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Abstract. We study the large time behavior of solutions to the dissipative Korteweg-de Vrie equations $u_t + u_{xxx} + |D|^\alpha u + uu_x = 0$ with $0 < \alpha < 2$. We find v such that $u-v$ decays like $t^{-r(\alpha)}$ as $t \to \infty$ in various Sobolev norm.

Keywords : KdV-like equations, dissipative dispersive equations, large time behavior
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1 Introduction

In this paper we study the asymptotic behavior of solutions to the following dissipative KdV equations

$$\begin{cases} u_t + u_{xxx} + |D|^\alpha u + uu_x = 0, & t \in \mathbb{R}_+, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{dKdV}$$

with $0 < \alpha < 2$ and where $|D|^\alpha$ is the Lévy operator defined through its Fourier transform by $\widehat{|D|^\alpha \varphi}(\xi) = |\xi|^\alpha \widehat{\varphi}(\xi)$. Here $u = u(t,x)$ is a real-valued function.

The (dKdV) equations are dissipative versions of the well-known KdV equation

$$u_t + u_{xxx} + uu_x = 0 \tag{1.1}$$

which have been extensively studied. Equation (1.1) is completely integrable and there exists an infinite sequence of conserved quantities. For sufficiently smooth initial data, we know that global in time solutions exist and can be asymptotically written as a sum of traveling wave solutions, called solitons, see [18], [14].

Concerning the pure dissipative equation

$$u_t + |D|^\alpha u + uu_x = 0, \tag{1.2}$$

it has been proposed to model a variety of physical phenomena, such that the growth of molecular interfaces (cf. [12]). Also, in [7], Jourdain, Méléard and Woyczynski pointed out the main interest of equation (1.2) in probability theory. Biler, Funaki and Woyczynski proved in [3] several local and global well-posedness results, in particular in the general setting $0 < \alpha \leq 2$, they obtained weak solutions of (1.2). Using the Fourier splitting method first introduced by Schonbek in [17], they showed that regular solutions satisfy the estimate

$$\|u(t)\|_{L^2} \leq c(1+t)^{-1/2\alpha} \tag{1.3}$$

for all $t > 0$. This result was improved by Biler, Karch and Woyczynski [4] in the case of a diffusion operator of the form $-\partial_x^2 + |D|^\alpha$. See also [11] for asymptotic results concerning (1.2) with $1 < \alpha < 2$.

Let us turn back to the (dKdV) equation. The Cauchy problem (dKdV) with $0 \leq \alpha \leq 2$ has been shown to be globally well-posed in the Sobolev spaces $H^s(\mathbb{R})$ for all $s > -3/4$ and furthermore, the solution $u(t)$ belongs to $H^\infty(\mathbb{R})$ for any $t > 0$ (cf. [15]). When $\alpha = 1/2$, (dKdV) models the evolution of the free surface for shallow water waves damped by viscosity, see [16]. When $\alpha = 2$, (dKdV) is the so-called KdV-Burgers equation which models the propagation of weakly nonlinear dispersive long waves in some contexts when dissipative effects occur (see [16]). In the case $\alpha = 0$, (dKdV) reads

$$u_t + u_{xxx} + u + uu_x = 0 \tag{1.4}$$

and it is easy to get the decay rate for the L^2-norm of the solution. Indeed, multiplying (1.4) by u and integrating over $\mathbb{R}$ give for regular solutions the equality

$$\frac{1}{2}\partial_t \int_{-\infty}^{\infty} u^2(t,x)dx + \int_{-\infty}^{\infty} u^2(t,x)dx = 0,$$

and it follows immediately that

$$\|u(t)\|_{L^2} = O(e^{-t}) \text{ as } t \to \infty.$$

Now consider the KdV-Burgers equation ((dKdV) with $\alpha = 2$). In a sharp contrast with what occurs for (1.4), Amick, Bona and Schonbek [1] proved that if $u_0 \in L^1(\mathbb{R}) \cap H^2(\mathbb{R})$, then the corresponding solution satisfies

$$\|u(t)\|_{L^2} \leq c(1+t)^{-1/4} \tag{1.5}$$

and furthermore, this estimate is optimal for a generic class of functions. The proof of this result is based on a subtle use of the Hopf-Cole transformation. Later, Karch [10] improved this result by showing that the asymptotic profile of the solution with a mass M is given by the fundamental solution U_M of the viscous Burgers equation (eq. (1.2) with $\alpha = 2$)

