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MEASURING THE ROUGHNESS OF RANDOM PATHS BY INCREMENT RATIOS

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Abstract A statistic based on increment ratios (IR) is defined and studied for measuring the roughness of random paths. The main advantages of this statistic are invariance with respect to smooth additive or multiplicative trends and applicability to infinite variance processes. The existence of the IR statistic limit (called the IR-roughness) is closely related to the existence of a tangent process. Three particular cases where the IR-roughness exists and is explicitly computed are considered. Firstly, for a diffusion process with smooth diffusion and drift coefficients, the IR-roughness coincides with the IR-roughness of a Brownian motion and its convergence rate is obtained. Secondly, the case of rough Gaussian processes is studied in detail under general assumptions which do not require stationarity conditions. Thirdly, the IR-roughness of a Lévy process with α -stable tangent process is established and can be used to estimate the fractional order $\alpha \in (0, 2)$ following a central limit theorem.

1. Introduction and the main results. It is well-known that random functions are typically “rough” (non-differentiable), which raise the question of determining and measuring roughness. Probably, the most studied roughness measures is the Hausdorff dimension and the p -variation index. There exists a considerable literature on statistical estimation of these and related quantities from a discrete grid. Hence, different estimators of the Hausdorff dimension have been studied, as the box-counting estimator (see Hall and Wood, 1993 for stationary Gaussian processes or Lévy-Véhel and Peltier, 1994, for Gaussian processes with stationary increments). To our knowledge, the H -variation estimator, where H is a measurable function, was first proposed by Guyon and Leon (1989) for stationary Gaussian processes where central and non-central limit theorems are established following the Hermite rank of H and the asymptotic local properties of the variogram and its second derivative. Further studies provided a continuation of this seminal paper in different ways. Istas and Lang (1997) studied the case of generalized quadratic variations for Gaussian processes with stationary increments, the papers of Coeurjolly (2001 and 2005) studied respectively the case of ℓ^p -variations for fractional Brownian motion and the case of ℓ^2 -variations for multifractional Brownian motion, and types of quadratic variations are often introduced. A recent paper of Coeurjolly (2007) deals with L -variations (based on linear combinations of empirical quantiles) for Gaussian locally self-similar processes. An estimator counting the number level crossings was also investigated by Feuerverger *et al.* (1994) for stationary Gaussian processes.

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In the present paper we introduce a new characteristic of roughness, defined as a sum of ratios of consecutive increments. For a real-valued function $f = (f(t), t \in [0, 1])$, define recursively

$$\begin{aligned}\Delta_j^{1,n} f &:= f\left(\frac{j+1}{n}\right) - f\left(\frac{j}{n}\right), \\ \Delta_j^{p,n} f &:= \Delta_j^n \Delta_j^{p-1,n} f = \sum_{i=0}^p (-1)^{p-i} \binom{p}{i} f\left(\frac{j+i}{n}\right)\end{aligned}$$

denote its p -order increment, $p = 1, 2, \dots$, $j = 0, 1, \dots, n-p$. Let

$$(1.1) \quad R^{p,n}(f) := \frac{1}{n-p} \sum_{k=0}^{n-p-1} \frac{|\Delta_k^{p,n} f + \Delta_{k+1}^{p,n} f|}{|\Delta_k^{p,n} f| + |\Delta_{k+1}^{p,n} f|},$$

with the convention $\frac{0}{0} := 1$. In particular,

$$(1.2) \quad R^{1,n}(f) = \frac{1}{n-1} \sum_{k=0}^{n-2} \frac{\left|f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right) + f\left(\frac{k+2}{n}\right) - f\left(\frac{k+1}{n}\right)\right|}{\left|f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)\right| + \left|f\left(\frac{k+2}{n}\right) - f\left(\frac{k+1}{n}\right)\right|}.$$

Note the ratio on the right-hand side of (1.1) is either 1 or less than 1 depending on whether the consecutive increments $\Delta_k^{p,n} f$ and $\Delta_{k+1}^{p,n} f$ have same signs or different signs; moreover, in the latter case, this ratio generally is small whenever the increments are similar in magnitude (“cancel each other”). Clearly, $0 \leq R^{p,n}(f) \leq 1$ for any f, n, p . Thus if $\lim R^{p,n}(f)$ exists when $n \rightarrow \infty$, the quantity $R^{p,n}(f)$ can be used to estimate this limit which is a “mean roughness” also called *IR-roughness of order p* . And this can be extended to sample paths of very general random processes, e.g. stationary processes, processes with stationary and nonstationary increments, and even \mathbb{L}^q -processes with $q < 1$.

Let us describe the main results of this paper. Sec. 2 derives some general results on asymptotic behavior of this estimator. Proposition 2.1 says that, for a sufficiently smooth function f , the limit $\lim_{n \rightarrow \infty} R^{p,n}(f) = 1$. In the most of the paper, $f = X$ is a random process. Following Dobrushin (1980), we say that $X = (X_t, t \in \mathbb{R})$ has a *small scale limit* $Y^{(t_0)}$ at point $t_0 \in \mathbb{R}$ if there exist a normalization $A^{(t_0)}(\lambda) \rightarrow \infty$ when $\lambda \rightarrow 0$ and a random process $Y^{(t_0)} = (Y_\tau^{(t_0)}, \tau \geq 0)$ such that

$$(1.3) \quad A^{(t_0)}(\lambda) (X_{t_0+\tau\lambda} - X_{t_0}) \xrightarrow[\lambda \rightarrow 0]{f.d.d.} Y_\tau^{(t_0)},$$

where $\xrightarrow{f.d.d.}$ stands for weak convergence of finite dimensional distributions. A related definition is given in Falconer (2002, 2003) who called the limit process $Y^{(t_0)}$ a *tangent process* (at t_0). See also Benassi *et al.* (1997). In many cases, the normalization $A^{(t_0)}(\lambda) = \lambda^{H(t_0)}$, where $0 < H(t_0) < 1$ and the limit tangent process $Y^{(t_0)}$ is self-similar with index $H(t_0)$ (Falconer, 2003 or Dobrushin, 1980). Proposition 2.2 states that if X satisfies a similar condition to (1.3), then the statistic $R^{p,n}(X)$ converges to the integral

$$(1.4) \quad R^{p,n}(X) \xrightarrow[n \rightarrow \infty]{\mathcal{P}} \int_0^1 \mathbb{E} \left[\frac{|\Delta_0^p Y^{(t)} + \Delta_1^p Y^{(t)}|}{|\Delta_0^p Y^{(t)}| + |\Delta_1^p Y^{(t)}|} \right] dt,$$

in probability, where $\Delta_j^p Y^{(t)} = \Delta_j^{p,1} Y^{(t)} = \sum_{i=0}^p (-1)^{p-i} \binom{p}{i} Y_{j+i}^{(t)}$, $j = 0, 1$ is the corresponding increment of the tangent process $Y^{(t)}$ at $t \in [0, 1)$. In the particular case when X has stationary

increments, relation (1.4) becomes

$$(1.5) \quad R^{p,n}(X) \xrightarrow[n \rightarrow \infty]{\mathcal{P}} \mathbb{E} \left[\frac{|\Delta_0^p Y + \Delta_1^p Y|}{|\Delta_0^p Y| + |\Delta_1^p Y|} \right].$$

Sec. 3 discusses the convergence in (1.4) for *diffusion processes* X admitting a stochastic differential $dX = a_t dB(t) + b_t dt$, where B is a standard Brownian motion and $(a_t), (b_t)$ are random (adapted) functions. It is clear that under general regularity conditions on the diffusion and drift coefficients $(a_t), (b_t)$, the process X admits the same local Hölder exponent as B at each point $t_0 \in (0, 1)$ and therefore the IR-roughness of X in (1.4) should not depend on these coefficients and should coincide with the corresponding limit for $X = B$. This is indeed the case since the tangent process of X at t is easily seen to be $Y^{(t)} = a_t B$ and the multiplicative factor a_t cancels in the numerator and the denominator of the fraction inside the expectation in (1.4). See Proposition 3.1 for details, where the convergence rate $O(n^{1/3})$ (a.s.) in (1.4) with the explicit limit value $\Lambda_p(1/2)$ (see below) obtained also when $X = B$ is established.

Considerable attention is given to the asymptotic behavior of the statistic $R^{p,n}(X)$ for “*fractal*” *Gaussian processes* (see Sec. 4). In such a frame, *fractional Brownian motion* (fBm in the sequel) is a typical example. Indeed, if X is a fBm with parameter $H \in (0, 1)$, then X is also its self tangent process for any $t \in [0, 1]$ and (see Sec. 4):

$$(1.6) \quad R^{p,n}(X) \xrightarrow[n \rightarrow \infty]{a.s.} \Lambda_p(H) \quad (p = 1, 2),$$

where

$$(1.7) \quad \Lambda_p(H) := \Lambda_0(\rho_p(H)),$$

$$(1.8) \quad \Lambda_0(r) := \frac{1}{\pi} \arccos(-r) + \frac{1}{\pi} \sqrt{\frac{1+r}{1-r}} \log\left(\frac{2}{1+r}\right),$$

$$(1.9) \quad \rho_p(H) := \text{corr}(\Delta_0^p B_H, \Delta_1^p B_H),$$

and where $\Delta_j^1 B_H = B_H(j+1) - B_H(j)$, $\Delta_j^2 B_H = B_H(j+2) - 2B_H(j+1) + B_H(j)$ ($j \in \mathbb{Z}$) are respective increments of fBm. Moreover,

$$(1.10) \quad \sqrt{n}(R^{p,n}(X) - \Lambda_p(H)) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, \Sigma_p(H)) \quad \text{if} \quad \begin{cases} p = 1, & 0 < H < 3/4, \\ p = 2, & 0 < H < 1. \end{cases}$$

The asymptotic variances $\Sigma_p(H)$ are given by

$$(1.11) \quad \Sigma_p(H) := \sum_{j \in \mathbb{Z}} \text{cov} \left(\frac{|\Delta_0^p B_H + \Delta_1^p B_H|}{|\Delta_0^p B_H| + |\Delta_1^p B_H|}, \frac{|\Delta_j^p B_H + \Delta_{j+1}^p B_H|}{|\Delta_j^p B_H| + |\Delta_{j+1}^p B_H|} \right).$$

The graphs of $\Lambda_p(H)$ and $\sqrt{\Sigma_p(H)}$ ($p = 1, 2$) are given in Figures 1 and 3 below.

The difference in the range of the parameter H for $p = 1$ and $p = 2$ in the central limit theorem in (1.10) are due to the fact that the second order increment process $(\Delta_j^2 B_H, j \in \mathbb{Z})$ is a short memory stationary Gaussian process for any $H \in (0, 1)$, in contrast to the first order increment process

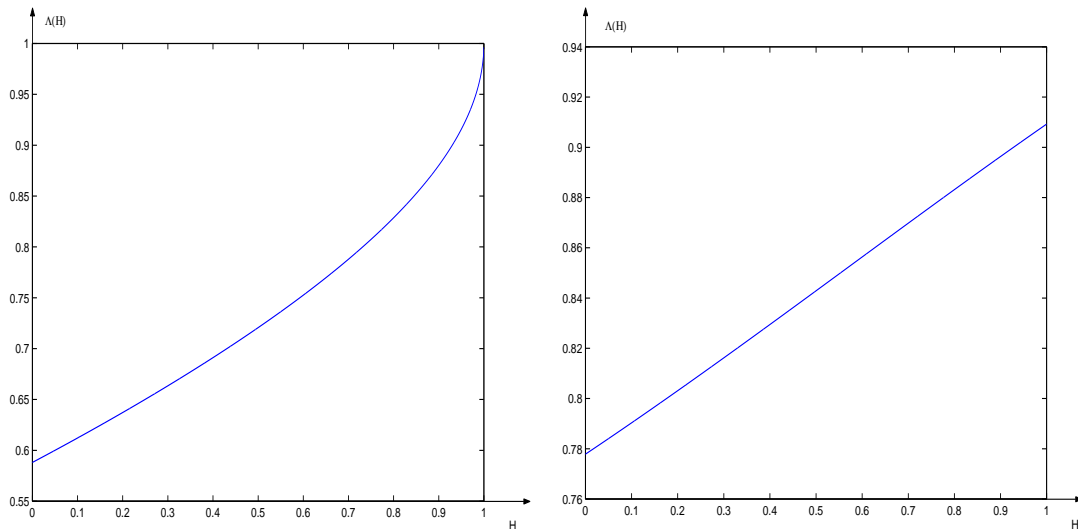


FIGURE 1. The graphs of $\Lambda_1(H)$ (left) and $\Lambda_2(H)$ (right).

