Proof of Theorem 2 in ”High-gain observers with updated high-gain and homogeneous correction terms”

Vincent Andrieu, Laurent Praly, Alessandro Astolfi

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V. Andrieu[a] L. Praly[b] A. Astolfi[c]

[a]*LAAS-CNRS, University of Toulouse, 31077 Toulouse, France*
vincent.andrieu@gmail.com

[b]*CAS, École des Mines de Paris, Fontainebleau, France*
praly@cas.ensmp.fr

[c]*EEE Dept, Imperial College, London, UK*
DISP, University of Rome Tor Vergata, Roma, Italy
a.astolfi@ic.ac.uk

Abstract

The aim of this note is to prove Theorem 2 in (Andrieu at al., 2007).

Key words: High-gain observers, Homogeneity in the bi-limit, Dynamic scaling.

1 Introduction

We consider a system, with state $\mathfrak{X} = (\mathcal{x}_1, \ldots, \mathcal{x}_n)$ in $\mathbb{R}^n$ described by :

$$\dot{\mathfrak{X}} = \mathfrak{A}(t)\,\mathcal{S}\,\mathfrak{X} \quad , \qquad y = \mathcal{x}_1 \ , \tag{1}$$

where y is the output, $\mathcal{S}$ is the left shift matrix defined as :

$$\mathcal{S}\,\mathfrak{X} = (\mathcal{x}_2, \ldots, \mathcal{x}_n, 0)^T \ ,$$

and $\mathfrak{A}(t)$ is a known time varying diagonal matrix $\mathfrak{A}(t) = \mathrm{diag}(\mathfrak{a}_1(t), \ldots, \mathfrak{a}_n(t))$, where the $\mathfrak{a}_i$ are assumed to satisfy :

$$0 < \underline{\mathfrak{a}} \leq \mathfrak{a}_i(t) \leq \overline{\mathfrak{a}} \quad \forall t \ . \tag{2}$$

After selecting $d_0 = 0$ and d_∞ arbitrarily in $\left[0, \frac{1}{n-1}\right)$, the system (1) is homogeneous in the bi-limit if and only if we choose the weights $r_0 = (r_{0,1}, \ldots, r_{0,n})$ and $r_\infty = (r_{\infty,1}, \ldots, r_{\infty,n})$ as :

$$r_{0,i} = 1 \ , \ r_{\infty,i} = 1 - d_\infty\,(n - i) \ . \tag{3}$$

In (Andrieu et al., 2006), a new observer was proposed for system (1) for the particular case where $\mathfrak{a}_i(t) = 1$. Its design is done recursively together with the one of an appropriate error Lyapunov function W which is homogeneous in the bi-limit (see below for the definition of homogeneity in the bi-limit).

In (Andrieu et al., 2007), we combine this tool with gain updating to obtain a new high-gain observer. To do so we use an extra property on W (see (5) below) which is a counterpart of (Praly, 2003, equation (16)) or (Krishnamurthy et al., 2003, Lemma A1). The fact that it can be obtained with also the presence of $\mathfrak{A}$ is stated in the following result.

Theorem 2 *Given d_∞ in $[0, \frac{1}{n-1})$, let d_W be a positive real number satisfying $d_W \geq 2 + d_\infty$ and $\mathfrak{B} = diag(\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$ with $\mathfrak{b}_j > 0$. If (2) holds, there exist a vector field $K : \mathbb{R} \to \mathbb{R}^n$ which is homogeneous in the bi-limit with associated weights r_0 and r_∞, and a positive definite, proper and C^1 function $W : \mathbb{R}^n \to \mathbb{R}_+$, homogeneous in the bi-limit with associated triples (r_0, d_W, W_0) and $(r_\infty, d_W, W_\infty)$, such that the following holds.*

(1) The functions W_0 and W_∞ are positive definite and proper and, for each j in $\{1, \ldots, n\}$, the function $\frac{\partial W}{\partial e_j}$ is homogeneous in the bi-limit with approximating functions $\frac{\partial W_0}{\partial e_j}$ and $\frac{\partial W_\infty}{\partial e_j}$.