$$u_t - u_{xx} + uu_x = 0$$

with the same mass. More precisely, we have

$$t^{(1-1/p)/2}\|u(t) - U_M(t)\|_{L^p} \to 0 \quad \text{as } t \to \infty$$

for each $p \in [1, \infty]$. In other words, we can say that for large times, the dispersion is negligible compared to dissipation and nonlinearity effects. His method of proof is based on a scaling argument. This kind of behavior was also heuristically observed by Dix in [6]. He called this situation the "balanced case" because both dissipation and nonlinearity contributions appear in the long time behavior of the solution, this is formally expressed by the relation $\alpha = 2$.

In the present paper we study the so-called "asymptotically weak nonlinearity case" $\alpha < 2$. For a large class of equations, solution of the nonlinear problem asymptotically looks like solution of the corresponding linear problem (with same initial data). One of the goals of this article is to show that similar behaviors occur for (dKdV) with $0 < \alpha < 2$.

Following the works of Karch [9], we shall mainly work on the integral formulation of (dKdV) :

$$u(t) = S_\alpha(t) * u_0 - \frac{1}{2}\int_0^t S_\alpha(t-s) * \partial_x u^2(s)ds \tag{1.6}$$

valid for any sufficiently regular solution, and where $S_\alpha(t)$ is defined by

$$S_\alpha(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\xi}e^{(i\xi^3 - |\xi|^\alpha)t}d\xi, \quad t > 0.$$

First, using the properties of the generalized heat kernel, we give a complete asymptotic expansion of the free solution $S_\alpha(t)*u_0$. After deriving the decay rates estimates of the solution in various Sobolev norms $\|\cdot\|$, we show that $\|u(t) - S_\alpha(t) * u_0\|$ is bounded by $ct^{-r(\alpha)}$, $r(\alpha) > 0$. Next, we improve this result by finding terms $w = w(t,x)$ such that $\|u(t) - S_\alpha(t) * u_0 - w(t)\|$ decays to zero faster than $t^{-r(\alpha)}$.

Notation. The notation to be used are standard. The letter c denotes a constant which may change at each occurrence. For $p \in [1,\infty]$ we define the Lebesgue space $L^p(\mathbb{R})$ by its norm $\|f\|_{L^p} = \left(\int_{-\infty}^{\infty} |f(x)|^p dx\right)^{1/p}$ with the usual modification for $p = \infty$. If $f = f(t,x)$ is a space-time function, the L^p-norm of f will be taken in the x-variable. For $j \geq 0$ and $p \in [1,\infty]$, the Sobolev spaces $H^{p,j}(\mathbb{R})$ and $\dot{H}^{p,j}(\mathbb{R})$ are respectively endowed with the norms $\|f\|_{H^{p,j}} = \|f\|_{L^p} + \|\partial_x^j f\|_{L^p}$ and $\|f\|_{\dot{H}^{p,j}} = \|\partial_x^j f\|_{L^p}$. When $p = 2$, we simplify by the notation $H^j(\mathbb{R})$ and $\dot{H}^j(\mathbb{R})$. If $f \in \mathcal{S}'(\mathbb{R})$, we define its Fourier transform by setting $\hat{f}(\xi) = \mathcal{F}f(\xi) = \int_{-\infty}^{\infty} e^{-ix\xi} f(x)dx$.
We introduce G_α, the fundamental solution of the equation $u_t + |D|^\alpha u = 0$, i.e.

$$G_\alpha(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\xi} e^{-t|\xi|^\alpha} d\xi, \quad t > 0.$$

It is clear that G_α has the self-similarity property

$$G_\alpha(t,x) = t^{-1/\alpha} G_\alpha(1, xt^{-1/\alpha}), \quad x \in \mathbb{R}, t > 0. \tag{1.7}$$

On the other hand, we know that $G_\alpha(t) \in H^{p,j}(\mathbb{R})$ for any $p \in [1,\infty]$ and $j \geq 0$, see for instance [13].
Finally, for $f \in L^1(x^j dx)$, $j \in \mathbb{N}$, we set $\mathcal{M}_j(f) = \int_{-\infty}^{\infty} f(x)x^j dx$.

