defining $\tilde{S}(s, x) = \sum_{|\mathbf{k}|=1}^N \sum_{0 \leq k_2+k_3 \leq |\mathbf{k}|} c_{(|\mathbf{k}|-(k_2+k_3), k_2, k_3)} \int_0^x I(|\mathbf{k}|, k_2, k_3, s, z) dz$, we have

$$\sqrt{t} \left(\frac{S_t(x)}{t} - c_0 \right) = \frac{1}{\sqrt{t}} \sum_{k=1}^{[t]} \int_{(k-1)}^k \tilde{S}(s, x) ds + \frac{1}{\sqrt{t}} \int_{[t]}^t \tilde{S}(s, x) ds. \tag{55}$$

But when introducing the time shift operator θ_h of the field \mathbf{W} , we have

$$\int_{(k-1)}^k \tilde{S}(s, x) ds = \theta_{k-1} \circ \int_0^1 \tilde{S}(s, x) ds,$$

hence (55) appears as a sum of m -dependent random variables having second moment, to which the Hoeffding-Robbins Theorem can be applied, providing that

$$\sqrt{t} \left(\frac{S_t(x)}{t} - c_0 \right) \Rightarrow \mathcal{N}(0, \sigma^2(x)), \quad \sigma^2(x) \text{ being defined in (54)}. \tag{56}$$

Now let us prove the L^2 -approximation of $N_{sp}(u, v_1, v_2, t)$ by $N_{sp}^Q(u, v_1, v_2, t)$ defined in (51), i.e. that

$$\limsup_{Q \rightarrow \infty} \limsup_{t \rightarrow \infty} \mathbb{E} \left([\sqrt{t} (N_{sp}(u, v_1, v_2, t) - N_{sp}^Q(u, v_1, v_2, t))]^2 \right) = 0.$$

Indeed, we can write

$$\sqrt{t} (N_{sp}(u, v_1, v_2, t) - N_{sp}^Q(u, v_1, v_2, t)) = \frac{1}{\sqrt{t}} \sum_{k=1}^{[t]} \theta_{k-1} \circ Y_Q(M) + \frac{1}{\sqrt{t}} I_2(t), \tag{57}$$

where

$$I_2(t) := \sum_{l=Q+1}^{\infty} \sum_{0 \leq n+m \leq l} \frac{H_{l-(n+m)}(u) \varphi(u) g_{n,m}}{(l-(n+m))!} \int_t^{[t]} \int_0^M H_{l-(n+m)}(\partial_x W(s, z)) H_m \left(\frac{\eta_1(s, z)}{\sqrt{\lambda_1}} \right) H_n \left(\frac{\eta_2(s, z)}{\sqrt{\lambda_2}} \right) dz ds,$$

and

$$Y_Q(M) = \sum_{l=Q+1}^{\infty} \sum_{0 \leq n+m \leq l} \frac{H_{l-(n+m)}(u) \varphi(u) g_{n,m}}{(l-(n+m))!} \int_0^1 \int_0^M H_{l-(n+m)}(\partial_x W(s, z)) H_m \left(\frac{\eta_1(s, z)}{\sqrt{\lambda_1}} \right) H_n \left(\frac{\eta_2(s, z)}{\sqrt{\lambda_2}} \right) dz ds.$$

The first term in (57) is a sum of m -dependent random variables whose asymptotic variance tends to zero as Q tends to infinity; the second term in (57) tends also to zero as $t \rightarrow \infty$ uniformly in Q since we have

$$\begin{aligned} \text{Var}(I_2(t)) &\leq \text{Var}[(t - [t])N_{sp}(u, v_1, v_2, t - [t])] = \text{Var}\left(\int_0^{t-[t]} \tilde{N}_{sp}(s, u, v_1, v_2) ds\right) \\ &\leq \mathbb{E}\left[\left(\int_0^{t-[t]} \tilde{N}_{sp}(s, u, v_1, v_2) ds\right)^2\right] \leq (t - [t])^2 \mathbb{E}[(N_{sp}(0, u, v_1, v_2))^2] \\ &\leq \mathbb{E}[(N_{sp}(0, u, v_1, v_2))^2] = O(1). \end{aligned}$$

To conclude to the CLT, it is enough to write $N_{sp}^Q(u, v_1, v_2, t) := \frac{1}{t} \int_0^t \int_0^M F_Q(\mathbf{Y}(s, z)) dz ds$, with $F_Q = F \circ \Sigma^{1/2}$, $c_0 = \mathbb{E}N_{sp}(u, v_1, v_2, t) = \gamma_0$, and to choose $x = M$; the asymptotic normality follows then from (53) and (56). \square

Remark: note that the proportion $\frac{N_{sp}(u, v_1, v_2, t)}{N_{[0, M]}}$ of the number of specular points with a given velocity (e.g. between $-v_2$ and v_1) among the number of specular points can be shown as in [13] for the number of maxima, to converge to a Gaussian r.v.