(2) There exist two positive real numbers c_1 and c_2 such that we have, for all (t, E) in $\mathbb{R} \times \mathbb{R}^n$,

$$\frac{\partial W}{\partial E}(E)\,\mathfrak{A}(t)\,(\mathcal{S}\,E + K(e_1)) \tag{4}$$
$$\leq -c_1 \left(W(E) + W(E)^{\frac{d_W + d_\infty}{d_W}} \right) \ ,$$

$$\frac{\partial W}{\partial E}(E)\,\mathfrak{B}\,E \;\geq\; c_2\,W(E)\;, \tag{5}$$

The proof of this Theorem was omitted in (Andrieu et al., 2007) due to space limitation and is given in Section 3. Section 2 gives some prerequisite needed to address this proof.

2 Some prerequisite

The proof of this Theorem needs some prerequisite. Indeed, we recall the definition of homogeneity in the bi-limit, introduced in (Andrieu et al., 2006), and give some related properties.

Given a vector $r = (r_1, \ldots, r_n)$ in $(\mathbb{R}_+/\{0\})^n$, we define the dilation of a vector x in $\mathbb{R}^n$ as

$$\lambda^r \diamond x \;=\; (\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n)^T \;.$$

Definition 1 (Homogeneity in the 0-limit)

- *A continuous function $\phi\ :\mathbb{R}^n \to \mathbb{R}$ is said homogeneous in the 0-limit with associated triple (r_0, d_0, ϕ_0), where r_0 in $(\mathbb{R}_+/\{0\})^n$ is the weight, d_0 in $\mathbb{R}_+$ the degree and $\phi_0\ :\mathbb{R}^n \to \mathbb{R}$ the approximating function, respectively, if ϕ_0 is continuous and not identically zero and, for each compact set C in $\mathbb{R}^n$ and each $\varepsilon > 0$, there exists λ^* such that we have :*

$$\max_{x\in C}\left|\frac{\phi(\lambda^{r_0}\diamond x)}{\lambda^{d_0}} - \phi_0(x)\right| \;\leq\; \varepsilon \quad \forall\lambda\in(0,\lambda^*].$$

- *A vector field $f = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$ is said homogeneous in the 0-limit with associated triple (r_0, d_0, f_0), where $f_0 = \sum_{i=1}^n f_{0,i}\frac{\partial}{\partial x_i}$, if, for each i in $\{1,\ldots,n\}$, the function f_i is homogeneous in the 0-limit with associated triple $(r_0, d_0 + r_{0,i}, f_{0,i})$* [1] .

Definition 2 (Homogeneity in the ∞-limit)

- *A continuous function $\phi\ :\mathbb{R}^n \to \mathbb{R}$ is said homogeneous in the ∞-limit with associated triple $(r_\infty, d_\infty, \phi_\infty)$ where r_∞ in $(\mathbb{R}_+/\{0\})^n$ is the weight, d_∞ in $\mathbb{R}_+$ the degree and $\phi_\infty\ :\mathbb{R}^n \to \mathbb{R}$ the approximating function, respectively, if ϕ_∞ is continuous and not identically zero and, for each compact set C in $\mathbb{R}^n$ and each $\varepsilon > 0$, there exists λ^* such that we have :*

$$\max_{x\in C}\left|\frac{\phi(\lambda^{r_\infty}\diamond x)}{\lambda^{d_\infty}} - \phi_\infty(x)\right| \;\leq\; \varepsilon \quad \forall\lambda\in[\lambda^*,+\infty)\;.$$

- *A vector field $f = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$ is said homogeneous in the ∞-limit with associated triple $(r_\infty, d_\infty, f_\infty)$, with $f_\infty = \sum_{i=1}^n f_{\infty,i}\frac{\partial}{\partial x_i}$, if, for each i in $\{1,\ldots,n\}$, the function f_i is homogeneous in the ∞-limit with associated triple $(r_\infty, d_\infty + r_{\infty,i}, f_{\infty,i})$.*

Definition 3 (Homogeneity in the bi-limit) *A continuous function $\phi\ :\mathbb{R}^n \to \mathbb{R}$ (or a vector field f) is said homogeneous in the bi-limit if it is homogeneous in the 0-limit and homogeneous in the ∞-limit.*

The following propositions are proved, or are direct consequences of results, in (Andrieu et al., 2006).