2 Main results

As we are going to show, the solution of (dKdV) can be approximated by the solution of the corresponding linear equation. We first give a complete asymptotic expansion of $S_\alpha(t) * u_0$, which will be used in the proof of the main theorem.

Theorem 2.1. *Let $p \in [1,\infty]$ and $j, N \in \mathbb{N}$. Then for all $t \geq 1$ and $u_0 \in L^1((1+|x|)^{N+1}dx)$,*

$$\Big\| S_\alpha(t) * u_0 - \sum_{n=0}^{N} \frac{(-1)^n}{n!} \mathcal{M}_n(u_0) \partial_x^n G_\alpha(t) - \sum_{k=1}^{N} \frac{t^k}{k!} \sum_{\ell=0}^{N-1} \frac{(-1)^\ell}{\ell!} \mathcal{M}_\ell(u_0) \partial_x^\ell (-\partial_x)^{3k} G_\alpha(t) \Big\|_{\dot{H}^{p,j}}$$
$$\leq ct^{-(1-1/p)/\alpha - j/\alpha - (N+1)/\alpha} \tag{2.1}$$

Remark 2.1. *When $N=0$, the sum $\sum_{k=1}^{N}$ in (2.1) has to be understood as 0, and thus (2.1) reads*

$$\|S_\alpha(t)*u_0-\mathcal{M}_0(u_0)G_\alpha(t)\|_{\dot{H}^{p,j}}\leq ct^{-(1-1/p)/\alpha-j/\alpha-1/\alpha}. \tag{2.2}$$

If $N=1$, we have the following asymptotic expansion for $S_\alpha(t)*u_0$,

$$\begin{aligned}\|S_\alpha(t)*u_0-\mathcal{M}_0(u_0)G_\alpha(t)+\mathcal{M}_1(u_0)\partial_xG_\alpha(t)+t\mathcal{M}_0(u_0)\partial_x^3G_\alpha(t)\|_{\dot{H}^{p,j}}&\\ \leq ct^{-(1-1/p)/\alpha-j/\alpha-2/\alpha}&.\end{aligned}$$

Remark 2.2. *The term $\sum_{n=0}^{N}\frac{(-1)^n}{n!}\mathcal{M}_n(u_0)\partial_x^nG_\alpha(t)$ in (2.1) corresponds to the asymptotic expansion of $G_\alpha(t)*u_0$, solution to the generalized heat equation $u_t+|D|^\alpha u=0$. The other terms are due to the dispersive effects and appear only for $N\geq 1$.*

Now we consider the nonlinear equation (dKdV) with $0<\alpha<2$. Throughout this paper, we make the following assumptions :

$$u_0\in L^1(\mathbb{R})\cap L^2(\mathbb{R}), \tag{2.3}$$

$$u\in C(]0,\infty[;H^\infty(\mathbb{R})), \tag{2.4}$$

$$\text{if } u_0\in H^j(\mathbb{R}),\ \text{then } \sup_{t\geq 0}\|\partial_x^ju(t)\|_{L^2}<\infty. \tag{2.5}$$

For $u_0\in L^2(\mathbb{R})$, existence of global solutions satisfying (2.4) was proved for example in [15]. Moreover, if $u_0\in H^j(\mathbb{R})$, it was shown that the solution is continuous from $[0,\infty[$ to $H^j(\mathbb{R})$. In Section 4, we will show that assumption (2.5) is verified for such solutions when $u_0\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$, at least in the case $\alpha>1$.

Theorem 2.2. *Let $p\in[2,\infty]$ and $j\in\mathbb{N}$. Assume that $u_0\in H^{j+1}(\mathbb{R})\cap L^1(\mathbb{R})$ and (2.4)-(2.5) hold true. Then we have*

$$\|u(t)\|_{\dot{H}^{p,j}}\leq c(1+t)^{-(1-1/p)/\alpha-j/\alpha},\quad t>0. \tag{2.6}$$

When $j=0$, (2.6) is valid for all $p\in[1,\infty]$.

Next we find the first term in the asymptotic expansion of the solution.