3.2 Twinkles

Longuet-Higgins has shown in [17] that the specular points evolve until a certain time up to the vanishing of the curvature. The number of images seen by the observer is not constant, specular points may appear or disappear, the images move. The creation or annihilation of specular points may be called a twinkle. At such a twinkle, the water surface is not only oriented to reflect light into the eye, but is curved so as to focus it there, which can be mathematically translated as, when introducing the function $f(t, x) = W(t, x) + \frac{1}{2}\kappa x^2$,

$$\begin{cases} Y_1(t, x) = \partial_x f(t, x) = 0 & : \text{ to have a specular point} \\ Y_2(t, x) = \partial_{xx} f(t, x) = 0 & : \text{ to have a singularity in the curvature,} \end{cases}$$

where $W(t, x)$ is the process whose spectral representation is given in (38).

We are interested in the number N_{TW} of twinkles.

Let \mathbf{Y} denotes the vectorial process $\mathbf{Y} = (Y_1(t, x), Y_2(t, x))$ such that:

$$\begin{cases} Y_1(t, x) = 0 \\ Y_2(t, x) = 0. \end{cases}$$

Let us suppose that the process W has fourth order continuous derivatives and let us M_{pq} denotes the mixed spectral moment:

$$M_{pq} = \int_{-\infty}^{\infty} \lambda^{p+q/2} f(\lambda) d\lambda.$$

Then we have

Proposition 2 *The mean number of twinkles at fixed time t and as $x \rightarrow \infty$, is given by*

$$\mathbb{E}[N_{TW}] = \frac{t}{\sqrt{2\pi^3}} \frac{\sqrt{M_{60}(M_{40}M_{22} - M_{31}^2)}}{\kappa M_{40}} e^{-\frac{\kappa^2}{2M_{40}}} \left[e^{-\frac{a^2}{2}} + a \int_0^a e^{-\frac{v^2}{2}} dv \right], \quad (58)$$

where $a := \frac{M_{31}}{\sqrt{M_{40}(M_{40}M_{22} - M_{31}^2)}}$.

We get back the same formula as Longuet-Higgins ([17], p.853), except to a factor 2 that corresponds again to consider the integral in x over $(-\infty, \infty)$.

Proof.

We will need a multidimensional Rice formula to count the number of roots of a nonlinear system of equations having the same number of equations and variables. Such a formula, available in Cabaña ([7], p.80) and also in Azais & Wschebor ([3]), leads to

$$\begin{aligned} \mathbb{E}[N_{TW}] &= \\ & \int_0^t \int_0^x \tilde{p}_{(s,u)}(0,0) \mathbb{E}[|\partial_{xx}f(s,u)\partial_{xt}f(s,u) - \partial_{xt}f(s,u)\partial_{xx}f(s,u)| \mid \partial_x f(s,u) = 0, \partial_{xx}f(s,u) = 0] du ds \\ &= \int_0^t \int_0^x p(-\kappa u, -\kappa) \mathbb{E}[|\partial_{xt}W(s,u)\partial_{xx}W(s,u)| \mid \partial_x W(s,u) = -\kappa u, \partial_{xx}W(s,u) = -\kappa] du ds, \end{aligned} \quad (59)$$

where $\tilde{p}_{(s,u)}(u_1, u_2)$ is the joint density of the random vector $(\partial_x f, \partial_{xx}f)$ and $p(u_1, u_2)$ is the joint density of $(\partial_x W, \partial_{xx}W)$.

Note that

$$\mathbb{E}[\partial_x W(s,u)\partial_{xt}W(s,u)] = \mathbb{E}[\partial_x W(s,u)\partial_{xx}W(s,u)] = 0, \quad \mathbb{E}[\partial_{xx}W(s,u)\partial_{xt}W(s,u)] = M_{31}, \quad \mathbb{E}[\partial_x W\partial_{xx}W(s,u)] = -M_{40}.$$

