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Abstract A statistic based on increment ratios (IR) and related to zero crossings of increment sequence is defined and studied for measuring the roughness of random paths. The main advantages of this statistic are robustness to smooth additive and multiplicative trends and applicability to infinite variance processes. The existence of the IR statistic limit (called the IR-roughness below) is closely related to the existence of a tangent process. Three particular cases where the IR-roughness exists and is explicitly computed are considered. Firstly, for a diffusion process with smooth diffusion and drift coefficients, the IR-roughness coincides with the IR-roughness of a Brownian motion and its convergence rate is obtained. Secondly, the case of rough Gaussian processes is studied in detail under general assumptions which do not require stationarity conditions. Thirdly, the IR-roughness of a Lévy process with $\alpha-$stable tangent process is established and can be used to estimate the fractional parameter $\alpha \in (0,2)$ following a central limit theorem.

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1. Introduction and the main results

It is well-known that random functions are typically "rough" (non-differentiable), which raise the question of determining and measuring roughness. Probably, the most studied roughness measures are the Hausdorff dimension and the $p-$variation index. There exists a considerable literature on statistical estimation of these and related quantities from a discrete grid. Hence, different estimators of the Hausdorff dimension have been studied, as the box-counting estimator (see Hall and Wood, 1993 for stationary Gaussian processes or Lévy-Véhel and Peltier, 1994, for Gaussian processes with stationary increments). To our knowledge, the H-variation estimator, where H is a measurable function, was first proposed by Guyon and Leon (1989) for stationary Gaussian processes where central and non-central limit theorems are established following the Hermite rank of H and the asymptotic local properties of the variogram and its second derivative. Further studies provided a continuation of this seminal paper in different ways. Istas and Lang (1997) studied generalized quadratic variations of Gaussian processes with stationary increments. Coeurjolly (2001 and 2005) studied ℓ^p-variations of fractional Brownian motion and ℓ^2-variations of multifractional Brownian motion. Coeurjolly (2007) discussed L-variations based on linear combinations of empirical quantiles for Gaussian locally self-similar processes. An estimator counting the number level crossings was investigated by Feuerverger *et al.* (1994) for stationary Gaussian processes.

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In the present paper we introduce a new characteristic of roughness, defined as a sum of ratios of consecutive increments. For a real-valued function $f = (f(t), t \in [0,1])$, define recursively

$$\begin{aligned} \Delta_j^{1,n} f &:= f(\frac{j+1}{n}) - f(\frac{j}{n}), \\ \Delta_j^{p,n} f &:= \Delta_j^{1,n} \Delta_j^{p-1,n} f = \sum_{i=0}^{p} (-1)^{p-i} \binom{p}{i} f(\frac{j+i}{n}), \end{aligned} \tag{1.1}$$

so that $\Delta_j^{p,n} f$ denotes the $p-$order increment of f at $\frac{j}{n}$, $p = 1, 2, \cdots$, $j = 0, 1, \cdots, n-p$. Let

$$R^{p,n}(f) := \frac{1}{n-p} \sum_{k=0}^{n-p-1} \frac{\left|\Delta_k^{p,n} f + \Delta_{k+1}^{p,n} f\right|}{|\Delta_k^{p,n} f| + |\Delta_{k+1}^{p,n} f|}, \tag{1.2}$$

with the convention $\frac{0}{0} := 1$. In particular,

$$R^{1,n}(f) = \frac{1}{n-1} \sum_{k=0}^{n-2} \frac{\left|f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right) + f\left(\frac{k+2}{n}\right) - f\left(\frac{k+1}{n}\right)\right|}{\left|f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)\right| + \left|f\left(\frac{k+2}{n}\right) - f\left(\frac{k+1}{n}\right)\right|}. \tag{1.3}$$

Note the ratio on the right-hand side of (1.2) is either 1 or less than 1 depending on whether the consecutive increments $\Delta_k^{p,n} f$ and $\Delta_{k+1}^{p,n} f$ have same signs or different signs; moreover, in the latter case, this ratio generally is small whenever the increments are similar in magnitude ("cancel each other"). Clearly, $0 \le R^{p,n}(f) \le 1$ for any f, n, p. Thus if $\lim R^{p,n}(f)$ exists when $n \to \infty$, the quantity $R^{p,n}(f)$ can be used to estimate this limit which represents the "mean roughness of f" also called the *$p-$th order IR-roughness of* f below. We show below that these definitions can be extended to sample paths of very general random processes, e.g. stationary processes, processes with stationary and nonstationary increments, and even $\mathbb{L}^q$-processes with $q < 1$.

Let us describe the main results of this paper. Section 2 derives some general results on asymptotic behavior of this estimator. Proposition 2.1 says that, for a sufficiently smooth function f, the limit $\lim_{n\to\infty} R^{p,n}(f) = 1$. In the most of the paper, $f = X$ is a random process. Following Dobrushin (1980), we say that $X = (X_t,\, t \in \mathbb{R})$ has a *small scale limit $Y^{(t_0)}$ at point $t_0 \in \mathbb{R}$* if there exist a normalization $A^{(t_0)}(\delta) \to \infty$ when $\delta \to 0$ and a random process $Y^{(t_0)} = (Y_\tau^{(t_0)}, \tau \ge 0)$ such that

$$A^{(t_0)}(\delta)\,(X_{t_0+\tau\delta} - X_{t_0}) \xrightarrow[\delta\to 0]{f.d.d.} Y_\tau^{(t_0)}, \tag{1.4}$$

where $\xrightarrow{f.d.d.}$ stands for weak convergence of finite dimensional distributions. A related definition is given in Falconer (2002, 2003) who called the limit process $Y^{(t_0)}$ a *tangent process* (at t_0). See also Benassi *et al.* (1997). In many cases, the normalization $A^{(t_0)}(\delta) = \delta^{H(t_0)}$, where $0 < H(t_0) < 1$ and the limit tangent process $Y^{(t_0)}$ is self-similar with index $H(t_0)$ (Falconer, 2003 or Dobrushin, 1980). Proposition 2.2 states that if X satisfies a similar condition to (1.4), then the statistic $R^{p,n}(X)$ converges to the integral

$$R^{p,n}(X) \xrightarrow[n\to\infty]{\mathcal{P}} \int_0^1 \mathrm{E}\left[\frac{|\Delta_0^p Y^{(t)} + \Delta_1^p Y^{(t)}|}{|\Delta_0^p Y^{(t)}| + |\Delta_1^p Y^{(t)}|}\right] \mathrm{d}t, \tag{1.5}$$

in probability, where $\Delta_j^p Y^{(t)} = \Delta_j^{p,1} Y^{(t)} = \sum_{i=0}^p (-1)^{p-i} \binom{p}{i} Y_{j+i}^{(t)}$, $j = 0, 1$ is the corresponding increment of the tangent process $Y^{(t)}$ at $t \in [0, 1)$. In the particular case when X has stationary increments, relation (1.5) becomes

$$R^{p,n}(X) \xrightarrow[n\to\infty]{\mathcal{P}} \mathrm{E}\left[\frac{|\Delta_0^p Y + \Delta_1^p Y|}{|\Delta_0^p Y| + |\Delta_1^p Y|}\right]. \tag{1.6}$$

Section 3 discusses the convergence in (1.5) for *diffusion processes* X admitting a stochastic differential $\mathrm{d}X = a_t \mathrm{d}B(t) + b_t \mathrm{d}t$, where B is a standard Brownian motion and $(a_t), (b_t)$ are random (adapted) functions. It is clear that under general regularity conditions on the diffusion and drift coefficients $(a_t), (b_t)$, the process

X admits the same local Hölder exponent as B at each point $t_0 \in (0,1)$ and therefore the IR-roughness of X in (1.5) should not depend on these coefficients and should coincide with the corresponding limit for $X = B$. This is indeed the case since the tangent process of X at t is easily seen to be $Y^{(t)} = a_t B$ and the multiplicative factor a_t cancels in the numerator and the denominator of the fraction inside the expectation in (1.5). See Proposition 3.1 for details, where the convergence rate $O(n^{1/3})$ (a.s.) in (1.5) with explicit limit values $\Lambda_p(1/2)$ is established for diffusions X and $p = 1, 2$.

Considerable attention is given to the asymptotic behavior of the statistic $R^{p,n}(X)$ for *"fractal" Gaussian processes* (see Section 4). In such a frame, *fractional Brownian motion* (fBm in the sequel) is a typical example. Indeed, if X is a fBm with parameter $H \in (0,1)$, then X is also its self tangent process for any $t \in [0,1]$ and (see Section 4):

$$R^{p,n}(X) \quad \underset{n\to\infty}{\overset{a.s.}{\longrightarrow}} \quad \Lambda_p(H) \qquad (p = 1, 2), \tag{1.7}$$

where

$$\Lambda_p(H) := \lambda(\rho_p(H)), \tag{1.8}$$

$$\lambda(r) := \frac{1}{\pi}\arccos(-r) + \frac{1}{\pi}\sqrt{\frac{1+r}{1-r}}\log\left(\frac{2}{1+r}\right), \tag{1.9}$$

$$\rho_p(H) := \operatorname{corr}\big(\Delta_0^p B_H, \Delta_1^p B_H\big), \tag{1.10}$$

and where $\Delta_j^1 B_H = B_H(j+1) - B_H(j)$, $\Delta_j^2 B_H = B_H(j+2) - 2B_H(j+1) - B_H(j)$ $(j \in \mathbb{Z})$ are respective increments of fBm. Moreover,

$$\sqrt{n}\big(R^{p,n}(X) - \Lambda_p(H)\big) \underset{n\to\infty}{\overset{\mathcal{D}}{\longrightarrow}} \mathcal{N}(0, \Sigma_p(H)) \quad \text{if} \quad \begin{cases} p = 1, & 0 < H < 3/4, \\ p = 2, & 0 < H < 1, \end{cases} \tag{1.11}$$

where $\underset{n\to\infty}{\overset{\mathcal{D}}{\longrightarrow}}$ stands for weak convergence of probability distributions. The asymptotic variances $\Sigma_p(H)$ in (1.11) are given by

$$\Sigma_p(H) := \sum_{j\in\mathbb{Z}} \operatorname{cov}\left(\frac{\big|\Delta_0^p B_H + \Delta_1^p B_H\big|}{\big|\Delta_0^p B_H\big| + \big|\Delta_1^p B_H\big|}, \frac{\big|\Delta_j^p B_H + \Delta_{j+1}^p B_H\big|}{\big|\Delta_j^p B_H\big| + \big|\Delta_{j+1}^p B_H\big|}\right). \tag{1.12}$$

The graphs of $\Lambda_p(H)$ and $\sqrt{\Sigma_p(H)}$ $(p = 1, 2)$ are given in Figures 1 and 2 below.

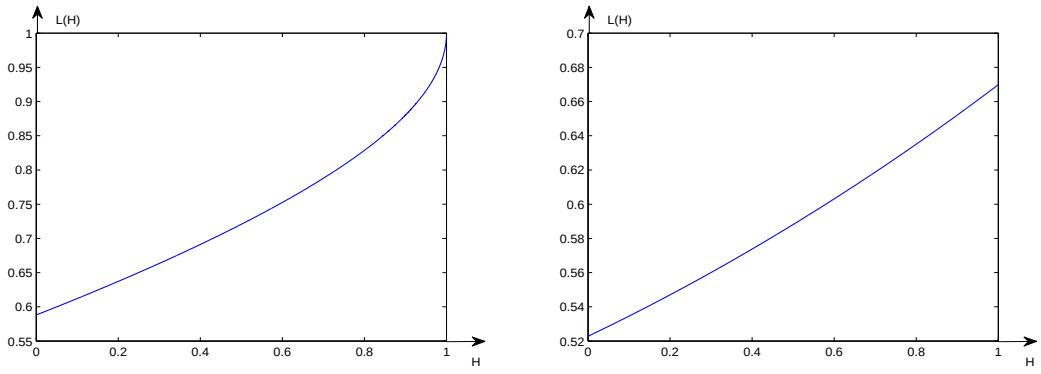

Figure 1. The graphs of $\Lambda_1(H)$ (left) and $\Lambda_2(H)$ (right).

The difference in the range of the parameter H for $p = 1$ and $p = 2$ in the central limit theorem in (1.11) are due to the fact that the second order increment process $\big(\Delta_j^2 B_H, j \in \mathbb{Z}\big)$ is a short memory stationary

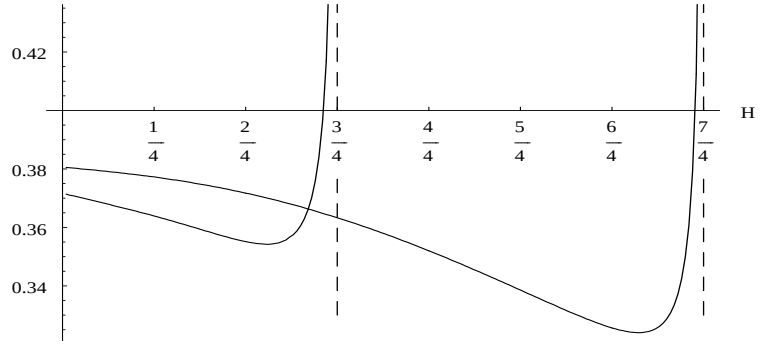

Figure 2. The graphs of $\sqrt{\Sigma_p(H)}$, $p=1$ (with a pole at 3/4) and $p=2$ (with a pole at 7/4) (from Stoncelis and Vaičiulis, 2008, with kind permission of the authors)

Gaussian process for any $H \in (0,1)$, in contrast to the first order increment process $\left(\Delta_j^1 B_H, j \in \mathbb{Z}\right)$ which has long memory for $H > 3/4$.

Generalizations of (1.7) and (1.11) to Gaussian processes having nonstationary increments are proposed in Section 4. Roughly speaking, $R^{p,n}(X)$, $p=1,2$ converge a.s. and satisfy a central limit theorem, provided for any $t \in [0,1]$ the process X admits a fBm with parameter $H(t)$ as a tangent process (more precise assumptions **(A.1)**, **(A.1)′** and $\mathbf{(A.2)}_p$ are provided in Section 4). In such frames, the limits in (1.7) are $\int_0^1 \Lambda_p(H(t))\mathrm{d}t$ instead of $\Lambda_p(H)$ and the asymptotic variances in (1.11) also change. The case of Gaussian processes with stationary increments is discussed in detail and the results are used to define a $\sqrt{n}-$consistent estimator of H, under semiparametric assumptions on the asymptotic behavior of the variogram or the spectral density. Bardet and Surgailis (2010) study a punctual estimator of $H(t_0)$ obtained from a localization around $t_0 \in (0,1)$ of the statistic $R^{2,n}(X)$.

The main advantages of estimators of the type (1.2) involving a scaling invariant function of increments seem to be the following. Firstly, the estimator $R^{p,n}(X)$ essentially depends on local regularity of the process X and not on possible "multiplicative and additive factors" such as diffusion and drift coefficients in Section 3 or smoothly multiplicative and additive trended Gaussian processes, see Proposition 4.1 of Section 4. This property is important when dealing with financial data involving heteroscedasticity and volatility clustering. Such a robustness property (also satisfied by the estimators based on generalized quadratic variations of wavelet coefficients) represents a clear advantage versus classical parametric Whittle or semi-parametric log-periodogram estimators. Secondly, the estimators in (1.2) are bounded functionals and have finite moments of any order. Section 5 discusses jump Lévy processes, with the Lévy measure regularly varying of fractional index $\alpha \in (0,2)$ at the origin. Using a modification of (1.2), we define a $\sqrt{n}-$consistent estimator of α, together with a central limit theorem, in a very general semiparametric frame. This result is new and interesting because there exist very few papers providing consistent estimators of α (to our knowledge, the only comparable results have been established in (Belomestny, 2010) and (Ait Sahalia and Jacod, 2009) in a financial and somewhat different context). Finally, in the Gaussian case, using the approximation formula provided in Remark 4.3, an estimator of H based on $R^{2,n}(X)$ can be extremely simply computed:

$$\widehat{H}_n^{(2)} \simeq \frac{1}{0.1468}\Big(\frac{1}{n-2}\sum_{k=0}^{n-3}\frac{\left|X_{\frac{k+2}{n}} - 2X_{\frac{k+1}{n}} + X_{\frac{k}{n}} + X_{\frac{k+3}{n}} - 2X_{\frac{k+2}{n}} + X_{\frac{k+1}{n}}\right|}{\left|X_{\frac{k+2}{n}} - 2X_{\frac{k+1}{n}} + X_{\frac{k}{n}}\right| + \left|X_{\frac{k+3}{n}} - 2X_{\frac{k+2}{n}} + X_{\frac{k+1}{n}}\right|}. - 0.5174\Big).$$

In the R language, if `X` is the vector $\left(X_{\frac{1}{n}}, X_{\frac{2}{n}}, \cdots, X_1\right)$,

```
H2n = (mean(abs(diff(diff(X[-1])) + diff(diff(X[-length(X)]))))
       /(abs(diff(diff(X[-1]))) + abs(diff(diff(X[-length(X)])))) - 0.5174)/0.1468.
```

Therefore its computation is very fast and does not require any tuning parameters such as the scales for

estimators based on quadratic variations or wavelet coefficients. The convergence rate of our estimator is $\sqrt{n}$ as for the parametric Whittle or the generalized quadratic variation estimators and hence it is more accurate than most of other well-known semi-parametric estimators (log-periodogram, local Whittle or wavelet based estimators).

Estimators of the form (1.2) can also be applied to discrete time (sequences) instead of continuous time processes (functions). For instance Surgailis *et al.* (2008) extended the statistic $R^{2,n}(X)$ to discrete time processes and used it to test for $I(d)$ behavior ($-1/2 < d < 5/4$) of observed time series. Vaičiulis (2009) considered estimation of the tail index of i.i.d. observations using an increment ratio statistic.