Proposition 1 *Let η and μ be two continuous homogeneous in the bi-limit functions with weights r_0 and r_∞, degrees $d_{\eta,0}$, $d_{\eta,\infty}$ and $d_{\mu,0}$, $d_{\mu,\infty}$, and continuous approximating functions η_0, η_∞, μ_0, μ_∞.*

(1) The function $x \mapsto \eta(x)\mu(x)$ is homogeneous in the bi-limit with associated triples $(r_0, d_{\eta,0}+d_{\mu,0}, \eta_0\,\mu_0)$ and $(r_\infty, d_{\eta,\infty} + d_{\mu,\infty}, \eta_\infty\,\mu_\infty)$.

(2) If the degrees satisfy $d_{\eta,0} \geq d_{\mu,0}$ and $d_{\eta,\infty} \leq d_{\mu,\infty}$ and the functions μ, μ_0 and μ_∞ are positive definite then there exists a positive real number c satisfying :

$$\eta(x) \;\leq\; c\,\mu(x) \quad ,\ \forall\, x \;\in\; \mathbb{R}^n\;.$$

Proposition 2 *If $\phi\ :\mathbb{R}^n \to \mathbb{R}$ and $\zeta\ :\mathbb{R}\to\mathbb{R}$ are homogeneous in the 0-limit functions, with weights $r_{\phi,0}$ and $r_{\zeta,0}$, degrees $d_\phi = r_{\zeta,0}$ and d_ζ in $\mathbb{R}_+$, and approximating functions ϕ_0 and ζ_0, then $\zeta\circ\phi$ is homogeneous in the 0-limit with weight $r_{\phi,0}$, degree d_ζ, and approximating function $\zeta_0\circ\phi_0$. The same result holds for the cases of homogeneity in the ∞-limit and in the bi-limit.*

Proposition 3 *Let $\phi\ :\mathbb{R}\to\mathbb{R}$ be a bijective homogeneous in the 0-limit function with associated triple $\left(1, d_0, \phi_0\,x^{d_0}\right)$ with $\phi_0 \neq 0$ and $d_0 > 0$. Then, the inverse function $\phi^{-1}\ :\mathbb{R}\to\mathbb{R}$ is homogeneous in the 0-limit function with associated triple $\left(1, \frac{1}{d_0}, \left(\frac{x}{\phi_0}\right)^{\frac{1}{d_0}}\right)$. The same result holds for the cases of homogeneity in the ∞-limit and in the bi-limit.*

Proposition 4 *If the function ϕ is homogeneous in the 0-limit with associated triple (r_0, d_0, ϕ_0), then the function $\Phi_i(x) = \int_0^{x_i}\phi(x_1,\ldots,x_{i-1},s,x_n)\,ds$ is homogeneous in the 0-limit with associated triple $(r_0, d_0 + r_{0,i}, \Phi_{i,0})$, where the approximating function is given by*

$$\Phi_{i,0}(x) \;=\; \int_0^{x_i}\phi_0(x_1,\ldots,x_{i-1},s,x_n)\,ds\;.$$

The same result holds for the cases of homogeneity in the ∞-limit and in the bi-limit.

Proposition 5 *Suppose η and μ are two functions homogeneous in the bi-limit, with weights r_0 and r_∞, degrees d_0 and d_∞, and such that the approximating functions, denoted η_0 and η_∞, and, μ_0 and μ_∞ are continuous. If $\mu(x) \geq 0$ and*

$$\begin{array}{rcl}\{\,x\in\mathbb{R}^n\setminus\{0\}\;,\quad \mu(x)=0\,\} & \Rightarrow & \eta(x)>0\;,\\ \{\,x\in\mathbb{R}^n\setminus\{0\}\;,\quad \mu_0(x)=0\,\} & \Rightarrow & \eta_0(x)>0\;,\\ \{\,x\in\mathbb{R}^n\setminus\{0\}\;,\quad \mu_\infty(x)=0\,\} & \Rightarrow & \eta_\infty(x)>0\;,\end{array}$$

then there exists a strictly positive real number k^* such that, for all $k \geq k^*$, the functions $\eta(x) + k\,\mu(x)$, $\eta_0(x) + k\,\mu_0(x)$ and $\eta_\infty(x) + k\,\mu_\infty(x)$ are positive definite.