By using the regression model:

$$\begin{cases} \partial_{xt}W(s,u) = v_1\partial_x W(s,u) + \beta_1\partial_{xx}W(s,u) + e_1 \\ \partial_{xx}W(s,u) = v_2\partial_x W(s,u) + \beta_2\partial_{xx}W(s,u) + e_2, \end{cases}$$

where (e_1, e_2) is a Gaussian vector independent of $(\partial_x W(s,u), \partial_{xx}W(s,u))$, and $v_1 = 0$, $\beta_1 = \frac{M_{31}}{M_{40}}$, $v_2 = -\frac{M_{40}}{M_{20}}$ and $\beta_2 = 0$, the equation (59) becomes:

$$\begin{aligned} \mathbb{E}[N_{TW}] &= \int_0^t \int_0^x p(-\kappa u, -\kappa) \mathbb{E}\left[\left|\left(\frac{M_{31}}{M_{40}}\kappa + e_1\right)\left(\frac{M_{40}}{M_{20}}\kappa u + e_2\right)\right|\right] du ds \\ &= \frac{e^{-\frac{\kappa^2}{2M_{40}}}}{2\pi\sqrt{M_{20}M_{40}}} \int_0^t \int_0^x e^{-\frac{\kappa^2 u^2}{2M_{20}}} \mathbb{E}\left[\left|\left(\frac{M_{31}}{M_{40}}\kappa + e_1\right)\left(\frac{M_{40}}{M_{20}}\kappa u + e_2\right)\right|\right] du ds \\ &= t \frac{e^{-\frac{\kappa^2}{2M_{40}}}}{2\pi\sqrt{M_{20}M_{40}}} \mathbb{E}\left[\left|\frac{M_{31}}{M_{40}}\kappa + e_1\right|\right] \int_0^x e^{-\frac{\kappa^2 u^2}{2M_{20}}} \mathbb{E}\left[\left|\frac{M_{40}}{M_{20}}\kappa u + e_2\right|\right] du. \end{aligned}$$

On one hand, since $\sigma_1^2 := \mathbb{E}[e_1^2] = \frac{M_{22}M_{40} - M_{31}^2}{M_{40}}$, and $\sigma_2^2 := \mathbb{E}[e_2^2] = \frac{M_{60}M_{20} - M_{40}^2}{M_{20}}$, then

$$\mathbb{E}\left|\frac{M_{31}}{M_{40}}\kappa + e_1\right| = \sigma_1\sqrt{\frac{2}{\pi}} \left[\frac{M_{31}}{M_{40}\sigma_1} \kappa \int_0^{\frac{M_{31}}{M_{40}\sigma_1}\kappa} e^{-\frac{v^2}{2}} dv + e^{-\frac{(\frac{M_{31}}{M_{40}\sigma_1})^2 \kappa^2}{2}} \right].$$

On the other hand, we have

$$\begin{aligned} \int_0^x e^{-\frac{\kappa^2 u^2}{2M_{20}}} \mathbb{E}\left|\frac{M_{40}}{M_{20}}\kappa u + e_2\right| du &= \int_0^x e^{-\frac{\kappa^2 u^2}{2M_{20}}} \sigma_2\sqrt{\frac{2}{\pi}} \left[\frac{M_{40}}{M_{20}\sigma_2} \kappa u \int_0^{\frac{M_{40}}{M_{20}\sigma_2} \kappa u} e^{-\frac{v^2}{2}} dv + e^{-\frac{(\frac{M_{40}}{M_{20}\sigma_2})^2 \kappa^2 u^2}{2}} \right] du \\ &\xrightarrow{x \rightarrow \infty} \sigma_2\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\frac{\kappa^2 u^2}{2M_{20}}} \left[\frac{M_{40}}{M_{20}\sigma_2} \kappa u \int_0^{\frac{M_{40}}{M_{20}\sigma_2} \kappa u} e^{-\frac{v^2}{2}} dv + e^{-\frac{(\frac{M_{40}}{M_{20}\sigma_2})^2 \kappa^2 u^2}{2}} \right] du \\ &= \sigma_2\sqrt{\frac{2}{\pi}} \left[\left(\frac{M_{40}^2}{\sigma_2^2 M_{20}} + 1\right) \int_0^\infty e^{-\frac{\kappa^2 u^2}{2M_{20}}} \left(1 + \frac{M_{40}}{\sigma_2^2 M_{20}}\right) du \right] \\ &= \frac{1}{\kappa} (M_{40}^2 + \sigma_2^2 M_{20})^{1/2}. \end{aligned}$$

Hence we obtain

$$\mathbb{E}[N_{TW}] = \frac{t}{\kappa} \frac{e^{-\frac{\kappa^2}{2M_{40}}}}{\sqrt{2\pi^3}\sqrt{M_{20}M_{40}}} \sigma_1 \left[\frac{M_{31}}{M_{40}\sigma_1} \kappa \int_0^{\frac{M_{31}}{M_{40}\sigma_1}\kappa} e^{-\frac{v^2}{2}} dv + e^{-\frac{(\frac{M_{31}}{M_{40}\sigma_1})^2 \kappa^2}{2}} \right] (M_{40}^2 + \sigma_2^2 M_{20})^{1/2},$$

which gives (58), with $a := \frac{M_{31}}{M_{40}\sigma_1}$. \square .

The computation of the variance and to provide a CLT will be subject to further work.

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