$(\Delta_j^1 B_H, j \in \mathbb{Z})$. Generalizations of (1.6) and (1.10) to Gaussian processes having nonstationary increments are proposed in Sec. 4. Roughly speaking, $R^{p,n}(X)$, $p = 1, 2$ converge a.s. and satisfy a central limit theorem if for any $t \in [0, 1]$ the process X admits a fBm with parameter $H(t)$ as a tangent process (more precise assumptions **(A.1)**, **(A.1)'** and **(A.2)_p** are provided in Sec. 4). In such frames, the limits in (1.6) are $\int_0^1 \Lambda_p(H(t))dt$ instead of $\Lambda_p(H)$ and the asymptotic variances in (1.10) also change. The case of Gaussian processes with stationary increments is discussed in detail and the results are used to define a \sqrt{n} -consistent estimator of H , under semiparametric assumptions on the asymptotic behavior of the variogram or the spectral density. Bardet and Surgailis (2009) study a punctual estimator of $H(t_0)$ obtained from a localization around $t_0 \in (0, 1)$ of the statistic $R^{2,n}(X)$.

The main advantages of estimators of the type (1.1) (involving a scaling invariant function of increments) versus generalized quadratic variation, log-periodogram or wavelet based estimators seem to be the following. Firstly, the estimators $R^{p,n}(X)$ essentially depend on local regularity of processes and not on possible “multiplicative and additive factors” such as diffusion and drift coefficients in Sec. 3 or smoothly multiplicative and additive trended Gaussian processes (see Corollary 4.2 of Sec. 4). This property is important when dealing with financial data involving heteroscedasticity and volatility clustering.

Secondly, the estimators in (1.1) are bounded functionals and have finite moments of any order. Sec. 5 discusses jump Lévy processes, with the Lévy measure regularly varying of fractional index $\alpha \in (0, 2)$ at the origin. Using a modification of (1.1), we define a \sqrt{n} -consistent estimator of α , together with a central limit theorem, in a very general semiparametric frame. This result is new and interesting because there exist very few papers providing consistent estimators of α (to our knowledge, the only comparable results have been established in Belomestny, 2009 or Ait Sahalia and Jacod, 2009, in a financial and different frame).

Estimators of the form (1.1) can also be applied to discrete time (sequences) instead of continuous time processes (functions). For instance Surgailis *et al.* (2008) extended the statistic $R^{2,n}(X)$

to discrete time processes and used it to test for $I(d)$ behavior ($-1/2 < d < 5/4$) of observed time series. Vaičiulis (2009) considered estimation of the tail index of i.i.d. observations using an increment ratio statistic.

The paper is organized as follows. Sec. 2 discusses some general (consistency) properties of the estimators $R^{p,n}(X)$. Sec. 3 deals with the case when X is a diffusion. The case of Gaussian processes X is considered in Sec. 4 while the case of Lévy processes is studied in Sec. 5. Sec. 6-7 contain proofs and other derivations and graphs.

2. Some asymptotic results. It is easy to prove that $R^{1,n}(f) \xrightarrow[n \rightarrow \infty]{} 1$ if f is continuously differentiable on $[0, 1]$ and the derivative f' does not vanish on $[0, 1]$ except maybe for a finite number of points. Moreover, it is obvious that $R^{1,n}(f) = 1$ if f is monotone on $[0, 1]$: the IR-roughness of a monotone function is the same than for a smooth function which is not a surprising result (the same occurs for other measures of roughness, Hölder exponent or Hausdorff dimension for instance). We conjecture that $R^{p,n}(f) \rightarrow 1$ for any $p \geq 1$ and $f : [0, 1] \rightarrow \mathbb{R}$ which is $(p - 1)$ times differentiable and the derivative $f^{(p-1)}$ has bounded variation on $[0, 1]$ with the support $\text{supp}(f^{(p-1)}) = [0, 1]$. However, we can prove a weaker result.

Proposition 2.1 *Let f be $(p - 1)$ -times continuously differentiable with $f^{(p-1)}$ being absolutely continuous on $[0, 1]$ having the Radom–Nikodym derivative $g = (f^{(p-1)})'$. Assume that $g \neq 0$ a.e. in $[0, 1]$. Then $R^{p,n}(f) \xrightarrow[n \rightarrow \infty]{} 1$.*

Proof. We restrict the proof to the case $p = 2$ since the general case is analogous. Assume n is large enough and for a given $t \in (0, 1)$, let $k_n(t) \in \{0, \dots, n - 2\}$ be chosen so that $t \in [k_n(t)/n, (k_n(t) + 1)/n)$, therefore $k_n(t) = [nt] - 1$. We claim that for a.e. $t \in (0, 1)$

$$(2.1) \quad \lim_{n \rightarrow \infty} n^2 \Delta_{k_n(t)}^{2,n} f = g(t), \quad \lim_{n \rightarrow \infty} n^2 \Delta_{k_n(t)+1}^{2,n} f = g(t).$$

Using the fact that the function $(x_1, x_2) \mapsto \frac{|x_1+x_2|}{|x_1|+|x_2|}$ is continuous on $\mathbb{R}^2 \setminus \{(0, 0)\}$, we obtain

$$(2.2) \quad h^{2,n}(t) := \frac{|n^2 \Delta_{k_n(t)}^{2,n} f + n^2 \Delta_{k_n(t)+1}^{2,n} f|}{|n^2 \Delta_{k_n(t)}^{2,n} f| + |n^2 \Delta_{k_n(t)+1}^{2,n} f|} \xrightarrow[n \rightarrow \infty]{} \frac{|g(t) + g(t)|}{|g(t)| + |g(t)|} = 1$$

for a.e. $t \in (0, 1)$, where we used the fact that $g(t) \neq 0$ a.e. Since for $n \geq 2$, $R^{2,n}(f)$ can be written as $R^{2,n}(f) = \frac{n}{n-p} \int_0^1 h^{2,n}(t) dt$, relation $R^{p,n}(f) \xrightarrow[n \rightarrow \infty]{} 1$ follows by the dominated convergence theorem and the fact that $0 \leq h^{2,n}(t) \leq 1$.

Relations (2.1) can be proved using the Lebesgue–Vitali theorem (see Shilov and Gurevich, 1967, Ch. 4, Theorem 3), as follows. Consider the signed measure μ on Borel subsets of $[0, 1/2]^2$ given by

$$\mu(A) = \int_A g(x_1 + x_2) dx_1 dx_2.$$

Note $\Delta_k^{2,n} f = \mu((k/2n, (k+1)/2n] \times (k/2n, (k+1)/2n])$, ($k = 0, \dots, n - 2$). Since rectangles $[a_1, a_1 + h] \times [a_2, a_2 + h]$, ($0 \leq a_i < a_i + h \leq 1/2, i = 1, 2$) form a Vitali system on $[0, 1/2]^2$, the above mentioned Lebesgue–Vitali theorem implies that

$$(2.3) \quad \phi_n(t_1, t_2) := n^2 \mu \left(\left(\frac{k_n(t_1)}{2n}, \frac{k_n(t_1)+1}{2n} \right] \times \left(\frac{k_n(t_2)}{2n}, \frac{k_n(t_2)+1}{2n} \right] \right) \xrightarrow[n \rightarrow \infty]{} g \left(\frac{t_1+t_2}{2} \right)$$

a.e. in $[0, 1]^2$. Taking into account the form of the limiting function in (2.3), it follows the convergence $n^2 \Delta_{k_n(t)}^{2,n} f = \phi_n(t, t) \xrightarrow[n \rightarrow \infty]{} g(t)$ a.e. on $[0, 1]$. This proves the first relation in (2.1) and the second one follows analogously. \square

Let us turn now to the case when $f(t) = X_t$, $t \in [0, 1]$ is a random process. Now and in all the sequel, $R^{p,n}(X)$ is denoted $R^{p,n}$. Below we formulate a general condition for the convergence of $R^{p,n}$ to a deterministic limit.

Assumption (A): For a.e. pairs $(t_1, t_2) \in (0, 1)^2$, $t_1 \neq t_2$, for $i = 1, 2$ there exist:

- (i) normalizations $A^{(t_i)}(\lambda) \rightarrow \infty$ ($\lambda \rightarrow 0$),
- (ii) (mutually) independent random processes $Y^{(t_i)} = (Y^{(t_i)}(\tau), \tau \in [0, 1])$, satisfying the condition: $\forall \tau \in (0, 1]$, $P(|Y^{(t_i)}(\tau)| > 0) = 1$;

such that for $\lambda \rightarrow 0$, $s_1 \rightarrow t_1$, $s_2 \rightarrow t_2$

$$(2.4) \quad \left(A^{(t_1)}(\lambda)(X_{s_1+\lambda\tau} - X_{s_1}), A^{(t_2)}(\lambda)(X_{s_2+\lambda\tau} - X_{s_2}) \right) \xrightarrow{f.d.d.} (Y^{(t_1)}(\tau), Y^{(t_2)}(\tau)).$$

REMARK 2.1 Relation (2.4) implies the existence of a joint small scale limit $(Y^{(t_1)}, Y^{(t_2)})$ at a.e. pair $(t_1, t_2) \in (0, 1)$, with independent components $Y^{(t_1)}, Y^{(t_2)}$. Note Assumption (A) and Proposition 2.2 below are very general, in the sense that they do not assume any particular structure or distribution of X .

Proposition 2.2 *Let X satisfy Assumption (A). Then*

$$(2.5) \quad \mathbb{E} \left(R^{p,n} - \int_0^1 \mathbb{E} \left[\frac{|\Delta_0^p Y(t) + \Delta_1^p Y(t)|}{|\Delta_0^p Y(t)| + |\Delta_1^p Y(t)|} \right] dt \right)^2 \xrightarrow[n \rightarrow \infty]{} 0.$$

Proof. The statement follows from

$$(2.6) \quad \mathbb{E} R^{p,n} \xrightarrow[n \rightarrow \infty]{} \int_0^1 \mathbb{E} \left[\frac{|\Delta_0^p Y(t) + \Delta_1^p Y(t)|}{|\Delta_0^p Y(t)| + |\Delta_1^p Y(t)|} \right] dt \quad \text{and} \quad \mathbb{E} (R^{p,n} - \mathbb{E} R^{p,n})^2 \xrightarrow[n \rightarrow \infty]{} 0.$$

Write $\mathbb{E} R^{p,n} = \frac{n}{n-p} \int_0^1 \mathbb{E} h_X^{p,n}(t) dt$, where (c.f. (2.2))

$$\begin{aligned} h_X^{p,n}(t) &:= \frac{A^{(t)}(1/n) \left(|\Delta_{k_n(t)}^{p,n} X + \Delta_{k_n(t)+1}^{p,n} X \right)}{A^{(t)}(1/n) \left(|\Delta_{k_n(t)}^{p,n} X| + |\Delta_{k_n(t)+1}^{p,n} X \right)} \\ &\xrightarrow[n \rightarrow \infty]{\mathcal{D}} \frac{|\Delta_0^p Y(t) + \Delta_1^p Y(t)|}{|\Delta_0^p Y(t)| + |\Delta_1^p Y(t)|}, \end{aligned}$$

for a.e. $t \in (0, 1)$, according to Assumption (A) and the continuous mapping theorem. Whence and from $0 \leq h_X^{p,n} \leq 1$ (and therefore the Lebesgue theorem) the first relation in (2.6) follows. Moreover,

$$\mathbb{E} (R^{p,n} - \mathbb{E} R^{p,n})^2 = \left(\frac{n}{n-p} \right)^2 \int_0^1 \int_0^1 \mathbb{E} [h_X^{p,n}(t) h_X^{p,n}(t')] dt dt' - (\mathbb{E} R^{p,n})^2,$$

and with the same arguments as previously and the independence of $Y^{(t)}$ and $Y^{(t')}$ when $t \neq t'$,

$$\mathbb{E} [h_X^{p,n}(t) h_X^{p,n}(t')] \xrightarrow[n \rightarrow \infty]{} \mathbb{E} \left[\frac{|\Delta_0^p Y(t) + \Delta_1^p Y(t)|}{|\Delta_0^p Y(t)| + |\Delta_1^p Y(t)|} \cdot \frac{|\Delta_0^p Y(t') + \Delta_1^p Y(t')|}{|\Delta_0^p Y(t')| + |\Delta_1^p Y(t')|} \right] = \mathbb{E} h_X^{p,n}(t) \cdot \mathbb{E} h_X^{p,n}(t')$$

and therefore $\left(\frac{n}{n-p}\right)^2 \int_0^1 \int_0^1 \mathbb{E}[h_X^{p,n}(t)h_X^{p,n}(t')]dt dt' - \left(\int_0^1 \mathbb{E}h_X^{p,n}(t)dt\right)^2 \xrightarrow{n \rightarrow \infty} 0$, thereby proving the second relation in (2.6) and the proposition. \square

The following obvious but interesting corollary can also be added to this result. It proves that smooth multiplicative or additive trends do no change the \mathbb{L}^2 -asymptotic behavior of $R^{p,n}$.