[1] In the case of a vector field the degree d_0 can be negative as long as $d_0 + r_{0,i} \geq 0$, for all $1 \leq i \leq n$.

3 Proof of Theorem 2

The proof we propose here is an adaptation of the one in (Andrieu et al., 2006). It is done by induction. To do so we use notations with an index showing the value from which we start counting. For instance $E_i = (e_i, \dots, e_n)^T$ denotes a state vector in $\mathbb{R}^{n-i+1}$. $\mathcal{S}_i$ is the left shift matrix of dimension $n-i+1$, i.e.

$$\mathcal{S}_i E_i \;=\; (e_{i+1}, \dots, e_n, 0)^T \;.$$

Proposition 6 *Let d_W be a positive real number satisfying $d_W \geq 2 + d_\infty$. Suppose there exist a bounded continuous diagonal matrix function $\mathfrak{A}_{i+1}$, a homogeneous in the bi-limit vector field $K_{i+1} \; : \mathbb{R} \to \mathbb{R}^{n-i}$, and a positive definite, proper and C^1 function homogeneous in the bi-limit $W_{i+1} \; : \mathbb{R}^{n-i} \to \mathbb{R}_+$, with associated triples $(r_0, d_W, W_{i+1,0})$ and $(r_\infty, d_W, W_{i+1,\infty})$ such that the following holds :*

(1) the function $W_{i+1,0}$ and $W_{i+1,\infty}$ are positive definite and proper and for all j in $[i+1, n]$, the functions $\frac{\partial W_{i+1}}{\partial e_j}$ are homogeneous in the bi-limit with approximating functions $\frac{\partial W_{i+1,0}}{\partial e_j}$ and $\frac{\partial W_{i+1,\infty}}{\partial e_j}$.

(2) There exist positive real numbers c, $\mathfrak{b}_{i+1}$, $\mathfrak{b}_n$ such that for all E_{i+1} in $\mathbb{R}^{n-i}$:

$$\sum_{j=i+1}^{n} \mathfrak{b}_j \, \frac{\partial W_{i+1}}{\partial e_j}(E_{i+1}) \, e_j \;\geq\; c\, W_{i+1}(E_{i+1}) \;, \qquad (6)$$

$$\frac{\partial W_{i+1}}{\partial E_{i+1}}(E_{i+1}) \mathfrak{A}_{i+1}(t) \left(\mathcal{S}_{i+1} E_{i+1} + K_{i+1}(e_{i+1})\right) \qquad (7)$$
$$\leq -c \left(W_{i+1}(E_{i+1}) + W_{i+1}(E_{i+1})^{\frac{d_W + d_\infty}{d_W}} \right)$$

Then, for any positive real number $\mathfrak{b}_i$, and any continuous positive function α_i, bounded away from 0, there exist a homogeneous in the bi-limit vector field $K_i \; : \mathbb{R} \to \mathbb{R}^{n-i+1}$, and a positive definite, proper and C^1 function $W_i \; : \mathbb{R}^{n-i+1} \to \mathbb{R}_+$ homogeneous in the bi-limit with associated triples $(r_0, d_W, W_{i,0})$ and $(r_\infty, d_W, W_{i,\infty})$ such that the following holds :

(1) The functions $W_{i,0}$ and $W_{i,\infty}$ are positive definite and proper and for all j in $[i, n]$, the functions $\frac{\partial W_i}{\partial e_j}$ are homogeneous in the bi-limit with approximating functions $\frac{\partial W_{i,0}}{\partial e_j}$ and $\frac{\partial W_{i,\infty}}{\partial e_j}$.