Theorem 2.3. *Let $p \in [2,\infty]$ and $j \in \mathbb{N}$. We assume that $u_0 \in H^{j+3}(\mathbb{R}) \cap L^1(\mathbb{R})$ and that the solution u satisfies (2.4)-(2.5). Then, for all $t > 0$,*

$$\|u(t)-S_\alpha(t)*u_0\|_{\dot{H}^{p,j}} \le c \begin{cases} (1+t)^{(-(1-1/p)/\alpha-j/\alpha)-1/\alpha} & \text{for } \ 0<\alpha<1, \\ (1+t)^{(-(1-1/p)-j)-1}\log(1+t) & \text{for } \ \alpha=1, \\ (1+t)^{(-(1-1/p)/\alpha-j/\alpha)-(2/\alpha-1)} & \text{for } \ 1<\alpha<2. \end{cases}$$

In view of Theorems 2.2 and 2.3, it is clear that decay rate of $u(t)-S_\alpha(t)* u_0$ in $\dot{H}^{p,j}$-norm is better than when considering only $u(t)$. In order to find other terms in the asymptotic expansion, we need to consider separately the cases $0<\alpha<1$, $\alpha=1$ and $1<\alpha<2$.

When $0<\alpha<1$ or $\alpha=1$, the difference between the asymptotic behavior of the first and second term is subtle. For the first term, we have $\|u(t)-S_\alpha(t)\|_{\dot{H}^{p,j}} = O(t^{-(1-1/p)/\alpha-j/\alpha-1/\alpha})$ (when $\alpha<1$), whereas for the second one, say $w(t)$, we have $\|u(t)-S_\alpha(t)-w(t)\|_{\dot{H}^{p,j}} = o(t^{-(1-1/p)/\alpha-j/\alpha-1/\alpha})$. The following result holds for $\alpha \le 1$.

Theorem 2.4. *Suppose $p\in[2,\infty]$, $j\in\mathbb{N}$, $u_0 \in H^{j+3}(\mathbb{R})\cap L^1(\mathbb{R})$ and that (2.4)-(2.5) are verified.*

(i) If $0<\alpha<1$, then

$$t^{((1-1/p)/\alpha+j/\alpha)+1/\alpha}\left\|u(t)-S_\alpha(t)*u_0+\frac{1}{2}\Big(\int_0^\infty\int_{-\infty}^\infty u^2(s,y)dyds\Big)\partial_x G_\alpha(t)\right\|_{\dot{H}^{p,j}} \to 0 \tag{2.7}$$

as $t\to\infty$.

(ii) If $\alpha=1$, then

$$\frac{t^{(1-1/p)+j+1}}{\log t}\left\|u(t)-S_1(t)*u_0+\frac{M^2}{4\pi}(\log t)\partial_x G_1(t)\right\|_{\dot{H}^{p,j}} \to 0 \tag{2.8}$$

where $M=\mathcal{M}_0(u_0)=\int_{-\infty}^\infty u_0$.

Remark 2.3. *In the case $\alpha<1$, the integral $\int_0^\infty\int_{-\infty}^\infty u^2(s,y)dyds$ which appears in (2.7) is convergent due to Theorem 2.2 :*

$$\int_0^\infty\int_{-\infty}^\infty u^2(s,y)dyds = \int_0^\infty \|u(s)\|_{L^2}^2 ds \le c\int_0^\infty (1+s)^{-1/\alpha}ds < \infty.$$

Now we deal with the case $1<\alpha<2$. In this situation we get an asymptotic expansion of the solution at the rate $O(t^{-(1-1/p)/\alpha-j/\alpha-1/\alpha})$ (in

$\dot{H}^{p,j}$-norm, and for almost every α) but we need more than two terms in this expansion to derive it. The main idea is to use the successive terms $F^n(t)$ which appear in the Picard iterative scheme applied to the Duhamel formulation (1.6), i.e.

$$\begin{cases} F^0(t) = S_\alpha(t) * u_0, \\ F^{n+1}(t) = S_\alpha(t) * u_0 - \frac{1}{2}\int_0^t S_\alpha(t-s) * \partial_x (F^n(s))^2 ds. \end{cases}$$

Theorem 2.5. *Let $1 < \alpha < 2$, $p \in [2, \infty]$, $j \in \mathbb{N}$ and $u_0 \in H^{j+3}(\mathbb{R}) \cap L^1(\mathbb{R})$. Suppose that conditions (2.4) and (2.5) are satisfied.*

(i) If $\frac{2N+1}{N+1} < \alpha < \frac{2N+3}{N+2}$ for a $N \in \mathbb{N}$, then

$$\|u(t) - F^{N+1}(t)\|_{\dot{H}^{p,j}} \leq c(1+t)^{-(1-1/p)/\alpha - j/\alpha - 1/\alpha}.$$