Corollary 2.1 *Let $Z = (Z_t, t \in [0, 1])$ satisfy Assumption (A) and a, b be two continuously differentiable functions on $(0, 1)$. Define X such that $X_t = a(t)Z_t + b(t)$ for a.e. $t \in (0, 1)$. Then (2.5) holds.*

3. Diffusions. Let

$$(3.1) \quad X_t = X_0 + \int_0^t a_s dB(s) + \int_0^t b_s ds, \quad t \in [0, 1]$$

be a diffusion (or Itô's) process on \mathbb{R} . In (3.1), we assume the existence of a right-continuous filtration $\mathcal{F} = (\mathcal{F}_t, t \in [0, 1])$, a standard Brownian motion B adapted to \mathcal{F} ; moreover, $a_s, b_s, s \in [0, 1]$ are adapted random functions satisfying $\int_0^1 |b_s| ds < \infty$, $\int_0^1 a_s^2 ds < \infty$ a.s., and X_0 is a \mathcal{F}_0 -measurable r.v. Write $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot | \mathcal{F}_t]$ for the conditional expectation. Let $\Lambda_1(1/2) = \Lambda_0(\rho_1(1/2)) \simeq 0.7206$ and $\Lambda_2(1/2) = \Lambda_0(\rho_2(1/2)) \simeq 0.5881$.

Theorem 3.1 *Assume the following conditions: there exist random variables K_1, K_2 such that $0 < K_i < \infty$ a.s., and such that, for any sufficiently small $h > 0$ and any $0 \leq t < t+h \leq 1$, the following inequalities hold, a.s.:*

$$(3.2) \quad |a_t| \geq K_1, \quad \mathbb{E}_t b_{t+h}^2 \leq K_2 \quad \text{and} \quad \mathbb{E}_t (a_{t+h} - a_t)^2 \leq K_2 h.$$

Then

$$(3.3) \quad R^{p,n} - \Lambda_p(1/2) = O(n^{-1/3}) \quad \text{a.s.} \quad (p = 1, 2).$$

Proof. We restrict the proof to the case $p = 1$ since the case $p = 2$ is analogous. For notational simplicity, assume that n is odd. Define

$$(3.4) \quad \eta_n(k) := \frac{|\Delta_k^{1,n} X + \Delta_{k+1}^{1,n} X|}{|\Delta_k^{1,n} X| + |\Delta_{k+1}^{1,n} X|}, \quad \eta'_n(k) := \mathbb{E}_{k/n}[\eta_n(k)], \quad \eta''_n(k) := \eta_n(k) - \eta'_n(k)$$

and correspondingly write $R^{1,n} = R'_n + R''_n$, $R'_n := (n-1)^{-1} \sum_{k=0}^{n-2} \eta'_n(k)$, $R''_n := (n-1)^{-1} \sum_{k=0}^{(n-2)/2} \eta''_n(2k)$, $R''_{n2} := (n-1)^{-1} \sum_{k=0}^{(n-4)/2} \eta''_n(2k+1)$. As $(\eta''_n(2k), \mathcal{F}_{(2k+2)/n}, k = 0, \dots, (n-2)/2)$ is a martingale difference sequence, so by Burkholder's inequality,

$$\mathbb{E}(R''_{n1})^8 \leq C n^{-8} \left(\sum_{k=0}^{(n-2)/2} \mathbb{E}^{1/4}(\eta''_n(2k))^8 \right)^4 \leq C n^{-4}$$

and therefore

$$\sum_{n=1}^{\infty} \mathbb{P}(|R''_{n1}| > n^{-1/3}) \leq C \sum_{n=1}^{\infty} n^{8/3} n^{-4} < \infty,$$

implying $R''_{n1} = O(n^{-1/3})$ a.s. A similar fact holds for R''_{n2} . Thus, it remains to prove

$$(3.5) \quad R'_n - \Lambda_1(1/2) = O(n^{-1/3}) \quad a.s.$$

Observe

$$\eta'_n(k) - \Lambda_1(1/2) = \mathbf{E}_{k/n} \left[\frac{|Z_1(k) + \xi_1(k) + Z_2(k) + \xi_2(k)|}{|Z_1(k) + \xi_1(k)| + |Z_2(k) + \xi_2(k)|} \right] - \mathbf{E} \left[\frac{|Z_1(k) + Z_2(k)|}{|Z_1(k)| + |Z_2(k)|} \right],$$

where

$$\begin{aligned} Z_1(k) &:= n^{1/2} \Delta_k^{1,n} B, & Z_2(k) &:= n^{1/2} \Delta_{k+1}^{1,n} B, \\ \xi_1(k) &:= n^{1/2} \int_{k/n}^{(k+1)/n} \left(\frac{a_s}{a_{k/n}} - 1 \right) dB(s) + n^{1/2} \int_{k/n}^{(k+1)/n} \frac{b_s}{a_{k/n}} ds, \\ \xi_2(k) &:= n^{1/2} \int_{(k+1)/n}^{(k+2)/n} \left(\frac{a_s}{a_{k/n}} - 1 \right) dB(s) + n^{1/2} \int_{(k+1)/n}^{(k+2)/n} \frac{b_s}{a_{k/n}} ds. \end{aligned}$$

According to Lemma 3.2 below, $|\eta'_n(k) - \Lambda_1(1/2)| \leq 36 \max_{i=1,2} (\mathbf{E}_{k/n} \xi_i^2(k))^{1/3}$ and therefore

$$|R'_n - \Lambda_1(1/2)| \leq 36 \max\{(\mathbf{E}_{k/n} \xi_i^2(k))^{1/3} : i = 1, 2, k = 0, 1, \dots, n-1\}.$$

Whence, (3.5) follows from the following fact: there exists a r.v. $K < \infty$, independent of n and such that for any $n \geq 1$, $k = 0, \dots, n-1$, $i = 1, 2$

$$(3.6) \quad \mathbf{E}_{k/n} \xi_i^2(k) \leq K n^{-1}, \quad a.s.$$

Indeed, using (3.2),

$$\begin{aligned} \mathbf{E}_{k/n} \xi_1^2(k) &= n \int_{k/n}^{(k+1)/n} \mathbf{E}_{k/n} \left(\frac{a_s}{a_{k/n}} - 1 \right)^2 ds + n \mathbf{E}_{k/n} \left(\int_{k/n}^{(k+1)/n} \frac{b_s}{a_{k/n}} ds \right)^2 \\ &\leq n K_1^{-2} \int_{k/n}^{(k+1)/n} \mathbf{E}_{k/n} (a_s - a_{k/n})^2 ds + K_1^{-2} \int_{k/n}^{(k+1)/n} \mathbf{E}_{k/n} b_s^2 ds \\ &\leq K_2 K_1^{-2} n^{-1}, \quad a.s., \end{aligned}$$

and the bound (3.6) for $i = 2$ follows similarly. This proves (3.6) and Theorem 3.1, too. \square

Lemma 3.1 *Let $\psi(x_1, x_2) = |x_1 + x_2| / (|x_1| + |x_2|)$ ($x_1, x_2 \in \mathbb{R}$), and let $Z_i, i = 1, 2$ be independent $\mathcal{N}(0, 1)$ r.v.'s. Then for any random variables ξ_1, ξ_2 ,*

$$(3.7) \quad |\mathbf{E}\psi(Z_1 + \xi_1, Z_2 + \xi_2) - \mathbf{E}\psi(Z_1, Z_2)| \leq 20 \max_{i=1,2} (\mathbf{E}\xi_i^2)^{1/3}.$$

Proof. Let $\delta^2 := \max_{i=1,2} \mathbf{E}\xi_i^2$. If $\delta^2 \geq 1/2$, then (3.7) holds since the l.h.s. of (3.7) does not exceed 1. Let $\delta^2 < 1/2$ in the sequel. Write $U := \psi(Z_1 + \xi_1, Z_2 + \xi_2) - \psi(Z_1, Z_2) = U_\delta + U_\delta^c$,

$$\begin{aligned} U_\delta &:= U \mathbf{1}(A_\delta) = (\psi(Z_1 + \xi_1, Z_2 + \xi_2) - \psi(Z_1, Z_2)) \mathbf{1}(A_\delta), \\ U_\delta^c &:= U \mathbf{1}(A_\delta^c) = (\psi(Z_1 + \xi_1, Z_2 + \xi_2) - \psi(Z_1, Z_2)) \mathbf{1}(A_\delta^c), \end{aligned}$$

where $\mathbf{1}(A_\delta)$ is the indicator of the event

$$A_\delta := \{|Z_1| > \delta^{2/3}, |Z_2| > \delta^{2/3}, |\xi_1| < \delta^{2/3}/2, |\xi_2| < \delta^{2/3}/2\},$$

and $\mathbf{1}(A_\delta^c) = 1 - \mathbf{1}(A_\delta)$ is the indicator of the complementary event A^c . Clearly,

$$\begin{aligned} \mathbb{E}|U_\delta^c| &\leq 2(\mathbb{P}(|Z_1| < \delta^{2/3}) + \mathbb{P}(|Z_2| < \delta^{2/3}) + \mathbb{P}(|\xi_1| \geq \delta^{2/3}/2) + \mathbb{P}(|\xi_2| \geq \delta^{2/3}/2)) \\ &\leq \frac{4}{\sqrt{2\pi}} \delta^{2/3} + 2 \frac{\mathbb{E}\xi^2}{\delta^{4/3}} \leq 4 \delta^{2/3}. \end{aligned}$$

It remains to estimate $\mathbb{E}|U_\delta|$. By the mean value theorem,

$$|U_\delta| \leq \left(|\xi_1| \sup_D |\psi_{x_1}| + |\xi_2| \sup_D |\psi_{x_2}| \right) \mathbf{1}(A_\delta)$$

where

$$\begin{aligned} \sup_D |\psi_{x_1}| &:= \sup\{|\partial\psi(x_1, x_2)/\partial x_1| : |x_i - Z_i| \leq |\xi_i|, i = 1, 2\}, \\ \sup_D |\psi_{x_2}| &:= \sup\{|\partial\psi(x_1, x_2)/\partial x_2| : |x_i - Z_i| \leq |\xi_i|, i = 1, 2\}. \end{aligned}$$

Therefore

$$\begin{aligned} \mathbb{E}|U_\delta| &\leq \mathbb{E}^{1/2}(\xi_1^2) \mathbb{E}^{1/2}[(\sup_D |\psi_{x_1}|)^2 \mathbf{1}(A_\delta)] + \mathbb{E}^{1/2}(\xi_2^2) \mathbb{E}^{1/2}[(\sup_D |\psi_{x_2}|)^2 \mathbf{1}(A_\delta)] \\ &\leq \delta \left(\mathbb{E}^{1/2}[(\sup_D |\psi_{x_1}|)^2 \mathbf{1}(A_\delta)] + \mathbb{E}^{1/2}[(\sup_D |\psi_{x_2}|)^2 \mathbf{1}(A_\delta)] \right). \end{aligned}$$

Next,

$$|\psi_{x_i}(x_1, x_2)| = \frac{|\operatorname{sgn}(x_1 + x_2)(|x_1| + |x_2|) - (|x_1 + x_2|)\operatorname{sgn}(x_i)|}{(|x_1| + |x_2|)^2} \leq \frac{2}{|x_1| + |x_2|}.$$

Therefore

$$\begin{aligned} \sup_D |\psi_{x_i}|^2 \mathbf{1}(A_\delta) &\leq 4 \sup\{(|x_1| + |x_2|)^{-2} : |x_i - Z_i| \leq \delta^{2/3}/2, |Z_i| > \delta^{2/3}, i = 1, 2\} \\ &\leq 16 (|Z_1| + |Z_2|)^{-2} \mathbf{1}(|Z_i| > \delta^{2/3}, i = 1, 2) \end{aligned}$$

implying

$$\mathbb{E}[(\sup_D |\psi_{x_i}|)^2 \mathbf{1}(A_\delta)] \leq 16 \mathbb{E} \left[\frac{1}{(|Z_1| + |Z_2|)^2}; |Z_i| > \delta^{2/3}, i = 1, 2 \right] \leq C(\delta),$$

where

$$\begin{aligned} C(\delta) &\leq \frac{16}{2\pi} \int_{\{x_1^2 + x_2^2 > \delta^{4/3}\}} \frac{1}{x_1^2 + x_2^2} e^{-(x_1^2 + x_2^2)/2} dx_1 dx_2 \\ &\leq 16 \int_{\delta^{4/3}}^{\infty} r^{-1} e^{-r^2/2} dr \leq 16 (1 + (4/3) |\log \delta|). \end{aligned}$$

Hence $\mathbb{E}|U_\delta| \leq 8\delta(1 + (4/3) |\log \delta|)^{1/2}$, $\mathbb{E}|U_\delta^c| \leq 4\delta^{2/3}$. It remains to use $x(1 + (4/3) |\log x|)^{1/2} \leq 2x^{2/3}$ for all $0 < x \leq 1$. \square

Let us present some examples of Itô's processes X satisfying conditions (3.2).