(2) There exists a positive real number $\bar{c}$ such that for all E_i in $\mathbb{R}^{n-i+1}$,

$$\sum_{j=i}^{n} \mathfrak{b}_j \, \frac{\partial W_i}{\partial e_j}(E_i) \, e_j \;\geq\; \bar{c}\, W_i(E_i) \;. \qquad (8)$$

$$\frac{\partial W_i}{\partial E_i}(E_i) \, \mathfrak{A}_i(t) \; \left(\mathcal{S}_i E_i + K_i(e_i)\right) \leq \qquad (9)$$
$$-\bar{c} \left(W_i(E_i) + W_i(E_i)^{\frac{d_W + d_\infty}{d_W}} \right)$$

where $\mathfrak{A}_i$ is the diagonal matrix $diag(\alpha_i \quad \alpha_i \mathfrak{A}_{i+1})$.

Proof : The proof is divided in three steps.

1. Construction of the Lyapunov function. Consider the function $q_i : \mathbb{R} \to \mathbb{R}$ defined as [2]

$$q_i(s) = s + s^{\frac{r_{\infty,i+1}}{r_{\infty,i}}} \;.$$

This function is C^1, strictly increasing and onto. Also, with

$$\frac{r_{\infty,i} + d_\infty}{r_{\infty,i}} \;\geq\; 1 \quad , \qquad i \in \{1 \dots, n\} \;,$$

it is homogeneous in the bi-limit with associated triples $(1, 1, s)$ and $(r_{\infty,i}, r_{\infty,i+1}, s^{\frac{r_{\infty,i+1}}{r_{\infty,i}}})$. Its derivative q_i', is also homogeneous in the bi-limit with approximating functions 1 and $\frac{r_{\infty,i+1}}{r_{\infty,i}} \, s^{\frac{d_\infty}{r_{\infty,i}}}$. Using Proposition 3, we know that the inverse function q_i^{-1} of q_i is C^1 and homogeneous in the bi-limit with associated triples $(1, 1, s)$ and $(r_{\infty,i+1}, r_{\infty,i}, s^{\frac{r_{\infty,i}}{r_{\infty,i+1}}})$. Furthermore, since we have $d_W - 1 \leq \frac{d_W - r_{\infty,i}}{r_{\infty,i}}$, by picking the function ζ as

$$\zeta(s) \;=\; s^{d_W - 1} + s^{\frac{d_W - r_{\infty,i}}{r_{\infty,i}}} \quad ,$$

we obtain from Proposition 2 that the function :

$$s \;\mapsto\; q_i^{-1}(s)^{d_W - 1} \;+\; q_i^{-1}(s)^{\frac{d_W - r_{\infty,i}}{r_{\infty,i}}} \qquad (10)$$

is C^1 and homogeneous in the bi-limit with associated triples $(1, d_W - 1, s^{d_W - 1})$ and $(r_{\infty,i+1}, d_W - r_{\infty,i}, s^{\frac{d_W - r_{\infty,i}}{r_{\infty,i+1}}})$. Furthermore, since $d_W \geq 2 + d_\infty$, its derivative is homogeneous in the bi-limit with approximating functions :

$$(d_W - 1)|s|^{d_W - 2} \;, \qquad \frac{d_W - r_{\infty,i}}{r_{\infty,i+1}} \, |s|^{\frac{d_W - r_{\infty,i+1} - r_{\infty,i}}{r_{\infty,i+1}}} \;.$$

Let $W_i \; : \mathbb{R}^{n-i} \to \mathbb{R}_+$ be defined as [3]

$$W_i(E_i) \;=\; W_{i+1}(E_{i+1}) \;+\; \sigma_i \, V_i(\ell_i \, e_i, e_{i+1}) \;, \qquad (11)$$

with

$$V_i(s, e_{i+1}) = \int_{q_i^{-1}(e_{i+1})}^{s} v^{d_W - 1} - q_i^{-1}(e_{i+1})^{d_W - 1} dv$$
$$+ \int_{q_i^{-1}(e_{i+1})}^{s} v^{\frac{d_W - r_{\infty,i}}{r_{\infty,i}}} - q_i^{-1}(e_{i+1})^{\frac{d_W - r_{\infty,i}}{r_{\infty,i}}} dv \;,$$

where σ_i and ℓ_i are positive real numbers that will be defined later. W_i is positive definite and proper. Also, as $1 \geq r_{\infty,i}$, it is homogeneous in the bi-limit with weights r_0 and r_∞, and degrees $d_{W,0} = d_{W,\infty} = d_W$. The function given in (10) as well as its derivative being homogeneous in the bi-limit, we get with Proposition 4 that the functions $\frac{\partial W_i}{\partial e_j}$ are homogeneous in the bi-limit with approximating functions $\frac{\partial W_{i,0}}{\partial e_j}$ and $\frac{\partial W_{i,\infty}}{\partial e_j}$. Hence point

[2] Recall that we have : $r_{\infty,i} + d_\infty = r_{\infty,i+1} \leq 1$.