(ii) If $\alpha = \frac{2N+3}{N+2}$ for a $N \in \mathbb{N}$, then

$$\|u(t) - F^{N+1}(t)\|_{\dot{H}^{p,j}} \leq c(1+t)^{-(1-1/p)/\alpha - j/\alpha - 1/\alpha} \log(1+t).$$

Remark 2.4. *The results obtained in this paper for (dKdV) could be certainly adapted to more general dispersive dissipative equations taking the form*

$$u_t - |D|^r \partial_x u + |D|^\alpha u + \partial_x f(u) = 0, \tag{2.9}$$

where f is sufficiently smooth function behaving like $u|u|^{q-1}$ at the origin. Such general models were studied by Dix in [6]. Similar asymptotic expansion for solutions to (2.9) could be obtained in certain cases, when dissipation is not negligible in comparison with dispersion and nonlinearity :

$$\begin{cases} \alpha \leq r + 1, \\ 0 < \alpha < q. \end{cases}$$

The remainder of this paper is organized as follows. In Section 3, we derive linear estimates and prove Theorem 2.1. Uniform estimates of the nonlinear solution are obtained in Section 4. The decay rate (2.6) is established in Section 5. Finally, Section 6 is devoted to the proof of Theorems 2.3, 2.4 and 2.5.

3 Linear estimates

In this section, we prove some estimates related with $S_\alpha(t)$ and $G_\alpha(t)$. Our first lemma is a direct consequence of the self-similarity of G_α.

Lemma 3.1. *For any $p \in [1, \infty]$ and $j \in \mathbb{N}$,*

$$\|G_\alpha(t)\|_{\dot{H}^{p,j}} = ct^{-(1-1/p)/\alpha - j/\alpha}. \tag{3.1}$$

Proof. Equality (1.7) and a change of variables yield

$$\begin{aligned} \|G_\alpha(t)\|_{\dot{H}^{p,j}} &= \Big(\int_{-\infty}^{\infty} t^{-(j+1)p/\alpha} |\partial_x^j G_\alpha(1, xt^{-1/\alpha})|^p dx \Big)^{1/p} \\ &= t^{-(j+1)/\alpha} t^{1/\alpha p} \Big(\int_{-\infty}^{\infty} |\partial_x^j G_\alpha(1, y)|^p dy \Big)^{1/p}. \end{aligned}$$

The case $p = \infty$ is straightforward. □

Let us recall the following elementary result which will be extensively used in our future considerations. A proof of (3.3) can be found in [8].

Lemma 3.2. *If $1 \le k \le j$ and $f \in H^j(\mathbb{R})$, then*

$$\|f\|_{L^\infty}^2 \le \|f\|_{L^2} \|f_x\|_{L^2}, \quad \text{and} \quad \|\partial_x^k f\|_{L^2} \le \|f\|_{L^2}^{1-k/j} \|\partial_x^j f\|_{L^2}^{k/j}. \tag{3.2}$$

Moreover, for any $f \in L^2((1 + |x|)dx)$, one has

$$\|f\|_{L^1}^2 \le c\|f\|_{L^2} \|\partial_\xi \widehat{f}\|_{L^2}. \tag{3.3}$$

Next lemma describes the asymptotic behavior of $S_\alpha(t)$.

Lemma 3.3. *For any $p \in [1, \infty]$ and $j, N \in \mathbb{N}$,*

$$\Big\| S_\alpha(t) - \sum_{n=0}^{N} \frac{t^n}{n!} (-\partial_x)^{3n} G_\alpha(t) \Big\|_{\dot{H}^{p,j}} \le ct^{-(1-1/p)/\alpha - j/\alpha - (3/\alpha - 1)(N+1)}. \tag{3.4}$$

Proof. Setting $A(t) = S_\alpha(t) - \sum_{n=0}^{N} \frac{t^n}{n!} (-\partial_x)^{3n} G_\alpha(t)$, we obtain

$$\mathcal{F}(\partial_x^j A(t))(\xi) = (i\xi)^j e^{-t|\xi|^\alpha} \Big(e^{it\xi^3} - \sum_{n=0}^{N} \frac{t^n}{n!} (-i\xi)^{3n} \Big).$$

