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Martin Vohralík. Guaranteed and fully robust a posteriori error estimates for conforming discretizations of diffusion problems with discontinuous coefficients. 2008. hal-00235810v1

HAL Id: hal-00235810

<https://hal.science/hal-00235810v1>

Preprint submitted on 4 Feb 2008 (v1), last revised 21 Jul 2010 (v4)

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Guaranteed and fully robust a posteriori error estimates for conforming discretizations of diffusion problems with discontinuous coefficients *

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Abstract

We study in this paper a posteriori error estimates for H^1 -conforming numerical approximations of diffusion problems with a scalar, piecewise constant, and arbitrarily discontinuous diffusion coefficient. We derive estimators for the energy norm which give a guaranteed global upper bound in the sense that they feature no undetermined constants. Local lower bounds, up to constants independent of the diffusion coefficient, are also derived. In particular, no condition on the diffusion coefficient like its monotonous increasing along paths around mesh vertices is imposed, whence the present results are fully robust and include also the cases with singular solutions. The key requirement turns out to be that the diffusion coefficient is piecewise constant on dual cells associated with the vertices of an original simplicial mesh and that harmonic averaging is used in the numerical scheme definition. This is the usual case, e.g., for the cell-centered finite volume method, included in our analysis as well as the vertex-centered finite volume, finite difference, and continuous piecewise linear finite element ones. Our estimates consist of two estimators, one comparing a reconstructed $\mathbf{H}(\text{div})$ -conforming diffusive flux with the diffusive flux given by the piecewise linear representation of the approximate solution and one representing an improved residual. They are based on the local conservativity of all the studied methods on the dual grids, which we recall in the paper, as well as their mutual relations. Numerical experiments confirm the guaranteed upper bound, full robustness, and excellent efficiency of the presented estimators, which may still be improved by a negligible-cost local minimization.

Key words: finite volume method, finite element method, finite difference method, discontinuous coefficients, harmonic averaging, a posteriori error estimates, guaranteed upper bound, robustness

1 Introduction

We consider in this paper a model diffusion problem

$$-\nabla \cdot (a\nabla p) = f \quad \text{in } \Omega, \tag{1.1a}$$

$$p = 0 \quad \text{on } \partial\Omega, \tag{1.1b}$$

*This work was supported by the GdR MoMaS project “Numerical Simulations and Mathematical Modeling of Underground Nuclear Waste Disposal”, PACEN/CNRS, ANDRA, BRGM, CEA, EdF, IRSN, France.

where $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a polygonal (polyhedral) domain (open, bounded, and connected set), a is a scalar diffusion coefficient, and f is a source term. We shall derive here a posteriori error estimates for continuous piecewise linear finite element, vertex-centered finite volume, cell-centered finite volume, and finite difference approximations of this problem.

A posteriori error estimates for finite element discretization of a homogeneous form of (1.1a)–(1.1b) (with $a = 1$) have been a popular research subject starting from the Babuška and Rheinboldt work [3]. Several branches of estimators have been developed, such as the postprocessing (averaging) approach, cf. Zienkiewicz and Zhu [37], the residual approach, cf. Verfürth [30], or the equilibrated residual method, cf. Ainsworth and Oden [2]. The majority of the proposed estimators have been proved both reliable, i.e., yielding a global upper bound on the error between the exact and approximate solutions, and locally efficient, i.e., giving a local lower bound for the error as well. Let us recall that this means that all such estimators are equivalent with the error, and by consequence mutually equivalent. The only distinguishing point between such estimators are then the constants which figure in the equivalences. However, up to rare exceptions such as the works of Repin ([24] and further works), Destuynder and Métivet [11], Strouboulis *et al.* [27], Carstensen and Funken [8], Luce and Wohlmuth [20], Vejchodský [29], or Korotov [18], it is only shown that these constants are independent of the exact solution and of the mesh size but their actual value stays unknown. To remedy this inconvenience was the first motivation of this paper and we present here a posteriori error estimates which feature no undetermined constant in the upper bound.

Another important remark goes to the residual branch of a posteriori error estimates, which for (1.1a)–(1.1b) consist of two parts. The first one is the so-called residual estimator, of the form $c_a h_K \|f + \nabla \cdot (a \nabla p_h)\|_K$, where K is one element of the mesh, h_K is its diameter, p_h is the finite element approximate solution, c_a is a constant depending on a , and $\|\cdot\|$ denotes the L^2 -norm. The second one is the so-called mass balance estimator, typically of the form $c_a h_\sigma^{1/2} \llbracket a \nabla p_h \rrbracket \|_\sigma$ for each interior side (edge if $d = 2$, face if $d = 3$) σ , where $\llbracket \cdot \rrbracket$ is the “jump” operator given by equation (2.3) below. Remark now that in the case where a is piecewise constant, $\nabla \cdot (a \nabla p_h) = 0$, whence the residual estimator reduces to $c_a h_K \|f\|_K$, which we believe is not optimal at all. Hence the second motivation of this paper was to find a remedy to this situation.

The third and last motivation of the present paper was to give estimators which are robust with respect to discontinuous a . To the best of the author’s knowledge, such estimators are only known under the “monotonicity around vertices” condition on the distribution of the diffusion coefficient, see Bernardi and Verfürth [6], Petzoldt [22], or Ainsworth [1]. The above condition is however very restrictive and in particular excludes the physically interesting cases where regions with different diffusion coefficients meet in a checkerboard pattern and where the weak solution can present singularities.

We have in [33, 32, 13] recently derived a posteriori error estimates for mixed finite element, finite volume, and discontinuous Galerkin methods which avoid any undetermined constant in the upper bound and feature an improved residual of the form (in the lowest-order case) $c_a h_K \|f - f_K\|_K$, where f_K is the mean value of f over K . This result is based purely on the elementwise conservativity of these methods and not on the Galerkin orthogonality, which is the usual case for finite elements. We generalize in this paper this approach to the continuous piecewise linear finite element method on simplicial meshes (consisting of triangles if $d = 2$ and of tetrahedra if $d = 3$). We use the fact, still not well known in the finite element community, that this method is likewise locally conservative, and this on a dual mesh associated with the vertices of the original simplicial one. A (nonexhaustive) list of references relevant in this respect contains the works of Bank and Rose [4], Eymard *et al.* [14, Section III.12], and Hughes *et al.* [17], but we shall give all necessary results in Section 3 of this paper, after given the necessary preliminaries in Section 2.

We in fact recall or prove there equivalences between several classical numerical methods, namely the continuous piecewise linear finite element one on a simplicial mesh and vertex-centered finite volume, cell-centered finite volume, and finite difference ones on a dual mesh (under appropriate assumptions), so that our results apply to all these methods.

We next in Section 4 sketch the basic results of [33, 13], giving an optimal abstract framework for a posteriori error estimation, in order to prove our a posteriori error estimates in Section 5. For this purpose, we shall postprocess the original diffusive flux $-a\nabla p_h$, where p_h is the approximate numerical solution, into $\mathbf{t}_h \in \mathbf{H}(\text{div}, \Omega)$ defined in the lowest-order Raviart–Thomas–Nédélec space. This space is extensively used in the framework of the mixed finite element method, cf. [7, 26]. Ideas in this direction have already been proposed in the literature, cf. [24, 11, 20, 29, 18] and go back to the hypercircle method, see Syngé [28]. However, our approach differs from the above ones in particular in that we use the natural local conservativity of the finite element method over a dual grid explained above. Using \mathbf{t}_h , our residual estimator writes under the improved form $c_a h_D \|f - \nabla \cdot \mathbf{t}_h\|_D$, whereas the classical mass balance estimator is replaced by its lower bound $\|a^{1/2} \nabla p_h + a^{-1/2} \mathbf{t}_h\|_D$; here D is an element of the dual grid. Comparisons with the residual, equilibrated residual, averaging, functional, and equilibrated fluxes estimates then close Section 5.

The purpose of Section 6 is to prove the local efficiency and robustness of our estimators. A key role for the robustness turns out to be the use of harmonic averaging in both the scheme and the averaging in posteriori error estimates, recently introduced in the context of discontinuous Galerkin methods by Ern *et al.* [12, 13]. It appears that in order to get the full robustness of our estimators, harmonic averaging has to be used directly in the scheme definition, and this while aligning the discontinuities of the diffusion coefficient with the dual mesh. This contradicts the usual finite element practice, where the discontinuities are aligned with the original simplicial mesh, but it is in a complete agreement with the cell-centered finite volume (finite difference) approach, where the discontinuities are classically aligned with the volumes (of mass conservation), cf. Eymard *et al.* [14]. We discuss this issue in detail in Section 6.

Numerical experiments with the proposed estimates confirm all the theoretical results but show that the effectivity index (the ratio of the estimated and actual error), although largely improved over the classical residual estimates, does not approach the optimal value of one as it is the case in [33, 32]. By closer investigation, it turns out that whereas in mixed finite element or finite volume (discontinuous Galerkin) methods, the improved residual estimator represents a higher-order term, it is not the case here, where it on the contrary dominates the estimate. We propose a remedy to this situation, consisting in a very efficient and cheap local minimization, in Section 7. Numerical experiments presented in Section 8 confirm this improvement as well as the guaranteed upper bound and full robustness of our estimators.

Our error estimates do not require the mesh to be shape-regular and the datum f can be a general function; the usual requirement of shape-regularity and of polynomial data is only needed for the local efficiency proofs. Also, no saturation assumption, no convexity of Ω , and no additional regularity of the weak solution of (1.1a)–(1.1b) are needed in our setting. The homogeneous Dirichlet boundary condition is considered only for the sake of clarity of exposition; general boundary conditions can easily be taken in account, as we outline it in [35]. This paper is a detailed description of the results previously announced in [34]; some additional numerical experiments for the finite element method, together with another local minimization strategy, are then studied in [9], and extensions to the reaction–diffusion case in [10].

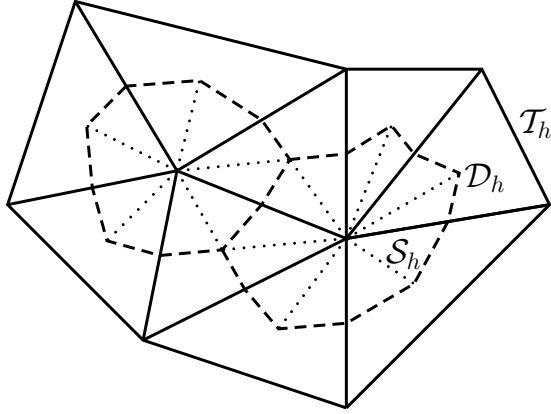


Figure 1: Original simplicial mesh \mathcal{T}_h , the associated dual mesh \mathcal{D}_h , and the fine simplicial mesh \mathcal{S}_h

2 Preliminaries

We give in this section the notation and assumptions, recall some inequalities which will be important in the sequel, and finally give details on the continuous problem (1.1a)–(1.1b).