[3] Compared to (Andrieu et al., 2006), σ_i is a new parameter introduced to obtain inequality (8).

1 of Proposition 6 is established.

2. Properties of the Lyapunov function. Let $J : \mathbb{R}^{n-i} \times \mathbb{R} \to \mathbb{R}$ be the function defined as :

$$J(E_{i+1}, s) = \mathfrak{b}_{i+1} \frac{\partial V_i}{\partial e_{i+1}}(s, e_{i+1})\, e_{i+1} + \mathfrak{b}_i \frac{\partial V_i}{\partial s}(s, e_{i+1})\, s \ .$$

The functions W_{i+1} and J are homogeneous in the bi-limit with associated weights 1 and $r_{\infty,i}$ for s and 1 and $r_{\infty,j}$ for e_j, $j \geq i+1$, and degrees $d_{W,0} = d_{W,\infty} = d_W$. By assumption W_{i+1} is positive definite and the same holds for its homogeneous approximations in the 0-limit and in the ∞-limit and we have :

$$J(0,s) = \mathfrak{b}_i \left[|s|^{d_W} + |s|^{\frac{d_W}{r_{\infty,i}}} \right] > 0 \quad \forall s \neq 0 \ .$$

It follows that the assumptions of Proposition 5 are satisfied with $\mu = W_{i+1}$ and $\eta = J$. Hence, with c given in (6), there exists a positive real number σ_i such that the functions $cW_{i+1,0} + \sigma_i J_0$, $cW_{i+1,\infty} + \sigma_i J_\infty$ and $cW_{i+1} + \sigma_i J$ are continuous and positive definite in (E_{i+1}, s). But then, from Proposition 1.2, there exists a positive real number $\bar{c}$ satisfying :

$$\tfrac{1}{\bar{c}} \left[cW_{i+1} + \sigma_i J \right] \geq W_i \ .$$

Since assumption (6) gives readily, for all E_i in $\mathbb{R}^{n-i+1}$,

$$\sum_{j=n}^{i} \mathfrak{b}_j \frac{\partial W_i}{\partial e_j}(E_i)\, e_j \geq c\, W_{i+1}(E_{i+1}) + \sigma_i\, J(E_{i+1}, \ell_i\, e_i) \ ,$$

we have established inequality (8) of Proposition 6.

3. Construction of the vector field K_i. Given a real number ℓ_i, we define the vector field K_i as :

$$K_i(e_i) = \begin{pmatrix} -q_i(\ell_i e_i) \\ K_{i+1}(q_i(\ell_i e_i)) \end{pmatrix}$$

With Propositions 1 and 2 and the properties we have established from q_i, it is a homogeneous in the bi-limit vector field.

We show now that by selecting ℓ_i large enough we can satisfy (9). We have :

$$\frac{\partial W_i}{\partial E_i}(E_i)\, \mathfrak{A}_i(t)\ (\mathcal{S}_i(E_i) + K_i(e_i)) = \alpha_i(t) \left[T_2(t, E_{i+1}, \ell_i e_i) + \ell_i\, T_1(E_{i+1}, \ell_i e_i) \right] \ . \tag{12}$$

with the notations :

$$T_1(E_{i+1}, s) = \sigma_i \frac{\partial V_i}{\partial s}(s, e_{i+1})(e_{i+1} - q_i(s))$$

$$T_2(t, E_{i+1}, s) = \left[\frac{\partial W_{i+1}}{\partial E_{i+1}}(E_{i+1}) + \sigma_i \frac{\partial V_i}{\partial E_{i+1}}(s, e_{i+1}) \right] \times \mathfrak{A}_{i+1}(t)\ (\mathcal{S}_{i+1}\, E_{i+1} + K_{i+1}(q_i(s)))$$