Using the Taylor expansion of the exponential function, we have

$$\Big| e^{it\xi^3} - \sum_{n=0}^{N} \frac{(it\xi^3)^n}{n!} \Big| \le \frac{(t|\xi|^3)^{N+1}}{(N+1)!}.$$

Thus, Plancherel theorem and the change of variables $\xi = t^{-1/\alpha}\eta$ give

$$\begin{aligned}
\|\partial_x^j A(t)\|_{L^2}^2 &\le c\int_{-\infty}^{\infty} |\xi|^{2j} e^{-2t|\xi|^\alpha} (t|\xi|^3)^{2(N+1)} d\xi \\
&= ct^{2(N+1)} \int_{-\infty}^{\infty} |\xi|^{2(j+3N+3)} e^{-2t|\xi|^\alpha} d\xi \\
&= ct^{-1/\alpha-2j/\alpha-2(3/\alpha-1)(N+1)},
\end{aligned}$$

which yields the result for $p = 2$. Now the case $p = \infty$ follows immediately from (3.2). When $p = 1$, we use estimate (3.3). One has

$$\begin{aligned}
\|\partial_\xi \mathcal{F}(\partial_x^j A(t))\|_{L^2} \le c\Big(\int_{-\infty}^{\infty} \Big[&|j\xi^{j-1}(t\xi^3)^{N+1}|^2 + |t\xi^{j+\alpha-1}(t\xi^3)^{N+1}|^2 \\
&+ |\xi|^{2j}\Big|3it\xi^2 e^{it\xi^3} - \sum_{n=0}^{N} \frac{3n(it)^n \xi^{3n-1}}{n!}\Big|^2\Big] e^{-2t|\xi|^\alpha} d\xi\Big)^{1/2} \\
\le\ & ct^{N+1}\Big(\int_{-\infty}^{\infty} j|\xi|^{2(j-1+3(N+1))} e^{-2t|\xi|^\alpha} d\xi\Big)^{1/2} \\
&+ ct^{N+2}\Big(\int_{-\infty}^{\infty} |\xi|^{2(j+\alpha-1+3(N+1))} e^{-2t|\xi|^\alpha} d\xi\Big)^{1/2} \\
&+ ct\Big(\int_{-\infty}^{\infty} |\xi|^{2(j+2)} |t\xi^3|^{2N} e^{-2t|\xi|^\alpha} d\xi\Big)^{1/2} \\
\le\ & ct^{-1/2\alpha-j/\alpha-(3/\alpha-1)(N+1)+1/\alpha}.
\end{aligned}$$

It follows that (3.4) holds true for $p = 1$ and then for all $p \in [1, \infty]$ by interpolation. □

Lemma 3.4. *For all* $p \in [2, \infty]$ *and* $j \in \mathbb{N}$,

$$\|S_\alpha(t)\|_{\dot{H}^{p,j}} \le ct^{-(1-1/p)/\alpha-j/\alpha} \tag{3.5}$$

and

$$\|S_\alpha(t)\|_{\dot{H}^{1,j}} \le ct^{-j/\alpha}(1 + t^{1-3/\alpha}).$$

Proof. For $p = 2$, $\|S_\alpha(t)\|_{\dot{H}^j} = \|G_\alpha(t)\|_{\dot{H}^j} = ct^{-1/2\alpha-j/\alpha}$. Then (3.5) follows by the first inequality in (3.2) and by interpolation. Concerning the L^1-norm, (3.4) with $N = 0$ and (3.1) provide

$$\|S_\alpha(t)\|_{\dot{H}^{1,j}} \le \|S_\alpha(t) - G_\alpha(t)\|_{\dot{H}^{1,j}} + \|G_\alpha(t)\|_{\dot{H}^{1,j}} \le ct^{-j/\alpha}(1 + t^{1-3/\alpha}).$$

□

Now we state a decomposition lemma for convolution products.