2.1 Meshes and notation

We shall work in this paper with triangulations \mathcal{T}_h which for all $h > 0$ consist of closed simplices such that $\bar{\Omega} = \bigcup_{K \in \mathcal{T}_h} K$ and which are conforming (matching), i.e., such that if $K, L \in \mathcal{T}_h$, $K \neq L$, then $K \cap L$ is either an empty set or a common face, edge, or vertex of K and L . Let h_K denote the diameter of K and let $h := \max_{K \in \mathcal{T}_h} h_K$. We denote by \mathcal{E}_h the set of all sides of \mathcal{T}_h , by $\mathcal{E}_h^{\text{int}}$ the set of interior, by $\mathcal{E}_h^{\text{ext}}$ the set of exterior, and by \mathcal{E}_K the set of all the sides of an element $K \in \mathcal{T}_h$; h_σ stands for the diameter of $\sigma \in \mathcal{E}_h$. We finally denote by \mathcal{V}_h ($\mathcal{V}_h^{\text{int}}$) the set of all (interior) vertices of \mathcal{T}_h and put, for $V \in \mathcal{V}_h$ and $K \in \mathcal{T}_h$, respectively,

$$\mathcal{T}_V := \{L \in \mathcal{T}_h; L \cap V \neq \emptyset\}, \quad (2.1)$$

$$\mathcal{T}_K := \{L \in \mathcal{T}_h; L \cap K \neq \emptyset\}. \quad (2.2)$$

We shall also consider dual partitions \mathcal{D}_h of Ω such that $\bar{\Omega} = \bigcup_{D \in \mathcal{D}_h} D$ and such that for each $V \in \mathcal{V}_h$, $V \in D_V$ for exactly one $D_V \in \mathcal{D}_h$. The notation V_D stands inversely for the vertex associated with a given $D \in \mathcal{D}_h$ and we use $\mathcal{D}_h^{\text{int}}, \mathcal{D}_h^{\text{ext}}$ to denote the dual volumes associated with vertices from $\mathcal{V}_h^{\text{int}}, \mathcal{V}_h^{\text{ext}}$, respectively. Next, \mathcal{F}_h stands for all sides of \mathcal{D}_h and $\mathcal{F}_h^{\text{int}}$ ($\mathcal{F}_h^{\text{ext}}$) for all interior (exterior) sides of \mathcal{D}_h . We shall always suppose that D_V lies in the interior of the polygon/polyhedron given by \mathcal{T}_V for all $V \in \mathcal{V}_h$ and that $\mathcal{E}_h^{\text{int}} \cap \mathcal{F}_h^{\text{int}}$ has a zero $(d-1)$ -dimensional Lebesgue measure. An example of such a partition is given in Figure 1; more details on different \mathcal{D}_h considered will be given in Section 3.

Finally, in order to define our a posteriori error estimates, we will need a second conforming simplicial triangulation \mathcal{S}_h of Ω . The basic requirement is that the interiors of the elements of \mathcal{S}_h do not intersect and sides of \mathcal{T}_h and \mathcal{D}_h (\mathcal{S}_h is a conforming refinement of both \mathcal{T}_h and \mathcal{D}_h). For the local efficiency proofs of our estimators, we will later need the assumption that $\{\mathcal{S}_h\}_h$ are shape-regular in the sense that there exists a constant $\kappa_S > 0$ such that $\min_{K \in \mathcal{S}_h} |K|/h_K^d \geq \kappa_S$ for all $h > 0$. One can easily construct a local triangulation \mathcal{S}_D of each $D \in \mathcal{D}_h$ as shown in Figure 1 and then put $\mathcal{S}_h := \bigcup_{D \in \mathcal{D}_h} \mathcal{S}_D$. We will use the notation \mathcal{G}_h for all sides of \mathcal{S}_h and $\mathcal{G}_h^{\text{int}}$ ($\mathcal{G}_h^{\text{ext}}$) for all interior (exterior) sides of \mathcal{S}_h .

Next, for $K \in \mathcal{T}_h$, \mathbf{n} will always denote its exterior normal vector; we shall also employ the notation \mathbf{n}_σ for a normal vector of a side $\sigma \in \mathcal{E}_h$, whose orientation is chosen arbitrarily but fixed for interior sides and coinciding with the exterior normal of Ω for exterior sides. For $\sigma \in \mathcal{E}_h^{\text{int}}$ shared by $K, L \in \mathcal{T}_h$ such that \mathbf{n}_σ points from K to L and a function φ , we shall define the jump operator $[[\cdot]]$ by

$$[[\varphi]] := (\varphi|_K)|_\sigma - (\varphi|_L)|_\sigma. \quad (2.3)$$

We put $[[\varphi]]_\sigma = \varphi|_\sigma$ for any $\sigma \in \mathcal{E}_h^{\text{ext}}$. We next associate with each $K \in \mathcal{T}_h$ and each $\sigma \in \mathcal{E}_K$ a weight $\omega_{K,\sigma}$ such that

$$0 \leq \omega_{K,\sigma} \leq 1 \quad \forall K \in \mathcal{T}_h, \forall \sigma \in \mathcal{E}_K, \quad (2.4a)$$

$$\omega_{K,\sigma} + \omega_{L,\sigma} = 1 \quad \forall \sigma = \sigma_{K,L} \in \mathcal{E}_h^{\text{int}}, \quad (2.4b)$$

$$\omega_{K,\sigma} = 1 \quad \forall \sigma \in \mathcal{E}_h^{\text{ext}} \text{ and } K \in \mathcal{T}_h \text{ such that } \sigma \in \mathcal{E}_K. \quad (2.4c)$$

For $\sigma = \sigma_{K,L} \in \mathcal{E}_h^{\text{int}}$, we define the weighted average operator $\{\cdot\}_\omega$ by

$$\{\varphi\}_\omega := \omega_{K,\sigma}(\varphi|_K)|_\sigma + \omega_{L,\sigma}(\varphi|_L)|_\sigma, \quad (2.5)$$

whereas for $\sigma \in \mathcal{E}_h^{\text{ext}}$, $\{\varphi\}_\omega := \varphi|_\sigma$. We use the same type of notation also for the meshes \mathcal{D}_h and \mathcal{S}_h .

Finally, we shall be working below with numerical methods whose approximate solution can be represented by continuous piecewise linear functions on \mathcal{T}_h , with value 0 at the boundary of Ω . The basis of this space, denoted by X_h , is spanned by the classical pyramidal functions ψ_V , $V \in \mathcal{V}_h^{\text{int}}$, such that $\psi_V(U) = \delta_{VU}$, $U \in \mathcal{V}_h$, δ being the Kronecker delta.

In what concerns functional notation, we denote by $(\cdot, \cdot)_S$ the L^2 -scalar product on S and by $\|\cdot\|_S$ the associated norm; when $S = \Omega$, the index is dropped off. We mean by $|S|$ the Lebesgue measure of S , by $|\sigma|$ the $(d-1)$ -dimensional Lebesgue measure of $\sigma \subset \mathbb{R}^{d-1}$, and in particular by $|\mathbf{s}|$ the length of a segment \mathbf{s} . Next, $H^1(S)$ is the Sobolev space of functions with square-integrable weak derivatives and $H_0^1(S)$ is its subspace of functions with traces vanishing on ∂S . Finally, $\mathbf{H}(\text{div}, S)$ is the space of functions with square-integrable weak divergences, $\mathbf{H}(\text{div}, S) = \{\mathbf{v} \in \mathbf{L}^2(S); \nabla \cdot \mathbf{v} \in L^2(S)\}$, and $\langle \cdot, \cdot \rangle_{\partial S}$ stands for the appropriate duality pairing on ∂S .

2.2 Assumptions

We shall suppose in this paper that $f(\mathbf{x}) \in L^2(\Omega)$ and that $a(\mathbf{x})$ is a piecewise constant scalar-valued function. We may in particular consider cases where a is piecewise constant on the triangulation \mathcal{T}_h and cases where a is piecewise constant on the dual partition \mathcal{D}_h . In all cases we suppose that for all $K \in \mathcal{T}_h$, there exist two positive constants $c_{a,K}$ and $C_{a,K}$ such that $c_{a,K} \leq a(\mathbf{x}) \leq C_{a,K}$ for all $\mathbf{x} \in K$. Similar notation will be used also for $D \in \mathcal{D}_h$ or for \mathcal{T}_K , $K \in \mathcal{T}_h$. Finally, we suppose that $c_{a,\Omega} \leq c_{a,K}$ and $C_{a,K} \leq C_{a,\Omega}$ for all $K \in \mathcal{T}_h$, for some positive constants $c_{a,\Omega}$ and $C_{a,\Omega}$.

2.3 Poincaré and Friedrichs inequalities

Let D be a polygon or a polyhedron. The Poincaré inequality states that

$$\|\varphi - \varphi_D\|_D^2 \leq C_{P,D} h_D^2 \|\nabla \varphi\|_D^2 \quad \forall \varphi \in H^1(D), \quad (2.6)$$

where φ_D is the mean of φ over D given by $\varphi_D := (\varphi, 1)_D / |D|$ and where the constant $C_{P,D}$ can for each convex D be evaluated as $1/\pi^2$, cf. [21, 5]. To evaluate $C_{P,D}$ for nonconvex elements D is

more complicated but it still can be done, cf. Eymard *et al.* [14, Lemma 10.2] or Carstensen and Funken [8, Section 2].

The Friedrichs inequality states that

$$\|\varphi\|_D^2 \leq C_{F,D,\partial\Omega} h_D^2 \|\nabla\varphi\|_D^2 \quad \forall \varphi \in H^1(D) \text{ such that } \varphi = 0 \text{ on } \partial\Omega \cap \partial D \neq \emptyset. \quad (2.7)$$

As long as $\partial\Omega$ is such that there exists a vector $\mathbf{b} \in \mathbb{R}^d$ such that for almost all $\mathbf{x} \in D$, the first intersection of $\mathcal{B}_{\mathbf{x}}$ and ∂D lies in $\partial\Omega$, where $\mathcal{B}_{\mathbf{x}}$ is the straight semi-line defined by the origin \mathbf{x} and the vector \mathbf{b} , $C_{F,D,\partial\Omega} = 1$, cf. [31, Remark 5.8]. To evaluate $C_{F,D,\partial\Omega}$ in the general case is more complicated but it still can be done, cf. [31, Remark 5.9] or Carstensen and Funken [8, Section 3].

2.4 Continuous problem

We define a bilinear form \mathcal{B} by

$$\mathcal{B}(p, \varphi) := (a\nabla p, \nabla\varphi) \quad p, \varphi \in H_0^1(\Omega) \quad (2.8)$$

and the corresponding energy norm by

$$\|\varphi\|^2 := \mathcal{B}(\varphi, \varphi). \quad (2.9)$$

The weak formulation of problem (1.1a)–(1.1b) is then to find $p \in H_0^1(\Omega)$ such that

$$\mathcal{B}(p, \varphi) = (f, \varphi) \quad \forall \varphi \in H_0^1(\Omega). \quad (2.10)$$

3 Local conservativity of finite elements and their relation to finite volumes and finite differences

The purpose of this section is to recall several relations between the continuous piecewise linear finite element, vertex-centered finite volume (also termed finite volume element or box), cell-centered finite volume, and finite difference methods for problem (1.1a)–(1.1b). A particular consequence is the local conservativity of the finite element method on a dual grid under appropriate conditions.

3.1 Definitions

We start by giving the definitions.

Definition 3.1 (Finite element method). *The finite element method for problem (1.1a)–(1.1b) reads: find $p_h \in X_h$ such that*

$$(a\nabla p_h, \nabla\psi_V)_{\mathcal{T}_V} = (f, \psi_V)_{\mathcal{T}_V} \quad \forall V \in \mathcal{V}_h^{\text{int}}. \quad (3.1)$$

Definition 3.2 (Vertex-centered finite volume method). *Let the dual grid \mathcal{D}_h consist of polygonal/polyhedral dual volumes and let a be piecewise constant on \mathcal{T}_h so that a is not double-valued on $\mathcal{F}_h^{\text{int}}$. Then the vertex-centered finite volume method for problem (1.1a)–(1.1b) reads: find $p_h \in X_h$ such that*

$$-\langle a\nabla p_h \cdot \mathbf{n}, 1 \rangle_{\partial D} = (f, 1)_D \quad \forall D \in \mathcal{D}_h^{\text{int}}. \quad (3.2)$$

Definition 3.3 (Weighted cell-centered finite volume method). Let \mathcal{D}_h be the Voronoï grid given by the vertices from \mathcal{V}_h , cf. Eymard et al. [14] (this requires that the vertices $V \in \mathcal{V}_h^{\text{ext}}$ are suitably placed so that $\bar{\Omega} = \bigcup_{D \in \mathcal{D}_h} D$). Let next $\mathcal{N}(D)$ denote the set of “neighbors” of $D \in \mathcal{D}_h$, i.e., of such $E \in \mathcal{D}_h$ that $\sigma_{D,E} := \partial D \cap \partial E$ is such that $|\sigma_{D,E}| \neq 0$; in such a case, let $d_{D,E}$ stand for the Euclidean distance of the associated vertices V_D and V_E . Let finally a be piecewise constant on \mathcal{D}_h . Then the weighted cell-centered finite volume method for problem (1.1a)–(1.1b) reads: find $p_h = \sum_{D \in \mathcal{D}_h} p_D \psi_{V_D}$, with $p_D = 0$ for all $D \in \mathcal{D}_h^{\text{ext}}$ so that $p_h \in X_h$, such that

$$- \sum_{E \in \mathcal{N}(D)} \{a\}_\omega \frac{|\sigma_{D,E}|}{d_{D,E}} (p_E - p_D) = (f, 1)_D \quad \forall D \in \mathcal{D}_h^{\text{int}}. \quad (3.3)$$

Here, two basic choices for the weights in $\{a\}_\omega$ on a side $\sigma = \sigma_{D,E} \in \mathcal{F}_h^{\text{int}}$ exist:

$$\omega_{D,\sigma} = \omega_{E,\sigma} = \frac{1}{2}, \quad (3.4)$$

which corresponds to the arithmetic averaging, and

$$\omega_{D,\sigma} = \frac{a_E}{a_D + a_E}, \quad \omega_{E,\sigma} = \frac{a_D}{a_D + a_E}, \quad (3.5)$$

which corresponds to the harmonic averaging.