REMARK 1.1 The referee noted that the IR-roughness might be connected to the level crossing index (see Feuerverger *et al.*, 1994). To our surprise, such a connection indeed exists as explained below. Let $Y_n(t)$, $t \in [0, 1-\frac{1}{n}]$ be the linear interpolation of the "differenced" sequence $\Delta_j^{1,n}X = X(\frac{j+1}{n}) - X(\frac{j}{n})$, $j = 0, 1, \cdots, n-1$:

$$Y_n(t) = n\left[(\frac{j+1}{n} - t)\Delta_j^{1,n}X + (t - \frac{j}{n})\Delta_{j+1}^{1,n}X\right], \qquad t \in [\frac{j}{n}, \frac{j+1}{n}),$$

$j = 0, 1, \cdots, n-2$. Then, using Figure 3 as a proof,

$$\begin{aligned} R^{1,n}(X) &= \frac{n}{n-1}\sum_{j=0}^{n-2}\Big|\text{meas}\Big\{t \in [\frac{j}{n}, \frac{j+1}{n}) : Y_n(t) > 0\Big\} - \text{meas}\Big\{t \in [\frac{j}{n}, \frac{j+1}{n}) : Y_n(t) < 0\Big\}\Big| \quad (1.13) \\ &= \frac{n}{n-1}\sum_{j=0}^{n-2}\Big|\int_{\frac{j}{n}}^{\frac{j+1}{n}}(\mathbf{1}(Y_n(t) > 0) - \mathbf{1}(Y_n(t) < 0))\mathrm{d}t\Big|. \end{aligned}$$

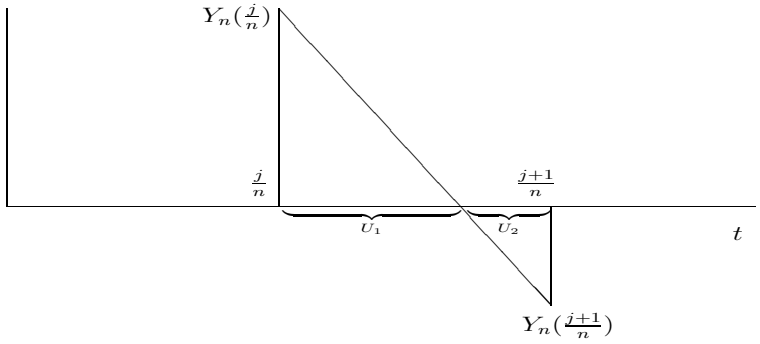

Figure 3. The proof of (1.13): follows by $\frac{|Y_n(\frac{j}{n})+Y_n(\frac{j+1}{n})|}{|Y_n(\frac{j}{n})|+|Y_n(\frac{j+1}{n})|} = n|U_1 - U_2|$.

Let $\psi(x_1, x_2) := |x_1 + x_2|/(|x_1| + |x_2|)$, $\psi_0(x_1, x_2) := \mathbf{1}(x_1x_2 \geq 0)$. Clearly, the two quantities $1 - \psi\big(Y_n(\frac{j}{n}), Y_n(\frac{j+1}{n})\big)$ and $1 - \psi_0\big(Y_n(\frac{j}{n}), Y_n(\frac{j+1}{n})\big)$ both are strictly positive if and only if Y_n crosses the zero level in the interval $[\frac{j}{n}, \frac{j+1}{n})$ but the former quantity measures not only the fact but also the "depth" of the crossing so that $1 - \psi\big(Y_n(\frac{j}{n}), Y_n(\frac{j+1}{n})\big)$ attains its maximal value 1 in the case of a "perfect" crossing in the middle of the interval $[\frac{j}{n}, \frac{j+1}{n})$ (see Figure 3).

It seems that similar asymptotic results can be obtained for $R_0^{p,n}(X) := \frac{1}{n-p}\sum_{k=0}^{n-p-1}\psi_0\big(\Delta_k^{p,n}X, \Delta_{k+1}^{p,n}X\big)$ measuring the number of zero crossings of the increment sequence $\Delta_k^{p,n}X, k = 0, 1, \cdots, n-p$ and other similar statistics obtained by replacing the functions ψ or ψ_0 by other scaling invariant functions. Let us note that $R_0^{1,n}(X)$ is related to the zero-crossings' counting statistic studied in Ho and Sun (1987) for stationary Gaussian time series. Also note that the Hermite rank of ψ_0 is 2 and that the corresponding limit function $\lambda_0(r) = \frac{1}{\pi}\arccos(-r)$ is strictly increasing on the interval $(-1, 1)$ similarly as the function $\lambda(r)$ in (1.9). On the other hand, while the statistic $R_0^{p,n}(X)$ is certainly of interest, the statistic $R^{p,n}(X)$ seems preferable to it for the reasons explained above. In particularly, in the case of symmetric Lévy processes X

with independent increments studied in Section 5, the latter statistic leads to an estimator of the fractional index while the former statistic can be easily shown to converge to 1/2.

The paper is organized as follows. Section 2 discusses some general (consistency) properties of the estimators $R^{p,n}(X)$. Section 3 deals with the case when X is a diffusion. The case of Gaussian processes X is considered in Section 4 while the case of Lévy processes is studied in Section 5. Section 6 contains proofs and other derivations.

Below, we write C for generic constants which may change from line to line.

2. Some asymptotic results

The definition of $R^{p,n}f$ in (1.2) can be extended to more general increments (the so-called generalized variations). Consider a filter $a := (a_0, \cdots, a_q) \in \mathbb{R}^{q+1}$ such that there exists $p \in \mathbb{N}, p \leq q$ satisfying

$$\sum_{\ell=0}^{q} \ell^i a_\ell = 0 \quad \text{for } i = 0, \cdots, p-1 \quad \text{and} \quad \sum_{\ell=0}^{q} \ell^p a_\ell \neq 0. \tag{2.1}$$

The class of such filters will be denoted $\mathcal{A}(p,q)$. For $n \in \mathbb{N}^* := \{1, 2, \cdots\}$ and a function $f = (f(t), t \in [0,1])$, define the generalized variations of f by

$$\Delta_j^{a,n} f \ := \ \sum_{\ell=0}^{q} a_\ell f(\frac{j+\ell}{n}), \qquad j = 0, 1, \cdots, n-q. \tag{2.2}$$

A particular case of (2.2) corresponding to $q = p \geq 1$, $a_\ell = (-1)^{p-\ell}\binom{p}{\ell}$ is the $p-$order increment $\Delta_j^{p,n} f$ in (1.1). For a filter $a \in \mathcal{A}(p,q)$, let

$$R^{a,n}(f) \ := \ \frac{1}{n-q} \sum_{k=0}^{n-q-1} \frac{\left|\Delta_k^{a,n} f + \Delta_{k+1}^{a,n} f\right|}{|\Delta_k^{a,n} f| + |\Delta_{k+1}^{a,n} f|}. \tag{2.3}$$

It is easy to prove that $R^{1,n}(f) \underset{n\to\infty}{\longrightarrow} 1$ if f is continuously differentiable on $[0,1]$ and the derivative f' does not vanish on $[0,1]$ except maybe for a finite number of points. Moreover, it is obvious that $R^{1,n}(f) = 1$ if f is monotone on $[0,1]$: the IR-roughness of a monotone function is the same as of a smooth function, which is not surprising since a similar fact holds for other measures of roughness such as the $p-$variation index or the Hausdorff dimension.

We conjecture that $R^{p,n}(f) \to 1$ and $R^{a,n}(f) \to 1$ for any $q \geq p \geq 1$, $a \in \mathcal{A}(p,q)$ and $f : [0,1] \to \mathbb{R}$ which is $(p-1)$ times differentiable and the derivative $f^{(p-1)}$ has bounded variation on $[0,1]$ with the support $\mathrm{supp}(f^{(p-1)}) = [0,1]$. However, we can prove a weaker result.

Proposition 2.1 *Let f be $(p-1)$-times continuously differentiable ($p \geq 1$) with $f^{(p-1)}$ being absolutely continuous on $[0,1]$ having the Radom–Nikodym derivative $g = \left(f^{(p-1)}\right)'$. Assume that $g \neq 0$ a.e. in $[0,1]$. Then $R^{p,n}(f) \underset{n\to\infty}{\longrightarrow} 1$ and $R^{a,n}(f) \underset{n\to\infty}{\longrightarrow} 1$ for any $a \in \mathcal{A}(p,q)$, $q \geq p$.*

Proof. We restrict the proof to the case $p = 2$ since the general case is analogous. Using summation by parts, we can rewrite $\Delta_j^{a,n} f$ as

$$\Delta_j^{a,n} f \ = \ \sum_{i=0}^{q} b_i \, \Delta_{i+j}^{2,n} f, \tag{2.4}$$

where $b_i := \sum_{k=0}^{i} \sum_{\ell=0}^{k} a_\ell$, $i = 0, 1, \cdots, q$, $b_{q-1} = b_q = 0$ and $\bar{b} := \sum_{i=0}^{q} b_i = \frac{1}{2}\sum_{i=1}^{q} i^2 a_i \neq 0$ in view of the assumption $a \in \mathcal{A}(2,q)$.

Assume n is large enough and for a given $t \in (0,1)$, let $k_n(t) \in \{0, \cdots, n-2\}$ be chosen so that $t \in [k_n(t)/n, (k_n(t)+1)/n)$, therefore $k_n(t) = [nt] - 1$. We claim that for a.e. $t \in (0,1)$

$$\lim_{n\to\infty} n^2 \Delta^{a,n}_{k_n(t)} f \;=\; \bar{b}\, g(t), \quad \lim_{n\to\infty} n^2 \Delta^{a,n}_{k_n(t)+1} f \;=\; \bar{b}\, g(t). \tag{2.5}$$

Using the fact that the function $(x_1, x_2) \mapsto \frac{|x_1+x_2|}{|x_1|+|x_2|}$ is continuous on $\mathbb{R}^2 \backslash \{(0,0)\}$, we obtain

$$h^{a,n}(t) \;:=\; \frac{\left| n^2 \Delta^{a,n}_{k_n(t)} f + n^2 \Delta^{a,n}_{k_n(t)+1} f \right|}{\left| n^2 \Delta^{a,n}_{k_n(t)} f \right| + \left| n^2 \Delta^{a,n}_{k_n(t)+1} f \right|} \;\underset{n\to\infty}{\longrightarrow}\; \frac{|\bar{b}\, g(t) + \bar{b}\, g(t)|}{|\bar{b}\, g(t)| + |\bar{b}\, g(t)|} \;=\; 1 \tag{2.6}$$

for a.e. $t \in (0,1)$, where we used the fact that $\bar{b}\, g(t) \neq 0$ a.e. Since for $n \geq q$, $R^{a,n}(f)$ can be written as $R^{a,n}(f) = \frac{n}{n-q} \int_0^1 h^{a,n}(t) \mathrm{d}t$, relation $R^{a,n}(f) \underset{n\to\infty}{\longrightarrow} 1$ follows by the dominated convergence theorem and the fact that $0 \leq h^{a,n}(t) \leq 1$.

Relations (2.5) can be proved using the Lebesgue–Vitali theorem (see Shilov and Gurevich, 1967, Ch. 4, §10, Theorem 1), as follows. Consider the signed measure μ on Borel subsets of $[0, 1/2]^2$ given by

$$\mu(A) = \int_A g(x_1 + x_2) \mathrm{d}x_1 \mathrm{d}x_2.$$

Note $\Delta^{2,n}_k f \;=\; \mu((k/2n, (k+2)/2n] \times (k/2n, (k+2)/2n]),\ (k = 0, \cdots, n-2)$. Since rectangles $[x_1, x_1 + h] \times [x_2, x_2 + h],\ (0 \leq x_i < x_i + h \leq 1/2, i = 1, 2)$ form a Vitali system on $[0, 1/2]^2$, the above mentioned Lebesgue–Vitali theorem implies that

$$\phi_n(t_1, t_2) := n^2 \mu \left(\left(\tfrac{k_n(t_1)}{2n}, \tfrac{k_n(t_1)+2}{2n} \right] \times \left(\tfrac{k_n(t_2)}{2n}, \tfrac{k_n(t_2)+2}{2n} \right] \right) \underset{n\to\infty}{\longrightarrow} g\left(\tfrac{t_1+t_2}{2} \right) \tag{2.7}$$

a.e. in $[0,1]^2$. Taking into account the form of the measure μ and the limiting function in (2.7), it follows the convergence $n^2 \Delta^{2,n}_{k_n(t)} f = \phi_n(t,t) \underset{n\to\infty}{\longrightarrow} g(t)$ a.e. on $[0,1]$. Next, for any fixed $i = 0, 1, \cdots$, the sequence of rectangles $(\frac{k_n(t_1)+i}{2n}, \frac{k_n(t_1)+i+2}{2n}] \times (\frac{k_n(t_2)+i}{2n}, \frac{k_n(t_2)+i+2}{2n}],\ n = 1, 2, \cdots$ is regularly contracting to $(t_1, t_2) \in (0,1)^2$ in the sense of (Shilov and Gurevich, 1967, Ch. 4, §10). Hence, using the lemma on p. 214 of the above monograph, it follows that $n^2 \mu\big(\big(\frac{k_n(t_1)+i}{2n}, \frac{k_n(t_1)+i+2}{2n} \big] \times \big(\frac{k_n(t_2)+i}{2n}, \frac{k_n(t_2)+i+2}{2n} \big] \big) \underset{n\to\infty}{\longrightarrow} g\big(\frac{t_1+t_2}{2} \big)$ a.e. in $[0,1]^2$, implying

$$n^2 \Delta^{2,n}_{k_n(t)+i} f \underset{n\to\infty}{\longrightarrow} g(t) \qquad \text{a.e. on } [0,1], \quad \text{for any } \; i = 0, 1, \cdots.$$

Together with (2.4), this proves (2.5) and the proposition. □

Let us turn now to the case when $f(t) = X_t$, $t \in [0,1]$ is a random process. Now and in all the sequel, $R^{p,n}(X)$, $R^{a,n}(X)$ are denoted $R^{p,n}$, $R^{a,n}$, respectively. Below we formulate a general condition for the convergence of $R^{p,n}$ and $R^{a,n}$ to a deterministic limit.

Assumption (A): For a.e. pairs $(t_1, t_2) \in (0,1)^2, t_1 \neq t_2$, for $i = 1, 2$ there exist:

(i) normalizations $A^{(t_i)}(\delta) \to \infty \;\; (\delta \to 0)$,

(ii) (mutually) independent random processes $Y^{(t_i)} = (Y^{(t_i)}(\tau), \tau \in [0,1])$,

such that for $\delta \to 0, s_1 \to t_1, s_2 \to t_2$

$$\Big(A^{(t_1)}(\delta)(X_{s_1+\delta\tau} - X_{s_1}), A^{(t_2)}(\delta)(X_{s_2+\delta\tau} - X_{s_2}) \Big) \;\overset{f.d.d.}{\longrightarrow}\; (Y^{(t_1)}(\tau), Y^{(t_2)}(\tau)). \tag{2.8}$$

Remark 2.1 Relation (2.8) implies the existence of a joint small scale limit $(Y^{(t_1)}, Y^{(t_2)})$ at a.e. pair $(t_1, t_2) \in (0,1)$, with independent components $Y^{(t_1)}, Y^{(t_2)}$. Note Assumption (A) and Proposition 2.2 below are very general, in the sense that they do not assume any particular structure or distribution of X.

Proposition 2.2 *Let $a = (a_0, \cdots, a_q) \in \mathcal{A}(p,q)$, $1 \le p \le q$ be a filter and let X satisfy Assumption (A). Assume in addition that $\mathrm{P}(|\Delta_j^a Y^{(t)}| > 0) = 1$, $j = 0, 1$ for a.e. $t \in (0,1)$, where $\Delta_j^a Z \equiv \Delta_j^{a,1} Z = \sum_{\ell=0}^q a_\ell Z(j+\ell)$. Then*

$$\mathrm{E}\Big(R^{a,n} - \int_0^1 \mathrm{E}\left[\frac{|\Delta_0^a Y^{(t)} + \Delta_1^a Y^{(t)}|}{|\Delta_0^a Y^{(t)}| + |\Delta_1^a Y^{(t)}|}\right] \mathrm{d}t\Big)^2 \underset{n\to\infty}{\longrightarrow} 0. \tag{2.9}$$

Proof. The statement follows from

$$\mathrm{E}R^{a,n} \underset{n\to\infty}{\longrightarrow} \int_0^1 \mathrm{E}\left[\frac{|\Delta_0^a Y^{(t)} + \Delta_1^a Y^{(t)}|}{|\Delta_0^a Y^{(t)}| + |\Delta_1^a Y^{(t)}|}\right] \mathrm{d}t \quad \text{and} \quad \mathrm{E}\left(R^{a,n} - \mathrm{E}R^{a,n}\right)^2 \underset{n\to\infty}{\longrightarrow} 0. \tag{2.10}$$

Write $\mathrm{E}R^{a,n} = \frac{n}{n-q} \int_0^1 \mathrm{E}h_X^{a,n}(t)\mathrm{d}t$, where (c.f. (2.6))

$$\begin{aligned} h_X^{a,n}(t) \quad &:= \quad \frac{A^{(t)}(1/n)\left(\left|\Delta_{k_n(t)}^{a,n} X + \Delta_{k_n(t)+1}^{a,n} X\right|\right)}{A^{(t)}(1/n)\left(\left|\Delta_{k_n(t)}^{a,n} X\right| + \left|\Delta_{k_n(t)+1}^{a,n} X\right|\right)} \\ &\underset{n\to\infty}{\overset{\mathcal{D}}{\longrightarrow}} \quad \frac{|\Delta_0^a Y^{(t)} + \Delta_1^a Y^{(t)}|}{|\Delta_0^a Y^{(t)}| + |\Delta_1^a Y^{(t)}|} \;=:\; h_Y^a(t), \end{aligned}$$

for a.e. $t \in (0,1)$, according to Assumption (A) and the continuous mapping theorem. Whence and from $0 \le h_X^{a,n} \le 1$ using the Lebesgue dominated convergence theorem the first relation in (2.10) follows. Moreover,

$$\mathrm{E}(R^{a,n} - \mathrm{E}R^{a,n})^2 \;=\; \Big(\frac{n}{n-q}\Big)^2 \int_0^1\int_0^1 \mathrm{E}\big[h_X^{a,n}(t)h_X^{a,n}(t')\big]\mathrm{d}t\mathrm{d}t' - \big(\mathrm{E}R^{a,n}\big)^2,$$

and with the same arguments as previously and the independence of $Y^{(t)}$ and $Y^{(t')}$ when $t \ne t'$,

$$\mathrm{E}\big[h_X^{a,n}(t)h_X^{a,n}(t')\big] \underset{n\to\infty}{\longrightarrow} \mathrm{E}\left[\frac{|\Delta_0^a Y^{(t)} + \Delta_1^a Y^{(t)}|}{|\Delta_0^a Y^{(t)}| + |\Delta_1^a Y^{(t)}|} \cdot \frac{|\Delta_0^a Y^{(t')} + \Delta_1^a Y^{(t')}|}{|\Delta_0^a Y^{(t')}| + |\Delta_1^a Y^{(t')}|}\right] \;=\; \mathrm{E}h_Y^a(t) \cdot \mathrm{E}h_Y^a(t')$$

and therefore $\Big(\frac{n}{n-q}\Big)^2 \int_0^1\int_0^1 \mathrm{E}\big[h_X^{a,n}(t)h_X^{a,n}(t')\big]\mathrm{d}t\mathrm{d}t' - \Big(\int_0^1 \mathrm{E}h_X^{a,n}(t)\mathrm{d}t\Big)^2 \underset{n\to\infty}{\longrightarrow} 0$, thereby proving the second relation in (2.10) and the proposition. □

The following easy but interesting corollary can also be added to this result. It proves that smooth multiplicative or additive trends do no change the $\mathbb{L}^2$-asymptotic behavior of $R^{a,n}$. Let $\mathcal{C}^p[0,1]$ denote the class of all $p-$times continuously differentiable functions on $[0,1]$.