EXAMPLE 3.1 Let $(X_t, t \in [0, 1])$ be a Markov process satisfying a stochastic equation

$$(3.8) \quad X_t = x_0 + \int_0^t a(X_s) dB(s) + \int_0^t b(X_s) ds,$$

where $x_0 \in \mathbb{R}$ is nonrandom, $a(x), b(x), x \in \mathbb{R}$ are real measurable functions and B is a standard Brownian motion. Let $\mathcal{F}_t := \sigma\{B(s), s \leq t\}, 0 \leq t \leq 1$ be the natural filtration. Assume that

$$(3.9) \quad |a(x) - a(y)| \leq K|x - y|, \quad |b(x) - b(y)| \leq K|x - y| \quad (x, y \in \mathbb{R})$$

for some constant $K < \infty$. Then equation (3.8) admits a unique adapted solution; see e.g. Gikhman and Skorohod (1969). Let $a_t = a(X_t), b_t = b(X_t)$. Assume in addition that $|a(x)| \geq K_1, (x \in \mathbb{R})$ for some nonrandom constant $K_1 > 0$. Then the first inequality in (3.2) is trivially satisfied; moreover, the second and third relations in (3.2) are also satisfied, with $K_2 = C(1 + \sup_{0 \leq t \leq 1} X_t^2) < \infty$ and $K_3 = C$, where C is nonrandom and depends on the constant K in (3.9) only.

EXAMPLE 3.2 Let $X_t := g(t, B(t))$, where B is a standard Brownian motion and $g(t, x)$ is a (jointly) continuous function on $[0, 1] \times \mathbb{R}$, having continuous partial derivatives $g_t(t, x) := \partial g(t, x)/\partial t$, $g_x(t, x) := \partial g(t, x)/\partial x$, $g_{xx}(t, x) = \partial^2 g(t, x)/\partial x^2$. By Itô's lemma,

$$dX_t = g_x(t, B(t))dB(t) + \left(g_t(t, B(t)) + \frac{1}{2}g_{xx}(t, B(t)) \right) dt,$$

so that X admits the representation (3.1) with $a_t = g_x(t, B(t)), b_t = g_t(t, B(t)) + \frac{1}{2}g_{xx}(t, B(t))$ and the same filtration as in the previous example. Assume that

$$|g_x(t, x)| \geq K_1, \quad |g_x(s, y) - g_x(t, x)| \leq K(|s - t|^{1/2} + |y - x|),$$

for all $(t, x), (s, y) \in [0, 1] \times \mathbb{R}$ and some constants $0 < K_1, K < \infty$. Then X satisfies the conditions in (3.2).

4. Gaussian processes.

4.1. Assumptions. Let $X = (X_t, t \in [0, 1])$ be a Gaussian process, with zero mean. Without loss of generality, assume $X_0 = 0$. Define $\sigma_{p,n}^2(k)$, the variance of $\Delta_k^{p,n}X$, and $\rho_{p,n}(k)$, the correlation coefficient between $\Delta_k^{p,n}X$ and $\Delta_{k+1}^{p,n}X$, i.e.

$$(4.1) \quad \sigma_{p,n}^2(k) := \mathbb{E} \left[(\Delta_k^{p,n}X)^2 \right],$$

$$(4.2) \quad \rho_{p,n}(k) := \frac{\mathbb{E}[\Delta_k^{p,n}X \Delta_{k+1}^{p,n}X]}{\sigma_{p,n}(k)\sigma_{p,n}(k+1)}.$$

Let $B_H = (B_H(t), t \in \mathbb{R})$ be a fractional Brownian motion (fBm) with parameter $0 < H < 1$, i.e., a Gaussian process with zero mean and covariance such that $\mathbb{E}B_H(s)B_H(t) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H})$. Its p -th order increments $(\Delta_j^p B_H, j \in \mathbb{Z})$ form a stationary Gaussian process, for any $p \geq 1$. In particular, the covariance function of $\Delta_j B_H \equiv \Delta_j^1 B_H = B_H(j+1) - B_H(j)$ and $\Delta_j^2 B_H = B_H(j+2) - 2B_H(j+1) + B_H(j)$ can be explicitly calculated:

$$(4.3) \quad \mathbb{E}[\Delta_0 B_H \Delta_j B_H] = 2^{-1} (|j+1|^{2H} + |j-1|^{2H} - 2|j|^{2H}),$$

$$(4.4) \quad \mathbb{E}[\Delta_0^2 B_H \Delta_j^2 B_H] = 2^{-1} (-|j+2|^{2H} + 4|j+1|^{2H} - 6|j|^{2H} + 4|j-1|^{2H} - |j-2|^{2H}).$$

From Taylor expansion,

$$\begin{aligned} \mathbb{E}[\Delta_0 B_H \Delta_j B_H] &\sim 2H(2H-1)j^{2H-2}, \\ \mathbb{E}[\Delta_0^2 B_H \Delta_j^2 B_H] &\sim 2H(2H-1)(2H-2)(2H-3)j^{2H-4}, \end{aligned}$$

as $j \rightarrow \infty$, and therefore the first increment, $(\Delta_j B_H)$, has a summable covariance if and only if $0 < H < 3/4$, while the second increment, $(\Delta_j^2 B_H)$, has a summable covariance for any $0 < H < 1$.

Let $\mathbb{N}^* = \{1, 2, \dots\}$. Introduce the following conditions:

(A.1) There exist continuous functions $H(t) \in (0, 1)$ and $c(t) > 0$ for $t \in [0, 1]$ such that $\forall j \in \mathbb{N}^*$

$$(4.5) \quad \lim_{n \rightarrow \infty} \sup_{t \in (0,1)} \left| \frac{\mathbb{E}(X_{([nt]+j)/n} - X_{[nt]/n})^2}{(j/n)^{2H(t)}} - c(t) \right| = 0, \quad \text{with}$$

$$(4.6) \quad \lim_{n \rightarrow \infty} \sup_{t \in (0,1)} |H(t) - H(t + \frac{1}{n})| \log n = 0.$$

(A.1)' There exist continuous functions $H(t) \in (0, 1)$ and $c(t) > 0$ for $t \in [0, 1]$ such that $\forall j \in \mathbb{N}^*$

$$(4.7) \quad \lim_{n \rightarrow \infty} \sup_{t \in (0,1)} \sqrt{n} \left| \frac{\mathbb{E}(X_{([nt]+j)/n} - X_{[nt]/n})^2}{(j/n)^{2H(t)}} - c(t) \right| = 0, \quad \text{with}$$

$$(4.8) \quad \lim_{n \rightarrow \infty} \sup_{t \in (0,1)} |H(t) - H(t + \frac{1}{n})| \sqrt{n} \log n = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \sup_{t \in (0,1)} |c(t) - c(t + \frac{1}{n})| \sqrt{n} = 0.$$

(A.2)_p There exist $d > 0$, $\gamma > 1/2$ and $0 \leq \theta < \gamma/2$ such that for any $1 \leq k < j \leq n$ with $n \in \mathbb{N}^*$

$$(4.9) \quad \left| \mathbb{E}[\Delta_k^{p,n} X \Delta_j^{p,n} X] \right| \leq d \sigma_{p,n}(k) \sigma_{p,n}(j) \cdot n^\theta \cdot |j - k|^{-\gamma}.$$

A straightforward application of Assumption **(A.1)** (or **(A.1)'**) implies that $\sqrt{c(t)} B_{H(t)}$ is the tangent process of X for all $t \in (0, 1)$ and more precisely:

Property 4.1 *Assumptions **(A.1)**, **(A.1)'** respectively imply that, for any $j \in \mathbb{Z}$ and $p = 1, 2$,*

$$(4.10) \quad \lim_{n \rightarrow \infty} \sup_{t \in (0,1)} \left| \frac{\mathbb{E}[\Delta_{[nt]}^{p,n} X \Delta_{j+[nt]}^{p,n} X]}{\mathbb{E}[\Delta_0^{p,n} B_{H(t)} \Delta_j^{p,n} B_{H(t)}]} - c(t) \right| = 0,$$

$$(4.11) \quad \lim_{n \rightarrow \infty} \sqrt{n} \sup_{t \in (0,1)} \left| \frac{\mathbb{E}[\Delta_{[nt]}^{p,n} X \Delta_{j+[nt]}^{p,n} X]}{\mathbb{E}[\Delta_0^{p,n} B_{H(t)} \Delta_j^{p,n} B_{H(t)}]} - c(t) \right| = 0.$$

Moreover, for any $t \in (0, 1)$ and $p = 1, 2$

$$\left(n^{H(t)} \Delta_{j+[nt]}^{p,n} X \right)_{j \in \mathbb{Z}} \xrightarrow[n \rightarrow \infty]{f.d.d.} \left(\sqrt{c(t)} \Delta_j^p B_{H(t)} \right)_{j \in \mathbb{Z}}.$$

Assumption **(A.1)** can be characterized as *uniform local self-similarity of (X_t)* (the uniformity refers to the supremum over $t \in (0, 1)$ in (4.5)). Note that for X having stationary increments and variogram $V(t) = \mathbb{E}X_t^2$, Assumption **(A.1)** is reduces to $V(t) \sim ct^{2H}$ ($c > 0, 0 < H < 1$). For $j = 0, 1$, relation (4.10) implies that the variance and the $1/n$ -lag correlation coefficient of $\Delta_{[nt]}^{p,n} X$ satisfy for all $t \in (0, 1)$,

$$(4.12) \quad \sigma_{1,n}^2([nt]) \underset{n \rightarrow \infty}{\sim} c(t) \sigma_1^2(H(t)) = c(t) \mathbb{E}[(\Delta_0 B_{H(t)})^2] = c(t) \left(\frac{1}{n} \right)^{2H(t)},$$

$$(4.13) \quad \rho_{1,n}([nt]) \underset{n \rightarrow \infty}{\longrightarrow} \rho_1(H(t)) = \text{corr}(B_{H(t)}(1), B_{H(t)}(2) - B_{H(t)}(1)) = 2^{2H(t)-1} - 1,$$

$$(4.14) \quad \sigma_{2,n}^2([nt]) \underset{n \rightarrow \infty}{\sim} c(t) \sigma_2^2(H(t)) = c(t) \mathbb{E}[(\Delta_0^2 B_{H(t)})^2] = c(t) (4 - 4^{H(t)}) \left(\frac{1}{n} \right)^{2H(t)},$$

$$(4.15) \quad \begin{aligned} \rho_{2,n}([nt]) &\underset{n \rightarrow \infty}{\longrightarrow} \rho_2(H(t)) = \text{corr}(B_{H(t)}(2) - 2B_{H(t)}, B_{H(t)}(3) - 2B_{H(t)}(2) + B_{H(t)}(1)) \\ &= \frac{-3^{2H(t)} + 2^{2H(t)+2} - 7}{8 - 2^{2H(t)+1}} \end{aligned}$$

(see (7.1)). Moreover, relations (4.12)–(4.15) hold uniformly in $t \in (0, 1)$. Condition (4.6) is a technical condition which implies (and is "almost" equivalent to) the continuity of the function $t \rightarrow H(t)$. Assumption $(\mathbf{A.1})'$ is a sharper convergence condition than Assumption $(\mathbf{A.1})$ required for establishing central limit theorems.

Condition (4.9) specifies a nonasymptotic inequality satisfied by the correlation of increments $\Delta_k^{p,n} X$. The particular case of stationary processes allows to better understand this point. Indeed, if (X_t) has stationary increments, the covariance of the stationary process $(\Delta_k^{p,n} X, k \in \mathbb{Z})$ is completely determined by the variogram $V(t)$, e.g.

$$(4.16) \quad \mathbb{E}[\Delta_k^{1,n} X \Delta_j^{1,n} X] = \frac{1}{2} \left\{ V\left(\frac{k-j+1}{n}\right) + V\left(\frac{k-j-1}{n}\right) - 2V\left(\frac{k-j}{n}\right) \right\}.$$

In the "most regular" case, when $X = B_H$ is a fBm and therefore $V(t) = t^{2H}$, it is easy to check that assumption $(\mathbf{A.2})_2$ holds with $\theta = 0$ and $\gamma = 4 - 2H > 2$ ($0 < H < 1$), while $(\mathbf{A.2})_1$ with $\theta = 0, \gamma = 2 - 2H$ is equivalent to $H < 3/4$ because of the requirement $\gamma > 1/2$. However, for $X = B_H$, $(\mathbf{A.2})_1$ holds with appropriate $\theta > 0$ in the wide region $0 < H < 7/8$, by choosing $\theta < 2 - 2H$ arbitrarily close to $2 - 2H$ and then $\gamma < 2 - 2H + \theta$ arbitrary close to $4 - 4H$. A similar choice of parameters θ and γ allows to satisfy $(\mathbf{A.2})_p$ for more general X with stationary increments and variogram $V(t) \sim ct^{2H}$ ($t \rightarrow 0$), under additional regularity conditions on $V(t)$ (see below).

Property 4.2 *Let X have stationary increments and variogram $V(t) \sim ct^{2H}$ ($t \rightarrow 0$), with $c > 0$, $H \in (0, 1)$.*

(i) *Assume, in addition, that $0 < H < 7/8$ and $|V''(t)| \leq Ct^{-\kappa}$ ($0 < t < 1$), for some $C > 0$ and $4 - 4H > \kappa \geq 2 - 2H, \kappa > 1/2$. Then assumption $(\mathbf{A.2})_1$ holds.*

(ii) *Assume, in addition, that $|V^{(4)}(t)| \leq Ct^{-\kappa}$ ($0 < t < 1$), for some $C > 0$ and $8 - 4H > \kappa \geq 4 - 2H$. Then assumption $(\mathbf{A.2})_2$ holds.*

The following property provides a sufficient condition for $(\mathbf{A.2})_p$ in spectral terms, which does not require differentiability of the variogram.