Definition 3.4 (Weighted vertex-centered finite volume method). Let the dual grid \mathcal{D}_h consist of polygonal/polyhedral dual volumes. Then we can design a weighted vertex-centered finite volume method for problem (1.1a)–(1.1b) as follows: find $p_h \in X_h$ such that

$$- \langle \{a\}_\omega \nabla p_h \cdot \mathbf{n}, 1 \rangle_{\partial D} = (f, 1)_D \quad \forall D \in \mathcal{D}_h^{\text{int}}. \quad (3.6)$$

Remark 3.5 (Arithmetic/harmonic averaging in the vertex-centered finite volume method). We first remark that when a is piecewise constant on \mathcal{T}_h , the above definition coincides with the standard Definition 3.2, which is known to lead to arithmetic-like averaging of a . When, however, a is piecewise constant on \mathcal{D}_h , then as in the cell-centered finite volume case, the two basic choices for the weights in $\{a\}_\omega$, (3.4) and (3.5), lead respectively to arithmetic and harmonic averaging of a .

Definition 3.6 (Finite element method with harmonic averaging). Let the dual grid \mathcal{D}_h consist of polygonal/polyhedral dual volumes and let a be piecewise constant on \mathcal{D}_h . Let us define \tilde{a} by

$$\tilde{a}|_K = \left(\frac{(a^{-1}, 1)_K}{|K|} \right)^{-1} \quad \forall K \in \mathcal{T}_h. \quad (3.7)$$

Then we can define a finite element method with harmonic averaging for problem (1.1a)–(1.1b) as: find $p_h \in X_h$ such that

$$(\tilde{a} \nabla p_h, \nabla \psi_V)_{\mathcal{T}_V} = (f, \psi_V)_{\mathcal{T}_V} \quad \forall V \in \mathcal{V}_h^{\text{int}}. \quad (3.8)$$

Remark 3.7 (Arithmetic/harmonic averaging in the finite element method). We remark that the difference between the matrices of (3.1) and (3.8) corresponds to the difference between the matrices of the piecewise linear nonconforming finite element method and that of the hybridization of the lowest-order Raviart–Thomas mixed finite element method in that the first ones use the arithmetic and the second ones use the harmonic averaging of the diffusion coefficient a , see [15, Remark 3.2 and Lemma 8.1]. Remark in particular that by Definitions 3.1 and 3.6, one has in the finite element method the choice between the arithmetic and the harmonic averaging as in the finite volume one.

3.2 Equivalences

We are now ready to recall several equivalence results between the above methods.

Lemma 3.8 (Equivalence between matrices of finite elements and vertex-centered finite volumes). *Let $D \in \mathcal{D}_h$ have Lipschitz-continuous boundaries and let $|\sigma \cap D| = |\sigma|/d$ for each $\sigma \in \mathcal{E}_h^{\text{int}}$ with a vertex $V_D \in \mathcal{V}_h^{\text{int}}$ and the associated $D \in \mathcal{D}_h^{\text{int}}$. Let, moreover, a be piecewise constant on \mathcal{T}_h . Then, for all $p_h \in X_h$,*

$$(a\nabla p_h, \nabla \psi_{V_D})_{\mathcal{T}_{V_D}} = -\langle a\nabla p_h \cdot \mathbf{n}, 1 \rangle_{\partial D} \quad \forall D \in \mathcal{D}_h^{\text{int}}. \quad (3.9)$$

Proof. Employing the Green theorem and the finite elements basis functions form, see [4, Lemma 3] for $d = 2$. \square

Lemma 3.9 (Equivalence between matrices of finite elements and cell-centered finite volumes). *Let $d = 2$, let \mathcal{T}_h be Delaunay, that is let the circumcircle of each triangle does not contain any vertex in its interior, and let, moreover, no circumcenters of boundary triangles lie outside the domain Ω . Let \mathcal{D}_h be constructed using the orthogonal bisectors of the edges of \mathcal{T}_h and let $a = 1$. Then, for all $p_h \in X_h$,*

$$(\nabla p_h, \nabla \psi_{V_D})_{\mathcal{T}_{V_D}} = - \sum_{E \in \mathcal{N}(D)} \frac{|\sigma_{D,E}|}{d_{D,E}} (p_E - p_D) \quad \forall D \in \mathcal{D}_h^{\text{int}}.$$

Proof. See [14, Section III.12]. \square

Remark 3.10 (Relation between finite elements and cell-centered finite volumes if $d = 3$). *We remark that the above lemma does not generalize to three space dimensions, see, e.g., Letniowski [19] or Putti and Cordes [23].*

Lemma 3.11 (Equivalence between right-hand sides of finite elements and finite volumes). *Let $|D \cap K| = |K|/(d+1)$ for each $D \in \mathcal{D}_h^{\text{int}}$ and each $K \in \mathcal{T}_{V_D}$. Let, moreover, f be piecewise constant on \mathcal{T}_h . Then*

$$(f, \psi_{V_D})_{\mathcal{T}_{V_D}} = (f, 1)_D \quad \forall D \in \mathcal{D}_h^{\text{int}}. \quad (3.10)$$

Proof. Straightforward using the condition $|D \cap K| = |K|/(d+1)$ for $D \in \mathcal{D}_h^{\text{int}}$ and $K \in \mathcal{T}_{V_D}$ and a quadrature formula for linear functions on simplices. \square

3.3 Consequences

The following corollaries are obvious consequences of the previous lemmas.

Corollary 3.12 (Equivalence between finite elements and vertex-centered finite volumes). *Let the assumptions of Lemmas 3.8 and 3.11 be verified. Then the finite element method given by Definition 3.1 and the vertex-centered finite volume methods given by Definitions 3.2 and 3.4 produce the same discrete systems.*

Corollary 3.13 (Local conservativity of the finite element method on dual grids). *Let the assumptions of Lemmas 3.8 and 3.11 be verified. Then the finite element method given by Definition 3.1 is locally conservative over the dual grid $\mathcal{D}_h^{\text{int}}$.*

Corollary 3.14 (Equivalence between weighted cell- and vertex-centered finite volumes). *Let $d = 2$, let \mathcal{T}_h be Delaunay, let, moreover, no circumcenters of boundary triangles lie outside the domain Ω , and let \mathcal{D}_h be constructed using the orthogonal bisectors of the edges of \mathcal{T}_h . Let next a be piecewise constant on \mathcal{D}_h . Then the weighted cell-centered finite volume method given by Definition 3.3 and the weighted vertex-centered finite volume method given by Definition 3.4 produce the same discrete systems.*

3.4 Remarks

We finish this section by some additional remarks.

Remark 3.15 (Local conservativity of the finite element method). *Corollary 3.13 should be understood in the following sense: First of all, equation (3.2) states that the sum of fluxes entering/leaving $D \in \mathcal{D}_h^{\text{int}}$ equals the sources on this element. Secondly, rewriting $-\langle a\nabla p_h \cdot \mathbf{n}, 1 \rangle_{\partial D}$ as $-\sum_{E \in \mathcal{N}(D)} \langle a\nabla p_h \cdot \mathbf{n}, 1 \rangle_{\sigma_{D,E}}$ and noticing that the quantity $a\nabla p_h \cdot \mathbf{n}$ is single-valued on $\sigma_{D,E}$ under the given assumptions, local mass balance, in the sense that the mass leaving from one element (D) enters its neighbor (E), is likewise satisfied. Consequently, the finite element method is well locally mass conservative on $\mathcal{D}_h^{\text{int}}$, even if it is not locally mass conservative on \mathcal{T}_h . Remark finally that the above assertions are only valid exactly if in particular a and f are piecewise constant on \mathcal{T}_h . In the general case, local mass conservativity on \mathcal{D}_h only holds up to a numerical quadrature.*

Remark 3.16 (Choice of the dual grids). *In the above developments, a large freedom is left in what concerns the actual choice of the dual grids \mathcal{D}_h . The basic and most frequently used grid satisfying both the assumptions of Lemmas 3.8 and 3.11 is given by straight lines connecting the triangle barycentres through the midpoints of the edges of \mathcal{T}_h if $d = 2$ and similarly if $d = 3$.*

Remark 3.17 (Finite difference method). *Let \mathcal{D}_h consist of squares if $d = 2$ and cubes if $d = 3$. Then the finite difference method for problem (1.1a)–(1.1b) coincides with cell-centered finite volume one given by Definition 3.3, cf. Eymard et al. [14].*

Remark 3.18 (Tensor-valued diffusion coefficients). *In problem (1.1a)–(1.1b), we could also consider a tensor-valued diffusion coefficient \mathbf{A} in place of the scalar-valued a . Definitions 3.1 and 3.6 would in this case contain \mathbf{A} in place of a and similarly for Definitions 3.2 and 3.4. Then, for \mathbf{A} piecewise constant on \mathcal{T}_h , Lemma 3.8 still holds true and similarly for Corollaries 3.12 and 3.13.*

Remark 3.19 (Generalizations). *In [16], the above relations between finite elements and finite volumes employing dual grids have been used in order to construct a simple, stable, and consistent scheme enabling the discretization of strongly nonlinear convection–diffusion–reaction problems on nonmatching grids containing nonconvex elements with one unknown per element and positive definite systems, offering the arithmetic and harmonic averaging choices. In this reference, other remarks on the relations between finite elements and finite volumes are stated.*

4 Optimal abstract framework for a posteriori error estimation

We recall in this section the basic results of [33, 13], giving an optimal abstract framework for a posteriori error estimation in problem (1.1a)–(1.1b). The first result is the following abstract upper bound:

Theorem 4.1 (Abstract a posteriori error estimate). *Let p be the weak solution of problem (1.1a)–(1.1b) given by (2.10) and let $p_h \in H_0^1(\Omega)$ be arbitrary. Then*

$$\| \| p - p_h \| \| \leq \inf_{\mathbf{t} \in \mathbf{H}(\text{div}, \Omega)} \sup_{\varphi \in H_0^1(\Omega), \| \varphi \| = 1} \{ (f - \nabla \cdot \mathbf{t}, \varphi) - (a\nabla p_h + \mathbf{t}, \nabla \varphi) \}. \quad (4.1)$$

Proof. We first notice that

$$\| \| p - p_h \| \| = \mathcal{B} \left(p - p_h, \frac{p - p_h}{\| \| p - p_h \| \|} \right)$$

by (2.9). Clearly, as $\varphi := (p - p_h)/\|p - p_h\| \in H_0^1(\Omega)$, we immediately have $\mathcal{B}(p, \varphi) = (f, \varphi)$ by (2.10). Using this we obtain, for an arbitrary $\mathbf{t} \in \mathbf{H}(\text{div}, \Omega)$ and employing the Green theorem,

$$\begin{aligned} \mathcal{B}(p - p_h, \varphi) &= (f, \varphi) - (a\nabla p_h, \nabla \varphi) = (f, \varphi) - (a\nabla p_h + \mathbf{t}, \nabla \varphi) + (\mathbf{t}, \nabla \varphi) \\ &= (f - \nabla \cdot \mathbf{t}, \varphi) - (a\nabla p_h + \mathbf{t}, \nabla \varphi). \end{aligned}$$