Corollary 2.1 *Let $a \in \mathcal{A}(p,q)$ and X satisfy the conditions of Proposition 2.2 with $A^{(t)}(\delta) = O(\delta^{-1})$ $(\delta \to 0)$ for each $t \in [0,1]$. Assume that $\alpha \in \mathcal{C}^1[0,1]$, $\beta \in \mathcal{C}^p[0,1]$, $\inf_{t\in[0,1]} \alpha(t) > 0$ and $\sup_{t\in[0,1]} |X(t)| < \infty$ a.s. Define Z such that $Z_t = \alpha(t)\, X_t + \beta(t)$, $t \in [0,1]$. Then (2.9) holds with $R^{p,n} = R^{p,n}(X)$ replaced by $R^{p,n}(Z)$.*

Proof. We consider here $p \ge 2$ but the case $p = 1$ can be easily obtained. With $\alpha(\frac{k+j}{n}) = \alpha(\frac{k}{n}) + \frac{j}{n}\alpha'(\frac{k}{n})\big(1 + o(1)\big)$ and $\Delta_k^{a,n}\beta = O\big(\frac{1}{n^2}\big)$, for a.e. $t \in (0,1)$ and $k = k_n(t)$ as defined in the proof of Proposition 2.1, we deduce that

$$\Delta_k^{a,n} Z \;=\; \alpha\big(\frac{k}{n}\big)\Delta_k^{a,n} X + \frac{1}{n}\alpha'\big(\frac{k}{n}\big)\Delta_k^{a',n} X + o\big(\frac{1}{n}\big) \sup_{t\in(0,1)} |X(t)| + O\big(\frac{1}{n^2}\big),$$

with $a' = (ja_j)_{0\le j\le q} \in \mathcal{A}(p-1,q)$. Therefore,

$$\begin{aligned} A^{(t)}\big(\frac{1}{n}\big)\Delta_{k_n(t)}^{a,n} Z \quad &= \quad \alpha\big(\frac{k_n(t)}{n}\big)A^{(t)}\big(\frac{1}{n}\big)\Delta_{k_n(t)}^{a,n} X + \frac{1}{n}O_p\big(A^{(t)}\big(\frac{1}{n}\big)\Delta_{k_n(t)}^{a',n} X\big) + o_p\big(\frac{1}{n}A^{(t)}\big(\frac{1}{n}\big)\big) \\ &\underset{n\to\infty}{\overset{\mathcal{D}}{\longrightarrow}} \quad \alpha(t)\Delta_0^a Y^{(t)}. \end{aligned}$$

In a similar way, for a.e. pairs $(t,t') \in (0,1)^2$, $t \neq t'$, we can verify the joint convergence in distribution of r.v.'s $A^{(t)}(\frac{1}{n})\Delta^{a,n}_{k_n(t)}Z$, $A^{(t)}(\frac{1}{n})\Delta^{a,n}_{k_n(t)+1}Z$, $A^{(t')}(\frac{1}{n})\Delta^{a,n}_{k_n(t')}Z, A^{(t')}(\frac{1}{n})\Delta^{a,n}_{k_n(t')+1}Z$ to the limiting r.v.'s $\alpha(t)\Delta^a_0 Y^{(t)}, \alpha(t)\Delta^a_1 Y^{(t)}, \alpha(t')\Delta^a_0 Y^{(t')}, \alpha(t')\Delta^a_1 Y^{(t')}$. Now, the statement of the corollary follows by the argument at the end of the proof of Proposition 2.2. □

REMARK 2.2 By definition, the statistics $R^{p,n}$ and $R^{a,n}$ for $a \in \mathcal{A}(p,q)$, $1 \leq p \leq q$ are invariant with respect to additive polynomial trends of order less than p; in particular, $R^{3,n}$ is insensitive to a quadratic trend while $R^{2,n}$ does not have this property. On the other hand, Corollary 2.1 (see also Proposition 4.1) says that under weak additional conditions on X, any sufficiently smooth additive or multiplicative trends do not affect the limit of $R^{p,n}$ as soon as $p \geq 1$. In the important special case when the limit process $Y^{(t)} = B_H$ in Assumption (A) and (2.9) is a fractional Brownian motion with parameter $H \in (0,1)$ independent of t, the statistic $R^{p,n}$ converges in mean square to the expectation $\mathrm{E}\frac{|\Delta^p_0 B_H + \Delta^p_1 B_H|}{|\Delta^p_0 B_H| + |\Delta^p_1 B_H|} = \lambda(\rho_p(H))$, c.f. (1.8)-(1.10). Numerical computations show that the correlation coefficient $\rho_p(H)$ is a monotone function of H for any $p \geq 1$ and tends to constant value -1 on the interval $(0,1)$ as p increases. Therefore, for larger values of p, the range of $\lambda(\rho_p(H))$ is rather small and $R^{p,n}$ seems less capable to estimate H. A final reason for our concentrating on the "lower-order" statistics $R^{p,n}$, $p = 1,2$ in the rest of the paper is the fact that $R^{2,n}$ satisfies the central limit theorem in (1.11) on the whole interval $H \in (0,1)$.

3. Diffusions

Let

$$X_t \;=\; X_0 + \int_0^t a_s \mathrm{d}B(s) + \int_0^t b_s \mathrm{d}s, \qquad t \in [0,1] \tag{3.1}$$

be a diffusion (or Itô's) process on $\mathbb{R}$. In (3.1), we assume the existence of a right-continuous filtration $\mathcal{F} = (\mathcal{F}_t, t \in [0,1])$, a standard Brownian motion B adapted to $\mathcal{F}$; moreover, $a_s, b_s, s \in [0,1]$ are adapted random functions satisfying $\int_0^1 |b_s|\mathrm{d}s < \infty$, $\int_0^1 a_s^2 \mathrm{d}s < \infty$ a.s., and X_0 is a $\mathcal{F}_0-$measurable r.v. Write $\mathrm{E}_t[\cdot] = \mathrm{E}[\cdot|\mathcal{F}_t]$ for the conditional expectation. Let $\Lambda_1(1/2) = \lambda(\rho_1(1/2)) \simeq 0.7206$ and $\Lambda_2(1/2) = \lambda(\rho_2(1/2)) \simeq 0.5881$. The proof of the following Lemma 3.1 is given in Annexe.

Lemma 3.1 *Let $\psi(x_1,x_2) := |x_1+x_2|/(|x_1|+|x_2|)$ $(x_1,x_2 \in \mathbb{R})$, and let $Z_i, i = 1,2$ be independent $\mathcal{N}(0,1)$ r.v.'s. Then for any random variables ξ_1, ξ_2,*

$$|\mathrm{E}\psi(Z_1+\xi_1, Z_2+\xi_2) - \mathrm{E}\psi(Z_1,Z_2)| \;\leq\; 20 \max_{i=1,2} \left(\mathrm{E}\xi_i^2\right)^{1/3}. \tag{3.2}$$

Theorem 3.1 *Assume the following conditions: there exist random variables K_1, K_2 such that $0 < K_i < \infty$ a.s., and such that, for any sufficiently small $h > 0$ and any $0 \leq t < t+h \leq 1$, the following inequalities hold, a.s.:*

$$|a_t| \;\geq\; K_1, \qquad \mathrm{E}_t b^2_{t+h} \;\leq\; K_2 \quad \textit{and} \quad \mathrm{E}_t(a_{t+h} - a_t)^2 \;\leq\; K_2 h. \tag{3.3}$$

Then

$$R^{p,n} - \Lambda_p(1/2) \;=\; O(n^{-1/3}) \quad \textit{a.s.} \quad (p = 1,2). \tag{3.4}$$

Proof. We restrict the proof to the case $p = 1$ since the case $p = 2$ is analogous. For notational simplicity, assume that n is odd. Define

$$\eta_n(k) := \frac{|\Delta^{1,n}_k X + \Delta^{1,n}_{k+1} X|}{|\Delta^{1,n}_k X| + |\Delta^{1,n}_{k+1} X|}, \quad \eta'_n(k) := \mathrm{E}_{k/n}[\eta_n(k)], \quad \eta''_n(k) := \eta_n(k) - \eta'_n(k) \tag{3.5}$$

and correspondingly write $R^{1,n} = R'_n + R''_n,\ R'_n := (n-1)^{-1}\sum_{k=0}^{n-2}\eta'_n(k),\ R''_{n1} := (n-1)^{-1}\sum_{k=0}^{(n-2)/2}\eta''_n(2k)$, $R''_{n2} := (n-1)^{-1}\sum_{k=0}^{(n-4)/2}\eta''_n(2k+1)$. As $(\eta''_n(2k), \mathcal{F}_{(2k+2)/n}, k = 0, \dots, (n-2)/2)$ is a martingale difference sequence, so by Burkholder's inequality,

$$\mathrm{E}(R''_{n1})^8 \le Cn^{-8}\left(\sum_{k=0}^{(n-2)/2}\mathrm{E}^{1/4}(\eta'_n(2k))^8\right)^4 \le Cn^{-4}$$

and therefore

$$\sum_{n=1}^{\infty}\mathrm{P}(|R''_{n1}| > n^{-1/3}) \le C\sum_{n=1}^{\infty} n^{8/3}n^{-4} < \infty,$$

implying $R''_{n1} = O(n^{-1/3})$ a.s. A similar fact holds for R''_{n2}. Thus, it remains to prove

$$R'_n - \Lambda_1(1/2) = O(n^{-1/3}) \qquad a.s. \tag{3.6}$$

Observe

$$\eta'_n(k) - \Lambda_1(1/2) \;=\; \mathrm{E}_{k/n}\left[\frac{|Z_1(k)+\xi_1(k)+Z_2(k)+\xi_2(k)|}{|Z_1(k)+\xi_1(k)|+|Z_2(k)+\xi_2(k)|}\right] - \mathrm{E}\left[\frac{|Z_1(k)+Z_2(k)|}{|Z_1(k)|+|Z_2(k)|}\right],$$

where

$$\begin{aligned}
Z_1(k) &:= n^{1/2}\Delta_k^{1,n}B, \qquad Z_2(k) := n^{1/2}\Delta_{k+1}^{1,n}B,\\
\xi_1(k) &:= n^{1/2}\int_{k/n}^{(k+1)/n}\Big(\frac{a_s}{a_{k/n}}-1\Big)\mathrm{d}B(s) + n^{1/2}\int_{k/n}^{(k+1)/n}\frac{b_s}{a_{k/n}}\mathrm{d}s,\\
\xi_2(k) &:= n^{1/2}\int_{(k+1)/n}^{(k+2)/n}\Big(\frac{a_s}{a_{k/n}}-1\Big)\mathrm{d}B(s) + n^{1/2}\int_{(k+1)/n}^{(k+2)/n}\frac{b_s}{a_{k/n}}\mathrm{d}s.
\end{aligned}$$

According to Lemma 3.1 above, $|\eta'_n(k) - \Lambda_1(1/2)| \le 36\max_{i=1,2}\left(\mathrm{E}_{k/n}\xi_i^2(k)\right)^{1/3}$ and therefore

$$|R'_n - \Lambda_1(1/2)| \;\le\; 36\ \max\{(\mathrm{E}_{k/n}\xi_i^2(k))^{1/3} : i = 1, 2, k = 0, 1, \dots, n-1\}.$$

Whence, (3.6) follows from the following fact: there exists a r.v. $K < \infty$, independent of n and such that for any $n \ge 1,\ k = 0, \dots, n-1,\ i = 1, 2$

$$\mathrm{E}_{k/n}\xi_i^2(k) \;\le\; K\,n^{-1}, \quad \text{a.s.} \tag{3.7}$$

Indeed, using (3.3),

$$\begin{aligned}
\mathrm{E}_{k/n}\xi_1^2(k) &= n\int_{k/n}^{(k+1)/n}\mathrm{E}_{k/n}\Big(\frac{a_s}{a_{k/n}}-1\Big)^2\mathrm{d}s + n\mathrm{E}_{k/n}\Big(\int_{k/n}^{(k+1)/n}\frac{b_s}{a_{k/n}}\mathrm{d}s\Big)^2\\
&\le nK_1^{-2}\int_{k/n}^{(k+1)/n}\mathrm{E}_{k/n}\Big(a_s - a_{k/n}\Big)^2\mathrm{d}s + K_1^{-2}\int_{k/n}^{(k+1)/n}\mathrm{E}_{k/n}b_s^2\mathrm{d}s\\
&\le K_2K_1^{-2}n^{-1}, \qquad \text{a.s.},
\end{aligned}$$

and the bound (3.7) for $i = 2$ follows similarly. This proves (3.7) and Theorem 3.1, too. □

Let us present some examples of Itô's processes X satisfying conditions (3.3).

Example 3.1 Let $(X_t,\ t \in [0,1])$ be a Markov process satisfying a stochastic equation

$$X_t = x_0 + \int_0^t a(X_s)\mathrm{d}B(s) + \int_0^t b(X_s)\mathrm{d}s, \tag{3.8}$$

where $x_0 \in \mathbb{R}$ is nonrandom, $a(x), b(x), x \in \mathbb{R}$ are real measurable functions and B is a standard Brownian motion. Let $\mathcal{F}_t := \sigma\{B(s), s \le t\}, 0 \le t \le 1$ be the natural filtration. Assume that

$$|a(x) - a(y)| \le K|x - y|, \qquad |b(x) - b(y)| \le K|x - y| \qquad (x, y \in \mathbb{R}) \tag{3.9}$$

for some constant $K < \infty$. Then equation (3.8) admits a unique adapted solution; see e.g. Gikhman and Skorohod (1969). Let $a_t = a(X_t), b_t = b(X_t)$. Assume in addition that $|a(x)| \ge K_1, \ (x \in \mathbb{R})$ for some nonrandom constant $K_1 > 0$. Then the first inequality in (3.3) is trivially satisfied; moreover, the second and third relations in (3.3) are also satisfied, with $K_2 = C(1 + \sup_{0 \le t \le 1} X_t^2) < \infty$ and $K_3 = C$, where C is nonrandom and depends on the constant K in (3.9) only.

EXAMPLE 3.2 Let $X_t := g(t, B(t))$, where B is a standard Brownian motion and $g(t, x)$ is a (jointly) continuous function on $[0, 1] \times \mathbb{R}$, having continuous partial derivatives $g_t(t, x) := \partial g(t, x)/\partial t$, $g_x(t, x) := \partial g(t, x)/\partial x$, $g_{xx}(t, x) = \partial^2 g(t, x)/\partial x^2$. By Itô's lemma,

$$\mathrm{d}X_t = g_x(t, B(t))\mathrm{d}B(t) + \left(g_t(t, B(t)) + \frac{1}{2} g_{xx}(t, B(t)) \right) \mathrm{d}t,$$

so that X admits the representation (3.1) with $a_t = g_x(t, B(t)), b_t = g_t(t, B(t)) + \frac{1}{2} g_{xx}(t, B(t))$ and the same filtration as in the previous example. Assume that

$$|g_x(t, x)| \ge K_1, \qquad |g_x(s, y) - g_x(t, x)| \le K(|s - t|^{1/2} + |y - x|),$$

for all $(t, x), (s, y) \in [0, 1] \times \mathbb{R})$ and some constants $0 < K_1, K < \infty$. Then X satisfies the conditions in (3.3).

4. Gaussian processes

4.1. Assumptions

Let $X = (X_t, \, t \in [0, 1])$ be a Gaussian process, with zero mean. Without loss of generality, assume $X_0 = 0$. Define $\sigma^2_{p,n}(k)$, the variance of $\Delta^{p,n}_k X$, and $\rho_{p,n}(k)$, the correlation coefficient between $\Delta^{p,n}_k X$ and $\Delta^{p,n}_{k+1} X$, *i.e.*

$$\sigma^2_{p,n}(k) \;:=\; \mathrm{E}\Big[\left(\Delta^{p,n}_k X\right)^2 \Big], \qquad \rho_{p,n}(k) \;:=\; \frac{\mathrm{E}\big[\Delta^{p,n}_k X \; \Delta^{p,n}_{k+1} X\big]}{\sigma_{p,n}(k)\sigma_{p,n}(k+1)}. \tag{4.1}$$

Let $B_H = (B_H(t), t \in \mathbb{R})$ be a fractional Brownian motion (fBm) with parameter $0 < H < 1$, i.e., a Gaussian process with zero mean and covariance such that $\mathrm{E}B_H(s)B_H(t) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t - s|^{2H}\right)$. Its $p-$th order increments $\left(\Delta^p_j B_H, j \in \mathbb{Z}\right)$ form a stationary Gaussian process, for any $p \ge 1$. In particular, the covariance function of $\Delta_j B_H \equiv \Delta^1_j B_H = B_H(j+1) - B_H(j)$ and $\Delta^2_j B_H = B_H(j+2) - 2B_H(j+1) + B_H(j)$ can be explicitly calculated:

$$\mathrm{E}\big[\Delta_0 B_H \Delta_j B_H\big] \;=\; 2^{-1}\left(|j+1|^{2H} + |j-1|^{2H} - 2|j|^{2H}\right), \tag{4.2}$$

$$\mathrm{E}\big[\Delta^2_0 B_H \Delta^2_j B_H\big] \;=\; 2^{-1}\left(-|j+2|^{2H} + 4|j+1|^{2H} - 6|j|^{2H} + 4|j-1|^{2H} - |j-2|^{2H}\right). \tag{4.3}$$

From Taylor expansion,

$$\begin{aligned} \mathrm{E}\big[\Delta_0 B_H \Delta_j B_H\big] &\sim 2H(2H-1)j^{2H-2}, \\ \mathrm{E}\big[\Delta^2_0 B_H \Delta^2_j B_H\big] &\sim 2H(2H-1)(2H-2)(2H-3)j^{2H-4}, \end{aligned}$$

as $j \to \infty$, and therefore the first increment, $(\Delta_j B_H)$, has a summable covariance if and only if $0 < H < 3/4$, while the second increment, $(\Delta^2_j B_H)$, has a summable covariance for any $0 < H < 1$.