Lemma 3.5. *Let* $p \in [1, \infty]$ *and* $N \in \mathbb{N}$. *For any* $h \in L^1((1+|x|)^{N+1}dx)$ *and* $g \in C^{N+1}(\mathbb{R}) \cap H^{p,N+1}(\mathbb{R})$,

$$\Big\| g * h - \sum_{n=0}^{N} \frac{(-1)^n}{n!} \mathcal{M}_n(h) \partial_x^n g \Big\|_{L^p} \leq c \|\partial_x^{N+1} g\|_{L^p} \|h\|_{L^1(|x|^{N+1}dx)}.$$

Proof. It is an easy consequence of the Taylor formula as well as Young inequality. ☐

Applying Lemma 3.5 with $g = \partial_x^j G_\alpha(t)$ and using estimate (3.1), we deduce the

Corollary 3.1. *If* $p \in [1, \infty]$ *and* $j, N \in \mathbb{N}$, *then*

$$\Big\| G_\alpha(t) * h - \sum_{n=0}^{N} \frac{(-1)^n}{n!} \mathcal{M}_n(h) \partial_x^n G_\alpha(t) \Big\|_{\dot{H}^{p,j}} \leq c t^{-(1-1/p)/\alpha - j/\alpha - (N+1)/\alpha} \|h\|_{L^1(|x|^{N+1}dx)}$$

for any $h \in L^1((1+|x|)^{N+1}dx)$.

We are now in a position to prove Theorem 2.1.

Proof of Theorem 2.1. By the triangle inequality,

$$\begin{aligned}
&\Big\| S_\alpha(t) * u_0 - \sum_{n=0}^{N} \frac{(-1)^n}{n!} \mathcal{M}_n(u_0) \partial_x^n G_\alpha(t) - \sum_{k=1}^{N} \frac{t^k}{k!} \sum_{\ell=0}^{N-1} \frac{(-1)^\ell}{\ell!} \mathcal{M}_\ell(u_0) \partial_x^\ell (-\partial_x)^{3k} G_\alpha(t) \Big\|_{\dot{H}^{p,j}} \\
&\leq \Big\| S_\alpha(t) * u_0 - G_\alpha(t) * u_0 - \sum_{k=1}^{N} \frac{t^k}{k!} (-\partial_x)^{3k} G_\alpha(t) * u_0 \Big\|_{\dot{H}^{p,j}} \\
&\quad + \Big\| G_\alpha(t) * u_0 - \sum_{n=0}^{N} \frac{(-1)^n}{n!} \mathcal{M}_n(u_0) \partial_x^n G_\alpha(t) \Big\|_{\dot{H}^{p,j}} \\
&\quad + \sum_{k=1}^{N} \frac{t^k}{k!} \Big\| (-\partial_x)^{3k} G_\alpha(t) * u_0 - \sum_{\ell=0}^{N-1} \frac{(-1)^\ell}{\ell!} \mathcal{M}_\ell(u_0) \partial_x^\ell (-\partial_x)^{3k} G_\alpha(t) \Big\|_{\dot{H}^{p,j}} \\
&:= I + II + III.
\end{aligned}$$

I is estimated with the help of (3.4),

$$
\begin{aligned}
I &= \left\| \partial_x^j \Big(S_\alpha(t) - \sum_{k=0}^{N} \frac{t^k}{k!} (-\partial_x)^{3k} G_\alpha(t) \Big) * u_0 \right\|_{L^p} \\
&\leq \left\| \partial_x^j \Big(S_\alpha(t) - \sum_{k=0}^{N} \frac{t^k}{k!} (-\partial_x)^{3k} G_\alpha(t) \Big) \right\|_{L^p} \|u_0\|_{L^1} \\
&\leq ct^{-(1-1/p)/\alpha - j/\alpha - (3/\alpha - 1)(N+1)} \leq ct^{-(1-1/p)/\alpha - j/\alpha - (N+1)/\alpha},
\end{aligned}
$$

since $\alpha < 2$. Concerning II, we use Corollary 3.1 as follows :

$$
II \leq ct^{-(1-1/p)/\alpha - j/\alpha - (N+1)/\alpha} \|u_0\|_{L^1(|x|^{N+1}dx)}.
$$

Finally for the term III, Corollary 3.1 allows us to conclude

$$
\begin{aligned}
III &\leq \sum_{k=1}^{N} \frac{t^k}{k!} \left\| \partial_x^{3k+j} \Big(G_\alpha(t) * u_0 - \sum_{\ell=0}^{N-1} \frac{(-1)^\ell}{\ell!} \mathcal{M}_\ell(u_0) \partial_x^\ell G_\alpha(t) \Big) \right\|_{L^p} \\
&\leq ct^{-(1-1/p)/\alpha - j/\alpha - N/\alpha} \sum_{k=1}^{N} t^{(1-3/\alpha)k} \\
&\leq ct^{-(1-1/p)/\alpha - j/\alpha - N/\alpha + 1 - 3/\alpha} \\
&\leq ct^{-(1-1/p)/\alpha - j/\alpha - (N+1)/\alpha}.
\end{aligned}
$$

□

4 Uniform estimates of solutions to (dKdV)

We begin by the proof of Theorem 2.2 in the case $j = 0$ and $p = 1$.