From here, it is enough to note that $\|\varphi\| = 1$ and that $\mathbf{t} \in \mathbf{H}(\text{div}, \Omega)$ was chosen arbitrary to conclude the proof. \square

Theorem 4.1 gives a framework for optimal a posteriori error estimate, as shows the following theorem:

Theorem 4.2 (Global efficiency of the abstract estimate). *Let p be the weak solution of problem (1.1a)–(1.1b) given by (2.10) and let $p_h \in H_0^1(\Omega)$ be arbitrary. Let the a posteriori error estimate be given by Theorem 4.1. Then*

$$\inf_{\mathbf{t} \in \mathbf{H}(\text{div}, \Omega)} \sup_{\varphi \in H_0^1(\Omega), \|\varphi\|=1} \{(f - \nabla \cdot \mathbf{t}, \varphi) - (a\nabla p_h + \mathbf{t}, \nabla \varphi)\} \leq \|p - p_h\|.$$

Proof. It suffices to put $\mathbf{t} = -a\nabla p$ and to use the Schwarz inequality and the fact that $\|\varphi\| = 1$. \square

It follows from the above that not only that the abstract a posteriori error estimate of Theorem 4.1 gives a guaranteed upper bound (features no undetermined constant), but that it is also globally efficient (gives a global lower bound), is fully robust with respect to data (the efficiency constant does not depend on it), and is exact (the effectivity index is equal to 1). Finally, note that at this stage, no additional assumptions like a polynomial form of the data or the shape regularity of the mesh are needed.

5 Guaranteed a posteriori error estimates

Our purpose is now to find a locally computable version of the abstract a posteriori estimate of the previous section. We first present a general a posteriori error estimate, applicable to any conforming method under the condition that a suitable $\mathbf{t} \in \mathbf{H}(\text{div}, \Omega)$ can be found. We then propose a way of construction of this \mathbf{t} for the weighted vertex-centered finite volume method (3.6) and the finite element method (3.1) and conclude by some remarks and generalizations.

5.1 A general guaranteed a posteriori error estimate for conforming methods

The following is a very general version of an a posteriori error estimate for any conforming method:

Theorem 5.1 (A general guaranteed a posteriori error estimate). *Let p be the weak solution of problem (1.1a)–(1.1b) given by (2.10) and let $p_h \in H_0^1(\Omega)$ be arbitrary. Let next \mathcal{D}_h be a general partition of Ω such as those described in Section 2.1 and let $\mathbf{t}_h \in \mathbf{H}(\text{div}, \Omega)$ be arbitrary but such that*

$$(\nabla \cdot \mathbf{t}_h, 1)_D = (f, 1)_D \quad \forall D \in \mathcal{D}_h^{\text{int}}. \quad (5.1)$$

Then

$$\|p - p_h\| \leq \left\{ \sum_{D \in \mathcal{D}_h} (\eta_{R,D} + \eta_{DF,D})^2 \right\}^{\frac{1}{2}},$$

where the diffusive flux estimator $\eta_{\text{DF},D}$ is given by

$$\eta_{\text{DF},D} := \|a^{\frac{1}{2}}\nabla p_h + a^{-\frac{1}{2}}\mathbf{t}_h\|_D \quad D \in \mathcal{D}_h, \quad (5.2)$$

and the residual estimator $\eta_{\text{R},D}$ is given by

$$\eta_{\text{R},D} := m_{D,a} \|f - \nabla \cdot \mathbf{t}_h\|_D \quad D \in \mathcal{D}_h, \quad (5.3)$$

where

$$m_{D,a}^2 := C_{\text{P},D} \frac{h_D^2}{c_{a,D}} \quad D \in \mathcal{D}_h^{\text{int}}, \quad (5.4a)$$

$$m_{D,a}^2 := C_{\text{F},D,\partial\Omega} \frac{h_D^2}{c_{a,D}} \quad D \in \mathcal{D}_h^{\text{ext}}, \quad (5.4b)$$

with $C_{\text{P},D}$ the constant from the Poincaré inequality (2.6) and $C_{\text{F},D,\partial\Omega}$ the constant from the Friedrichs inequality (2.7).

Proof. Put $\mathbf{t} = \mathbf{t}_h$ in Theorem 4.1. Note that, for each $D \in \mathcal{D}_h^{\text{int}}$,

$$(f - \nabla \cdot \mathbf{t}_h, \varphi)_D = (f - \nabla \cdot \mathbf{t}_h, \varphi - \varphi_D)_D \leq \eta_{\text{R},D} \|\varphi\|_D,$$

using (5.1), the Poincaré inequality (2.6), the Schwarz inequality, and the definition (2.9) of the energy norm. We cannot use a similar approach also for $D \in \mathcal{D}_h^{\text{ext}}$ since there is no local conservativity supposed in (5.1) on these dual volumes. On the other hand, however, $\varphi = 0$ on $\partial D \cap \partial\Omega$, whence

$$(f - \nabla \cdot \mathbf{t}_h, \varphi)_D \leq \eta_{\text{R},D} \|\varphi\|_D$$

for each $D \in \mathcal{D}_h^{\text{ext}}$, using the Friedrichs inequality (2.7), the Schwarz inequality, and the definition (2.9) of the energy norm. Finally, $-(a\nabla p_h + \mathbf{t}, \nabla \varphi)_D \leq \eta_{\text{DF},D} \|\varphi\|_D$ is immediate using the fact that a is positive and scalar and the Schwarz inequality. Hence it now suffices to use the Cauchy–Schwarz inequality and to notice that $\|\varphi\| = 1$ in order to conclude the proof. \square

In general, we will look for a suitable \mathbf{t}_h in the lowest-order Raviart–Thomas–Nédélec space $\mathbf{RTN}(\mathcal{S}_h)$ defined over the fine simplicial mesh \mathcal{S}_h introduced in Section 2.1. The space $\mathbf{RTN}(\mathcal{S}_h) \subset \mathbf{H}(\text{div}, \Omega)$ is a space of vector functions having on each $K \in \mathcal{S}_h$ the form $(a_K + d_K x, b_K + d_K y)^t$ if $d = 2$ and $(a_K + d_K x, b_K + d_K y, c_K + d_K z)^t$ if $d = 3$. Note that the requirement $\mathbf{RTN}(\mathcal{S}_h) \subset \mathbf{H}(\text{div}, \Omega)$ imposes the continuity of the normal trace across all $\sigma \in \mathcal{G}_h^{\text{int}}$ and recall that $\mathbf{v} \cdot \mathbf{n}_\sigma$ is a constant for all $\sigma \in \mathcal{G}_h$ and that these side fluxes also represent the degrees of freedom of $\mathbf{RTN}(\mathcal{S}_h)$. For more details, we refer to [7, 26].

5.2 Guaranteed estimate for the weighted vertex-centered finite volume method

We show here how to construct a suitable $\mathbf{t}_h \in \mathbf{RTN}(\mathcal{S}_h)$ for the weighted vertex-centered finite volume method (3.6).

We simply define $\mathbf{t}_h \in \mathbf{RTN}(\mathcal{S}_h)$ by

$$\mathbf{t}_h \cdot \mathbf{n}_\sigma = -\{a\nabla p_h \cdot \mathbf{n}_\sigma\}_\omega \quad \forall \sigma \in \mathcal{G}_h. \quad (5.5)$$

We first note that $\mathbf{t}_h \cdot \mathbf{n}_\sigma$ is given directly by $-a\nabla p_h \cdot \mathbf{n}_\sigma$ for such $\sigma \in \mathcal{G}_h$ where there is no jump neither in a , nor in ∇p_h , using the conditions (2.4). If $a = 1$, then these are all the sides $\sigma \in \mathcal{G}_h$

which are in the interior of some $K \in \mathcal{T}_h$ or at the boundary of Ω . The same holds true if a is piecewise constant on \mathcal{T}_h , but if a is piecewise constant on \mathcal{D}_h , this set is restricted to all the sides $\sigma \in \mathcal{G}_h$ which are simultaneously in the interior of some $K \in \mathcal{T}_h$ and $D \in \mathcal{D}_h$ or at the boundary of Ω . For the moment, we do not specify the weights in (5.5), but it will turn out that only the choice (3.5) will lead to robust estimates. The following is the key necessary result:

Lemma 5.2 (Reconstructed diffusion residual for the weighted vertex-centered finite volume method (3.6)). *Let $p_h \in X_h$ be given by the weighted vertex-centered finite volume method (3.6), with any weights satisfying (2.4). Let \mathbf{t}_h be given by (5.5), where the weights in (5.5) are the same as those used in (3.6) for all $\sigma \in \mathcal{F}_h^{\text{int}}$ if a is discontinuous and piecewise constant on \mathcal{D}_h . If a is piecewise constant on \mathcal{T}_h (comprising the case $a = 1$), let the weights in (5.5) be arbitrary but satisfying (2.4). Then*

$$(\nabla \cdot \mathbf{t}_h, 1)_D = (f, 1)_D \quad \forall D \in \mathcal{D}_h^{\text{int}}.$$

Proof. The local conservativity of the weighted vertex-centered finite volume method (3.6) and the definition (5.5) of \mathbf{t}_h imply that

$$\langle \mathbf{t}_h \cdot \mathbf{n}, 1 \rangle_{\partial D} = (f, 1)_D \quad \forall D \in \mathcal{D}_h^{\text{int}},$$

noticing that $\{a\}_\omega \nabla p_h \cdot \mathbf{n} = \{a \nabla p_h \cdot \mathbf{n}\}_\omega$ for all $\sigma \in \mathcal{G}_h^{\text{int}}$, since all such sides lie in the interior of some $K \in \mathcal{T}_h$, where ∇p_h is constant. The assertion of the lemma now follows by the Green theorem. \square

We are now ready to summarize our a posteriori error estimate for this method:

Theorem 5.3 (Guaranteed a posteriori error estimate for the weighted vertex-centered finite volume method (3.6)). *Let p be the weak solution of problem (1.1a)–(1.1b) given by (2.10) and let p_h be its weighted vertex-centered finite volume approximation given by (3.6), with any weights satisfying (2.4). Let \mathbf{t}_h be given by (5.5) with the weights discussed in Lemma 5.2. Then the a posteriori error estimate of Theorem 5.1 holds true.*

5.3 Guaranteed estimate for the finite element method

We give in this section a version of our a posteriori error estimate for the finite element method.