Introduce the following conditions:

(A.1) There exist continuous functions $H(t) \in (0,1)$ and $c(t) > 0$ for $t \in [0,1]$ such that $\forall j \in \mathbb{N}^*$

$$\lim_{n\to\infty} \sup_{t\in(0,1)} \left| \frac{\mathrm{E}\big(X_{([nt]+j)/n} - X_{[nt]/n}\big)^2}{(j/n)^{2H(t)}} - c(t) \right| \;=\; 0, \quad \text{with} \tag{4.4}$$

$$\lim_{n\to\infty} \sup_{t\in(0,1)} \big|H(t) - H(t+\frac{1}{n})\big| \log n = 0. \tag{4.5}$$

(A.1)′ There exist continuous functions $H(t) \in (0,1)$ and $c(t) > 0$ for $t \in [0,1]$ such that $\forall j \in \mathbb{N}^*$

$$\lim_{n\to\infty} \sup_{t\in(0,1)} \sqrt{n} \left| \frac{\mathrm{E}\big(X_{([nt]+j)/n} - X_{[nt]/n}\big)^2}{(j/n)^{2H(t)}} - c(t) \right| \;=\; 0, \quad \text{with} \tag{4.6}$$

$$\lim_{n\to\infty} \sup_{t\in(0,1)} \big|H(t) - H(t+\frac{1}{n})\big|\sqrt{n} \log n = 0 \quad \text{and} \quad \lim_{n\to\infty} \sup_{t\in(0,1)} \big|c(t) - c(t+\frac{1}{n})\big|\sqrt{n} = 0. \tag{4.7}$$

(A.2)$_p$ There exist $d > 0$, $\gamma > 1/2$ and $0 \le \theta < \gamma/2$ such that for any $1 \le k < j \le n$ with $n \in \mathbb{N}^*$

$$\Big|\mathrm{E}\big[\Delta_k^{p,n} X \; \Delta_j^{p,n} X\big]\Big| \;\le\; d\; \sigma_{p,n}(k)\sigma_{p,n}(j) \cdot n^{\theta} \cdot |j-k|^{-\gamma}. \tag{4.8}$$

A straightforward application of Assumption **(A.1)** (or **(A.1)′**) implies that $\sqrt{c(t)}B_{H(t)}$ is the tangent process of X for all $t \in (0,1)$ and more precisely:

Property 4.1 *Assumptions* **(A.1)**, **(A.1)′** *respectively imply that, for any* $j \in \mathbb{Z}$ *and* $p = 1, 2$,

$$\lim_{n\to\infty} \sup_{t\in(0,1)} \left| \frac{\mathrm{E}\big[\Delta_{[nt]}^{p,n} X \; \Delta_{j+[nt]}^{p,n} X\big]}{\mathrm{E}\big[\Delta_0^{p,n} B_{H(t)} \; \Delta_j^{p,n} B_{H(t)}\big]} - c(t) \right| \;=\; 0, \tag{4.9}$$

$$\lim_{n\to\infty} \sqrt{n} \sup_{t\in(0,1)} \left| \frac{\mathrm{E}\big[\Delta_{[nt]}^{p,n} X \; \Delta_{j+[nt]}^{p,n} X\big]}{\mathrm{E}\big[\Delta_0^{p,n} B_{H(t)} \; \Delta_j^{p,n} B_{H(t)}\big]} - c(t) \right| \;=\; 0. \tag{4.10}$$

Moreover, for any $t \in (0,1)$ *and* $p = 1, 2$

$$\Big(n^{H(t)} \Delta_{j+[nt]}^{p,n} X\Big)_{j\in\mathbb{Z}} \;\overset{\substack{f.d.d.\\ n\to\infty}}{\longrightarrow}\; \Big(\sqrt{c(t)}\, \Delta_j^p B_{H(t)}\Big)_{j\in\mathbb{Z}}.$$

Assumption **(A.1)** can be characterized as *uniform local self-similarity of* (X_t) (the uniformity refers to the supremum over $t \in (0,1)$ in (4.4)). Note that for X having stationary increments and variogram $V(t) = \mathrm{E}X_t^2$, Assumption **(A.1)** reduces to $V(t) \sim ct^{2H}$ $(c > 0, 0 < H < 1)$. For $j = 0, 1$, relation (4.9) implies that for any $t \in (0,1)$, the variance and the $(1/n)-$lag correlation coefficient of $\Delta_{[nt]}^{p,n} X$ satisfy the following relations:

$$\sigma_{1,n}^2([nt]) \;\underset{n\to\infty}{\sim}\; c(t)\sigma_1^2(H(t)) = c(t)\,\mathrm{E}\big[(\Delta_0 B_{H(t)})^2\big] = c(t)\Big(\frac{1}{n}\Big)^{2H(t)}, \tag{4.11}$$

$$\rho_{1,n}([nt]) \;\underset{n\to\infty}{\longrightarrow}\; \rho_1(H(t)) = \;\mathrm{corr}(B_{H(t)}(1), B_{H(t)}(2) - B_{H(t)}(1)) = 2^{2H(t)-1} - 1, \tag{4.12}$$

$$\sigma_{2,n}^2([nt]) \;\underset{n\to\infty}{\sim}\; c(t)\sigma_2^2(H(t)) = c(t)\,\mathrm{E}\big[(\Delta_0^2 B_{H(t)})^2\big] = c(t)(4 - 4^{H(t)})\Big(\frac{1}{n}\Big)^{2H(t)}, \tag{4.13}$$

$$\begin{aligned} \rho_{2,n}([nt]) \;&\underset{n\to\infty}{\longrightarrow}\; \rho_2(H(t)) = \;\mathrm{corr}(B_{H(t)}(2) - 2B_{H(t)}, B_{H(t)}(3) - 2B_{H(t)}(2) + B_{H(t)}(1)) \\ &= \quad \frac{-3^{2H(t)} + 2^{2H(t)+2} - 7}{8 - 2^{2H(t)+1}} \end{aligned} \tag{4.14}$$

(see (6.31)). Moreover, relations (4.11)–(4.14) hold uniformly in $t \in (0,1)$. Condition (4.5) is a technical condition which implies (and is "almost equivalent" to) the continuity of the function $t \to H(t)$. Assumption $(\mathbf{A.1})'$ is a sharper convergence condition than Assumption **(A.1)** required for establishing central limit theorems.

Condition (4.8) specifies a nonasymptotic inequality satisfied by the correlation of increments $\Delta_k^{p,n}X$. The particular case of stationary processes allows to better understand this point. Indeed, if (X_t) has stationary increments, the covariance of the stationary process $(\Delta_k^{p,n}X, k \in \mathbb{Z})$ is completely determined by the variogram $V(t)$, e.g.

$$\mathrm{E}\big[\Delta_k^{1,n}X\ \Delta_j^{1,n}X\big] = \frac{1}{2}\Big\{V\Big(\frac{k-j+1}{n}\Big) + V\Big(\frac{k-j-1}{n}\Big) - 2V\Big(\frac{k-j}{n}\Big)\Big\}. \tag{4.15}$$

In the "most regular" case, when $X = B_H$ is a fBm and therefore $V(t) = t^{2H}$, it is easy to check that assumption $\mathbf{(A.2)}_2$ holds with $\theta = 0$ and $\gamma = 4-2H > 2$ $(0 < H < 1)$, while $\mathbf{(A.2)}_1$ with $\theta = 0, \gamma = 2-2H$ is equivalent to $H < 3/4$ because of the requirement $\gamma > 1/2$. However, for $X = B_H$, $\mathbf{(A.2)}_1$ holds with appropriate $\theta > 0$ in the wider region $0 < H < 7/8$, by choosing $\theta < 2-2H$ arbitrarily close to $2-2H$ and then $\gamma < 2-2H+\theta$ arbitrary close to $4-4H$. A similar choice of parameters θ and γ allows to satisfy $\mathbf{(A.2)}_p$ for more general X with stationary increments and variogram $V(t) \sim ct^{2H}$ $(t \to 0)$, under additional regularity conditions on $V(t)$ (see below).

Property 4.2 *Let X have stationary increments and variogram $V(t) \sim ct^{2H}$ $(t \to 0)$, with $c > 0$, $H \in (0,1)$. (i) Assume, in addition, that $0 < H < 7/8$ and $|V''(t)| \le Ct^{-\kappa}$ $(0 < t < 1)$, for some $C > 0$ and $4-4H > \kappa \ge 2-2H, \kappa > 1/2$. Then assumption $\mathbf{(A.2)}_1$ holds.*

(ii) Assume, in addition, that $|V^{(4)}(t)| \le Ct^{-\kappa}$ $(0 < t < 1)$, for some $C > 0$ and $8-4H > \kappa \ge 4-2H$. Then assumption $\mathbf{(A.2)}_2$ holds.

The following property provides a sufficient condition for $\mathbf{(A.2)}_p$ in spectral terms, which does not require differentiability of the variogram.

Property 4.3 *Let X be a Gaussian process having stationary increments and the spectral representation (see for instance Cramèr and Leadbetter, 1967)*

$$X_t \ = \ \int_{\mathbb{R}} \big(\mathrm{e}^{\mathrm{i}t\xi} - 1\big)\, f^{1/2}(\xi)\, W(\mathrm{d}\xi), \qquad \textit{for all}\ \ t \in \mathbb{R}, \tag{4.16}$$

where $W(\mathrm{d}x) = \overline{W(-\mathrm{d}x)}$ is a complex-valued Gaussian white noise with zero mean and variance $\mathrm{E}\,|W(\mathrm{d}x)|^2 = \mathrm{d}x$ and f is a non-negative even function called the spectral density of X such that

$$\int_{\mathbb{R}} \big(1 \wedge |\xi|^2\big)\, f(\xi)\, \mathrm{d}\xi \ < \ \infty. \tag{4.17}$$

Moreover, assume that f is differentiable on (K, ∞) and

$$f(\xi) \sim c\,\xi^{-2H-1} \quad (\xi \to \infty), \qquad |f'(\xi)| \le C\,\xi^{-2H-2} \quad (\xi > K) \tag{4.18}$$

for some constants $c, C, K > 0$. Then X satisfies assumption $\mathbf{(A.2)}_1$ for $0 < H < 3/4$ and assumption $\mathbf{(A.2)}_2$ for $0 < H < 1$.

4.2. Limit theorems

Before establishing limit theorems for the statistics $R^{p,n}$ for Gaussian processes, recall that λ is given in (1.9) and with $\rho_p(H)$ in (6.31) one has

$$\int_0^1 \Lambda_p(H(t))\mathrm{d}t \ = \int_0^1 \lambda(\rho_p(H(t)))\mathrm{d}t \ = \ \int_0^1 \mathrm{E}\left[\frac{\big|\Delta_0^p B_{H(t)} + \Delta_1^p B_{H(t)}\big|}{\big|\Delta_0^p B_{H(t)}\big| + \big|\Delta_1^p B_{H(t)}\big|}\right]\mathrm{d}t.$$

Straightforward computations show that Assumptions $\mathbf{(A.1)}$ and $\mathbf{(A.2)}_p$ imply Assumption $\mathbf{(A)}$ with $A^{(t)}(\delta) = \delta^{-H(t)}$, $Y^{(t)} = \sqrt{c(t)}\, B_{H(t)}$. Therefore Proposition 2.2 ensures the convergence (in $\mathbb{L}^2$) of the statistics $R^{p,n}$ to $\int_0^1 \Lambda_p(H(t))\mathrm{d}t$. Bardet and Surgailis (2009) proved a.s. convergence in Theorem 4.1, below, using a general moment bound for functions of multivariate Gaussian processes (see Lemma 1 in Section 6). A sketch of this proof can be found in Section 6.

Theorem 4.1 *Let X be a Gaussian process satisfying Assumptions* **(A.1)** *and* **(A.2)**$_p$. *Then,*

$$R^{p,n} \xrightarrow[n\to\infty]{a.s.} \int_0^1 \Lambda_p(H(t))\mathrm{d}t \qquad (p=1,2). \tag{4.19}$$

Corollary 4.1 *Assume that X is a Gaussian process having stationary increments, whose variogram satisfies the conditions of Properties 4.2 or 4.3. Then*

$$R^{p,n} \xrightarrow[n\to\infty]{a.s.} \Lambda_p(H) \qquad (p=1,2). \tag{4.20}$$

The following Theorem 4.2 is also established in Bardet and Surgailis (2009). Its proof (see a sketch of this proof in Section 6) uses a general central limit theorem for Gaussian subordinated nonstationary triangular arrays (see Theorem 1 in Section 6). Note that the Hermite rank of $\psi(x_1,x_2)=|x_1+x_2|/(|x_1|+|x_2|)$ is 2 and this explains the difference between the cases $p=1$ and $p=2$ in Theorem 4.2: in the first case, the inequalities in (6.12) for $(\boldsymbol{Y}_n(k))$ as defined in (6.9)-(6.10) hold only if $\sup_{t\in[0,1]} H(t) < 3/4$, while in the latter case these inequalities hold for $0 < \sup_{t\in[0,1]} H(t) < 1$. A similar fact is true also for the estimators based on generalized quadratic variations, see Istas and Lang (1997), Coeurjolly (2001).

Theorem 4.2 *Let X be a Gaussian process satisfying assumptions* **(A.1)**$'$ *and* **(A.2)**$_p$, *with $\theta=0$. Moreover, assume additionally* $\sup_{t\in[0,1]} H(t) < 3/4$ *if $p=1$. Then, for $p=1,2$,*

$$\sqrt{n}\Big(R^{p,n} - \int_0^1 \Lambda_p(H(t))\mathrm{d}t\Big) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}\Big(0, \int_0^1 \Sigma_p(H(\tau))\mathrm{d}\tau\Big), \tag{4.21}$$

with $\Lambda_p(H)$ and $\Sigma_p(H)$ given in (1.8) and (1.12), respectively.

The following proposition shows that the previous theorems are satisfied when smooth multiplicative and additive trends are considered.

Proposition 4.1 *Let $Z=(Z_t=\alpha(t)X_t+\beta(t),\, t\in[0,1])$, where $X=(X_t, t\in[0,1])$ is a zero mean Gaussian process and α,β are deterministic continuous functions on $[0,1]$ with $\inf_{t\in[0,1]}\alpha(t)>0$.*

(i) Let X satisfy assumptions of Theorem 4.2 and $\alpha\in\mathcal{C}^p[0,1], \beta\in\mathcal{C}^p[0,1]$. Then the statement of Theorem 4.2 holds with X replaced by Z.

(ii) Let X satisfy assumptions of Theorem 4.1 and $\alpha\in\mathcal{C}^1[0,1], \beta\in\mathcal{C}^1[0,1]$. Then the statement of Theorem 4.1 holds with X replaced by Z.

Remark 4.1 A version of the central limit theorem in (4.21) is established in Bardet and Surgailis (2009) with $\int_0^1 \Lambda_p(H(t))\mathrm{d}t$ replaced by $\mathrm{E}R^{p,n}$ under weaker assumption than **(A.1)**$'$ or even **(A.1)**: only properties (4.11)-(4.12) (for $p=1$) and (4.13)-(4.14) (for $p=2$), in addition to **(A.2)**$_p$ with $\theta=0$, are required.

The particular case of Gaussian processes having stationary increments can also be studied:

Corollary 4.2 *Assume that X is a Gaussian process having stationary increments and there exist $c>0$, $C>0$ and $0<H<1$ such that at least one of the two following conditions (a), (b) hold:*

(a) variogram $V(t)=c\,t^{2H}\big(1+o(t^{1/2})\big)$ *for* $t\to 0$ *and* $|V^{(2p)}(t)|\le Ct^{2H-2p}$ *for all* $t\in(0,1]$;

(b) spectral density f satisfies (4.17), (4.18) and $f(\xi)=c\xi^{-2H-1}\big(1+o(\xi^{-1/2})\big)$ $(\xi\to\infty)$.

Then:

$$\sqrt{n}\big(R^{p,n}-\Lambda_p(H)\big) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}\Big(0,\Sigma_p(H)\Big) \qquad \text{if} \quad \begin{cases} p=1, & 0<H<3/4, \\ p=2, & 0<H<1. \end{cases} \tag{4.22}$$

Moreover, with the expression of $s_2^2(H)$ given in Section 6,

$$\sqrt{n}\Big(\Lambda_2^{-1}(R^{2,n})-H\Big) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}\big(0,s_2^2(H)\big). \tag{4.23}$$

Therefore, $\widehat{H}_n = \Lambda_2^{-1}(R^{2,n})$ is an estimator of the parameter H following a central limit theorem with a convergence rate $\sqrt{n}$ under semi-parametric assumptions. Similar results were obtained by Guyon and Leon (1989) and Istas and Lang (1997) for generalized quadratic variations under less general assumptions.

REMARK 4.2 In the context of Corollary 4.2 and $H \in (3/4, 1)$, we expect that $R^{1,n}$ follows a nongaussian limit distribution with convergence rate n^{2-2H}.

REMARK 4.3 Figure 1 exhibits that $H \mapsto \Lambda_2(H)$ is nearly linear and is well-approximated by $.1468H + .5174$. Consequently, $\int_0^1 \Lambda_2(H(t))\mathrm{d}t \approx .1468\bar{H} + .5174$, where $\bar{H} = \int_0^1 H(t)\mathrm{d}t$ is the mean value of the function $H(\cdot)$.

Another interesting particular case of Theorem 4.2 leads to a punctual estimator of the function $H(t)$ from a localization of the statistic $R^{2,n}$. For $t_0 \in (0,1)$ and $\alpha \in (0,1)$, define

$$R_\alpha^{2,n}(t_0) := \frac{1}{2n^\alpha} \sum_{k=[nt_0-n^\alpha]}^{[nt_0+n^\alpha]} \frac{\left|\Delta_k^{2,n}X + \Delta_{k+1}^{2,n}X\right|}{|\Delta_k^{2,n}X| + |\Delta_{k+1}^{2,n}X|}.$$

This estimator is studied in Bardet and Surgailis (2010) and compared to the estimator based on generalized quadratic variations discussed in Benassi *et al.* (1998) and Coeurjolly (2005).

4.3. Examples

Below we provide some concrete examples of Gaussian processes which admit a fBm as the tangent process. For some examples the hypotheses of Theorems 4.1-4.2 and the subsequent corollaries are satisfied. For other examples, the verification of our hypotheses (in particular, of the crucial covariance bound $(\mathbf{A.2})_p$) remains an open problem and will be discussed elsewhere.

EXAMPLE 4.1 *Fractional Brownian motion (fBm).* As noted above, a fBm $X = B_H$ satisfies $\mathbf{(A.1)'}$ as well as $\mathbf{(A.2)}_1$ (for $0 < H < 3/4$ if $\theta = 0$ and $0 < H < 7/8$ if $0 < \theta < 2 - 2H$ with θ arbitrary close to $2 - 2H$ and therefore $\gamma < 2 - 2H + \theta$ arbitrary close to $4 - 4H$ may satisfy $\gamma > 1/2$) and $\mathbf{(A.2)}_2$ (for $0 < H < 1$), with $H(t) \equiv H$, $c(t) \equiv c$. Therefore, for fBm both Theorems 4.1 (the almost sure convergence, satisfied for $0 < H < 7/8$ when $p = 1$ and for $0 < H < 1$ when $p = 2$) and 4.2 (the central limit theorem, satisfied for $0 < H < 3/4$ when $p = 1$ and for $0 < H < 1$ when $p = 2$)) apply. Obviously, a fBm also satisfies the conditions of Corollary 4.2. Thus, the rate of convergence of the estimator $\Lambda_2^{-1}(R^{2,n}) =: \widehat{H}_n$ of H is $\sqrt{n}$. But in such a case the self-similarity property of fBm allows to use in this case asymptotically efficient Whittle or maximum likelihood estimators (see Fox and Taqqu, 1987, or Dahlhaus, 1989). However, for a fBm with a continuously differentiable multiplicative and additive trends, which leads to a semi-parametric context, the convergence rate of $\widehat{H}_n$ is still $\sqrt{n}$ while parametric estimators cannot be applied.