Property 4.3 *Let X be a Gaussian process having stationary increments and the spectral representation (see for instance Cramèr and Leadbetter, 1967)*

$$(4.17) \quad X_t = \int_{\mathbb{R}} (e^{it\xi} - 1) f^{1/2}(\xi) W(d\xi), \quad \text{for all } t \in \mathbb{R},$$

where $W(dx) = \overline{W(-dx)}$ is a complex-valued Gaussian white noise with zero mean and variance $\mathbb{E}|W(dx)|^2 = dx$ and f is a non-negative even function called the spectral density of X such that

$$(4.18) \quad \int_{\mathbb{R}} (1 \wedge |\xi|^2) f(\xi) d\xi < \infty.$$

Moreover, assume that f is differentiable on (K, ∞) and

$$(4.19) \quad f(\xi) \sim c\xi^{-2H-1} \quad (\xi \rightarrow \infty), \quad |f'(\xi)| \leq C\xi^{-2H-2} \quad (\xi > K)$$

for some constants $c, C, K > 0$. Then X satisfies assumption $(\mathbf{A.2})_1$ for $0 < H < 3/4$ and assumption $(\mathbf{A.2})_2$ for $0 < H < 1$.

4.2. *Limit theorems.* Before establishing limit theorems for the statistics $R^{p,n}$ for Gaussian processes, recall that Λ_0 is given in (1.8) and with $\rho_p(H)$ in (7.1) one has

$$\int_0^1 \Lambda_p(H(t))dt = \int_0^1 \Lambda_0(\rho_p(H(t)))dt = \int_0^1 \mathbb{E} \left[\frac{|\Delta_0^p B_{H(t)} + \Delta_1^p B_{H(t)}|}{|\Delta_0^p B_{H(t)}| + |\Delta_1^p B_{H(t)}|} \right] dt.$$

Straightforward computations show that Assumptions **(A.1)** (existence of tangent process $Y^{(t)}$) and normalization $A^{(t)}(\lambda) = \lambda^{-H(t)}$ and **(A.2)_p** (independence of $Y^{(t)}$ and $Y^{(t')}$) imply Assumption **(A)** with $Y^{(t)} = \sqrt{c(t)} B_{H(t)}$. Therefore Proposition 2.2 ensures the convergence (in \mathbb{L}^2) of the statistics $R^{p,n}$ to $\int_0^1 \Lambda_p(H(t))dt$. Bardet and Surgailis (2009) proved a.s. convergence in Theorem 4.1, below, using a general moment bound for multidimensional Gaussian processes.

Theorem 4.1 *Let X be a Gaussian process satisfying Assumptions **(A.1)** and **(A.2)_p**. Then,*

$$(4.20) \quad R^{p,n} \xrightarrow[n \rightarrow \infty]{a.s.} \int_0^1 \Lambda_p(H(t))dt \quad (p = 1, 2).$$

Corollary 4.1 *Assume that X is a Gaussian process having stationary increments, whose variogram satisfies the conditions of Properties 4.2 or 4.3. Then*

$$(4.21) \quad R^{p,n} \xrightarrow[n \rightarrow \infty]{a.s.} \Lambda_p(H) \quad (p = 1, 2).$$

The following Theorem 4.2 is also established in Bardet and Surgailis (2009). Its proof uses a general central limit theorem for Gaussian subordinated nonstationary triangular arrays.

Theorem 4.2 *Let X be a Gaussian process satisfying assumptions **(A.1)'** and **(A.2)_p**, with $\theta = 0$. Moreover, assume additionally $\sup_{t \in [0,1]} H(t) < 3/4$ if $p = 1$. Then, for $p = 1, 2$,*

$$(4.22) \quad \sqrt{n} \left(R^{p,n} - \int_0^1 \Lambda_p(H(t))dt \right) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N} \left(0, \int_0^1 \Sigma_p(H(\tau))d\tau \right),$$

with $\Lambda_p(H)$ and $\Sigma_p(H)$ given in (1.7) and (1.11), respectively.

Once again, both the previous theorems are satisfied when smooth multiplicative and additive trends are considered. This is an important advantage of the statistic $R^{p,n}$ compared to statistics as quadratic variations.

Corollary 4.2 *Let $Z = (Z_t, t \in [0, 1])$ satisfy assumptions of Theorem 4.1 (respectively Theorem 4.2) and a, b be two continuously differentiable functions on $(0, 1)$ and $a(t) > 0, t \in [0, 1]$. Define X such that $X_t = a(t) Z_t + b(t)$ for a.e. $t \in (0, 1)$. Then Theorem 4.1 (respectively Theorem 4.2) holds.*

REMARK 4.1 A version of the central limit theorem in (4.22) is established in Bardet and Surgailis (2009) with $\int_0^1 \Lambda_p(H(t))dt$ replaced by $\mathbb{E}R^{p,n}$ under weaker assumption than **(A.1)'** or even **(A.1)**: only properties (4.12) and (4.13) are required.

The particular case of Gaussian processes having stationary increments can also be studied:

Corollary 4.3 *Assume that X is a Gaussian process having stationary increments and there exist $c > 0$, $C > 0$ and $0 < H < 1$ such that at least one of the two following conditions (a), (b) hold:*

- (a) *variogram* $V(t) = ct^{2H}(1 + o(t^{1/2}))$ for $t \rightarrow 0$ and $|V^{(2p)}(t)| \leq Ct^{2H-2p}$ for all $t \in (0, 1]$;
 (b) *spectral density f satisfies (4.18), (4.19) and $f(\xi) = c\xi^{-2H-1}(1 + o(\xi^{-1/2}))$ ($\xi \rightarrow \infty$).*

Then:

$$(4.23) \quad \sqrt{n}(R^{p,n} - \Lambda_p(H)) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, \Sigma_p(H)) \quad \text{if} \quad \begin{cases} p = 1, & 0 < H < 3/4, \\ p = 2, & 0 < H < 1. \end{cases}$$

Moreover, with the expression of $s_2^2(H)$ given in Sec. 7,

$$(4.24) \quad \sqrt{n}(\Lambda_2^{-1}(R^{2,n}) - H) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, s_2^2(H)).$$

Therefore, $\widehat{H}_n = \Lambda_2^{-1}(R^{2,n})$ is an estimator of the parameter H following a central limit theorem with a convergence rate \sqrt{n} under semi-parametric assumptions. Similar results were obtained by Guyon and Leon (1989) and Istas and Lang (1997) for generalized quadratic variations under less general assumptions.

REMARK 4.2 In the context of Corollary 4.3 and $H \in (3/4, 1)$, one expects that that $R^{1,n}$ follows a nongaussian limit distribution with convergence rate n^{2-2H} .

REMARK 4.3 Figure 1 exhibits that $H \mapsto \Lambda_2(H)$ is nearly linear and is well-approximated by $.1468H + .5174$. Consequently, $\int_0^1 \Lambda_2(H(t))dt \approx .1468\bar{H} + .5174$, where $\bar{H} = \int_0^1 H(t)dt$ is the mean value of function $H(\cdot)$.

Another interesting particular case of Theorem 4.2 leads to a punctual estimator of the function $H(t)$ from a localization of the statistic $R^{2,n}$. For $t_0 \in (0, 1)$ and $\alpha \in (0, 1)$, define

$$R_\alpha^{2,n}(t_0) := \frac{1}{2n^\alpha} \frac{\sum_{k=[nt_0-n^\alpha]}^{[nt_0+n^\alpha]} |\Delta_k^{2,n}X + \Delta_{k+1}^{2,n}X|}{|\Delta_k^{2,n}X| + |\Delta_{k+1}^{2,n}X|}.$$

This estimator is studied in Bardet and Surgailis (2009) and compared to the estimator based generalized quadratic variations discussed in Benassi *et al.* (1998) and Coeurjolly (2005).

4.3. Examples. Below we provide some concrete examples of Gaussian processes which admit a fBm as the tangent process. For some examples the hypotheses of Theorems 4.1-4.2 and the subsequent corollaries are satisfied. For other examples, the verification of our hypotheses (in particular, of the crucial covariance bound $(\mathbf{A.2})_p$) remains an open problem and will be discussed elsewhere.

EXAMPLE 4.1 *Fractional Brownian motion (fBm).* As noted above, a fBm $X = B_H$ satisfies $(\mathbf{A.1})'$ as well as $(\mathbf{A.2})_1$ (for $0 < H < 7/8$) and $(\mathbf{A.2})_2$ (for $0 < H < 1$), with $H(t) \equiv H$, $c(t) \equiv c$. Therefore, for fBm both Theorems 3.1 (the almost sure convergence) and 3.2 (the central limit theorem) apply in the respective regions of the parameter H . Obviously, a fBm also satisfies the conditions of Corollary 4.3. Thus, the rate of convergence of the estimator $\Lambda_2^{-1}(R^{2,n}) =: \widehat{H}_n$ of H is \sqrt{n} . But in such a case the self-similarity property of fBm allows to use in this case asymptotically efficient Whittle or maximum likelihood estimators (see Fox and Taqqu, 1987, or Dahlhaus, 1989).

However, for a fBm with a continuously differentiable multiplicative and additive trends, which leads to a semi-parametric context, the convergence rate of \widehat{H}_n is still \sqrt{n} (but parametric estimators cannot be applied).

EXAMPLE 4.2 *Multiscale fractional Brownian motion* (see Bardet and Bertrand, 2007) defined as follows: for $\ell \in \mathbb{N}^*$, a (M_ℓ) -multiscale fractional Brownian motion $X = (X_t, t \in \mathbb{R})$ ((M_ℓ) -fBm for short) is a Gaussian process having stationary increments and a spectral density f such that

$$(4.25) \quad f(\xi) = \frac{\sigma_j^2}{|\xi|^{2H_j+1}} \mathbf{1}_{[\omega_j, \omega_{j+1})}(|\xi|) \quad \text{for all } \xi \in \mathbb{R}$$

with $\omega_0 := 0 < \omega_1 < \dots < \omega_\ell < \omega_{\ell+1} := \infty$, $\sigma_i > 0$ and $H_i \in \mathbb{R}$ for $i \in \{0, \dots, \ell\}$ with $H_0 < 1$ and $H_\ell > 0$. Therefore condition (4.19) of Property 4.3 is satisfied, with $K = \omega_\ell$ and $H = H_\ell$. Moreover, the condition $f(\xi) = c\xi^{-2H-1}(1 + o(\xi^{-1/2}))$ ($\xi \rightarrow \infty$) required in Corollary 4.3 is also checked with $H = H_\ell$. Consequently, the same conclusions as in the previous example apply for this process as well, in the respective regions determined by the parameter H_ℓ at high frequencies $x > \omega_\ell$ alone. The same result is also obtained for a more general process defined by $f(\xi) = c\xi^{-2H-1}$ for $|\xi| \geq \omega$ and condition (4.18) is only required elsewhere. Once again, such conclusions hold also in case of continuously differentiable multiplicative and additive trends.

EXAMPLE 4.3 *Multifractional Brownian motion* (mBm) (see Ayache *et al.*, 2005). A mBm $X = (X_t, t \in [0, 1])$ is a Gaussian process defined by

$$(4.26) \quad X_t = B_{H(t)}(t) = g(H(t)) \int_{\mathbb{R}} \frac{e^{itx} - 1}{|x|^{H(t)+1/2}} W(dx),$$

where $W(dx)$ is the same as in (4.17), $H(t)$ is a (continuous) function on $[0, 1]$ taking values in $(0, 1)$ and finally, $g(H(t))$ is a normalization such that $\text{EX}_t^2 = 1$. It is well-known that a mBm is locally asymptotically self-similar at each point $t \in (0, 1)$ having a fBm $B_{H(t)}$ as its tangent process at t (see Benassi *et al.*, 1997). This example is studied in more detail in Bardet and Surgailis (2009).