Theorem 5.4 (Guaranteed a posteriori error estimate for the finite element method (3.1)). *Let the dual mesh \mathcal{D}_h be such that $D \in \mathcal{D}_h$ have Lipschitz-continuous boundaries, let $|\sigma \cap D| = |\sigma|/d$ for each $\sigma \in \mathcal{E}_h^{\text{int}}$ with a vertex $V_D \in \mathcal{V}_h^{\text{int}}$ and the associated $D \in \mathcal{D}_h^{\text{int}}$, and let $|D \cap K| = |K|/(d+1)$ for each $D \in \mathcal{D}_h^{\text{int}}$ and each $K \in \mathcal{T}_{V_D}$. Let next a be piecewise constant on \mathcal{T}_h , let p be the weak solution of problem (1.1a)–(1.1b) given by (2.10), and let p_h be its finite element approximation given by (3.1). Let finally \mathbf{t}_h be given by (5.5) with any weights satisfying (2.4). Then*

$$\| \|p - p_h\| \| \leq \left\{ \sum_{D \in \mathcal{D}_h} (\eta_{R,D} + \eta_{DF,D})^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\text{Osc},K}^2 \right\}^{\frac{1}{2}},$$

where $\eta_{DF,D}$ is given by (5.2), $\eta_{R,D}$ is given by

$$\eta_{R,D} := m_{D,a} \| \tilde{f} - \nabla \cdot \mathbf{t}_h - (\tilde{f} - \nabla \cdot \mathbf{t}_h)_D \|_D \quad D \in \mathcal{D}_h^{\text{int}}, \quad (5.6a)$$

$$\eta_{R,D} := m_{D,a} \| f - \nabla \cdot \mathbf{t}_h \|_D \quad D \in \mathcal{D}_h^{\text{ext}} \quad (5.6b)$$

with $m_{D,a}$ given by (5.4a)–(5.4b) and \tilde{f} given on each $K \in \mathcal{T}_h$ by $(f, 1)_K/|K|$, and the data oscillation estimators $\eta_{\text{osc},K}$ are given by

$$\eta_{\text{osc},K} := (d+1)^{\frac{1}{2}} C_{C,K}^{\frac{1}{2}} \frac{h_K}{C_{a,\mathcal{T}_K}^{\frac{1}{2}}} \|f - \tilde{f}\|_K \quad K \in \mathcal{T}_h, \quad (5.7)$$

where $C_{C,K}$ is the constant for the interpolation error in the Clément interpolant given by $\varphi_h = \sum_{D \in \mathcal{D}_h^{\text{int}}} \varphi_D \psi_{V_D}$ with $\varphi_D := (\varphi, 1)_D/|D|$,

$$\|\varphi - \varphi_h\|_K^2 \leq C_{C,K} h_K^2 \|\nabla \varphi\|_{\mathcal{T}_K}^2, \quad (5.8)$$

cf. Carstensen and Funken [8].

Proof. Under the assumptions of the theorem, relations (3.9) and (3.10), with f replaced by \tilde{f} , hold true. Hence, taking into account the construction (5.5) of \mathbf{t}_h ,

$$(\nabla \cdot \mathbf{t}_h, 1)_D = (\tilde{f}, 1)_D + (f - \tilde{f}, \psi_{V_D})_{\mathcal{T}_{V_D}}$$

as in Lemma 5.2. Thus

$$\begin{aligned} (f - \nabla \cdot \mathbf{t}_h, \varphi) &= \sum_{D \in \mathcal{D}_h^{\text{int}}} \{(\tilde{f} - \nabla \cdot \mathbf{t}_h, \varphi - \varphi_D)_D + (f - \tilde{f}, \varphi)_D - (f - \tilde{f}, \varphi_D \psi_{V_D})_{\mathcal{T}_{V_D}}\} \\ &+ \sum_{D \in \mathcal{D}_h^{\text{ext}}} (f - \nabla \cdot \mathbf{t}_h, \varphi)_D = \sum_{D \in \mathcal{D}_h^{\text{int}}} (\tilde{f} - \nabla \cdot \mathbf{t}_h, \varphi - \varphi_D)_D \\ &+ (f - \tilde{f}, \varphi - \varphi_h) + \sum_{D \in \mathcal{D}_h^{\text{ext}}} (f - \nabla \cdot \mathbf{t}_h, \varphi)_D. \end{aligned}$$

The estimate for $D \in \mathcal{D}_h^{\text{ext}}$ is the same as that in the proof of Theorem 5.1 and similarly for $D \in \mathcal{D}_h^{\text{int}}$, noticing however that

$$(\tilde{f} - \nabla \cdot \mathbf{t}_h, \varphi - \varphi_D)_D = (\tilde{f} - \nabla \cdot \mathbf{t}_h - (\tilde{f} - \nabla \cdot \mathbf{t}_h)_D, \varphi - \varphi_D)_D \leq \eta_{R,D} \|\varphi\|_D$$

with $\eta_{R,D}$ given by (5.6a). As $(\tilde{f} - \nabla \cdot \mathbf{t}_h)_D$ is not necessarily zero in this case and noticing that $\|v - v_D\|_D \leq \|v\|_D$ for each $v \in L^2(D)$, we can only improve the estimate while subtracting $(\tilde{f} - \nabla \cdot \mathbf{t}_h)_D$. Finally, the last term is estimated using the Schwarz inequality, the inequality (5.8), the definition (2.9) of the energy norm, the Cauchy–Schwarz inequality, and finally noting that $\sum_{K \in \mathcal{T}_h} \|\varphi\|_{\mathcal{T}_K}^2 \leq (d+1) \sum_{K \in \mathcal{T}_h} \|\varphi\|_K^2$. \square

5.4 Remarks and generalizations

Remark 5.5 (Comparison with standard residual estimators). *The above estimates have three basic advantages in comparison with standard residual estimators, cf. Verfürth [30]. First of all, they feature no undetermined constant. We remark however that a straightforward evaluation of the constants $C_{P,D}$ and $C_{F,D,\partial\Omega}$ is only possible when $D \in \mathcal{D}_h^{\text{int}}$ are convex and $D \in \mathcal{D}_h^{\text{ext}}$ have suitable form, see Section 2.3. Likewise, in the finite element method when f is not piecewise constant on \mathcal{T}_h , $C_{C,K}$ should be evaluated—see in this respect also Remark 5.10 below. Next, the classical residual estimator $h_K \|f\|_K$ is replaced by its improved versions (5.3) or (5.6a)–(5.6b). Lastly, as it will be seen in Lemma 6.2 below, the diffusive flux estimator $\eta_{\text{DF},D}$ represents a lower bound for a sum of the classical edge mass balance estimators $h_K^{\frac{1}{2}} \|\llbracket a \nabla p_h \cdot \mathbf{n}_\sigma \rrbracket\|_\sigma$. The improved behavior of our estimators over the classical one for the finite element method is numerically studied in [9].*

Remark 5.6 (Comparison with the equilibrated residual method). *In the equilibrated residual method, cf. [2], one searches equilibrated fluxes expressing local conservativity over each $K \in \mathcal{T}_h$. Contrarily to this approach, our estimators are based on the natural conservativity of the finite element method over dual grids \mathcal{D}_h , leading immediately to a guaranteed and locally computable upper bound.*

Remark 5.7 (Comparison with the Zienkiewicz–Zhu averaging). *The similarity of our approach with the Zienkiewicz–Zhu [37] estimator relies in the fact that both contain what we call a diffusive flux estimator, where \mathbf{t}_h is produced from $-a\nabla p_h$ by averaging. Concerning the differences, first of all, in the Zienkiewicz–Zhu estimator, one averages punctual nodal values and the postprocessed diffusive flux \mathbf{t}_h is smooth (belongs to C^0). According to Theorem 4.1, there is, however, no need for this—only normal traces of \mathbf{t}_h should be continuous, i.e., $\mathbf{t}_h \in \mathbf{H}(\text{div}, \Omega)$ is enough, which is achieved in our approach by averaging the side normal components. Secondly, this estimator does not give a guaranteed upper bound since in particular the residual parts (5.3) or (5.6a)–(5.6b) are omitted. Thirdly, these two differences become fundamentally important when a is discontinuous, since in this case, in general, $-a\nabla p$ is not smooth and only belongs to $\mathbf{H}(\text{div}, \Omega)$, whence $\mathbf{t}_h \in \mathbf{H}(\text{div}, \Omega)$ is optimal, and the residual parts play a crucial role through the presence of the material coefficient $c_{a,D}^{\frac{1}{2}}$.*

Remark 5.8 (Comparison with functional a posteriori estimates). *Repin [24] or Korotov [18] use instead of Theorem 5.1 the estimate*

$$\| \|p - p_h\| \| \leq \frac{C_{F,\Omega}^{1/2} h_\Omega}{c_{a,\Omega}^{1/2}} \|f - \nabla \cdot \mathbf{t}_h\| + \|a^{\frac{1}{2}} \nabla p_h + a^{-\frac{1}{2}} \mathbf{t}_h\|,$$

which follows readily from Theorem 4.1 using the Cauchy–Schwarz inequality, the Friedrichs inequality, and the definition of the energy seminorm. Here p is the weak solution given by (2.10), $p_h \in H_0^1(\Omega)$ and $\mathbf{t}_h \in \mathbf{H}(\text{div}, \Omega)$ are arbitrary, $C_{F,\Omega}$ is the constant from the Friedrichs inequality (2.7) with $D = \Omega$, and h_Ω is the diameter of Ω . The advantage is that no particular construction of $\mathbf{t}_h \in \mathbf{H}(\text{div}, \Omega)$ has to be done and the estimate is thus fully scheme-independent. However, as no information from the computation is used, the residual term is in general too large by the presence of h_Ω instead of h_D which we find in Theorem 5.1. Secondly, the term $1/c_{a,\Omega}^{1/2}$ is also greatly unfavorable in comparison with $1/c_{a,D}^{1/2}$ found in our estimates.

Remark 5.9 (Comparison with the estimator of Luce and Wohlmuth [20]). *Our estimators are similar to those of Luce and Wohlmuth [20], in particular in that we construct the dual mesh \mathcal{D}_h and the second simplicial triangulation \mathcal{S}_h and a $\mathbf{t}_h \in \mathbf{RTN}(\mathcal{S}_h)$. However, our construction of \mathbf{t}_h is much simpler and, moreover, as we will see later, it will lead to full robustness with respect to discontinuous coefficients.*

Remark 5.10 (Data oscillation in the finite element method). *Note that whenever $f \in H^1(K)$ for all $K \in \mathcal{T}_h$, the data oscillation estimators $\eta_{\text{Osc},K}$ in the finite element method are superconvergent, as $\|f - \tilde{f}\|_K \leq 1/\pi h_K \|\nabla f\|_K$ by the Poincaré inequality (2.6) (here we have employed the convexity of simplices). Moreover, if f is piecewise constant on \mathcal{T}_h , they disappear completely, so that then the a posteriori error estimate of Theorem 5.4 coincides with that of Theorem 5.3, as the finite element method coincides with the weighted vertex-centered finite volume one, see Corollary 3.12.*

Remark 5.11 (Generalizations to other methods). *The a posteriori error estimates of Theorems 5.3 and 5.4 may be generalized easily to other methods discussed in Section 3 using the equivalence results stated therein.*

6 Local efficiency and robustness of the a posteriori error estimates

We prove in this section the local efficiency and robustness of our a posteriori error estimates. The results are presented for the weighted vertex-centered finite volume method (3.6) and generalizations are discussed later.

6.1 Local efficiency and robustness of the a posteriori error estimate for weighted vertex-centered finite volumes

The local efficiency of the a posteriori error estimate of Theorem 5.3 is stated in the following theorem:

Theorem 6.1 (Local efficiency and robustness of the a posteriori error estimate for the weighted vertex-centered finite volume method (3.6)). *Let a be piecewise constant on \mathcal{D}_h , let f be a piecewise polynomial of degree m , let p be the weak solution of problem (1.1a)–(1.1b) given by (2.10), and let p_h be its weighted vertex-centered finite volume approximation given by (3.6), with the weights given by (3.5). Let next \mathcal{S}_h be shape-regular and let the a posteriori error estimate be given by Theorem 5.3, with in particular \mathbf{t}_h given by (5.5) with the weights (3.5), $\eta_{R,D}$ given by (5.3), and $\eta_{DF,D}$ by (5.2). Then, for each $D \in \mathcal{D}_h$, there holds*

$$\eta_{DF,D} \leq C \| \|p - p_h\| \|_{\mathcal{T}_{V_D}}, \quad (6.1)$$

$$\eta_{R,D} \leq \tilde{C} \| \|p - p_h\| \|_{\mathcal{T}_{V_D}}, \quad (6.2)$$

where the constant C depends only on the space dimension d , on the shape regularity parameter $\kappa_{\mathcal{S}}$, and on the polynomial degree m of f and \tilde{C} in addition depends on the constant $C_{P,D}$ from the Poincaré inequality (2.6) if $D \in \mathcal{D}_h^{\text{int}}$ or on the constant $C_{F,D,\partial\Omega}$ from the Friedrichs inequality (2.7) if $D \in \mathcal{D}_h^{\text{ext}}$.