EXAMPLE 4.2 *Multiscale fractional Brownian motion* (see Bardet and Bertrand, 2007) defined as follows: for $\ell \in \mathbb{N}^*$, a (M_ℓ)-multiscale fractional Brownian motion $X = (X_t,\, t \in \mathbb{R})$ ((M_ℓ)- fBm for short) is a Gaussian process having stationary increments and a spectral density f such that

$$f(\xi) = \frac{\sigma_j^2}{|\xi|^{2H_j+1}}\, \mathbf{1}(\omega_j \le |\xi| < \omega_{j+1}) \quad \text{for all} \quad \xi \in \mathbb{R} \tag{4.24}$$

with $\omega_0 := 0 < \omega_1 < \cdots < \omega_\ell < \omega_{\ell+1} := \infty$, $\sigma_i > 0$ and $H_i \in \mathbb{R}$ for $i \in \{0, \cdots, \ell\}$ with $H_0 < 1$ and $H_\ell > 0$. Therefore condition (4.18) of Property 4.3 is satisfied, with $K = \omega_\ell$ and $H = H_\ell$. Moreover, the condition $f(\xi) = c\xi^{-2H-1}\big(1 + o(\xi^{-1/2})\big)$ $(\xi \to \infty)$ required in Corollary 4.2 is also checked with $H = H_\ell$. Consequently, the same conclusions as in the previous example apply for this process as well, in the respective regions determined by the parameter H_ℓ at high frequencies $x > \omega_\ell$ alone. The same result is also obtained for a more general process defined by $f(\xi) = c\xi^{-2H-1}$ for $|\xi| \ge \omega$ and condition (4.17) is only required elsewhere. Once again, such conclusions hold also in case of continuously differentiable multiplicative and additive trends.

EXAMPLE 4.3 *Multifractional Brownian motion* (mBm) (see Ayache *et al.*, 2005). A mBm $X = (X_t,\, t \in [0,1])$ is a Gaussian process defined by

$$X_t \;=\; B_{H(t)}(t) \;=\; g(H(t)) \int_{\mathbb{R}} \frac{\mathrm{e}^{\mathrm{i}tx}-1}{|x|^{H(t)+1/2}} W(\mathrm{d}x), \tag{4.25}$$

where $W(\mathrm{d}x)$ is the same as in (4.16), $H(t)$ is a (continuous) function on $[0,1]$ taking values in $(0,1)$ and finally, $g(H(t))$ is a normalization such that $\mathrm{E}X_t^2 = 1$. It is well-known that a mBm is locally asymptotically self-similar at each point $t \in (0,1)$ having a fBm $B_{H(t)}$ as its tangent process at t (see Benassi *et al.*, 1997). This example is studied in more detail in Bardet and Surgailis (2010).

EXAMPLE 4.4 *Time-varying fractionally integrated processes.* Philippe *et al.* (2006, 2008) introduced two classes of mutually inverse time-varying fractionally integrated filters with discrete time and studied long-memory properties of the corresponding filtered white noise processes. Surgailis (2008) extended these filters to continuous time and defined "multifractional" Gaussian processes $(X_t,\, t \ge 0)$ and $(Y_t, t \ge 0)$ as follows

$$X_t \;=\; \int_{\mathbb{R}} \Big\{ \int_0^t \frac{1}{\Gamma(H(\tau)-.5)} (\tau - s)_+^{H(\tau)-1.5} \mathrm{e}^{A_-(s,\tau)} \mathrm{d}\tau \Big\} \mathrm{d}B(s), \tag{4.26}$$

$$Y_t \;=\; \int_{\mathbb{R}} \frac{1}{\Gamma(H(s)+.5)} \Big\{ (t-s)_+^{H(s)-.5} \mathrm{e}^{-A_+(s,t)} - (-s)_+^{H(s)-.5} \mathrm{e}^{-A_+(s,0)} \Big\} \mathrm{d}B(s), \tag{4.27}$$

where $s_+^\alpha := s^\alpha \mathbf{1}(s>0)$, B is a Brownian motion,

$$A_-(s,t) := \int_s^t \frac{H(u)-H(t)}{t-u} \mathrm{d}u, \qquad A_+(s,t) := \int_s^t \frac{H(s)-H(v)}{v-s} \mathrm{d}v \qquad (s<t)$$

and where $H(t), t \in \mathbb{R}$ is a general function taking values in $(0,\infty)$ and satisfying some weak additional conditions. Surgailis (2008) studied small and large scale limits of (X_t) and (Y_t) and showed that these processes resemble a fBm with Hurst parameter $H = H(t)$ at each point $t \in \mathbb{R}$ (i.e., admit a fBm as a tangent process) similarly to the mBm in the previous example. The last paper also argues that these processes present a more natural generalization of fBm than the mBm and have nicer dependence properties of increments. We expect that the assumptions $(\mathbf{A}.\mathbf{1}), (\mathbf{A}.\mathbf{1})', (\mathbf{A}.\mathbf{2})_p$ can be verified for (4.26), (4.27); however, this question requires further work.

5. Processes with independent increments

In this section, we assume that $X = (X_t,\, t \ge 0)$ is a (homogeneous) Lévy process, with a.s. right continuous trajectories, $X_0 = 0$. It is well-known that if the generating triplet of X satisfies certain conditions (in particularly, if the Lévy measure ν behaves regularly at the origin with index $\alpha \in (0,2)$), then X has a tangent process Y which is $\alpha-$stable Lévy process. A natural question is to estimate the parameter α with the help of the introduced statistics $R^{p,n}$. Unfortunately, the limit of these statistics as defined in (1.5) through the tangent process depends also on the skewness parameter $\beta \in [-1,1]$ of the $\alpha-$stable tangent process Y and so this limit cannot be used for determining of α if β is unknown.

In order to avoid this difficulty, we shall slightly modified our ratio statistic, as follow. Observe first that the second differences $\Delta_k^{2,n} X$ of Lévy process have a *symmetric* distribution (in contrast to the first differences $\Delta_k^{1,n} X$ which are not necessary symmetric). For notational simplicity we shall assume in this section that n is *even*. The modified statistic

$$\tilde{R}^{2,n} \;:=\; \frac{1}{n/2-1} \sum_{k=0}^{(n-4)/2} \psi\big(\Delta_{2k}^{2,n} X, \Delta_{2k+2}^{2,n} X\big), \qquad \psi(x,y) := \frac{|x+y|}{|x|+|y|}$$

is written in terms of "disjoint" (independent) second order increments $(\Delta_{2k}^{2,n} X, \Delta_{2k+2}^{2,n} X)$ having a symmetric joint distribution. Instead of extending general result of Proposition 2.2 to $\tilde{R}^{2,n}$, we shall directly obtain its

convergence under suitable assumptions on X. Note first

$$\mathrm{E}\tilde{R}^{2,n} \;=\; \mathrm{E}\psi\big(X^{(2)}_{1/n} - X^{(1)}_{1/n}, X^{(4)}_{1/n} - X^{(3)}_{1/n}\big), \tag{5.1}$$

where $X^{(i)}, i = 1, \cdots, 4$ are independent copies of X. Note that $1/2 \le \mathrm{E}\tilde{R}^{2,n} \le 1$ since

$$\begin{aligned}\mathrm{E}\psi\big(X^{(2)}_{1/n} - X^{(1)}_{1/n}, X^{(4)}_{1/n} - X^{(3)}_{1/n}\big) &\ge \mathrm{P}(X^{(2)}_{1/n} - X^{(1)}_{1/n} \ge 0, X^{(4)}_{1/n} - X^{(3)}_{1/n} \ge 0)\\ &\qquad + \mathrm{P}(X^{(2)}_{1/n} - X^{(1)}_{1/n} < 0, X^{(4)}_{1/n} - X^{(3)}_{1/n} < 0)\\ &\ge \mathrm{P}^2(X^{(2)}_{1/n} - X^{(1)}_{1/n} \ge 0) + \mathrm{P}^2(X^{(2)}_{1/n} - X^{(1)}_{1/n} < 0) \ge 1/2.\end{aligned}$$

Proposition 5.1 *Let there exists a limit*

$$\lim_{n\to\infty} \mathrm{E}\tilde{R}^{2,n} = \tilde{\Lambda}. \tag{5.2}$$

Then

$$\tilde{R}^{2,n} \;\xrightarrow[n\to\infty]{a.s.}\; \tilde{\Lambda}. \tag{5.3}$$

Proof. Write $\tilde{R}^{2,n} = \mathrm{E}\tilde{R}^{2,n} + (n/2-1)^{-1}Q_n$, where Q_n is a sum of centered 1-dependent r.v.'s which are bounded by 1 in absolute value. Therefore $\mathrm{E}((n/2-1)^{-1}Q_n)^4 = O(n^{-2})$ and the a.s. convergence $(n/2-1)^{-1}Q_n \to 0$ follows by the Chebyshev inequality. □

Next we discuss conditions on X for the convergence in (5.2). Recall that the distribution of X_t is infinitely divisible and its characteristic function is given by

$$\mathrm{E}\mathrm{e}^{\mathrm{i}\theta X_t} = \exp\Big\{t\Big(\mathrm{i}\gamma\theta - \frac{1}{2}a^2\theta^2 + \int_{\mathbb{R}} \big(\mathrm{e}^{\mathrm{i}u\theta} - 1 - \mathrm{i}u\theta\mathbf{1}(|u|\le 1)\big)\nu(\mathrm{d}u)\Big)\Big\}, \quad \theta\in\mathbb{R}, \tag{5.4}$$

where $\gamma \in \mathbb{R}, a \ge 0$ and ν is a measure on $\mathbb{R}$ such that $\int_{\mathbb{R}} \min(u^2, 1)\nu(\mathrm{d}u) < \infty$. The triplet (a, γ, ν) is called the generating triplet of X (Sato (1999)). Let $X^{(i)}, i = 1, 2$ be independent copies of X. Note $W_t := X^{(1)}_t - X^{(2)}_t$ is a Lévy process having the characteristic function

$$\mathrm{E}\mathrm{e}^{\mathrm{i}\theta W_t} = \exp\Big\{t\Big(-a^2\theta^2 + 2\int_0^\infty \mathrm{Re}\big(1 - \mathrm{e}^{\mathrm{i}u\theta}\big)\mathrm{d}K(u)\Big)\Big\}, \quad \theta\in\mathbb{R}, \tag{5.5}$$

where

$$K(u) \;:=\; \nu((-\infty, -u] \cup [u, \infty))$$

is monotone nonincreasing on $(0, \infty)$. Introduce the following condition: there exist $0 < \alpha \le 2$ and $c > 0$ such that

$$K(u) \sim \frac{c}{u^\alpha}, \quad u \downarrow 0. \tag{5.6}$$

It is clear that if such number α exists then $\alpha := \inf\big\{r \ge 0 : \int_{|x|\le 1} |x|^r \nu(dx) < \infty\big\}$ is the so-called fractional order or the Blumenthal-Getoor index of the Lévy process X.
Let Z_α be a standard $\alpha-$stable r.v. with characteristic function $\mathrm{E}\mathrm{e}^{\mathrm{i}\theta Z_\alpha} = \mathrm{e}^{-|\theta|^\alpha}$ and $Z^{(i)}_\alpha, i = 1, 2, 3$ be independent copies of Z_α.

Proposition 5.2 *Assume either $a > 0$ or else, $a = 0$ and condition (5.6) with $0 < \alpha \le 2$ and $c > 0$. Then $t^{-1/\alpha}(X^{(1)}_t - X^{(2)}_t) \xrightarrow[t\to 0]{\mathcal{D}} \tilde{c}\, Z_\alpha$ with $\tilde{c}$ depending on c, and (5.2), (5.3) hold with*

$$\tilde{\Lambda} \equiv \tilde{\Lambda}(\alpha) := \mathrm{E}\psi\big(Z^{(1)}_\alpha, Z^{(2)}_\alpha\big).$$

Moreover, with $\tilde{\sigma}^2(\alpha) := 2\mathrm{var}\big(\psi\big(Z^{(1)}_\alpha, Z^{(2)}_\alpha\big)\big) + 4\mathrm{cov}\big(\psi\big(Z^{(1)}_\alpha, Z^{(2)}_\alpha\big), \psi\big(Z^{(2)}_\alpha, Z^{(3)}_\alpha\big)\big)$,

$$\sqrt{n}(\tilde{R}^{2,n} - \mathrm{E}\tilde{R}^{2,n}) \;\xrightarrow[n\to\infty]{\mathcal{D}}\; \mathcal{N}(0, \tilde{\sigma}^2(\alpha)). \tag{5.7}$$

Proof. Relation $t^{-1/\alpha}W_t = t^{-1/\alpha}(X_t^{(1)} - X_t^{(2)}) \xrightarrow[t\to 0]{\mathcal{D}} \tilde{c}\, Z_\alpha$ is an easy consequence of the assumptions of the proposition and the general criterion of weak convergence of infinitely divisible distributions in Sato (1991, Theorem 8.7). It implies (5.2) by the fact that ψ is a.e. continuous on $\mathbb{R}^2$. Since $\tilde{R}^{2,n}$ is a sum of 1-dependent stationary and bounded r.v.'s, the central limit theorem in (5.7) follows from convergence of the variance:

$$n\,\mathrm{var}(\tilde{R}^{2,n}) \to \tilde{\sigma}^2(\alpha); \tag{5.8}$$

see e.g. Berk (1973). Rewrite $\tilde{R}^{2,n} = (n/2-1)^{-1}\sum_{k=0}^{(n-4)/2}\tilde{\eta}_n(k)$, $\tilde{\eta}_n(k) := \psi\big(\Delta_{2k}^{2,n}X, \Delta_{2k+2}^{2,n}X\big)$. We have

$$n\,\mathrm{var}(\tilde{R}^{2,n}) = \frac{n}{n/2-1}\,\mathrm{var}(\tilde{\eta}_n(0)) + \frac{2n(n/2-2)}{(n/2-1)^2}\mathrm{cov}(\tilde{\eta}_n(0),\tilde{\eta}_n(1)),$$

where $\mathrm{var}(\tilde{\eta}_n(0)) \underset{n\to\infty}{\longrightarrow} \mathrm{var}\big(\psi\big(Z_\alpha^{(1)}, Z_\alpha^{(2)}\big)\big)$, $\mathrm{cov}(\tilde{\eta}_n(0),\tilde{\eta}_n(1)) \underset{n\to\infty}{\longrightarrow} \mathrm{cov}\big(\psi\big(Z_\alpha^{(1)}, Z_\alpha^{(2)}\big), \psi\big(Z_\alpha^{(2)}, Z_\alpha^{(3)}\big)\big)$ similarly as in the proof of (5.2) above. This proves (5.8) and the proposition. □

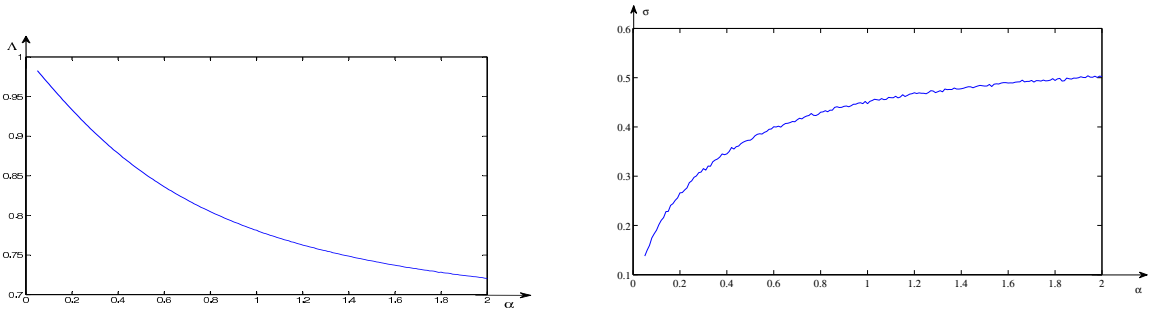

Figure 4. The graphs of $\alpha \mapsto \tilde{\Lambda}(\alpha) = \mathrm{E}\frac{|Z_\alpha^{(1)}+Z_\alpha^{(2)}|}{|Z_\alpha^{(1)}|+|Z_\alpha^{(2)}|}$ (left) and $\alpha \mapsto \tilde{\sigma}(\alpha)$ (right) for a process with independent increments.

The graph of $\tilde{\Lambda}(\alpha)$ is given in Figure 4. Note that $\tilde{\Lambda}(2) = \Lambda_1(1/2) \simeq 0.72$: this is the case of Brownian motion.

In order to evaluate the decay rate of the bias $\mathrm{E}\tilde{R}^{2,n} - \tilde{\Lambda}(\alpha)$ we need a uniform convergence rate in Lemma 5.1, below, for

$$\|F_n - G_\alpha\|_\infty := \sup_{x\in\mathbb{R}}|F_n(x) - G_\alpha(x)|, \quad F_n(x) := \mathrm{P}(n^{1/\alpha}W_{1/n} \le x), \quad G_\alpha(x) := \mathrm{P}(\tilde{Z}_\alpha \le x),$$

where $\tilde{Z}_\alpha := \tilde{c}Z_\alpha$ is the limiting $\alpha-$stable r.v. in Proposition 5.2 and $(W_t, t \ge 0)$ is the symmetric Levy process with characteristic function as in (5.5). The proof of Lemma 5.1 is given in Annexe.

Lemma 5.1 *(i) Let $a = 0$ and K satisfy (5.6). Denote $K_1(u) := K(u) - cu^{-\alpha}$, $|K_1|(u) := \int_u^\infty |\mathrm{d}K_1(v)|$, the variation of K_1 on $[u,\infty)$. Moreover, assume that there exist some constants $\beta, \delta > 0$ such that*

$$|K_1|(u) = O(u^{-(\alpha-\beta)_+}) \quad (u\to 0), \quad |K_1|(u) = O(u^{-\delta}) \quad (u\to\infty), \tag{5.9}$$

where $x_+ := \max(0,x)$. Then

$$\|F_n - G_\alpha\|_\infty = \begin{cases} O(n^{-\beta/\alpha}), & \text{if } \beta < \alpha, \\ O(n^{-1}\log n), & \text{if } \beta = \alpha, \\ O(n^{-1}), & \text{if } \beta > \alpha. \end{cases} \tag{5.10}$$

(ii) Let $a > 0$ and K satisfy

$$K(u) = O(u^{-\alpha}) \quad (u\to 0), \quad K(u) = O(u^{-\delta}) \quad (u\to\infty), \tag{5.11}$$

for some $0 \le \alpha < 2$, $\delta > 0$. Then

$$\|F_n - G_\alpha\|_\infty = \begin{cases} O(n^{-1+\alpha/2}), & \text{if } \alpha > 0, \\ O(n^{-1}\log n), & \text{if } \alpha = 0. \end{cases} \tag{5.12}$$

Proposition 5.3 *Assume either $a > 0$ or else $a = 0$ and condition (5.6). Then for any $\alpha \in (0, 2]$*

$$|\mathrm{E}\tilde{R}^{2,n} - \tilde{\Lambda}(\alpha)| \;\le\; 2C\|F_n - G_\alpha\|_\infty, \qquad C := \int_0^\infty (1+z)^{-2}\mathrm{d}z < \infty. \tag{5.13}$$

Proof. Let $\tilde{\psi}(x,y) := |x-y|/(x+y),\ x,y>0$, and let F_n, G_α be the same as in Lemma 5.1. Similarly as in Vaičiulis (2009, proof of Th. 1), write

$$\mathrm{E}\tilde{R}^{2,n} - \tilde{\Lambda}(\alpha) \;=\; 2\int_0^\infty\int_0^\infty \tilde{\psi}(x,y)(\mathrm{d}F_n(x)\mathrm{d}F_n(y) - \mathrm{d}G_\alpha(x)\mathrm{d}G_\alpha(y)) \;=\; 2(W_1+W_2),$$

where $W_1 := \int_0^\infty\int_0^\infty \tilde{\psi}(x,y)\mathrm{d}F_n(x)(\mathrm{d}F_n(y) - \mathrm{d}G_\alpha(y))$, $W_2 := \int_0^\infty\int_0^\infty \tilde{\psi}(x,y)\mathrm{d}G_\alpha(y)(\mathrm{d}F_n(x) - \mathrm{d}G_\alpha(x))$. Integrating by parts yields

$$\begin{aligned} |W_1| &= 2\int_0^\infty |x|\mathrm{d}F_n(x)\int_0^\infty |F_n(y) - G_\alpha(y)|\frac{\mathrm{d}y}{(x+y)^2} \\ &\le 2\|F_n - G_\alpha\|_\infty \int_0^\infty |x|\mathrm{d}F_n(x)\int_0^\infty \frac{\mathrm{d}y}{(x+y)^2} \;=\; C\|F_n - G_\alpha\|_\infty \end{aligned}$$

since $\int_0^\infty \mathrm{d}F_n(x) = 1/2$. A similar estimate holds for W_2. This proves (5.13). □

Propositions 5.2, 5.3, and Lemma 5.1, together with the Delta-method, yield the following corollary.