EXAMPLE 4.4 *Time-varying fractionally integrated processes*. Philippe *et al.* (2006, 2007) introduced two classes of mutually inverse time-varying fractionally integrated filters with discrete time and studied long-memory properties of the corresponding filtered white noise processes. Surgailis (2008) extended these filters to continuous time and defined ‘‘multifractional’’ Gaussian processes $(X_t, t \geq 0)$ and $(Y_t, t \geq 0)$ as follows

$$(4.27) \quad X_t = \int_{\mathbb{R}} \left\{ \int_0^t \frac{1}{\Gamma(H(\tau) - .5)} (\tau - s)_+^{H(\tau)-1.5} e^{A_-(s,\tau)} d\tau \right\} dB(s),$$

$$(4.28) \quad Y_t = \int_{\mathbb{R}} \frac{1}{\Gamma(H(s) + .5)} \left\{ (t - s)_+^{H(s)-.5} e^{-A_+(s,t)} - (-s)_+^{H(s)-.5} e^{-A_+(s,0)} \right\} dB(s),$$

where $s_+^\alpha := s^\alpha \mathbf{1}(s > 0)$, B is a Brownian motion,

$$A_-(s, t) := \int_s^t \frac{H(u) - H(t)}{t - u} du, \quad A_+(s, t) := \int_s^t \frac{H(s) - H(v)}{v - s} dv \quad (s < t)$$

and where $H(t), t \in \mathbb{R}$ is a general function taking values in $(0, \infty)$ and satisfying some weak additional conditions. Surgailis (2008) studied small and large scale limits of (X_t) and (Y_t) and showed that these processes resemble a fBm with Hurst parameter $H = H(t)$ at each point $t \in \mathbb{R}$

(i.e., admit a fBm as a tangent process) similarly to the mBm in the previous example. The last paper also argues that these processes present a more natural generalization of fBm than the mBm and have nicer dependence properties of increments. We expect that the assumptions $(\mathbf{A.1})$, $(\mathbf{A.1})'$, $(\mathbf{A.2})_p$ can be verified for (4.27), (4.28); however, this question requires further work.

5. Processes with independent increments. In this section, we assume that $X = (X_t, t \geq 0)$ is a (homogeneous) Lévy process, with a.s. right continuous trajectories, $X_0 = 0$. It is well-known that if the generating triplet of X satisfies certain conditions (in particular, if the Lévy measure ν behaves regularly at the origin with index $\alpha \in (0, 2)$), then X has a tangent process Y which is α -stable Lévy process. A natural question is to estimate the parameter α with the help of the introduced statistics $R^{p,n}$. Unfortunately, the limit of these statistics as defined in (1.5) through the tangent process depends also on the skewness parameter $\beta \in [-1, 1]$ of the α -stable tangent process Y and so this limit cannot be used for determining of α if β is unknown.

In order to avoid this difficulty, we shall slightly modified our ratio statistic, as follow. Observe first that the second differences $\Delta_k^{2,n} X$ of Lévy process have a *symmetric* distribution (in contrast to the first differences $\Delta_k^{1,n} X$ which are not necessary symmetric). For notational simplicity we shall assume in this section that n is *even*. The modified statistic

$$\tilde{R}^{2,n} := \frac{1}{n/2 - 1} \sum_{k=0}^{(n-4)/2} \psi(\Delta_{2k}^{2,n} X, \Delta_{2k+2}^{2,n} X), \quad \psi(x, y) := \frac{|x + y|}{|x| + |y|}$$

is written in terms of “disjoint” (independent) second order increments $(\Delta_{2k}^{2,n} X, \Delta_{2k+2}^{2,n} X)$ having a symmetric joint distribution. Instead of extending general result of Proposition 2.2 to $\tilde{R}^{2,n}$, we shall directly obtain its convergence under suitable assumptions on X . Note first

$$(5.1) \quad E\tilde{R}^{2,n} = E[\psi(X_{1/n}^{(2)} - X_{1/n}^{(1)}, X_{1/n}^{(4)} - X_{1/n}^{(3)})]$$

where $X^{(i)}, i = 1, \dots, 4$ are independent copies of X . Note that $1/2 \leq E\tilde{R}^{2,n} \leq 1$ since

$$\begin{aligned} E[\psi(X_{1/n}^{(2)} - X_{1/n}^{(1)}, X_{1/n}^{(4)} - X_{1/n}^{(3)})] &\geq P(X_{1/n}^{(2)} - X_{1/n}^{(1)} \geq 0, X_{1/n}^{(4)} - X_{1/n}^{(3)} \geq 0) \\ &\quad + P(X_{1/n}^{(2)} - X_{1/n}^{(1)} < 0, X_{1/n}^{(4)} - X_{1/n}^{(3)} < 0) \\ &\geq P^2(X_{1/n}^{(2)} - X_{1/n}^{(1)} \geq 0) + P^2(X_{1/n}^{(2)} - X_{1/n}^{(1)} < 0) \geq 1/2. \end{aligned}$$

Proposition 5.1 *Let there exists a limit*

$$(5.2) \quad \lim_{n \rightarrow \infty} E\tilde{R}^{2,n} = \tilde{\Lambda}.$$

Then

$$(5.3) \quad \tilde{R}^{2,n} \xrightarrow[n \rightarrow \infty]{a.s.} \tilde{\Lambda}.$$

Proof. Write $\tilde{R}^{2,n} = E\tilde{R}^{2,n} + (n/2 - 1)^{-1}Q_n$, where Q_n is a sum of centered 1-dependent r.v.'s which are bounded by 1 in absolute value. Therefore $E((n/2 - 1)^{-1}Q_n)^4 = O(n^{-2})$ and the a.s. convergence $(n/2 - 1)^{-1}Q_n \rightarrow 0$ follows by the Chebyshev inequality. \square

Next we discuss conditions on X for the convergence (5.2). Recall that the distribution of X_t is infinitely divisible and its characteristic function is given by

$$(5.4) \quad Ee^{i\theta X_t} = \exp \left\{ t \left(i\gamma\theta - \frac{1}{2}a^2\theta^2 + \int_{\mathbb{R}} (e^{iu\theta} - 1 - iu\theta I(|u| \leq 1))\nu(du) \right) \right\}, \quad \theta \in \mathbb{R},$$

where $\gamma \in \mathbb{R}$, $a \geq 0$ and ν is a measure on \mathbb{R} such that $\int_{\mathbb{R}} \min(u^2, 1)\nu(du) < \infty$. The triplet (a, γ, ν) is called the generating triplet of X (Sato (1999)). Let $X^{(i)}$, $i = 1, 2$ be independent copies of X . Note $W_t := X_t^{(1)} - X_t^{(2)}$ is a Lévy process having the characteristic function

$$(5.5) \quad \mathbb{E}e^{i\theta W_t} = \exp \left\{ t \left(-a^2\theta^2 + 2 \int_0^\infty \operatorname{Re}(1 - e^{iu\theta}) dK(u) \right) \right\}, \quad \theta \in \mathbb{R},$$

where

$$K(u) := \nu((-\infty, -u] \cup [u, \infty))$$

is monotone nonincreasing on $(0, \infty)$. Introduce the following condition: there exist $0 < \alpha \leq 2$ and $c > 0$ such that

$$(5.6) \quad K(u) \sim \frac{c}{u^\alpha}, \quad u \downarrow 0.$$

It is clear that if such number α exists then $\alpha := \inf \{r \geq 0 : \int_{|x| \leq 1} |x|^r \nu(dx) < \infty\}$ is the so-called fractional order or the Blumenthal-Gettoor index of the Lévy process X .

Let Z_α be a standard α -stable r.v. with characteristic function $\mathbb{E}e^{i\theta Z_\alpha} = e^{-|\theta|^\alpha}$ and $Z_\alpha^{(i)}$, $i = 1, 2, 3$ be independent copies of Z_α .

Proposition 5.2 *Assume either $a > 0$ or else, $a = 0$ and condition (5.6) with $0 < \alpha \leq 2$ and $c > 0$. Then $t^{-1/\alpha}(X_t^{(1)} - X_t^{(2)}) \xrightarrow[t \rightarrow 0]{\mathcal{D}} \tilde{c} Z_\alpha$ with \tilde{c} depending on c , and (5.2), (5.3) hold with*

$$\tilde{\Lambda} \equiv \tilde{\Lambda}(\alpha) := \mathbb{E}\psi(Z_\alpha^{(1)}, Z_\alpha^{(2)}).$$

Moreover, with $\tilde{\sigma}^2(\alpha) := 2\operatorname{var}(\psi(Z_\alpha^{(1)}, Z_\alpha^{(2)})) + 4\operatorname{cov}(\psi(Z_\alpha^{(1)}, Z_\alpha^{(2)}), \psi(Z_\alpha^{(2)}, Z_\alpha^{(3)}))$,

$$(5.7) \quad \sqrt{n}(\tilde{R}^{2,n} - \mathbb{E}\tilde{R}^{2,n}) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, \tilde{\sigma}^2(\alpha)).$$

Proof. Relation $t^{-1/\alpha}W_t = t^{-1/\alpha}(X_t^{(1)} - X_t^{(2)}) \xrightarrow[t \rightarrow 0]{\mathcal{D}} \tilde{c} Z_\alpha$ is an easy consequence of the assumptions of the proposition and the general criterion of weak convergence of infinitely divisible distributions in Sato (1991, Theorem 8.7). It implies (5.2) by the fact that ψ is a.e. continuous on \mathbb{R}^2 . Since $\tilde{R}^{2,n}$ is a sum of 1-dependent stationary and bounded r.v.'s, the central limit theorem in (5.7) follows from convergence of the variance:

$$(5.8) \quad n \operatorname{var}(\tilde{R}^{2,n}) \rightarrow \tilde{\sigma}^2(\alpha);$$

see e.g. Berk (1973). Rewrite $\tilde{R}^{2,n} = (n/2 - 1)^{-1} \sum_{k=0}^{(n-4)/2} \tilde{\eta}_n(k)$, $\tilde{\eta}_n(k) := \psi(\Delta_{2k}^{2,n} X, \Delta_{2k+2}^{2,n} X)$. We have

$$n \operatorname{var}(\tilde{R}^{2,n}) = \frac{n}{n/2 - 1} \operatorname{var}(\tilde{\eta}_n(0)) + \frac{2n(n/2 - 2)}{(n/2 - 1)^2} \operatorname{cov}(\tilde{\eta}_n(0), \tilde{\eta}_n(1)),$$

where $\operatorname{var}(\tilde{\eta}_n(0)) \xrightarrow[n \rightarrow \infty]{} \operatorname{var}(\psi(Z_\alpha^{(1)}, Z_\alpha^{(2)}))$, $\operatorname{cov}(\tilde{\eta}_n(0), \tilde{\eta}_n(1)) \xrightarrow[n \rightarrow \infty]{} \operatorname{cov}(\psi(Z_\alpha^{(1)}, Z_\alpha^{(2)}), \psi(Z_\alpha^{(2)}, Z_\alpha^{(3)}))$

similarly as in the proof of (5.2) above. This proves (5.8) and the proposition. \square

The graph of $\tilde{\Lambda}(\alpha)$ is given in Figure 2. Note that $\tilde{\Lambda}(2) = \Lambda_1(1/2) \simeq 0.72$: this is the case of Brownian motion.

In order to evaluate the decay rate of the bias $\mathbb{E}\tilde{R}^{2,n} - \tilde{\Lambda}(\alpha)$ we need a uniform convergence rate in Lemma 5.1, below, for

$$\|F_n - G_\alpha\|_\infty := \sup_{x \in \mathbb{R}} |F_n(x) - G_\alpha(x)|, \quad F_n(x) := \mathbb{P}(n^{1/\alpha} W_{1/n} \leq x), \quad G_\alpha(x) := \mathbb{P}(\tilde{Z}_\alpha \leq x),$$

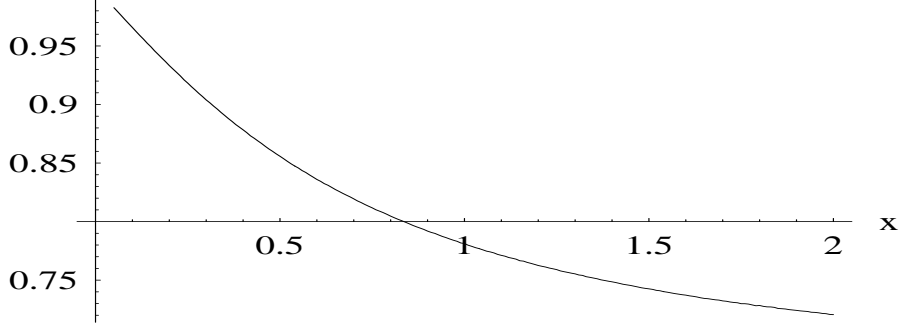


FIGURE 2. The graph of $\alpha \mapsto \tilde{\Lambda}(\alpha) = \mathbb{E} \frac{|Z_\alpha^{(1)} + Z_\alpha^{(2)}|}{|Z_\alpha^{(1)}| + |Z_\alpha^{(2)}|}$ for a process with independent increments.

where $\tilde{Z}_\alpha := \tilde{c}Z_\alpha$ is the limiting α -stable r.v. in Proposition 5.2 and $(W_t, t \geq 0)$ is the symmetric Levy process with characteristic function as in (5.5). The proof of Lemma 5.1 is given in Annexe.