The proof of Theorem 6.1 is decomposed into two parts:

Lemma 6.2 (Local efficiency of the diffusive flux estimator). *Let the assumptions of Theorem 6.1 be verified. Then (6.1) holds true.*

Proof. The proof follows the techniques of [30] and [12, 13]. Recall first the standard estimate

$$\|\mathbf{v}_h\|_K^2 \leq Ch_K \sum_{\sigma \in \mathcal{E}_K} \|\mathbf{v}_h \cdot \mathbf{n}\|_{\sigma}^2$$

valid for each $\mathbf{v}_h \in \mathbf{RTN}(K)$ and any simplex K . Here, and similarly in the rest of the proof, the constant C , not necessarily the same at each occurrence, depends only on d , $\kappa_{\mathcal{S}}$, and m .

Let now K be an arbitrary element in the simplicial mesh \mathcal{S}_D of a given $D \in \mathcal{D}_h$ and let us put $\mathbf{v}_h = a\nabla p_h + \mathbf{t}_h$. We have

$$\|a^{\frac{1}{2}}\nabla p_h + a^{-\frac{1}{2}}\mathbf{t}_h\|_K^2 = a_K^{-1} \|\mathbf{v}_h\|_K^2 \leq Ca_K^{-1} h_K \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{G}_h^{\text{int}}} \|\omega_{L,\sigma} [a\nabla p_h \cdot \mathbf{n}_{\sigma}]\|_{\sigma}^2,$$

where L denotes the neighboring element to K across $\sigma \in \mathcal{G}_h^{\text{int}}$, using that

$$(a\nabla p_h + \mathbf{t}_h)|_K \cdot \mathbf{n}_{\sigma} = (a\nabla p_h \cdot \mathbf{n}_{\sigma})|_K - \{a\nabla p_h \cdot \mathbf{n}_{\sigma}\}_{\omega} = \mathbf{n}_{\sigma} \cdot \mathbf{n} \omega_{L,\sigma} [a\nabla p_h \cdot \mathbf{n}_{\sigma}]$$

for $\sigma \in \mathcal{E}_K \cap \mathcal{G}_h^{\text{int}}$ and $(a\nabla p_h + \mathbf{t}_h)|_K \cdot \mathbf{n}_\sigma = 0$ for $\sigma \in \mathcal{E}_K \cap \mathcal{G}_h^{\text{ext}}$. Note that $\mathbf{n}_\sigma \cdot \mathbf{n} = \pm 1$ is only used as a sign determination.

Let us now consider a fixed $\sigma = \sigma_{K,L} \in \mathcal{E}_K \cap \mathcal{G}_h^{\text{int}}$. The estimate

$$h_K^{\frac{1}{2}} \|[a\nabla p_h \cdot \mathbf{n}_\sigma]\|_\sigma \leq C \sum_{M \in \{K,L\}} \|p - p_h\|_M$$

for the case where $a = 1$ is standard using the side and element bubble functions, the Green theorem, the inverse inequality, and the equivalence of norms on finite-dimensional spaces, cf. [30]. It implies

$$h_K^{\frac{1}{2}} \|[a\nabla p_h \cdot \mathbf{n}_\sigma]\|_\sigma \leq C \sum_{M \in \{K,L\}} a_M^{\frac{1}{2}} \|p - p_h\|_M.$$

It then follows that

$$\omega_{L,\sigma} a_K^{-\frac{1}{2}} h_K^{\frac{1}{2}} \|[a\nabla p_h \cdot \mathbf{n}_\sigma]\|_\sigma \leq C \sum_{M \in \{K,L\}} \omega_{L,\sigma} a_K^{-\frac{1}{2}} a_M^{\frac{1}{2}} \|p - p_h\|_M.$$

Finally, thanks to the definition (2.4) of $\omega_{L,\sigma}$, $\omega_{L,\sigma} a_K^{-\frac{1}{2}} a_M^{\frac{1}{2}} = \omega_{L,\sigma} \leq 1$ if $M = K$ and by (3.5), $\omega_{L,\sigma} a_K^{-\frac{1}{2}} a_M^{\frac{1}{2}} = a_K(a_K + a_L)^{-1} a_K^{-\frac{1}{2}} a_L^{\frac{1}{2}} \leq \frac{1}{2}$ if $M = L$, using the inequality $2ab \leq a^2 + b^2$.

Now finally, using the above results,

$$\begin{aligned} \eta_{\text{DF},D}^2 &= \sum_{K \in \mathcal{S}_D} \|a^{\frac{1}{2}} \nabla p_h + a^{-\frac{1}{2}} \mathbf{t}_h\|_K^2 \leq C \sum_{K \in \mathcal{S}_D} \sum_{\sigma_{K,L} \in \mathcal{E}_K \cap \mathcal{G}_h^{\text{int}}} a_K^{-1} h_K \omega_{L,\sigma_{K,L}}^2 \|[a\nabla p_h \cdot \mathbf{n}_{\sigma_{K,L}}]\|_{\sigma_{K,L}}^2 \\ &\leq C \sum_{K \in \mathcal{S}_D} \sum_{\sigma_{K,L} \in \mathcal{E}_K \cap \mathcal{G}_h^{\text{int}}} \sum_{M \in \{K,L\}} \|p - p_h\|_M^2 \leq C \|p - p_h\|_{T_D}^2, \end{aligned}$$

which was to be proved. \square

Lemma 6.3 (Local efficiency of the residual estimator). *Let the assumptions of Theorem 6.1 be verified. Then (6.2) holds true.*

Proof. Let us consider a fixed $D \in \mathcal{D}_h$. First,

$$\|f - \nabla \cdot \mathbf{t}_h\|_K \leq C a_K^{\frac{1}{2}} h_K^{-1} \|a^{\frac{1}{2}} \nabla p + a^{-\frac{1}{2}} \mathbf{t}_h\|_K$$

for each $K \in \mathcal{S}_D$ with C depending only on d , $\kappa_{\mathcal{S}}$, and m follows standardly by the Green theorem, the inverse inequality, the equivalence of norms on finite-dimensional spaces, and using the element bubble functions, cf. [30], so that

$$\|f - \nabla \cdot \mathbf{t}_h\|_D \leq C C_{a,D}^{\frac{1}{2}} h_D^{-1} \|a^{\frac{1}{2}} \nabla p + a^{-\frac{1}{2}} \mathbf{t}_h\|_D$$

holds true, using also the fact that $h_D / \min_{K \in \mathcal{S}_D} h_K$ is bounded by the shape-regularity of \mathcal{S}_h . Thus

$$h_D c_{a,D}^{-\frac{1}{2}} \|f - \nabla \cdot \mathbf{t}_h\|_D \leq C c_{a,D}^{-\frac{1}{2}} C_{a,D}^{\frac{1}{2}} \|a^{\frac{1}{2}} \nabla p + a^{-\frac{1}{2}} \mathbf{t}_h\|_D.$$

Next note that $c_{a,D}^{-\frac{1}{2}} C_{a,D}^{\frac{1}{2}} = 1$ for a piecewise constant on \mathcal{D}_h . Finally,

$$\|a^{\frac{1}{2}} \nabla p + a^{-\frac{1}{2}} \mathbf{t}_h\|_D \leq \|p - p_h\|_D + \|a^{\frac{1}{2}} \nabla p_h + a^{-\frac{1}{2}} \mathbf{t}_h\|_D$$

using the triangle inequality, which concludes the proof by virtue of the previously proved estimate (6.1). \square

We conclude this section by several remarks and comments on generalizations:

6.2 Remarks and generalizations

Remark 6.4 (Unconditioned robustness with respect to discontinuous a). *When a is piecewise constant on \mathcal{D}_h , when the harmonic averaging (3.5) has been used in the weighted vertex-centered finite volume method (3.6), and when the diffusive flux \mathbf{t}_h is likewise defined using harmonic averaging (3.5), equations (6.1)–(6.2) imply a full robustness of the estimators of Theorem 5.3 with respect to the diffusion coefficient a . In particular, no condition on the spatial distribution of the discontinuities in a is necessary, whereas in the previous results [6, 22, 1], a “monotonicity around vertices” condition on the distribution of the diffusion coefficient was always necessary.*

Remark 6.5 (Diffusion coefficient a piecewise constant on \mathcal{T}_h). *If a is piecewise constant on \mathcal{T}_h (whence the choice of the weights does not influence (3.6), cf. Remark 3.5) but harmonic averaging (3.5) has been used in order to define the diffusive flux \mathbf{t}_h , equation (6.1) still holds true, i.e., the diffusive flux estimator $\eta_{\text{DF},D}$ is still fully robust. It however follows from the proof of Lemma 6.3 that in equation (6.2), an additional factor $c_{a,D}^{-\frac{1}{2}} C_{a,D}^{\frac{1}{2}}$ appears, whence the residual estimator $\eta_{\text{R},D}$ is not robust in this case.*

Remark 6.6 (Local efficiency and robustness of the a posteriori error estimates for the finite element methods (3.1) and (3.8)). *If a is piecewise constant on \mathcal{T}_h and under the other assumptions of Lemmas 3.8 and 3.11, we have by Corollary 3.12 that the above remark holds true also for the finite element method (3.1). On the other hand, when a is piecewise constant on \mathcal{D}_h , the finite element method with harmonic averaging (3.8) leads to the same results as those of Theorem 6.1, up to the data oscillation estimators.*

Remark 6.7 (Comparison with discontinuous Galerkin methods). *In discontinuous Galerkin methods, similar results have been obtained for the diffusive flux estimator, see [12, 13]. In these methods, however, an additional nonconformity estimator appears, whose robustness can for the moment only be proved under the “monotonicity around vertices” condition on the distribution of the diffusion coefficient, cf. [1].*

Remark 6.8 (Comparison with the approach of [32] for cell-centered finite volume methods). *In the present paper, the results of the cell-centered finite volume method of Definition 3.3 are interpreted as a piecewise linear continuous function over a (Delaunay) triangular mesh \mathcal{T}_h associated with the Voronoï grid \mathcal{D}_h . In [32], a different definition (postprocessing) of the approximate solution, based on the finite volume fluxes and not using any other mesh than the given one, is used. One advantage of the present approach is that it, using harmonic averaging, leads to full robustness. Additional comparisons are presented in [35].*

7 Improvements by local minimization

We suggest here an improvement of the estimators studied in the previous sections, where $\mathbf{t}_h \in \text{RTN}(\mathcal{S}_h)$ was prescribed by (5.5). The motivation comes from Figure 2, which details the residual and diffusive flux estimates, given respectively by $\eta_{\text{R}} := \{\sum_{D \in \mathcal{D}_h} \eta_{\text{R},D}^2\}^{\frac{1}{2}}$ and $\eta_{\text{DF}} := \{\sum_{D \in \mathcal{D}_h} \eta_{\text{DF},D}^2\}^{\frac{1}{2}}$, for the example studied in the next section and complete estimates presented therein in Figure 7. We can clearly see that the residual estimator represents a major contribution while dominating the diffusive flux one. This is in a strict opposition to the case of mixed finite element, finite volume, and discontinuous Galerkin methods, cf. [33, 32, 13], where the residual estimator, defined in an analogous way, represents a higher-order term, only significant on rough grids. The explication of this phenomenon is indeed quite easy. In the above-cited methods, the

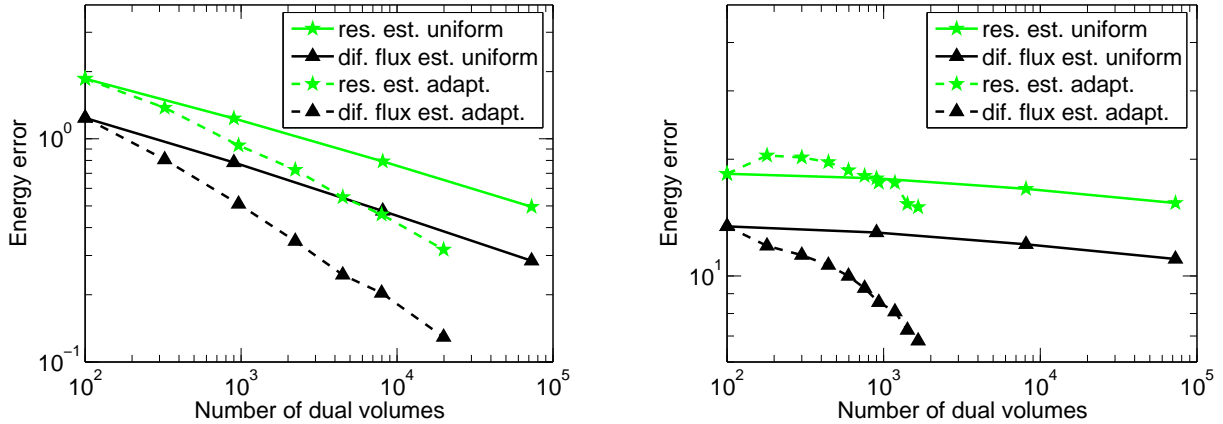


Figure 2: Comparison of residual and diffusive fluxes estimators in uniformly/adaptively refined meshes for $\alpha = 0.53544095$ (left) and $\alpha = 0.12690207$ (right)

vector field \mathbf{t}_h and formula (5.3) are given directly on the simplicial elements and (5.1) holds elementwise on \mathcal{T}_h with $\nabla \cdot \mathbf{t}_h$, a constant on each element, given by the mean value of f over K , whence the superconvergence by the Poincaré inequality (2.6) whenever f is piecewise H^1 on \mathcal{T}_h . In contrast to this situation, in the present setting, (5.1) is only valid on a set of elements \mathcal{S}_D .