Corollary 5.1 *Let a and K satisfy either the assumptions of Lemma 5.1(i) with $\beta > \alpha/2$, or the assumptions of Lemma 5.1(ii). Then*

$$\sqrt{n}(\tilde{R}^{2,n} - \tilde{\Lambda}(\alpha)) \;\xrightarrow[n\to\infty]{\mathcal{D}}\; \mathcal{N}(0, \tilde{\sigma}^2(\alpha)).$$

Moreover, if we define $\widehat{\alpha}_n := \tilde{\Lambda}^{-1}(\tilde{R}^{2,n})$, then

$$\sqrt{n}\big(\widehat{\alpha}_n - \alpha\big) \;\xrightarrow[n\to\infty]{\mathcal{D}}\; \mathcal{N}(0, \tilde{s}^2(\alpha)),$$

where $\tilde{s}^2(\alpha) := \big[\frac{\partial\tilde{\Lambda}}{\partial\alpha}(\alpha)\big]^{-2}\tilde{\sigma}^2(\alpha),\ 0 < \alpha \le 2$.

There exist very few papers concerning estimation of α in such a semiparametric frame. Nonparametric estimation of parameters of Lévy processes based on the empirical characteristic function has recently been considered in Neumann and Reiß (2009) and Gugushvili (2008), but the convergence rates there are $(\log n)^\kappa$ with $\kappa > 0$. Ait Sahalia and Jacod (2009) have proposed an estimator of the degree of activity of jumps (which is identical to the fractional order in the case of a Lévy process) in a general semimartingale framework using small increments of high frequency data. However from the generality of their model, the convergence rate of the estimator is not rate efficient (in fact smaller than $n^{1/5}$). A recent paper of Belomestny (2010) provides an efficient data-driven procedure to estimate α using a spectral approach but in a different semiparametric frame from ours. Thus, Corollary 5.1 appears as a new and interesting result since the estimator $\widehat{\alpha}_n$ follows a $\sqrt{n}$-central limit theorem.

6. Annexe : proofs

Proof of Lemma 3.1

Let $\delta^2 := \max_{i=1,2}\mathrm{E}\xi_i^2$. If $\delta^2 \ge 1/2$, then (3.2) holds since the l.h.s. of (3.2) does not exceed 1. Let $\delta^2 < 1/2$ in the sequel. Write $U := \psi(Z_1+\xi_1, Z_2+\xi_2) - \psi(Z_1,Z_2) = U_\delta + U_\delta^c$,

$$\begin{aligned} U_\delta &:= U\mathbf{1}(A_\delta) \;=\; (\psi(Z_1+\xi_1, Z_2+\xi_2) - \psi(Z_1,Z_2))\mathbf{1}(A_\delta), \\ U_\delta^c &:= U\mathbf{1}(A_\delta^c) \;=\; (\psi(Z_1+\xi_1, Z_2+\xi_2) - \psi(Z_1,Z_2))\mathbf{1}(A_\delta^c), \end{aligned}$$

where $\mathbf{1}(A_\delta)$ is the indicator of the event

$$A_\delta := \{|Z_1| > \delta^{2/3}, |Z_2| > \delta^{2/3}, |\xi_1| < \delta^{2/3}/2, |\xi_2| < \delta^{2/3}/2\},$$

and $\mathbf{1}(A_\delta^c) = 1 - \mathbf{1}(A_\delta)$ is the indicator of the complementary event A^c. Clearly,

$$\begin{aligned} \mathrm{E}|U_\delta^c| &\le 2(\mathrm{P}(|Z_1| < \delta^{2/3}) + \mathrm{P}(|Z_2| < \delta^{2/3}) + \mathrm{P}(|\xi_1| \ge \delta^{2/3}/2) + \mathrm{P}(|\xi_2| \ge \delta^{2/3}/2) \\ &\le \frac{4}{\sqrt{2\pi}}\,\delta^{2/3} + 2\,\frac{\mathrm{E}\xi^2}{\delta^{4/3}} \le 4\,\delta^{2/3}. \end{aligned}$$

It remains to estimate $\mathrm{E}|U_\delta|$. By the mean value theorem,

$$|U_\delta| \le \Big(|\xi_1| \sup_D |\psi_{x_1}| + |\xi_2| \sup_D |\psi_{x_2}|\Big)\,\mathbf{1}(A_\delta),$$

where

$$\begin{aligned} \sup_D |\psi_{x_1}| &:= \sup\{|\partial\psi(x_1,x_2)/\partial x_1| : |x_i - Z_i| \le |\xi_i|, i = 1,2\}, \\ \sup_D |\psi_{x_2}| &:= \sup\{|\partial\psi(x_1,x_2)/\partial x_2| : |x_i - Z_i| \le |\xi_i|, i = 1,2\}. \end{aligned}$$

Therefore

$$\begin{aligned} \mathrm{E}|U_\delta| &\le \mathrm{E}^{1/2}(\xi_1^2)\mathrm{E}^{1/2}\big[(\sup_D |\psi_{x_1}|)^2\mathbf{1}(A_\delta)\big] + \mathrm{E}^{1/2}(\xi_2^2)\mathrm{E}^{1/2}\big[(\sup_D |\psi_{x_2}|)^2\mathbf{1}(A_\delta)\big] \\ &\le \delta\left(\mathrm{E}^{1/2}\big[(\sup_D |\psi_{x_1}|)^2\mathbf{1}(A_\delta)\big] + \mathrm{E}^{1/2}\big[(\sup_D |\psi_{x_2}|)^2\mathbf{1}(A_\delta)\big]\right). \end{aligned}$$

Next,

$$|\psi_{x_i}(x_1,x_2)| = \frac{|\mathrm{sgn}(x_1+x_2)(|x_1|+|x_2|) - (|x_1+x_2|)\mathrm{sgn}(x_i)|}{(|x_1|+|x_2|)^2} \le \frac{2}{|x_1|+|x_2|}.$$

Therefore

$$\begin{aligned} \sup_D |\psi_{x_i}|^2\mathbf{1}(A_\delta) &\le 4\,\sup\big\{(|x_1|+|x_2|)^{-2} : |x_i - Z_i| \le \delta^{2/3}/2, |Z_i| > \delta^{2/3}, i = 1,2\big\} \\ &\le 16\,(|Z_1|+|Z_2|)^{-2}\mathbf{1}(|Z_i| > \delta^{2/3}, i = 1,2), \end{aligned}$$

implying

$$\mathrm{E}\big[(\sup_D |\psi_{x_i}|)^2\mathbf{1}(A_\delta)\big] \le 16\,\mathrm{E}\left[\frac{1}{(|Z_1|+|Z_2|)^2}; |Z_i| > \delta^{2/3}, i = 1,2\right] \le C(\delta),$$

where

$$\begin{aligned} C(\delta) &\le \frac{16}{2\pi}\int_{\{x_1^2+x_2^2 > \delta^{4/3}\}} \frac{1}{x_1^2+x_2^2}\mathrm{e}^{-(x_1^2+x_2^2)/2}\mathrm{d}x_1\mathrm{d}x_2 \\ &\le 16\int_{\delta^{4/3}}^\infty r^{-1}\mathrm{e}^{-r^2/2}\mathrm{d}r \le 16\,(1 + (4/3)|\log\delta|). \end{aligned}$$

Hence $\mathrm{E}|U_\delta| \le 8\,\delta(1 + (4/3)\,|\log\delta|)^{1/2}$, $\mathrm{E}|U_\delta^c| \le 4\,\delta^{2/3}$. It remains to use $x(1 + (4/3)\,|\log x|)^{1/2} \le 2x^{2/3}$ for all $0 < x \le 1$. □.

Proof of Property 4.1

We use the following identity: for any reals $x_1, \cdots, x_j$,

$$x_1x_j = \frac{1}{2}\Big\{\Big(\sum_{k=1}^j x_k\Big)^2 + \Big(\sum_{k=2}^{j-1} x_k\Big)^2 - \Big(\sum_{k=1}^{j-1} x_k\Big)^2 - \Big(\sum_{k=2}^j x_k\Big)^2\Big\}.$$

In particular,

$$\begin{aligned}\mathrm{E}\left[\Delta^{1,n}_{[nt]}X\Delta^{1,n}_{j+[nt]}X\right] &= \frac{1}{2}\Big\{\mathrm{E}\big(X_{j+1+[nt]}-X_{[nt]}\big)^2+\mathrm{E}\big(X_{j-1+[nt_*]}-X_{[nt_*]}\big)^2\\ &- \mathrm{E}\big(X_{j+[nt]}-X_{[nt]}\big)^2-\mathrm{E}\big(X_{j+[nt_*]}-X_{[nt_*]}\big)^2\Big\},\end{aligned}$$

where $t_* := t + (1/n)$ (so that $[nt_*] = [nt] + 1$). Then, using **(A.1)** and the notation u_n for a sequence tending to 0 as $n \to \infty$ uniformly in t and all $|j| < J$, where J is a fixed number, we obtain

$$\begin{aligned}\mathrm{E}\left[\Delta^{1,n}_{[nt]}X\Delta^{1,n}_{j+[nt]}X\right] &= \frac{1}{2}\Big\{c(t)\Big(\tfrac{j+1}{n}\Big)^{2H(t)}(1+u_n)+c(t_*)\Big(\tfrac{j-1}{n}\Big)^{2H(t_*)}(1+u_n)\\ &\quad - c(t)\Big(\tfrac{j}{n}\Big)^{2H(t)}(1+u_n)-c(t_*)\Big(\tfrac{j}{n}\Big)^{2H(t_*)}(1+u_n)\Big\}\\ &= \frac{c(t)}{2}\Big\{\Big(\tfrac{j+1}{n}\Big)^{2H(t)}+\Big(\tfrac{j-1}{n}\Big)^{2H(t)}-2\Big(\tfrac{j+1}{n}\Big)^{2H(t)}\Big\}(1+u_n)\end{aligned}$$

since $c(t_*) - c(t) = u_n$ and $\left(\frac{j}{n}\right)^{2H(t_*)} = \left(\frac{j}{n}\right)^{2H(t)}(1+u_n)$ follows from (4.5). This proves (4.9) for $p = 1$. Relation (4.9) for $p = 2$ follows analogously. Relation (4.10) also follows by the same argument and the fact that $c(t_*) - c(t) = u_n/\sqrt{n}$ and $\left(\frac{j}{n}\right)^{2H(t_*)} = \left(\frac{j}{n}\right)^{2H(t)}(1+u_n/\sqrt{n})$ hold in view of Assumption **(A.1)'**. Property 4.1 is proved. □

Proof of Property 4.2

With condition $V(t) \sim ct^{2H}$ $(t \to 0)$ in mind, inequality (4.8) reduces to

$$\begin{aligned}\big|V\left(\tfrac{k+1}{n}\right)+V\left(\tfrac{k-1}{n}\right)-2V\left(\tfrac{k}{n}\right)\big| &\le Cn^{-2H+\theta}k^{-\gamma} \quad (p=1,\ 2\le k\le n), \quad (6.1)\\ \big|V\left(\tfrac{k+2}{n}\right)-4V\left(\tfrac{k+1}{n}\right)+6V\left(\tfrac{k}{n}\right)-4V\left(\tfrac{k-1}{n}\right)+V\left(\tfrac{k-2}{n}\right)\big| &\le Cn^{-2H+\theta}k^{-\gamma} \quad (p=2,\ 4\le k\le n). \quad (6.2)\end{aligned}$$

The left hand side of (6.1) can be written and estimated as

$$\begin{aligned}\left|\int_0^{1/n}\int_0^{1/n}V''(t-s+(k/n))\mathrm{d}t\mathrm{d}s\right| &\le C\int_0^{1/n}\int_0^{1/n}|t-s+(k/n)|^{-\kappa}\mathrm{d}t\mathrm{d}s\\ &\le Cn^{\kappa-2}\int_0^1\int_0^1|t-s+k|^{-\kappa}\mathrm{d}t\mathrm{d}s\\ &\le Cn^{\kappa-2}k^{-\kappa} = Cn^{\theta-2H}k^{-\gamma}\end{aligned}$$

where $\gamma = \kappa > 1/2$ and $\theta = \kappa + 2H - 2 \in [0, \gamma/2)$ since $\kappa < 4 - 4H$. This proves part (i). Part (ii) follows similarly, by writing the left hand side of (6.2) as

$$\begin{aligned}\left|\int_0^{1/n}\cdots\int_0^{1/n}V^{(4)}(t-s+u-v+(k/n))\mathrm{d}t\mathrm{d}s\mathrm{d}u\mathrm{d}v\right| &\le C\int_0^{1/n}\cdots\int_0^{1/n}|t-s+u-v+(k/n)|^{-\kappa}\mathrm{d}t\mathrm{d}s\mathrm{d}u\mathrm{d}v\\ &\le Cn^{\kappa-4}k^{-\kappa} = Cn^{\theta-2H}k^{-\gamma},\end{aligned}$$

where $\gamma = \kappa > 1$ and $\theta = \kappa + 2H - 4 \in [0, \kappa/2)$ since $\kappa < 8 - 4H$. Property 4.2 is proved. □

Proof of Property 4.3

From (4.16) we have

$$\begin{aligned}\mathrm{E}\big[\Delta_0^{p,n}X\ \Delta_j^{p,n}X\big] &= \int_{\mathbb{R}}\left|\mathrm{e}^{\mathrm{i}(x/n)}-1\right|^{2p}\mathrm{e}^{\mathrm{i}x(j/n)}f(x)\mathrm{d}x = 2^{1+p}\int_0^\infty(1-\cos(x/n))^p\cos(xj/n)f(x)\mathrm{d}x\\ &= 2^{1+p}(I_1+I_2),\end{aligned}$$

with

$$\begin{aligned} |I_1| &= \Big|\int_0^K (1-\cos(x/n))^p \cos(xj/n) f(x)\mathrm{d}x\Big| \\ &= C\Big|\int_0^K (x/n)^{2p} f(x)\mathrm{d}x\Big| \;=\; \Big|Cn^{-2p}\int_0^K x^{2p} f(x)\mathrm{d}x\Big| \;\le\; Cn^{-2p}; \qquad (6.3) \\ \text{and}\quad I_2 &= n\int_{K/n}^\infty (1-\cos(x))^p \cos(xj) f(nx)\mathrm{d}x \\ &= (n/j)\int_{K/n}^\infty (1-\cos(x))^p f(nx)\mathrm{d}\sin(xj) \\ &= -(n/j)\int_{K/n}^\infty \sin(xj)\left((1-\cos(x))^p f(nx)\right)'_x \mathrm{d}x + O(n^{-2p}) \qquad (6.4) \end{aligned}$$

since $|(n/j)f(K)(1-\cos(K/n))^p \sin(jK/n)| \le C(n/j)(K/n)^{2p}|jK/n| \le Cn^{-2p}$ for any $K>0$ fixed.