But with (7), we get

$$\begin{aligned} T_2(t, E_{i+1}, s) = & -c \left(W_{i+1}(E_{i+1}) + W_{i+1}(E_{i+1})^{\frac{d_W + d_\infty}{d_W}} \right) \\ & + \frac{\partial W_{i+1}}{\partial E_{i+1}}(E_{i+1}) \mathfrak{A}_{i+1}(t) [K_{i+1}(q_i(s)) - K_{i+1}(e_{i+1})] \\ & + \sigma_i \frac{\partial V_i}{\partial e_{i+1}}(s, e_{i+1}) \mathfrak{A}_{i+1,i+1}(t) [e_{i+2} + K_{i+1,i+2}(q_i(s))] \ . \end{aligned}$$

Then, the function $\mathfrak{A}_{i+1}$ being bounded, say by c_A, we have :

$$T_2(t, E_{i+1}, s) \leq T_3(E_{i+1}, s) \tag{13}$$

with the notation,

$$\begin{aligned} T_3(E_{i+1}, s) = & -c \left(W_{i+1}(E_{i+1}) + W_{i+1}(E_{i+1})^{\frac{d_W + d_\infty}{d_W}} \right) \\ & + c_A \sum_{j=i+1}^{n} \left| \frac{\partial W_{i+1}}{\partial e_j}(E_{i+1}) \left(K_{i+1,j}(q_i(s)) - K_{i+1,j}(e_{i+1}) \right) \right| \\ & + c_A \left| \sigma_i \frac{\partial V_i}{\partial e_{i+1}}(s, e_{i+1}) [e_{i+2} + K_{i+1,i+2}(q_i(s))] \right| \ . \end{aligned}$$

The functions T_1 and T_3 are homogeneous in the bi-limit with weights r_0 and r_∞ for E_{i+1} and 1 and $r_{\infty,i}$ for s, degrees d_W and $d_\infty + d_W$, continuous approximating functions $T_{1,0}$ and $T_{1,\infty}$, and $T_{3,0}$ and $T_{3,\infty}$, with, in particular :

$$T_{1,0}(E_{i+1}, s) = \sigma_i\, (e_{i+1} - s) \left(s^{d_W - 1} - e_{i+1}^{d_W - 1} \right) \ ,$$

$$T_{1,\infty}(E_{i+1}, s) = \sigma_i\, (e_{i+1} - s^{\frac{r_{\infty,i+1}}{r_{\infty,i}}}) \times \left(s^{\frac{d_W - r_{\infty,i}}{r_{\infty,i}}} - e_{i+1}^{\frac{d_W - r_{\infty,i}}{r_{\infty,i+1}}} \right) \ .$$

As the function q_i^{-1} is strictly increasing and onto, the function $\frac{\partial V_i}{\partial s}(s, e_{i+1})$ has a unique zero at $q_i(s) = e_{i+1}$ and has the same sign as $q_i(s) - e_{i+1}$. It follows

$$T_1(E_{i+1}, s) \leq 0 \quad , \qquad \forall\, E_i \in \mathbb{R}_{n-i+1} \ ,$$

$$T_1(E_{i+1}, s) = 0 \qquad \Leftrightarrow \qquad q_i(s) = e_{i+1}$$

and similarly for the approximating functions $T_{1,0}$ and $T_{1,\infty}$. Since $\frac{\partial V_i}{\partial e_{i+1}}(s, e_{i+1})$ is zero for $q_i(s) = e_{i+1}$ and W_{i+1} is positive definite, we get

$$\{ E_{i+1} \neq 0\ ,\ T_1(E_{i+1}, s) = 0 \} \quad \Rightarrow \quad T_3(E_{i+1}, s) < 0 \ .$$

With Proposition 2, the same holds for the approximating functions. The assumptions of Proposition 5 being satisfied, there exists a positive real number ℓ_i^* such that, for all $\ell_i \geq \ell_i^*$ the function $T_3 + \ell_i\, T_1$ and its approximations are continuous and negative definite in (E_{i+1}, s).