Lemma 5.1 (i) *Let $a = 0$ and K satisfy (5.6). Denote $K_1(u) := K(u) - cu^{-\alpha}$, $|K_1|(u) := \int_u^\infty |dK_1(v)|$, the variation of K_1 on $[u, \infty)$. Moreover, assume that there exist some constants $\beta, \delta > 0$ such that*

$$(5.9) \quad |K_1|(u) = O(u^{-(\alpha-\beta)_+}) \quad (u \rightarrow 0), \quad |K_1|(u) = O(u^{-\delta}) \quad (u \rightarrow \infty),$$

where $x_+ = \max(0, x)$. Then

$$(5.10) \quad \|F_n - G_\alpha\|_\infty = \begin{cases} O(n^{-\beta/\alpha}), & \text{if } \beta < \alpha, \\ O(n^{-1} \log n), & \text{if } \beta = \alpha, \\ O(n^{-1}), & \text{if } \beta > \alpha. \end{cases}$$

(ii) *Let $a > 0$ and K satisfy*

$$(5.11) \quad K(u) = O(u^{-\alpha}) \quad (u \rightarrow 0), \quad K(u) = O(u^{-\delta}) \quad (u \rightarrow \infty),$$

for some $0 \leq \alpha < 2$, $\delta > 0$. Then

$$(5.12) \quad \|F_n - G_\alpha\|_\infty = \begin{cases} O(n^{-1+\alpha/2}), & \text{if } \alpha > 0, \\ O(n^{-1} \log n), & \text{if } \alpha = 0. \end{cases}$$

Proposition 5.3 *Assume either $a > 0$ or else $a = 0$ and condition (5.6). Then for any $\alpha \in (0, 2]$*

$$(5.13) \quad |\mathbb{E}\tilde{R}^{2,n} - \tilde{\Lambda}(\alpha)| \leq 2C\|F_n - G_\alpha\|_\infty, \quad C := \int_0^\infty (1+z)^{-2} dz < \infty.$$

Proof. Let $\psi(x, y) := |x - y|/(x + y)$, $x, y > 0$, and let F_n, G_α be the same as in Lemma 5.1. Similarly as in Vaičiulis (2009, proof of Th. 1), write

$$\mathbb{E}\tilde{R}^{2,n} - \tilde{\Lambda}(\alpha) = 2 \int_0^\infty \int_0^\infty \psi(x, y) (dF_n(x)dF_n(y) - dG_\alpha(x)dG_\alpha(y)) = 2(W_1 + W_2),$$

where $W_1 := \int_0^\infty \int_0^\infty \psi(x, y) dF_n(x)(dF_n(y) - dG_\alpha(y))$, $W_2 := \int_0^\infty \int_0^\infty \psi(x, y) dG_\alpha(y)(dF_n(x) - dG_\alpha(x))$. Integrating by parts yields

$$\begin{aligned} |W_1| &= 2 \int_0^\infty |x| dF_n(x) \int_0^\infty |F_n(y) - G_\alpha(y)| \frac{dy}{(x+y)^2} \\ &\leq 2 \|F_n - G_\alpha\|_\infty \int_0^\infty |x| dF_n(x) \int_0^\infty \frac{dy}{(x+y)^2} = C \|F_n - G_\alpha\|_\infty \end{aligned}$$

since $\int_0^\infty dF_n(x) = 1/2$. A similar estimate holds for W_2 . This proves (5.13). \square

Propositions 5.2, 5.3, and Lemma 5.1, together with the Delta-method, yield the following corollary.

Corollary 5.1 *Let a and K satisfy either the assumptions of Lemma 5.1(i) with $\beta > \alpha/2$, or the assumptions of Lemma 5.1(ii). Then*

$$\sqrt{n}(\tilde{R}^{2,n} - \tilde{\Lambda}(\alpha)) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, \tilde{\sigma}^2(\alpha)).$$

Moreover, if we define $\hat{\alpha}_n := \tilde{\Lambda}^{-1}(\tilde{R}^{2,n})$, then

$$\sqrt{n}(\hat{\alpha}_n - \alpha) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, \tilde{s}^2(\alpha)),$$

where $\tilde{s}^2(\alpha) := [\frac{\partial \tilde{\Lambda}}{\partial \alpha}(\alpha)]^{-2} \tilde{\sigma}^2(\alpha)$, $0 < \alpha \leq 2$.

There exist very few papers concerning estimation of α in such a semiparametric frame. Nonparametric estimation of parameters of Lévy processes based on the empirical characteristic function has recently been considered in Neumann and Reiß (2007) and Gugushvili (2008), but the convergence rates there are $(\log n)^\kappa$ with $\kappa > 0$. Ait Sahalia and Jacod (2009) have proposed an estimator of the degree of activity of jumps (which is identical to the fractional order in case of Lévy process) in a general semimartingale framework using small increments of high frequency data. However from the generality of their model, the convergence rate of the estimator is not rate efficient (in fact smaller than $n^{1/5}$). A very recent paper of Belomestny (2009) provides an efficient data-driven procedure to estimate α using a spectral approach but in a different semiparametric frame from ours. Thus, Corollary 5.1 appears as a new and interesting result since the estimator $\hat{\alpha}_n$ follows a \sqrt{n} -central limit theorem.

6. Annexe : proofs.

Proof of Property 4.1. We use the following identity: for any reals x_1, \dots, x_j ,

$$x_1 x_j = \frac{1}{2} \left\{ \left(\sum_{k=1}^j x_k \right)^2 + \left(\sum_{k=2}^{j-1} x_k \right)^2 - \left(\sum_{k=1}^{j-1} x_k \right)^2 - \left(\sum_{k=2}^j x_k \right)^2 \right\}.$$

In particular,

$$\begin{aligned} \mathbb{E} \left[\Delta_{[nt]}^{1,n} X \Delta_{j+[nt]}^{1,n} X \right] &= \frac{1}{2} \left\{ \mathbb{E}(X_{j+1+[nt]} - X_{[nt]})^2 + \mathbb{E}(X_{j-1+[nt_*]} - X_{[nt_*]})^2 \right. \\ &\quad \left. - \mathbb{E}(X_{j+[nt]} - X_{[nt]})^2 - \mathbb{E}(X_{j+[nt_*]} - X_{[nt_*]})^2 \right\}, \end{aligned}$$

where $t_* := t + (1/n)$ (so that $[nt_*] = [nt] + 1$). Then, using **(A.1)** and the notation u_n for a sequence tending to 0 as $n \rightarrow \infty$ uniformly in t and all $|j| < J$, where J is a fixed number, we obtain

$$\begin{aligned} \mathbb{E} \left[\Delta_{[nt]}^{1,n} X \Delta_{j+[nt]}^{1,n} X \right] &= \frac{1}{2} \left\{ c(t) \left(\frac{j+1}{n} \right)^{2H(t)} (1 + u_n) + c(t_*) \left(\frac{j-1}{n} \right)^{2H(t_*)} (1 + u_n) \right. \\ &\quad \left. - c(t) \left(\frac{j}{n} \right)^{2H(t)} (1 + u_n) - c(t_*) \left(\frac{j}{n} \right)^{2H(t_*)} (1 + u_n) \right\} \\ &= \frac{c(t)}{2} \left\{ \left(\frac{j+1}{n} \right)^{2H(t)} + \left(\frac{j-1}{n} \right)^{2H(t)} - 2 \left(\frac{j}{n} \right)^{2H(t)} \right\} (1 + u_n) \end{aligned}$$

since $c(t_*) - c(t) = u_n$ and $\left(\frac{j}{n} \right)^{2H(t_*)} = \left(\frac{j}{n} \right)^{2H(t)} (1 + u_n)$ follows from (4.6). This proves (4.10) for $p = 1$. Relation (4.10) for $p = 2$ follows analogously. Relation (4.11) also follows by the same argument and the fact that $c(t_*) - c(t) = u_n/\sqrt{n}$ and $\left(\frac{j}{n} \right)^{2H(t_*)} = \left(\frac{j}{n} \right)^{2H(t)} (1 + u_n/\sqrt{n})$ hold in view of Assumption **(A.1)**?. Property 4.1 is proved. \square

Proof of Property 4.2. With condition $V(t) \sim ct^{2H}$ ($t \rightarrow 0$) in mind, inequality (4.9) reduces to

$$\begin{aligned} (6.1) \quad & \left| V \left(\frac{k+1}{n} \right) + V \left(\frac{k-1}{n} \right) - 2V \left(\frac{k}{n} \right) \right| \leq Cn^{-2H+\theta} k^{-\gamma} \quad (p = 1, 2 \leq k \leq n), \\ (6.2) \quad & \left| 2V \left(\frac{k+2}{n} \right) - 4V \left(\frac{k+1}{n} \right) + 6V \left(\frac{k}{n} \right) - 4V \left(\frac{k-1}{n} \right) + V \left(\frac{k-2}{n} \right) \right| \leq Cn^{-2H+\theta} k^{-\gamma} \quad (p = 2, 4 \leq k \leq n). \end{aligned}$$

The left hand side of (6.1) can be written as

$$\begin{aligned} \left| \int_0^{1/n} \int_0^{1/n} V''(t-s+(k/n)) dt ds \right| &\leq C \int_0^{1/n} \int_0^{1/n} |t-s+(k/n)|^{-\kappa} dt ds \\ &\leq Cn^{\kappa-2} \int_0^1 \int_0^1 |t-s+k|^{-\kappa} dt ds \\ &\leq Cn^{\kappa-2} k^{-\kappa} = Cn^{\theta-2H} k^{-\gamma} \end{aligned}$$

where $\gamma = \kappa > 1/2$ and $\theta = \kappa + 2H - 2 \in [0, \gamma/2)$ since $\kappa < 4 - 4H$. This proves part (i). Part (ii) follows similarly, by writing the left hand side of (6.2) as

$$\begin{aligned} \left| \int_0^{1/n} \cdots \int_0^{1/n} V^{(4)}(t-s+u-v+(k/n)) dt ds du dv \right| &\leq C \int_0^{1/n} \cdots \int_0^{1/n} |t-s+u-v+(k/n)|^{-\kappa} dt ds du dv \\ &\leq Cn^{\kappa-4} k^{-\kappa} = Cn^{\theta-2H} k^{-\gamma}, \end{aligned}$$

where $\gamma = \kappa > 1$ and $\theta = \kappa + 2H - 4 \in [0, \kappa/2)$ since $\kappa < 8 - 4H$. Property 4.2 is proved. \square

Proof of Property 4.3. From (4.17) we have

$$\begin{aligned} \mathbb{E}[\Delta_0^{p,n} X \Delta_j^{p,n} X] &= \int_{\mathbb{R}} \left| e^{i(x/n)} - 1 \right|^{2p} e^{ix(j/n)} f(x) dx = 2^{1+p} \int_0^\infty (1 - \cos(x/n))^p \cos(xj/n) f(x) dx \\ &= 2^{1+p} (I_1 + I_2), \end{aligned}$$

with

$$\begin{aligned} (6.3) \quad |I_1| &= \left| \int_0^K (1 - \cos(x/n))^p \cos(xj/n) f(x) dx \right| \\ &= C \left| \int_0^K (x/n)^{2p} f(x) dx \right| = \left| Cn^{-2p} \int_0^K x^{2p} f(x) dx \right| \leq Cn^{-2p}; \end{aligned}$$

$$\begin{aligned}
 \text{and } I_2 &= n \int_{K/n}^{\infty} (1 - \cos(x))^p \cos(xj) f(nx) dx \\
 &= (n/j) \int_{K/n}^{\infty} (1 - \cos(x))^p f(nx) d \sin(xj) \\
 (6.4) \quad &= -(n/j) \int_{K/n}^{\infty} \sin(xj) ((1 - \cos(x))^p f(nx))'_x dx + O(n^{-2p})
 \end{aligned}$$

since $|(n/j)f(K)(1 - \cos(K/n))^p \sin(jK/n)| \leq C(n/j)(K/n)^{2p}|jK/n| \leq Cn^{-2p}$ for any $K > 0$ fixed.