Let $p=1$. The last integral can be rewritten as $\int_{K/n}^\infty \sin(xj)\left((1-\cos(x))f(nx)\right)'_x \mathrm{d}x = \tilde I_1 + \tilde I_2$, where

$$|\tilde I_1| \;\le\; \int_{K/n}^\infty |\sin(xj)\sin(x) f(nx)|\mathrm{d}x \;\le\; C\int_{K/n}^\infty |\sin(xj)\sin(x)|(nx)^{-2H-1}\mathrm{d}x \;\le\; \sum_{q=1}^3 I_{1q},$$

where we used the fact that $f(x)\le Cx^{-2H-1}$ $(x\ge K)$; see condition (4.18), and where

$$\begin{aligned} |I_{11}| &= \int_0^{1/j} |\sin(xj)\sin(x)|(nx)^{-2H-1}\mathrm{d}x \;\le\; Cj\int_0^{1/j} x^2(nx)^{-1-2H}\mathrm{d}x \;\le\; Cj^{2H-1}n^{-1-2H}, \\ |I_{12}| &= \int_{1/j}^1 |\sin(xj)\sin(x)|(nx)^{-2H-1}\mathrm{d}x \;\le\; C\int_{1/j}^1 x(nx)^{-1-2H}\mathrm{d}x \;\le\; Cj^{2H-1}n^{-1-2H}, \\ |I_{13}| &= \int_1^\infty |\sin(xj)\sin(x)|(nx)^{-2H-1}\mathrm{d}x \;\le\; C\int_1^\infty (nx)^{-1-2H}\mathrm{d}x \;\le\; Cn^{-1-2H}. \end{aligned}$$

Similarly, using (4.18),

$$|\tilde I_2| \;\le\; n\int_{K/n}^\infty |\sin(xj)(1-\cos(x))f'(nx)|\mathrm{d}x \;\le\; C\int_{K/n}^\infty |\sin(xj)(1-\cos(x)|(nx)^{-2H-2}\mathrm{d}x \;\le\; \sum_{q=1}^3 I_{2q},$$

where

$$\begin{aligned} I_{21} &= n\int_0^{1/j} |\sin(xj)(1-\cos(x))|(nx)^{-2H-2}\mathrm{d}x \;\le\; Cnj\int_0^{1/j} x^3(nx)^{-2-2H}\mathrm{d}x \;\le\; Cj^{2H-1}n^{-1-2H}, \\ I_{22} &= n\int_{1/j}^1 |\sin(xj)(1-\cos(x))|(nx)^{-2H-2}\mathrm{d}x \;\le\; Cn\int_{1/j}^1 x^2(nx)^{-2-2H}\mathrm{d}x \;\le\; Cj^{2H-1}n^{-1-2H}, \\ I_{23} &= n\int_1^\infty |\sin(xj)(1-\cos(x))|(nx)^{-2H-2}\mathrm{d}x \;\le\; Cn\int_1^\infty (nx)^{-2-2H}\mathrm{d}x \;\le\; Cn^{-1-2H}. \end{aligned}$$

We finally obtain, for $1\le j\le n$,

$$\Big|\mathrm{E}\big[\Delta_0^{1,n}X\;\Delta_j^{1,n}X\big]\Big| \;\le\; C(n/j)j^{2H-1}n^{-1-2H} + O(n^{-2}) \;\le\; Cn^{-2H}j^{2H-2}$$

implying (for $0<H<3/4$) $\mathbf{(A.2)}_1$ with $\theta=0$ and $\kappa = 2-2H > 1/2$. For $p=2$, the estimation of the integral in (6.4) is completely similar, resulting in the bound

$$\Big|\mathrm{E}\big[\Delta_0^{2,n}X\;\Delta_j^{2,n}X\big]\Big| \;\le\; C(n/j)n^{-1-2H} + O(n^{-4}) \;\le\; Cn^{-2H}j^{-1}$$

for any $0<H<1$, or $\mathbf{(A.2)}_2$ with $\theta=0$ and $\kappa = 1 > 1/2$. □

Sketch of the proof of Theorem 4.1

The proof of Theorem 4.1 is based on the moment inequality in Lemma 1, below, which extends a similar inequality in (Taqqu, 1977, Lemma 4.5) to vector-valued nonstationary Gaussian processes. The proof of Lemma 1 uses the diagram formula and is given in Bardet and Surgailis (2009). To formulate this lemma, we need the following definitions. Let $\boldsymbol{X}$ be a standard Gaussian vector in $\mathbb{R}^\nu\,(\nu \geq 1)$ and let $\mathbb{L}^2(\boldsymbol{X})$ denote the Hilbert space of measurable functions $f : \mathbb{R}^\nu \to \mathbb{R}$ satisfying $\|f\|^2 := \mathrm{E}(f(\boldsymbol{X}))^2 < \infty$. Let $\mathbb{L}^2_0(\boldsymbol{X}) = \{f \in \mathbb{L}^2(\boldsymbol{X}) : \mathrm{E}f(\boldsymbol{X}) = 0\}$. Let $(\boldsymbol{X}_1, \cdots, \boldsymbol{X}_N)$ be a collection of standardized Gaussian vectors $\boldsymbol{X}_t = (X_t^{(1)}, \cdots, X_t^{(\nu)}) \in \mathbb{R}^\nu$ having a joint Gaussian distribution in $\mathbb{R}^{\nu N}$. Let $\varepsilon \in [0,1]$ be a fixed number. Following Taqqu (1977), we call $(\boldsymbol{X}_1, \cdots, \boldsymbol{X}_N)$ *$\varepsilon-$standard* if $|\mathrm{E}X_t^{(u)}X_s^{(v)}| \leq \varepsilon$ for any $t \neq s, 1 \leq t, s \leq N$ and any $1 \leq u, v \leq \nu$. Finally, $\sum'$ denotes the sum over all distinct integers $1 \leq t_1, \cdots, t_p \leq N,\ t_i \neq t_j\ (i \neq j)$.

Lemma 1 *Let $(\boldsymbol{X}_1, \cdots, \boldsymbol{X}_N)$ be $\varepsilon-$standard Gaussian vector, $\boldsymbol{X}_t = (X_t^{(1)}, \cdots, X_t^{(\nu)}) \in \mathbb{R}^\nu\ (\nu \geq 1)$, and let $G_{j,t,N} \in \mathbb{L}^2(\boldsymbol{X}), 1 \leq j \leq p\ (p \geq 2), 1 \leq t \leq N$. For given integers $m, N \geq 1$, define*

$$Q_N \ := \ \max_{1\leq t\leq N} \sum_{1\leq s\leq N, s\neq t} \max_{1\leq u,v\leq \nu} |\mathrm{E}X_t^{(u)}X_s^{(v)}|^m. \tag{6.5}$$

Assume that for some integer $0 \leq \alpha \leq p$, the functions $G_{1,t,N}, \cdots, G_{\alpha,t,N}$ have a Hermite rank at least equal to m for any $N \geq 1, 1 \leq t \leq N$, and that $\varepsilon < \frac{1}{\nu p-1}$. Then

$$\sum{}' \mathrm{E}|G_{1,t_1,N}(\boldsymbol{X}_{t_1}) \cdots G_{p,t_p,N}(\boldsymbol{X}_{t_p})| \ \leq \ C(\varepsilon, p, m, \alpha, \nu) K N^{p-\frac{\alpha}{2}} Q_N^{\frac{\alpha}{2}},$$

where the constant $C(\varepsilon, p, m, \alpha, \nu)$ depends on $\varepsilon, p, m, \alpha, \nu$ only, and $K := \prod_{j=1}^p \max_{1\leq t\leq N} \|G_{j,t,N}\|$.

Sketch of the proof of Theorem 4.1. The convergence $\lim_{n\to\infty} \mathrm{E}R^{p,n} = \int_0^1 \Lambda_p(H(t))\mathrm{d}t$ is easy (see the proof of Proposition 2.2). Hence (4.19) follows from

$$\tilde{R}^{p,n} := R^{p,n} - \mathrm{E}R^{p,n} \ \xrightarrow[n\to\infty]{a.s.} \ 0. \tag{6.6}$$

Relation (6.6) follows from the Chebyshev Inequality and the following bound: there exist $C, \kappa > 1$ such that for any $n \geq 1$

$$\mathrm{E}\Big(\tilde{R}^{p,n}\Big)^4 \ \leq \ Cn^{-\kappa}. \tag{6.7}$$

By definition, $\tilde{R}^{p,n} = \frac{1}{n-p}\sum_{k=0}^{n-p-1} \tilde{\eta}_n(k)$, where $\tilde{\eta}_n(k) := \eta_n(k) - \mathrm{E}\eta_n(k)$ and $\eta_n(k) := \ \psi(\Delta_k^{p,n}X, \Delta_{k+1}^{p,n}X)$, $\psi(x,y) = |x+y|/(|x|+|y|)$ are nonlinear functions of Gaussian vectors $(\Delta_k^{p,n}X, \Delta_{k+1}^{p,n}X) \in \mathbb{R}^2$ having the Hermite rank 2; however, these vectors are not $\varepsilon-$standard and therefore Lemma 1 cannot be directly applied to estimate the l.h.s. of (6.7) (with $p = 1, \cdots, 4$, $\nu = 2$). To this end, we first need to "decimate" the sum $\tilde{R}^{p,n}$, as follows. (A similar "trick" was used in Csörgő and Mielniczuk (1996).) Let $\ell = [n^{\theta/\gamma}]$ be the sequence of integers increasing to ∞ (at a rate $o(n^{1/2})$ by condition $\theta < \gamma/2$) and write

$$\tilde{R}^{p,n} \ = \ \sum_{j=0}^{\ell-1} \tilde{R}_\ell^{p,n}(j) + o(1), \qquad \tilde{R}_\ell^{p,n}(j) \ := \ \frac{1}{n-1} \sum_{k=0}^{[(n-2-j)/\ell]} \tilde{\eta}_n(k\ell + j).$$

Then

$$\mathrm{E}\left(\tilde{R}^{p,n}\right)^4 \ \leq \ \ell^4 \max_{0\leq j<\ell} \mathrm{E}\left(\tilde{R}_\ell^{p,n}(j)\right)^4.$$

Write $\eta_n(k)$ as a (bounded) function in standardized Gaussian variables:

$$\eta_n(k) \ = \ f_{k,n}\left(\boldsymbol{Y}_n(k)\right), \tag{6.8}$$

where $\boldsymbol{Y}_n(k) = (Y_n^{(1)}(k), Y_n^{(2)}(k)) \in \mathbb{R}^2$,

$$Y_n^{(1)}(k) := \frac{\Delta_k^{p,n} X}{\sigma_{p,n}(k)}, \tag{6.9}$$

$$Y_n^{(2)}(k) := -\frac{\Delta_k^{p,n} X}{\sigma_{p,n}(k)} \frac{\rho_{p,n}(k)}{\sqrt{1-\rho_{p,n}^2(k)}} + \frac{\Delta_{k+1}^{p,n} X}{\sigma_{p,n}(k+1)} \frac{1}{\sqrt{1-\rho_{p,n}^2(k)}}, \tag{6.10}$$

$$\text{and} \quad f_{k,n}\left(x^{(1)}, x^{(2)}\right) := \psi\left(x^{(1)}, \frac{\sigma_{p,n}(k+1)}{\sigma_{p,n}(k)}\left(\rho_{p,n}(k)x^{(1)} + \sqrt{1-\rho_{p,n}^2(k)}x^{(2)}\right)\right), \tag{6.11}$$

where $\sigma_{p,n}^2(k)$, $\rho_{p,n}(k)$ are defined in (4.1). Then, for each k, $\boldsymbol{Y}_n(k) := (Y_n^{(1)}(k), Y_n^{(2)}(k))$ has a standard Gaussian distribution in $\mathbb{R}^2$ and $\tilde{\eta}_n(k) = f_{k,n}(\boldsymbol{Y}_n(k)) - \mathrm{E} f_{k,n}(\boldsymbol{Y}_n(k))$. Moreover, the vector $(\boldsymbol{Y}_n(k\ell+j),\, k = 0, 1, \cdots, [(n-2-j)/\ell]) \in \mathbb{R}^{2([(n-2-j)/\ell]+1)}$ is $\varepsilon-$standard provided ℓ is large enough. Now Lemma 1 can be used and it implies the bound (6.7) using Assumptions **(A.1)** and $\mathbf{(A.2)}_p$. The details of this proof can be found in Bardet et Surgailis (2009). □

Sketch of the proof of Theorem 4.2

The proof of Theorem 4.1 uses the following central limit theorem for Gaussian subordinated multidimensional triangular arrays. Theorem 1 is proved in Bardet and Surgailis (2009). It extends the earlier results in Breuer and Major (1983) and Arcones (1994). Below, similarly as in Lemma 1, $\boldsymbol{X} \in \mathbb{R}^\nu$ designates a standard Gaussian vector.

Theorem 1 *Let $(\boldsymbol{Y}_n(k))_{1\le k\le n, n\in\mathbb{N}}$ be a triangular array of standardized Gaussian vectors with values in $\mathbb{R}^\nu$, $\boldsymbol{Y}_n(k) = (Y_n^{(1)}(k), \cdots, Y_n^{(\nu)}(k))$, $\mathrm{E}Y_n^{(p)}(k) = 0$, $\mathrm{E}Y_n^{(p)}(k)Y_n^{(q)}(k) = \delta_{pq}$. For a given integer $m \ge 1$, introduce the following assumption: there exists a function $\rho : \mathbb{N} \to \mathbb{R}$ such that for any $1 \le p, q \le \nu$,*

$$\forall (j,k) \in \{1, \cdots, n\}^2, \quad \left|\mathrm{E}Y_n^{(p)}(j)Y_n^{(q)}(k)\right| \le |\rho(j-k)| \quad \textit{with} \quad \sum_{j\in\mathbb{Z}} |\rho(j)|^m < \infty. \tag{6.12}$$

Moreover, assume that for any $\tau \in [0,1]$ and any $J \in \mathbb{N}^*$,

$$(\boldsymbol{Y}_n([n\tau]+j))_{-J\le j\le J} \xrightarrow[n\to\infty]{\mathcal{D}} (\boldsymbol{W}_\tau(j))_{-J\le j\le J}, \tag{6.13}$$

where $(\boldsymbol{W}_\tau(j))_{j\in\mathbb{Z}}$ is a stationary Gaussian process taking values in $\mathbb{R}^\nu$ and depending on parameter $\tau \in (0,1)$. Let $\tilde{f}_{k,n} \in \mathbb{L}_0^2(\boldsymbol{X})$ $(n \ge 1, 1 \le k \le n)$ be a triangular array of functions all having Hermite rank at least m. Assume that there exists a $\mathbb{L}_0^2(\boldsymbol{X})-$valued continuous function $\tilde{\phi}_\tau, \tau \in [0,1]$, such that

$$\sup_{\tau\in[0,1]} \|\tilde{f}_{[\tau n],n} - \tilde{\phi}_\tau\|^2 = \sup_{\tau\in[0,1]} \mathrm{E}(\tilde{f}_{[\tau n],n}(\boldsymbol{X}) - \tilde{\phi}_\tau(\boldsymbol{X}))^2 \underset{n\to\infty}{\longrightarrow} 0. \tag{6.14}$$

Then, with $\sigma^2 = \displaystyle\int_0^1 \mathrm{d}\tau \Big(\sum_{j\in\mathbb{Z}} \mathrm{E}\big[\tilde{\phi}_\tau(\boldsymbol{W}_\tau(0))\, \tilde{\phi}_\tau(\boldsymbol{W}_\tau(j))\big]\Big) < \infty$,

$$n^{-1/2} \sum_{k=1}^n \tilde{f}_{k,n}(\boldsymbol{Y}_n(k)) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \sigma^2). \tag{6.15}$$

Sketch of the proof of Theorem 4.2. It suffices to show that

$$\sqrt{n}\Big|\mathrm{E}R^{n,p} - \int_0^1 \Lambda_p(H(t))\mathrm{d}t\Big| \underset{n\to\infty}{\longrightarrow} 0 \tag{6.16}$$

and

$$\sqrt{n}\big(R^{p,n} - \mathrm{E}R^{p,n}\big) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}\big(0, \int_0^1 \Sigma_p(H(\tau))\mathrm{d}\tau\big). \tag{6.17}$$

The proof of (6.16) uses Assumption **(A.1)'** or (4.10) and the easy fact that for Gaussian vectors $(Z_n^{(1)}, Z_n^{(2)}) \in \mathbb{R}^2$, $n \in \mathbb{N}$ with zero mean $\mathrm{E}Z_n^{(i)} \equiv 0$, $i = 1, 2, n \in \mathbb{N}$ and $\mathrm{E}(Z_0^{(1)})^2 = \mathrm{E}(Z_0^{(2)})^2 = 1$, $|\mathrm{E}Z_0^{(1)} Z_0^{(2)}| < 1$

$$\big|\mathrm{E}\psi(Z_0^{(1)} Z_0^{(2)}) - \mathrm{E}\psi(Z_n^{(1)}, Z_n^{(2)})\big| \ \le \ C \sum_{i,j=1}^{2} \big|\mathrm{E}Z_0^{(i)} Z_0^{(j)} - Z_n^{(i)} Z_n^{(j)}\big|. \tag{6.18}$$

The proof of (6.17) is deduced from Theorem 1 with the sequence of standardized Gaussian vectors $\boldsymbol{Y}_n(k) = (Y_n^{(1)}(k), Y_n^{(2)}(k))$ $(\nu = 2)$ given in (6.9)-(6.10) and the centered functions

$$\tilde{f}_{k,n}(x^{(1)}, x^{(2)}) := f_{k,n}(x^{(1)}, x^{(2)}) - \mathrm{E}f_{k,n}(\boldsymbol{Y}_n(k)), \qquad \tilde{\phi}_\tau(x^{(1)}, x^{(2)}) := \phi_\tau(x^{(1)}, x^{(2)}) - \mathrm{E}\phi_\tau(\boldsymbol{X})$$

with $f_{k,n} : \mathbb{R}^2 \to \mathbb{R}$ given in (6.11) and the (limit) function

$$\phi_\tau(x^{(1)}, x^{(2)}) := \psi\Big(x^{(1)}, \rho_p(H(\tau))x^{(1)} + \sqrt{1 - \rho_p^2(H(\tau))}x^{(2)}\Big).$$

Thanks to symmetry properties of these functions, it is clear that the Hermite rank of $\tilde{f}_{k,n}$ (for any k and n) and $\tilde{\phi}_\tau$ (for any $\tau \in [0, 1]$) is $m = 2$. Using Assumptions **(A.1)'** and $\mathbf{(A.2)}_p$ (with $\theta = 0, \gamma > 1/2$), one can show that the conditions of Theorem 1 are satisfied for the above $\tilde{f}_{n,k}, \tilde{\phi}_\tau$ and the limit process $(\boldsymbol{W}_\tau(j))_{j\in\mathbb{Z}}$ in (6.13) is written in terms of increments of fBm $(B_{H(\tau)}(j))_{j\in\mathbb{Z}}$:

$$\boldsymbol{W}_\tau(j) \ := \ \Big(\Delta_1^p B_{H(\tau)}(j), \big(-\rho_p(H(\tau))\Delta_1^p B_{H(\tau)}(j) + \Delta_1^p B_{H(\tau)}(j+1)\big) \big/ \sqrt{1 - \rho_p^2(H(\tau))}\Big)$$

having standardized uncorrelated components. The details of this proof can be found in Bardet and Surgailis (2009). □

Proof of Proposition 4.1

(i) Denote $\tilde{Z}_t := \alpha(t)X_t$, $R^{n,p} := R^{n,p}(Z), \tilde{R}^{n,p} := R^{n,p}(\tilde{Z})$. Clearly, part (i) (the CLT in (4.21) for trended process $Z = \alpha X + \beta$) follows from the following relations:

$$\sqrt{n}\,\big(\tilde{R}^{p,n} - \mathrm{E}\tilde{R}^{p,n}\big) \ \xrightarrow[n\to\infty]{\mathcal{D}} \ \mathcal{N}\big(0, \int_0^1 \Sigma_p(H(\tau))\mathrm{d}\tau\big), \tag{6.19}$$

$$\sqrt{n}\,\Big|\mathrm{E}\tilde{R}^{p,n} - \int_0^1 \Lambda_p(H(t))\mathrm{d}t\Big| \ \underset{n\to\infty}{\longrightarrow} \ 0, \tag{6.20}$$

$$\sqrt{n}\,\big|\mathrm{E}R^{n,p} - \mathrm{E}\tilde{R}^{n,p}\big| \ \underset{n\to\infty}{\longrightarrow} \ 0, \tag{6.21}$$

$$n\,\mathrm{var}(R^{n,p} - \tilde{R}^{n,p}) \ \underset{n\to\infty}{\longrightarrow} \ 0. \tag{6.22}$$

The central limit theorem in (6.19) follows from Theorem 4.2 and Remark 4.1 since Assumption $\mathbf{(A.2)}_p$ and the convergences (4.11)-(4.14) (with $c(t)$ replaced by $\alpha^2(t)c(t)$) can be easily verified for the process $\tilde{Z} = \alpha X$.