But then Proposition 1.2, with $\eta = W_i + W_i^{\frac{d_W + d_\infty}{d_W}}$ and $\mu = -(T_3 + \ell_i\, T_1)$, guarantees the existence of a positive real number number $\bar{c}$ satisfying :

$$-\tfrac{1}{\bar{c}} \left[T_3(E_{i+1}, \ell_i e_i) + \ell_i\, T_1(E_{i+1}, \ell_i e_i) \right] \geq W_i(E_i) + W_i(E_i)^{\frac{d_W + d_\infty}{d_W}} \ .$$

With (12) and (13), and since α_i is bounded away from 0, we have proved inequality (9) and completed the proof. $\square$

To construct the error Lyapunov function W and the vector field K, which prove Theorem 2, it is sufficient to iterate the construction proposed in Proposition 6 starting from

$$r_{\infty,n} = 1 \ , \quad \mathfrak{A}_n(t) = \frac{\mathfrak{a}_n(y)}{\mathfrak{a}_{n-1}(y)} \ ,$$
$$K_n(e_n) = -e_n \ - \ e_n^{\frac{r_{\infty,n}+d_\infty}{r_{\infty,n}}} \quad , \quad W_n(e_n) = |e_n|^{d_W} \ ,$$

where ℓ_n is any strictly positive positive real number. With (2), we get :

$$\frac{\underline{\mathfrak{a}}}{\overline{\mathfrak{a}}} \leq \frac{\mathfrak{a}_n(y)}{\mathfrak{a}_{n-1}(y)} \leq \frac{\overline{\mathfrak{a}}}{\underline{\mathfrak{a}}} \ , \ \mathfrak{b}_n \frac{\partial W_n}{\partial e_n}(E_n)\, e_n = \mathfrak{b}_n\, d_W \, |e_n|^{d_W} \ ,$$

and

$$\begin{aligned} &\tfrac{\partial W_n}{\partial e_n}(e_n)\tfrac{\mathfrak{a}_n(y)}{\mathfrak{a}_{n-1}(y)} K_n(e_n) \\ &\qquad = - \, d_W \, \tfrac{\mathfrak{a}_n(y)}{\mathfrak{a}_{n-1}(y)} \left(W_n(e_n) + W_n(e_n)^{\frac{d_W+d_\infty}{d_W}} \right) \ , \\ &\qquad \leq - \, d_W \, \tfrac{\underline{\mathfrak{a}}}{\overline{\mathfrak{a}}} \left(W_n(e_n) + W_n(e_n)^{\frac{d_W+d_\infty}{d_W}} \right) \ . \end{aligned}$$

Hence the assumptions of Proposition 6 are satified with $i+1 = n$.

We apply this Proposition recursively for $i+1$ ranging from n to 2 with, for $i = n-1, \ldots, 2$, $\alpha_i = \frac{\mathfrak{a}_i}{\mathfrak{a}_{i-1}}$ which lies in $[\frac{\underline{\mathfrak{a}}}{\overline{\mathfrak{a}}}, \frac{\overline{\mathfrak{a}}}{\underline{\mathfrak{a}}}]$, and $\alpha_1 = \mathfrak{a}_1 \geq \underline{\mathfrak{a}}$. In this way, we get

$$\begin{aligned} \mathfrak{A}_i \ &= \operatorname{diag}\left(\tfrac{\mathfrak{a}_i}{\mathfrak{a}_{i-1}}, \ldots, \tfrac{\mathfrak{a}_n}{\mathfrak{a}_{i-1}} \right) \quad \forall i \in \{n-1, \ldots, 2\} \ , \\ \mathfrak{A}_1 \ &= \operatorname{diag}\left(\mathfrak{a}_1, \ldots, \mathfrak{a}_n \right) \ . \end{aligned}$$

As a last comment, we remark that the idea of designing an observer recursively starting from x_n and going backwards towards x_1 is not new. It can be found in (Gauthier and Kupka, 2001, Lemma 6.2.1), (Praly and Jiang, 1998), (Shim and Seo, 2006) for instance.

Acknowledgements

This work has been performed while the first author was a member of the Control and Power Group, EEE Department of Imperial College London.

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