Let $p = 1$. The last integral can be rewritten as $\int_{K/n}^{\infty} \sin(xj) ((1 - \cos(x))f(nx))'_x dx = \tilde{I}_1 + \tilde{I}_2$, where

$$|\tilde{I}_1| \leq \int_{K/n}^{\infty} |\sin(xj) \sin(x) f(nx)| dx \leq C \int_{K/n}^{\infty} |\sin(xj) \sin(x)| (nx)^{-2H-1} dx \leq \sum_{q=1}^3 I_{1q},$$

where we used the fact that $f(x) \leq Cx^{-2H-1}$ ($x \geq K$); see condition (4.19), and where

$$\begin{aligned}
 |I_{11}| &= \int_0^{1/j} |\sin(xj) \sin(x)| (nx)^{-2H-1} dx \leq Cj \int_0^{1/j} x^2 (nx)^{-1-2H} dx \leq Cj^{2H-1} n^{-1-2H}, \\
 |I_{12}| &= \int_{1/j}^1 |\sin(xj) \sin(x)| (nx)^{-2H-1} dx \leq C \int_{1/j}^1 x (nx)^{-1-2H} dx \leq Cj^{2H-1} n^{-1-2H}, \\
 |I_{13}| &= \int_1^{\infty} |\sin(xj) \sin(x)| (nx)^{-2H-1} dx \leq C \int_1^{\infty} (nx)^{-1-2H} dx \leq Cn^{-1-2H}.
 \end{aligned}$$

Similarly, using (4.19),

$$|\tilde{I}_2| \leq n \int_{K/n}^{\infty} |\sin(xj)(1 - \cos(x))f'(nx)| dx \leq C \int_{K/n}^{\infty} |\sin(xj)(1 - \cos(x))| (nx)^{-2H-2} dx \leq \sum_{q=1}^3 I_{2q},$$

where

$$\begin{aligned}
 I_{21} &= n \int_0^{1/j} |\sin(xj)(1 - \cos(x))| (nx)^{-2H-2} dx \leq Cnj \int_0^{1/j} x^3 (nx)^{-2-2H} dx \leq Cj^{2H-1} n^{-1-2H}, \\
 I_{22} &= n \int_{1/j}^1 |\sin(xj)(1 - \cos(x))| (nx)^{-2H-2} dx \leq Cn \int_{1/j}^1 x^2 (nx)^{-2-2H} dx \leq Cj^{2H-1} n^{-1-2H}, \\
 I_{23} &= n \int_1^{\infty} |\sin(xj)(1 - \cos(x))| (nx)^{-2H-2} dx \leq Cn \int_1^{\infty} (nx)^{-2-2H} dx \leq Cn^{-1-2H}.
 \end{aligned}$$

We finally obtain, for $1 \leq j \leq n$,

$$\left| \mathbb{E}[\Delta_0^{1,n} X \Delta_j^{1,n} X] \right| \leq C(n/j)j^{2H-1}n^{-1-2H} + O(n^{-2}) \leq Cn^{-2H}j^{2H-2}$$

implying (for $0 < H < 3/4$) **(A.2)**₁ with $\theta = 0$ and $\kappa = 2 - 2H > 1/2$. For $p = 2$, the estimation of the integral in (6.4) is completely similar, resulting in the bound

$$\left| \mathbb{E}[\Delta_0^{2,n} X \Delta_j^{2,n} X] \right| \leq C(n/j)n^{-1-2H} + O(n^{-4}) \leq Cn^{-2H}j^{-1}$$

for any $0 < H < 1$, or **(A.2)**₂ with $\theta = 0$ and $\kappa = 1 > 1/2$. \square

Proof of Corollary 4.2. This corollary is obvious since Assumption **(A.1)** (with function ac instead c) and **(A.2)_p** are satisfied by X when they are satisfied by Z (using Taylor expansions). Moreover, if Z checks Assumption **(A.1)'**, then Z checks also the inequality (4.11) which suffices for the validity of the statement of Theorem 4.2. But then X checks also the inequality (4.11). \square

Proof of Corollary 4.3. (a) The argument at the end of the proof of Property 4.1 shows that V satisfies Assumption **(A.1)'**, while **(A.2)_p** follows from Property 4.2. Then the central limit theorem in (4.23) follows from Theorem 4.2.

(b) In this case, **(A.2)_p** follows from Property 4.3. Instead of verifying **(A.1)'**, it is simpler to directly verify condition (4.11) which suffices for the validity of the statement of Theorem 4.2. Using $f(\xi) = c\xi^{-2H-1}(1 + o(\xi^{-1/2}))$ ($\xi \rightarrow \infty$), similarly as in the proof of Property 4.3 for $j \in \mathbb{N}^*$ one obtains

$$\begin{aligned} & \sqrt{n} \left| n^{-2H} \mathbb{E}[\Delta_0^{p,n} X \Delta_j^{p,n} X] - c 2^{1+p} \int_0^\infty (1 - \cos(x))^p \cos(xj) x^{-2H-1} dx \right| \\ &= 2^{1+p} \left| \int_0^\infty (1 - \cos(x))^p \cos(xj) \times \sqrt{n} (n^{2H+1} (f(nx) - c(nx)^{-2H-1})) dx \right| \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

by Lebesgue Theorem, since $\int_0^\infty |(1 - \cos(x))^p \cos(xj) x^{-2H-3/2}| dx < \infty$. Therefore condition (4.11) is satisfied and Theorem 4.2 can be applied.

Finally, the function $H \mapsto \Lambda_2(H)$ is a $\mathcal{C}^1((0, 1))$ bijective function and from the Delta-method (see for instance Van der Vaart, 1998), the central limit theorem (4.24) is shown. \square

Proof of Lemma 5.1. (i) We use the following inequality (see, e.g. Ibragimov and Linnik, 1971, Th. 1.5.2):

$$(6.5) \quad \|F_n - G_\alpha\|_\infty \leq \frac{1}{2\pi} \int_{\mathbb{R}} \left| \frac{f_n(\theta) - g_\alpha(\theta)}{\theta} \right| d\theta,$$

where f_n, g_α are the characteristic functions of F_n, G_α , respectively. According to (5.6) and the definition of K_1 ,

$$f_n(\theta) - g_\alpha(\theta) = g_\alpha(\theta)(e^{-2n^{-1}I_n(\theta)} - 1),$$

where, with $v := \theta n^{1/\alpha}$,

$$I_n(\theta) := \int_0^\infty \operatorname{Re}(e^{iu} - 1) dK_1(u) = \int_0^{1/v} \dots + \int_{1/v}^\infty \dots =: I_1 + I_2.$$

If $v > 1$ then integrating by parts and using (5.9),

$$\begin{aligned} |I_2| &\leq 2|K_1|(1/v) = O(v^{(\alpha-\beta)_+}), \\ |I_1| &= \left| K_1(1/v) \operatorname{Re}(e^i - 1) - \int_0^{1/v} K_1(u) v \sin(uv) du \right| \\ &\leq 2|K_1(1/v)| + v^2 \int_0^{1/v} u |K_1(u)| du \\ &= O(v^{(\alpha-\beta)_+}) + v^2 (1/v)^{2-(\alpha-\beta)_+} \\ &= O(v^{(\alpha-\beta)_+}). \end{aligned}$$

Next, if $v \leq 1$, then similarly as above

$$\begin{aligned}
 |I_2| &\leq 2|K_1|(1/v) = O(v^\delta), \\
 |I_1| &= \left| K_1(1/v)\operatorname{Re}(e^i - 1) - \int_0^{1/v} K_1(u) v \sin(uv) du \right| \\
 &\leq 2|K_1(1/v)| + v^2 \int_0^{1/v} u|K_1(u)| du \\
 &= O(v^\delta) + Cv^2 \left(\int_0^1 uu^{-(\alpha-\beta)_+} du + \int_1^{1/v} u^{1-\delta} du \right) \\
 &= O(v^{\delta \wedge 2}).
 \end{aligned}$$

Therefore, for some constant C ,

$$|I_n(\theta)| \leq C \begin{cases} (\theta n^{1/\alpha})^{(\alpha-\beta)_+}, & \theta > 1/n^{1/\alpha}, \\ (\theta n^{1/\alpha})^{\delta \wedge 2}, & \theta \leq 1/n^{1/\alpha}. \end{cases}$$

Moreover, $g_\alpha(\theta) = e^{-c_1|\theta|^\alpha}$ for some $c_1 > 0$. Using these facts and (6.5) we obtain

$$\begin{aligned}
 \|F_n - G_\alpha\|_\infty &\leq C \left(\int_0^{1/n^{1/\alpha}} n^{-1} (\theta n^{1/\alpha})^{\delta \wedge 2} \frac{d\theta}{\theta} + \int_{1/n^{1/\alpha}}^\infty n^{-1} (\theta n^{1/\alpha})^{(\alpha-\beta)_+} e^{-c_1|\theta|^\alpha} \frac{d\theta}{\theta} \right) \\
 &\leq Cn^{-1} + C \frac{1}{n^{1-(\alpha-\beta)_+/\alpha}} \left(1 + \int_{1/n^{1/\alpha}}^1 \theta^{(\alpha-\beta)_+-1} d\theta \right).
 \end{aligned}$$

The above bound easily yields (5.10).

(ii) Follows similarly using (6.5) and the argument in (i) with K_1 replaced by K . \square

7. Appendix.

Computation of $\Lambda_0(r)$. From the definition of $\Lambda_0(r)$ and the change of variables $x_1 = a \cos \phi$, $x_2 = a \sin \phi$, with $|r| < 1$,

$$\begin{aligned}
 \Lambda_0(r) &= \frac{1}{2\pi\sqrt{1-r^2}} \int_{\mathbb{R}^2} \frac{|x_1 + x_2|}{|x_1| + |x_2|} e^{-\frac{1}{2(1-r^2)}(x_1^2 - 2rx_1x_2 + x_2^2)} dx_1 dx_2 \\
 &= \frac{\sqrt{1-r^2}}{\pi} \int_0^\pi \frac{|\cos \phi + \sin \phi|}{(|\cos \phi| + |\sin \phi|)(1-r \sin(2\phi))} d\phi \\
 &=: I_1 + I_2,
 \end{aligned}$$

where

$$\begin{aligned}
 I_1 &= \frac{\sqrt{1-r^2}}{\pi} \int_0^{\pi/2} \frac{1}{1-r \sin(2\phi)} d\phi \\
 &= \frac{\sqrt{1-r^2}}{\pi} \int_0^\infty \frac{1}{1+t^2-2rt} dt = \frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{r}{\sqrt{1-r^2}}\right) = \frac{1}{\pi} \arccos(-r); \\
 I_2 &= \frac{2\sqrt{1-r^2}}{\pi} \int_0^{\pi/4} \frac{\cos \phi - \sin \phi}{(\cos \phi + \sin \phi)(1+r \sin(2\phi))} d\phi = \frac{\sqrt{1-r^2}}{\pi} \int_0^1 \frac{1-t}{(1+t)(1+2rt+t^2)} dt \\
 &= \frac{\sqrt{1-r^2}}{\pi(1-r)} \log\left(\frac{2}{r+1}\right).
 \end{aligned}$$

The function $\Lambda_0(r)$ is monotone increasing on $[-1, 1]$; $\Lambda_0(1) = 1$, $\Lambda_0(-1) = 0$. It is easy to check that

$$(7.1) \quad \rho_1(H) = 2^{2H-1} - 1, \quad \rho_2(H) = \frac{-3^{2H} + 2^{2H+2} - 7}{8 - 2^{2H+1}}$$

are monotone increasing functions; $\rho_1(1) = 1, \rho_2(1) = 0$ so that $\Lambda_p(H) = \Lambda_0(\rho_p(H))$ for $p = 1, 2$ is also monotone for $H \in (0, 1)$.

Graphs of $\sqrt{\Sigma_p(H)}$ ($p = 1, 2$) (from Stoncelis and Vaičiulis, 2008, with kind permission of the authors).

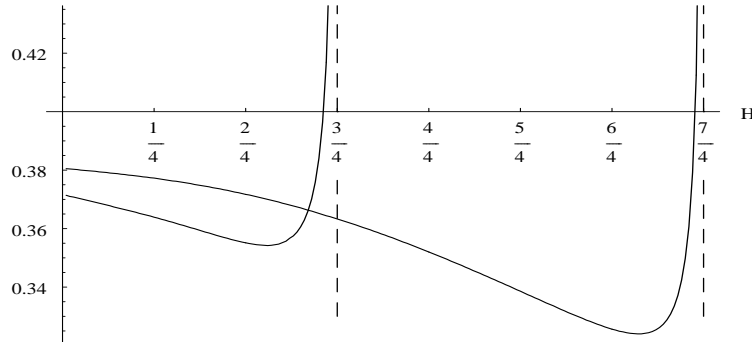


FIGURE 3. The graphs of $\sqrt{\Sigma_p(H)}$, $p = 1$ (with a pole at $3/4$) and $p = 2$ (with a pole at $7/4$)

Expression and graph of $s_2(H)$. From the Delta-method, $s_2^2(H) = \left[\frac{\partial}{\partial x} (\Lambda_2)^{-1}(\Lambda_2(H)) \right]^2 \Sigma_2(H)$ and therefore

$$s_2^2(H) = \left(\frac{\pi (8 - 2^{2H+1})^2 (1 - \rho_2(H)) \sqrt{1 - \rho_2^2(H)}}{(\log 2 - \log(1 + \rho_2(H))) (2^{2H+2} 9 \log 2 - 3^{2H} 16 \log 3 + 6^{2H} 4 \log(3/2))} \right)^2 \Sigma_2(H),$$

with the following approximated graph (using the numerical values of $\Sigma_2(H)$ in Stoncelis and Vaičiulis (2008)):

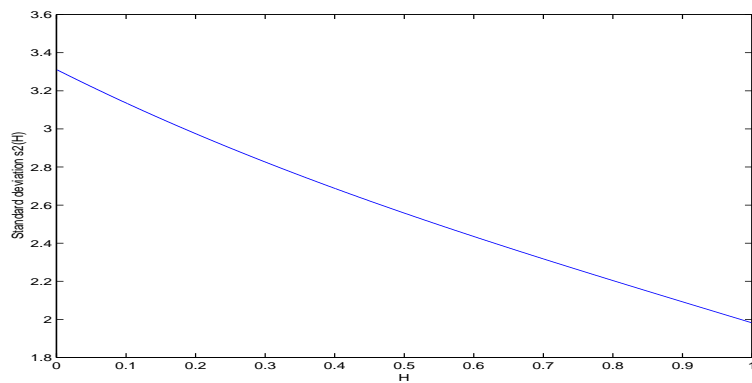


FIGURE 4. The graph of $\sqrt{s_2^2(H)}$

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