Let us turn to the proof of (6.19)-(6.22). For concreteness, let $p = 2$ in the rest of the proof. Since $\Delta_k^{2,n}\tilde{Z} = \alpha(k/n)\Delta_k^{2,n}X + 2\alpha'(k/n)n^{-1}\Delta_{k+1}^{1,n}X + O(1/n^2)X((k+2)/n)$, it follows easily from Assumption $(\mathbf{A.1})'$ and (4.10) (for X) that

$$\mathrm{E}\big[\Delta_{[nt]}^{2,n}\tilde{Z}\,\Delta_{j+[nt]}^{2,n}\tilde{Z}\big] \ = \ \alpha^2(t)\mathrm{E}\big[\Delta_{[nt]}^{2,n}X\,\Delta_{j+[nt]}^{2,n}X\big] + O(n^{-2H(t)-1} + n^{-H(t)-2})$$

implying (4.10) for $\tilde{Z}$ (with $c(t)$ replaced by $\alpha^2(t)c(t)$). Whence and using (4.10) and (6.18), relation (6.20) follows similarly as (6.16) above.

The proofs of (6.21)-(6.22) uses the following bounds from Bružaitė and Vaičiulis (2008, Lemma 1). Let $(U_1, U_2) \in \mathbb{R}^2$ be a Gaussian vector with zero mean, unit variances and a correlation coefficient ρ, $|\rho| < 1$. Then for any $b_1, b_2 \in \mathbb{R}$ and any $1/2 < a_1, a_2 < 2$

$$\big|\mathrm{E}\psi(a_1U_1 + b_1, a_2U_2 + b_2) - \mathrm{E}\psi(a_1U_1, a_2U_2)\big| \ \le \ C(b_1^2 + b_2^2), \tag{6.23}$$

$$\big|\mathrm{E}U_i\psi(a_1U_1 + b_1, a_2U_2 + b_2)\big| \ \le \ C(|b_1| + |b_2|), \quad i = 1, 2, \tag{6.24}$$

where the constant C depends only on ρ and does not depend on a_1, a_2, b_1, b_2. Using (6.23), Assumption $(\mathbf{A.1})'$ and the fact that $|\Delta_k^{2,n}\beta| \le Cn^{-2}$ we obtain

$$\big|\mathrm{E}R^{n,2} - \mathrm{E}\tilde{R}^{n,2}\big| \;\le\; Cn^{-1}\sum_{k=0}^{n-3}\big(n^{H(k/n)}n^{-2}\big)^2 \;=\; O(n^{-2}), \tag{6.25}$$

proving (6.21).

To prove (6.22), write $R^{n,2} = \frac{1}{n-2}\sum_{k=0}^{n-3}\eta_n(k)$, $\tilde{R}^{n,2} = \frac{1}{n-2}\sum_{k=0}^{n-3}\tilde{\eta}_n(k)$, where

$$\begin{aligned}\eta_n(k) &:= \psi\big(\Delta_k^{2,n}Z, \Delta_{k+1}^{2,n}Z\big) = f_{n,k}\big(\tilde{Y}_n^{(1)}(k) + \mu_n^{(1)}(k), \tilde{Y}_n^{(2)}(k) + \mu_n^{(2)}(k)\big),\\ \tilde{\eta}_n(k) &:= \psi\big(\Delta_k^{2,n}\tilde{Z}, \Delta_{k+1}^{2,n}\tilde{Z}\big) = f_{n,k}\big(\tilde{Y}_n^{(1)}(k), \tilde{Y}_n^{(2)}(k)\big),\end{aligned}$$

where standardized increments $\tilde{Y}_n^{(i)}(k), i = 1, 2$ are defined as in (6.9)-(6.10) with X replaced by $\tilde{Z}$, $f_{n,k}$ are defined in (6.11), and

$$\mu_n^{(1)}(k) := \frac{\Delta_k^{2,n}\beta}{\sigma_{2,n}(k)}, \quad \mu_n^{(2)}(k) := -\frac{\Delta_k^{2,n}\beta}{\sigma_{2,n}(k)}\frac{\rho_{2,n}(k)}{\sqrt{1-\rho_{2,n}^2(k)}} + \frac{\Delta_{k+1}^{2,n}\beta}{\sigma_{2,n}(k+1)}\frac{1}{\sqrt{1-\rho_{2,n}^2(k)}}.$$

Note, the $\tilde{\eta}_n(k)$'s and $\eta_n(k)$'s have the Hermite rank ≥ 2 and ≥ 1, respectively, since the Hermite coefficients of order 1 $c_n^{(i)}(k) := \mathrm{E}\big[Y_n^{(i)}(k)\eta_n(k)\big]$, $i = 1, 2$ of the $\eta_n(k)$'s are not zero in general. Using the bound in (6.24) and $|\Delta_k^{2,n}\beta| \le Cn^{-2}$ we obtain

$$|c_n^{(i)}(k)| \le C\big(|\mu_n^{(1)}(k)| + |\mu_n^{(1)}(k)|\big) \le Cn^{H(k/n)-2}. \tag{6.26}$$

Split $\eta_n(k) - \tilde{\eta}_n(k) - (\mathrm{E}\eta_n(k) - \mathrm{E}\tilde{\eta}_n(k)) = \zeta_n'(k) + \zeta_n''(k)$, where

$$\zeta_n'(k) := \sum_{i=1}^{2} c_n^{(i)}(k)Y_n^{(i)}(k), \quad \zeta_n''(k) := \eta_n(k) - \zeta_n'(k) - \tilde{\eta}_n(k) - (\mathrm{E}\eta_n(k) - \mathrm{E}\tilde{\eta}_n(k)). \tag{6.27}$$

Then $\mathrm{var}(R^{n,2} - \tilde{R}^{n,2}) \le 2(J_n' + J_n'')$, where $J_n' := \mathrm{E}\Big(\frac{1}{n-2}\sum_{k=0}^{n-3}\zeta_n'(k)\Big)^2$, $J_n'' := \mathrm{E}\Big(\frac{1}{n-2}\sum_{k=0}^{n-3}\zeta_n''(k)\Big)^2$. From (6.26) and Cauchy-Schwartz inequality, it follows $J_n' \le \frac{C}{(n-2)^2}\big((n-2)\sum_{k=0}^{n-3}n^{2H(k/n)-4}\big) = O(1/n^2) = o(1/n)$. Since the $\zeta_n''(k)$'s have Hermite rank ≥ 2, we can use the argument in the proof of Theorem 4.2 together with Arcones' inequality (Arcones, 1994) and the easy fact that $\mathrm{E}(\zeta_n''(k))^2 \to 0$ $(n \to \infty,\ k = 0, \cdots, n-3)$, to conclude that $J_n'' = o(1/n)$. This proves (6.22) and completes the proof of part (i).

(ii) Similarly as in the proof of part (i), we shall restrict ourselves to the case $p = 2$ for concreteness. We use the same notation and the decomposition of $R^{2,n}$ as in part (i):

$$R^{2,n} - \int_0^1 \Lambda_p(H(\tau))\mathrm{d}\tau \;=\; \Big(\tilde{R}^{2,n} - \int_0^1 \Lambda_p(H(\tau))\mathrm{d}\tau\Big) + \big(\mathrm{E}R^{n,p} - \mathrm{E}\tilde{R}^{n,p}\big) + Q_n, \tag{6.28}$$

where

$$Q_n \;:=\; R^{2,n} - \tilde{R}^{2,n} - \big(\mathrm{E}R^{n,p} - \mathrm{E}\tilde{R}^{n,p}\big) \;=\; \frac{1}{n-2}\sum_{k=0}^{n-3}\zeta_n'(k) + \frac{1}{n-2}\sum_{k=0}^{n-3}\zeta_n''(k) \;=:\; Q_n' + Q_n'';$$

see (6.27). Here, the a.s. convergence to zero of the first term on the r.h.s. of (6.28) follows from Theorem 4.1 since the conditions of this theorem for $\tilde{Z}$ are easily verified. The convergence to zero of the second term on the r.h.s. of (6.28) follows similarly as in (6.25), with the difference that $|\Delta_k^{2,n}\beta| \le Cn^{-1}$ since $\beta \in \mathcal{C}^1[0,1]$ and therefore

$$\big|\mathrm{E}R^{n,2} - \mathrm{E}\tilde{R}^{n,2}\big| \;\le\; Cn^{-1}\sum_{k=0}^{n-3}\big(n^{H(k/n)}n^{-1}\big)^2 \;=\; O(n^{-2(1-\sup_{t\in[0,1]}H(t))}) = o(1). \tag{6.29}$$

The proof of $Q_n'' \xrightarrow[n\to\infty]{a.s.} 0$ mimics that of Theorem 4.1 and relies on the fact that the $\zeta_n''(k)$'s have the Hermite rank ≥ 2 (see above). Relation $Q_n' \xrightarrow[n\to\infty]{a.s.} 0$ follows by the gaussianity of Q_n' and $\mathrm{E}(Q_n')^2 = O(n^{-\delta})$ for some $\delta > 0$. Since $|c_n^{(i)}(k)| \le Cn^{H(k/n)-1}$ in view of the first inequality in (6.26) and $|\Delta_k^{2,n}\beta| \le Cn^{-1}$, then $\mathrm{E}(Q_n')^2 \le Cn^{-2(1-\sup_{t\in[0,1]}H(t))}$ similarly as in (6.29) above. This proves part (ii) and the proposition. □

Proof of Corollary 4.2

(a) The argument at the end of the proof of Property 4.1 shows that V satisfies Assumption $(\mathbf{A}.\mathbf{1})'$, while $(\mathbf{A}.\mathbf{2})_p$ follows from Property 4.2. Then the central limit theorem in (4.22) follows from Theorem 4.2.

(b) In this case, $(\mathbf{A}.\mathbf{2})_p$ follows from Property 4.3. Instead of verifying $(\mathbf{A}.\mathbf{1})'$, it is simpler to directly verify condition (4.10) which suffices for the validity of the statement of Theorem 4.2. Using $f(\xi) = c\xi^{-2H-1}\big(1 + o(\xi^{-1/2})\big)$ $(\xi \to \infty)$, similarly as in the proof of Property 4.3 for $j \in \mathbb{N}^*$ one obtains

$$\begin{aligned}
&\sqrt{n}\,\Big|n^{-2H}\mathrm{E}\big[\Delta_0^{p,n} X\, \Delta_j^{p,n} X\big] - c\,2^{1+p}\int_0^\infty (1-\cos(x))^p \cos(xj)\, x^{-2H-1}\mathrm{d}x\Big| \\
&= \; 2^{1+p}\Big|\int_0^\infty (1-\cos(x))^p \cos(xj) \times \sqrt{n}\,\big(n^{2H+1}(f(nx) - c(nx)^{-2H-1})\big)\mathrm{d}x\Big| \;\underset{n\to\infty}{\longrightarrow}\; 0
\end{aligned}$$

by the Lebesgue dominated convergence theorem, since $\int_0^\infty \big|(1-\cos(x))^p \cos(xj)\, x^{-2H-3/2}\big|\mathrm{d}x < \infty$. Therefore condition (4.10) is satisfied and Theorem 4.2 can be applied.
Finally, the function $H \mapsto \Lambda_2(H)$ is a $\mathcal{C}^1(0,1)$ bijective function and from the Delta-method (see for instance Van der Vaart, 1998), the central limit theorem in (4.23) is shown. □

Proof of Lemma 5.1

(i) We use the following inequality (see, e.g. Ibragimov and Linnik, 1971, Theorem 1.5.2):

$$\|F_n - G_\alpha\|_\infty \le \frac{1}{2\pi}\int_{\mathbb{R}} \Big|\frac{f_n(\theta) - g_\alpha(\theta)}{\theta}\Big|\mathrm{d}\theta, \tag{6.30}$$

where f_n, g_α are the characteristic functions of F_n, G_α, respectively. According to (5.6) and the definition of K_1,

$$f_n(\theta) - g_\alpha(\theta) \;=\; g_\alpha(\theta)\big(\mathrm{e}^{-2n^{-1}I_n(\theta)} - 1\big),$$

where, with $v := \theta n^{1/\alpha}$,

$$I_n(\theta) \;:=\; \int_0^\infty \mathrm{Re}(\mathrm{e}^{\mathrm{i}u\,v} - 1)\mathrm{d}K_1(u) \;=\; \int_0^{1/v} \cdots + \int_{1/v}^\infty \cdots \;=:\; I_1 + I_2.$$

If $v > 1$ then integrating by parts and using (5.9),

$$\begin{aligned}
|I_2| &\le 2|K_1|(1/v) \;=\; O(v^{(\alpha-\beta)_+}), \\
|I_1| &= \Big|K_1(1/v)\mathrm{Re}(\mathrm{e}^{\mathrm{i}} - 1) - \int_0^{1/v} K_1(u)\, v\, \sin(uv)\,\mathrm{d}u\Big| \\
&\le 2|K_1(1/v)| + v^2\int_0^{1/v} u|K_1(u)|\mathrm{d}u \\
&= O(v^{(\alpha-\beta)_+}) + v^2(1/v)^{2-(\alpha-\beta)_+} = O(v^{(\alpha-\beta)_+}).
\end{aligned}$$

Next, if $v \le 1$, then similarly as above

$$\begin{aligned}
|I_2| &\le 2|K_1|(1/v) \;=\; O(v^\delta), \\
|I_1| &= \Big|K_1(1/v)\mathrm{Re}(\mathrm{e}^{\mathrm{i}} - 1) - \int_0^{1/v} K_1(u)\, v\, \sin(uv)\,\mathrm{d}u\Big| \\
&\le 2|K_1(1/v)| + v^2\int_0^{1/v} u|K_1(u)|\mathrm{d}u \\
&= O(v^\delta) + Cv^2\Big(\int_0^1 uu^{-(\alpha-\beta)_+}\mathrm{d}u + \int_1^{1/v} u^{1-\delta}\mathrm{d}u\Big) = O(v^{\delta\wedge 2}).
\end{aligned}$$

Therefore, for some constant C,

$$|I_n(\theta)| \le C \begin{cases} (\theta n^{1/\alpha})^{(\alpha-\beta)_+}, & \theta > 1/n^{1/\alpha}, \\ (\theta n^{1/\alpha})^{\delta\wedge 2}, & \theta \le 1/n^{1/\alpha}. \end{cases}$$

Moreover, $g_\alpha(\theta) = \mathrm{e}^{-c_1|\theta|^\alpha}$ for some $c_1 > 0$. Using these facts and (6.30) we obtain

$$\begin{aligned} \|F_n - G_\alpha\|_\infty &\le C\Big(\int_0^{1/n^{1/\alpha}} n^{-1}(\theta n^{1/\alpha})^{\delta\wedge 2}\frac{\mathrm{d}\theta}{\theta} + + \int_{1/n^{1/\alpha}}^\infty n^{-1}(\theta n^{1/\alpha})^{(\alpha-\beta)_+}\mathrm{e}^{-c_1|\theta|^\alpha}\frac{\mathrm{d}\theta}{\theta}\Big) \\ &\le Cn^{-1} + C\frac{1}{n^{1-(\alpha-\beta)_+/\alpha}}\Big(1 + \int_{1/n^{1/\alpha}}^1 \theta^{(\alpha-\beta)_+-1}\mathrm{d}\theta\Big). \end{aligned}$$

The above bound easily yields (5.10).

(ii) Follows similarly using (6.30) and the argument in (i) with K_1 replaced by K. □

Computation of $\lambda(r)$

From the definition of $\lambda(r)$ and the change of variables $x_1 = a\cos\phi$, $x_2 = a\sin\phi$, with $|r| < 1$,

$$\begin{aligned} \lambda(r) &= \frac{1}{2\pi\sqrt{1-r^2}}\int_{\mathbb{R}^2}\frac{|x_1+x_2|}{|x_1|+|x_2|}\mathrm{e}^{-\frac{1}{2(1-r^2)}(x_1^2-2rx_1x_2+x_2^2)}\mathrm{d}x_1\mathrm{d}x_2 \\ &= \frac{\sqrt{1-r^2}}{\pi}\int_0^\pi \frac{|\cos\phi+\sin\phi|}{(|\cos\phi|+|\sin\phi|)(1-r\sin(2\phi))}\mathrm{d}\phi \\ &=: I_1 + I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \frac{\sqrt{1-r^2}}{\pi}\int_0^{\pi/2}\frac{1}{1-r\sin(2\phi)}\mathrm{d}\phi \\ &= \frac{\sqrt{1-r^2}}{\pi}\int_0^\infty \frac{1}{1+t^2-2rt}\mathrm{d}t \;=\; \frac{1}{2} + \frac{1}{\pi}\arctan\Big(\frac{r}{\sqrt{1-r^2}}\Big) \;=\; \frac{1}{\pi}\arccos(-r); \\ I_2 &= \frac{2\sqrt{1-r^2}}{\pi}\int_0^{\pi/4}\frac{\cos\phi-\sin\phi}{(\cos\phi+\sin\phi)(1+r\sin(2\phi))}\mathrm{d}\phi \;=\; \frac{\sqrt{1-r^2}}{\pi}\int_0^1\frac{1-t}{(1+t)(1+2rt+t^2)}\mathrm{d}t \\ &= \frac{\sqrt{1-r^2}}{\pi(1-r)}\log\Big(\frac{2}{r+1}\Big). \end{aligned}$$

The function $\lambda(r)$ is monotone increasing on $[-1,1]$; $\lambda(1) = 1$, $\lambda(-1) = 0$. It is easy to check that

$$\rho_1(H) \;=\; 2^{2H-1} - 1, \qquad \rho_2(H) = \frac{-3^{2H} + 2^{2H+2} - 7}{8 - 2^{2H+1}} \tag{6.31}$$

are monotone increasing functions; $\rho_1(1) = 1, \rho_2(1) = 0$ so that $\Lambda_p(H) = \lambda(\rho_p(H))$ for $p = 1, 2$ is also monotone for $H \in (0,1)$.

Expression and graph of $s_2(H)$

From the Delta-method, $s_2^2(H) = \Big[\frac{\partial}{\partial x}(\Lambda_2)^{-1}(\Lambda_2(H))\Big]^2\Sigma_2(H)$ and therefore

$$s_2^2(H) = \Big(\frac{\pi\,(8-2^{2H+1})^2(1-\rho_2(H))\sqrt{1-\rho_2^2(H)}}{\big(\log 2 - \log(1+\rho_2(H))\big)\big(2^{2H+2}9\log 2 - 3^{2H}16\log 3 + 6^{2H}4\log(3/2)\big)}\Big)^2\Sigma_2(H),$$

with the approximated graph (using the numerical values of $\Sigma_2(H)$ in Stoncelis and Vaičiulis (2008)):

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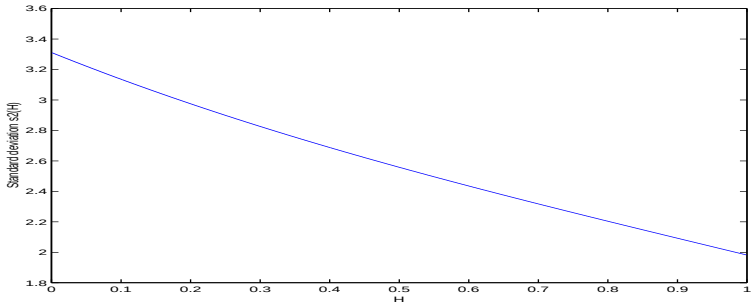

Figure 5. The graph of $\sqrt{s_2^2(H)}$

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