On the other hand, in the present setting, prescribing $\mathbf{t}_h \in \mathbf{RTN}(\mathcal{S}_h)$ by (5.5) is only a sufficient but not necessary condition for (5.1) to hold. In particular, it is enough to use the definition (5.5) on those sides of \mathcal{G}_h which are at the boundary of some $D \in \mathcal{D}_h^{\text{int}}$ to ensure the validity of (5.1), whereas $\mathbf{t}_h \cdot \mathbf{n}_\sigma$ can be chosen freely on the other sides. A particularly interesting feature is that all these sides are in the interior of some $D \in \mathcal{D}_h$ (or at the boundary of Ω when $D \in \mathcal{D}_h^{\text{ext}}$), so that their choice in one dual volume does not affect the choice in the neighboring one and can be done locally. Whereas a true local minimization has been studied in [9], we propose and numerically investigate here a different approach, where in particular no (local) linear system has to be solved.

Let now $D \in \mathcal{D}_h$ be fixed. The first step is to construct $\mathbf{t}_{1,D} \in \mathbf{RTN}(\mathcal{S}_D)$ given by (5.5). In the second one, we then construct $\mathbf{t}_{2,D} \in \mathbf{RTN}(\mathcal{S}_D)$ given by (5.5) only for such $\sigma \in \mathcal{G}_h$ which are at the boundary of some $D \in \mathcal{D}_h^{\text{int}}$ and such that $(\nabla \cdot \mathbf{t}_{2,D}, 1)_K = (f, 1)_K$ for all $K \in \mathcal{S}_D$. Note that as $(\nabla \cdot \mathbf{t}_{2,D}, 1)_D = \langle \mathbf{t}_{2,D} \cdot \mathbf{n}, 1 \rangle_{\partial D} = (f, 1)_D$ when $D \in \mathcal{D}_h^{\text{int}}$, this can be done without any (local) linear system solution by choosing the flux over one interior side and a sequential construction as $\sum_{K \in \mathcal{S}_D} (f, 1)_K = (f, 1)_D$. If $D \in \mathcal{D}_h^{\text{ext}}$, this argument is then replaced by the fact that we are free to choose the fluxes over the exterior sides. Now any $\mathbf{t}_D := \alpha \mathbf{t}_{1,D} + (1 - \alpha) \mathbf{t}_{2,D}$ obviously obeys (5.1) and we can minimize $\eta_D := \eta_{R,D} + \eta_{DF,D}$ as a function of this parameter. It turns out that it is much easier to minimize $\eta_{R,D}^2 + \eta_{DF,D}^2$, as this is a quadratic function of α , and the optimal value is easily found to be given by

$$\begin{aligned} & \alpha (\|a^{-\frac{1}{2}}(\mathbf{t}_{1,D} - \mathbf{t}_{2,D})\|_D^2 + m_{D,a}^2 \|\nabla \cdot (\mathbf{t}_{1,D} - \mathbf{t}_{2,D})\|_D^2) \\ &= -(a^{\frac{1}{2}} \nabla p_h + a^{-\frac{1}{2}} \mathbf{t}_{2,D}, a^{-\frac{1}{2}} (\mathbf{t}_{1,D} - \mathbf{t}_{2,D}))_D + m_{D,a}^2 (f - \nabla \cdot \mathbf{t}_{2,D}, \nabla \cdot (\mathbf{t}_{1,D} - \mathbf{t}_{2,D}))_D. \end{aligned}$$

As however this value does not necessarily minimize η_D (when it is uniquely defined by the above formula) but $\eta_{R,D}^2 + \eta_{DF,D}^2$, we finally propose as an improved estimator

$$\eta_D := \min\{\eta_D(\mathbf{t}_{1,D}), \eta_D(\mathbf{t}_{2,D}), \eta_D(\alpha \mathbf{t}_{1,D} + (1 - \alpha) \mathbf{t}_{2,D})\}. \quad (7.1)$$

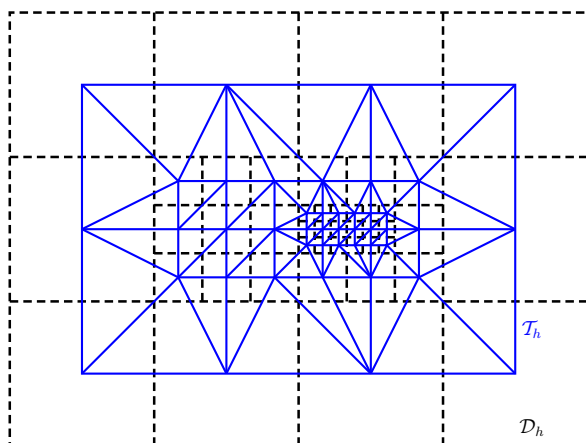


Figure 3: Example of a given nonmatching dual mesh \mathcal{D}_h and the corresponding primal triangular mesh \mathcal{T}_h

8 Numerical experiments

We present in this sections results of numerical experiments. We consider a model problem taken from [25], where $\Omega = (-1, 1) \times (-1, 1)$ is divided into four subdomains Ω_i along the Cartesian axes (the subregion $\{x > 0, y > 0\} \cap \Omega$ is denoted by Ω_1 and the subsequent numbering is done counterclockwise) and a is constant and equal to a^i in Ω_i . Supposing in addition that $f = 0$, analytical solution writing

$$p(r, \theta) = r^\alpha (a_i \sin(\alpha\theta) + b_i \cos(\alpha\theta))$$

in each Ω_i can be found. Here (r, θ) are the polar coordinates in Ω , a_i and b_i are constants depending on Ω_i , and α is a parameter. This solution is continuous across the interfaces but only the normal component of its flux $\mathbf{u} = -\mathbf{S}\nabla p$ is continuous; it exhibits a singularity at the origin and it only belongs to $H^{1+\alpha}(\Omega)$. We assume Dirichlet boundary conditions given by this solution and consider two sets of the coefficients. In the first one, $a^1 = a^3 = 5$, $a^2 = a^4 = 1$, $\alpha = 0.53544095$, and

$$\begin{aligned} a_1 &= 0.44721360; & b_1 &= 1.00000000; \\ a_2 &= -0.74535599; & b_2 &= 2.33333333; \\ a_3 &= -0.94411759; & b_3 &= 0.55555556; \\ a_4 &= -2.40170264; & b_4 &= -0.48148148. \end{aligned}$$

In the second one, $a^1 = a^3 = 100$, $a^2 = a^4 = 1$, $\alpha = 0.12690207$, and

$$\begin{aligned} a_1 &= 0.10000000; & b_1 &= 1.00000000; \\ a_2 &= -9.60396040; & b_2 &= 2.96039604; \\ a_3 &= -0.48035487; & b_3 &= -0.88275659; \\ a_4 &= 7.70156488; & b_4 &= -6.45646175. \end{aligned}$$

In order to get robust a posteriori error estimates, we know from Theorem 6.1 that a has to be piecewise constant on \mathcal{D}_h . If, however, we would first construct a simplicial mesh \mathcal{T}_h of Ω and then a dual grid \mathcal{D}_h as in Section 2.1, it would be very difficult to keep the dual mesh aligned with the inhomogeneities, especially for adaptive refinement. A possible solution is to first define the dual

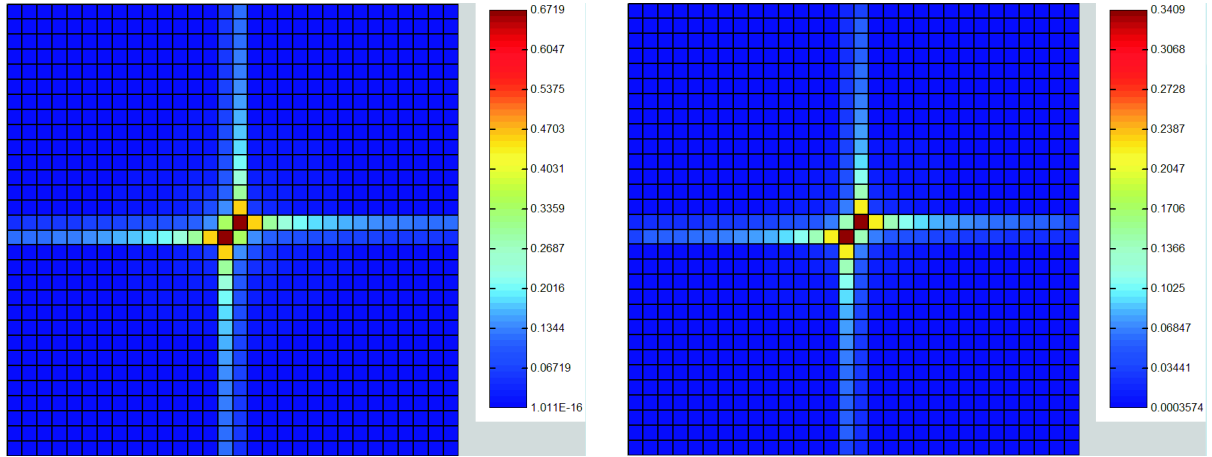


Figure 4: Estimated (left) and actual (right) error distribution on a uniformly refined mesh, $\alpha = 0.53544095$

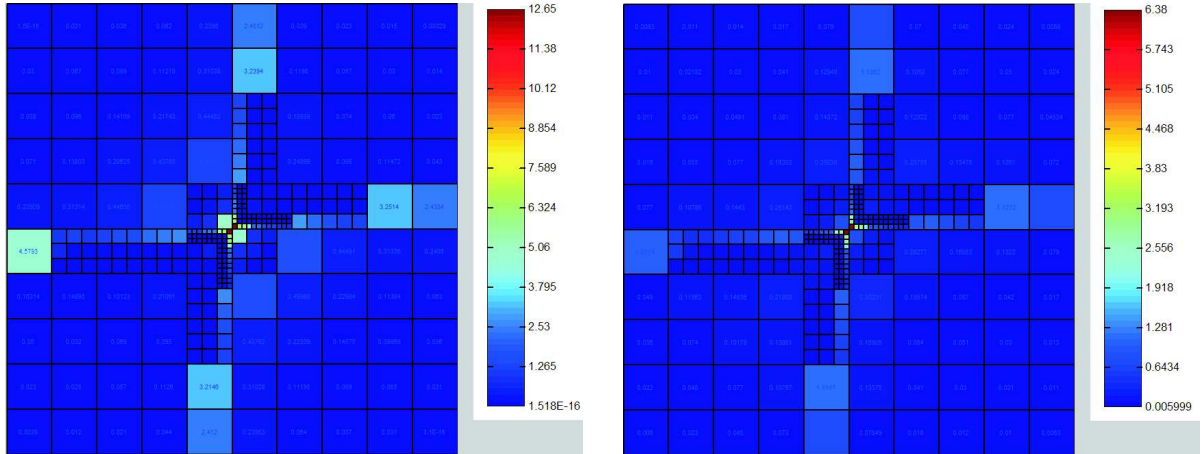


Figure 5: Estimated (left) and actual (right) error distribution on an adaptively refined mesh, $\alpha = 0.12690207$

mesh \mathcal{D}_h and only then the primal one \mathcal{T}_h . This type of approach has been proposed in, e.g., [16] and we employ it here. On the resulting couple of grids $\mathcal{D}_h, \mathcal{T}_h$, we then use the weighted vertex-centered finite volume method (3.6). Recall that on square grids (and their uniform refinements), this method is equivalent to the weighted cell-centered finite volume one, cf. Corollary 3.14, as well as to the finite difference one, cf. Remark 3.17. The advantage of the scheme (3.6) is that it can be used also when the original square grid has been locally refined (into a nonmatching grid) as in Figure 3. Note however that the symmetry of this scheme is then lost.

We in Figure 4 present the predicted and actual distribution of the error for $\alpha = 0.53544095$ and uniform mesh refinement, using the estimators of Theorem 5.3 (the interpolation error on nonhomogeneous Dirichlet boundary conditions is neglected). A similar comparison, this time for adaptive mesh refinement and $\alpha = 0.12690207$, is done in Figure 5. A square cell of the original dual mesh is refined into 9 identical subsquares if the estimated energy error is greater than 25% of the maximum of the estimators. We can see that in both cases the predicted error distribution is excellent and that in particular, the singularity at the origin is well detected. These results clearly illustrate the robust local lower bound of Theorem 6.1. We finally in Figure 6 give examples of

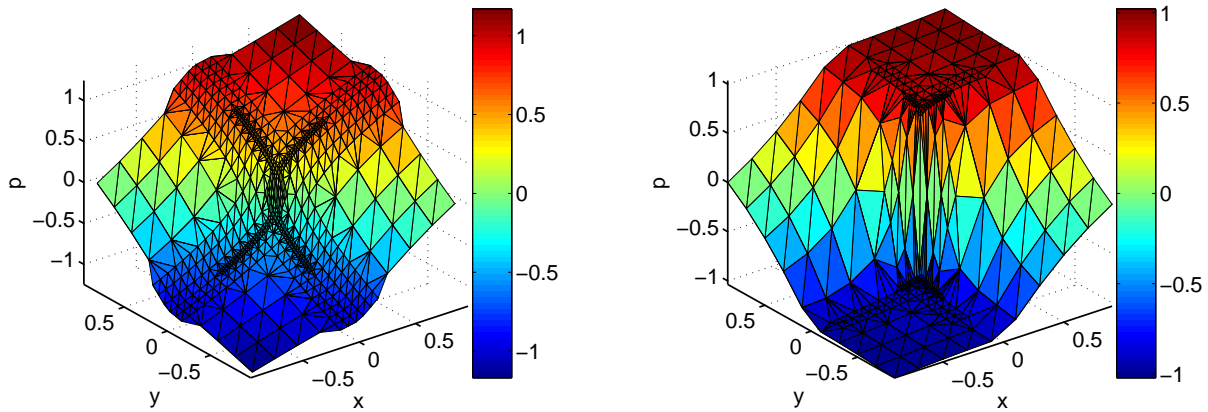


Figure 6: Approximate solutions on adaptively refined meshes, $\alpha = 0.53544095$ (left) and $\alpha = 0.12690207$ (right)

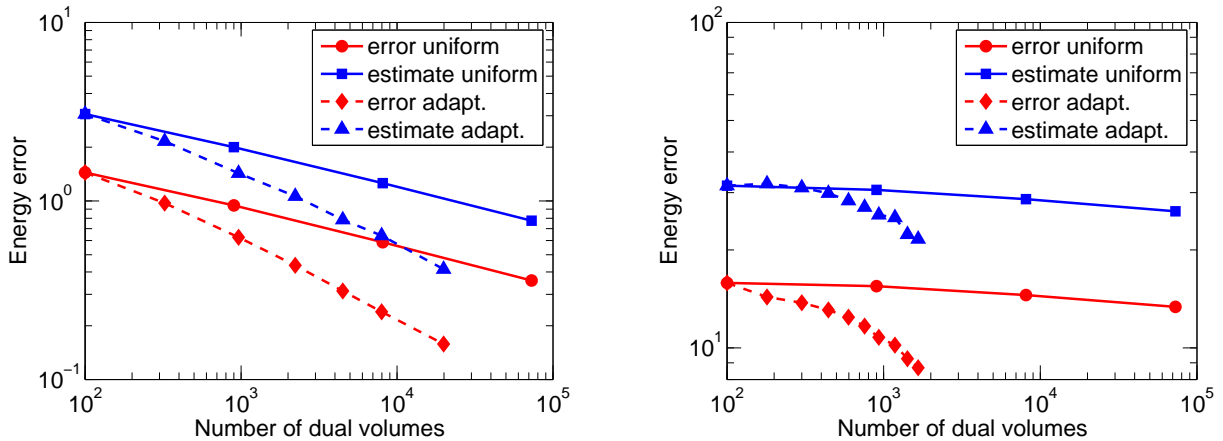


Figure 7: Estimated and actual error against the number of dual volumes in uniformly/adaptively refined meshes for $\alpha = 0.53544095$ (left) and $\alpha = 0.12690207$ (right)

the approximate solutions on the adaptively refined meshes in both cases; the strength of the singularity in the second case is quite obvious.

Knowing precisely the error distribution and refining adaptively the meshes, the next step is to check whether this leads to an increased efficiency of the calculations. This is in our case illustrated in Figure 7, from which it is evident that one can achieve a given precision with much fewer elements using adaptive mesh refinement based on our estimator. Here, the error in the energy norm (2.9) is approximated with a 7-point quadrature formula in each subtriangle $K \in \mathcal{S}_D$. In the code TALISMAN [36], which we use for numerical computations, at most 9 levels of refinement can be used. This technical limitation is the reason why we in the adaptive case and for $\alpha = 0.12690207$ only present results with a most 1660 dual volumes—this maximal refinement level is achieved near the origin but the maximal error is still located there. Finally, in Figure 8, we give the effectivity indices for the estimators of Theorem 5.3. We can clearly observe a confirmation of their robustness: whereas the inhomogeneity ratio rises from 5 to 100, the effectivity indices stay at the level of 2 for uniform refinement and are only slightly increased for adaptive refinement (since triangles with smaller angles exist in this case).

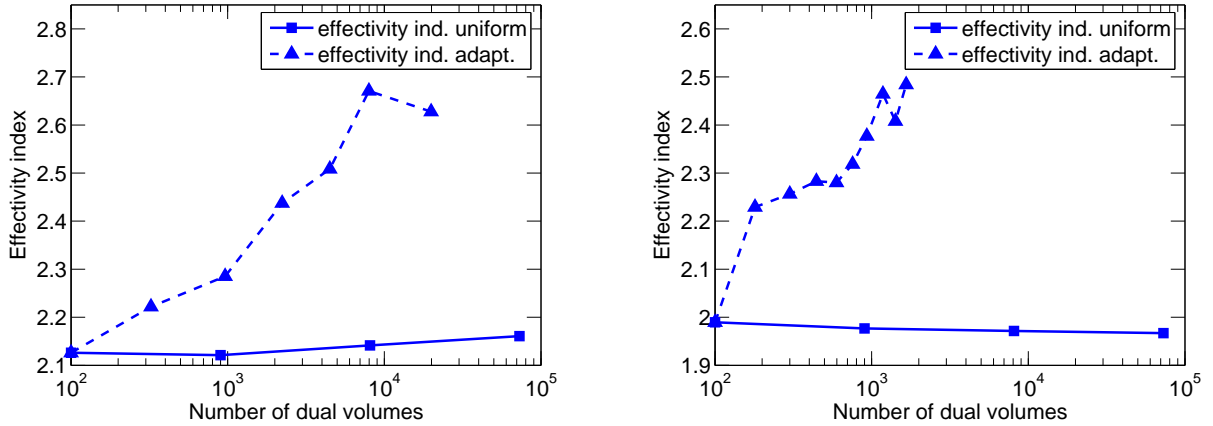


Figure 8: Effectivity indices against the number of dual volumes in uniformly/adaptively refined meshes for $\alpha = 0.53544095$ (left) and $\alpha = 0.12690207$ (right), original estimate of Theorem 5.3

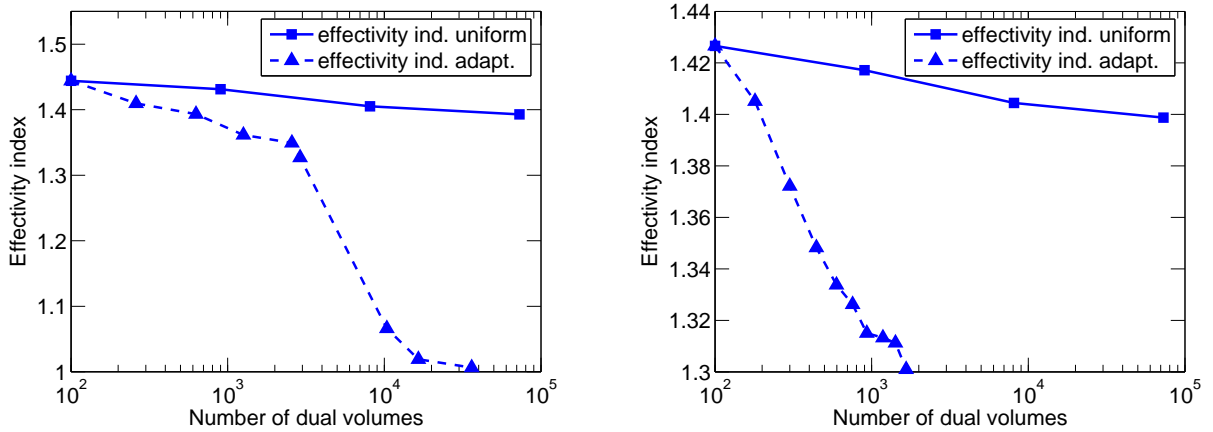


Figure 9: Effectivity indices against the number of dual volumes in uniformly/adaptively refined meshes for $\alpha = 0.53544095$ (left) and $\alpha = 0.12690207$ (right), estimate of Theorem 5.3 with local minimization (7.1)

We finally investigate the improvement of our error bound using the local minimization approach described in Section 7. It turns out that for $\alpha = 0.53544095$, the minimum in (7.1) is attained by the estimator $\eta_D(\mathbf{t}_{2,D})$; the residual estimators are actually equal to zero in this case since f is a constant. For $\alpha = 0.12690207$, the situation is similar in the majority of the cells but in some of them, the minimum in (7.1) is attained by the minimization estimator $\eta_D(\alpha \mathbf{t}_{1,D} + (1 - \alpha) \mathbf{t}_{2,D})$. The resulting effectivity indices are presented in Figure 9. We remark that in the adaptive refinement case, the cells marked for refinement were selected using the original estimators of Theorem 5.3. This approach gives better numerical results and is in coincidence with our theoretical results, since we only have the local efficiency for these original estimators. Note that for $\alpha = 0.12690207$, the adaptive refinement had to be again stopped before the singularity has been resolved for the technical reason indicated above and that a further decrease of the effectivity index in this case is likely. Finally, the presented results suggest that the estimator (7.1) is asymptotically exact, and this even in the case of discontinuous coefficients and singular solutions. A theoretical investigation of this observation will be a subject of some next work.

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