Lemma 4.1. *Let $u_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and u be a solution of (dKdV) satisfying (2.4). Then for all $t > 0$,*

$$
\|u(t)\|_{L^1} \leq \|u_0\|_{L^1}.
$$

Proof. Multiply (dKdV) by $\operatorname{sgn} u$ and then integrate over $\mathbb{R}$:

$$
\partial_t \|u(t)\|_{L^1} = -\int_{-\infty}^{\infty} (u_{xxx} + |D|^\alpha u + uu_x) \operatorname{sgn} u. \tag{4.1}
$$

We are going to show that for each $t > 0$, the right-hand side of (4.1) is negative. Note that assumption (2.4) means that for each $t > 0$, there exists $c = c(t)$ such that

$$
\forall j \geq 0, \ \|\partial_x^j u(t)\|_{L^2} \leq c. \tag{4.2}
$$

Since $-|D|^\alpha$ is the generator of contraction semigroup in $L^1(\mathbb{R})$, for each $u \in \mathcal{D}(-|D|^\alpha)$ (the domain of $-|D|^\alpha$),

$$\begin{aligned}
-\int_{-\infty}^{\infty} |D|^\alpha u \operatorname{sgn} u &= \lim_{s\to 0}\int_{-\infty}^{\infty} \frac{e^{-s|D|^\alpha}u - u}{s}\operatorname{sgn} u \\
&= \lim_{s\to 0}\frac{1}{s}\int_{-\infty}^{\infty}\left(e^{-s|D|^\alpha}u\operatorname{sgn} u - |u|\right) \\
&\le \limsup_{s\to 0}\frac{1}{s}\Big(\int_{-\infty}^{\infty}|e^{-s|D|^\alpha}u| - \int_{-\infty}^{\infty}|u|\Big) \\
&\le 0.
\end{aligned}$$

This last inequality is sometimes called Kato inequality, see [2]-[3]. To show that the other terms in the right-hand side of (4.1) are also negative, we introduce the following smooth regularization of the sgn function

$$\operatorname{sgn}_\eta(\xi) = \begin{cases} 1 & \text{if} \quad \xi > \eta\pi/2, \\ \sin(\xi/\eta) & \text{if} \quad |\xi| \le \eta\pi/2, \\ -1 & \text{if} \quad \xi < -\eta\pi/2. \end{cases}$$

Then, an integration by parts gives

$$-\int_{-\infty}^{\infty} uu_x \operatorname{sgn} u = -\lim_{\eta\to 0}\int_{-\infty}^{\infty} uu_x \operatorname{sgn}_\eta u = \frac{1}{2}\lim_{\eta\to 0}\int_{-\infty}^{\infty} u^2u_x \operatorname{sgn}_\eta' u.$$

On the other hand, sgn_η' has its support in $[-\eta\pi/2, \eta\pi/2]$ and $|\operatorname{sgn}_\eta'| \le 1/\eta$, hence setting $M_\eta = \{x : |u| < \eta\pi/2, u_x \ne 0\}$, one has $\operatorname{mes}(M_\eta) \to 0$ (mes denotes the Lebesgue measure) and

$$\Big|\int_{-\infty}^{\infty} u^2u_x . \operatorname{sgn}_\eta' u\Big| \le \frac{1}{\eta}\int_{M_\eta}|u^2u_x| \le c\eta\|u_x\|_{L^2}\Big(\int_{M_\eta}\Big)^{1/2} \to 0$$

as $\eta \to 0$ by (4.2). Thus $\int_{-\infty}^{\infty} uu_x \operatorname{sgn} u = 0$. We proceed similarly for the last term,

$$-\int_{-\infty}^{\infty} u_{xxx}\operatorname{sgn} u = -\lim_{\eta\to 0}\int_{-\infty}^{\infty} u_{xxx}\operatorname{sgn}_\eta u = \lim_{\eta\to 0}\int_{-\infty}^{\infty} u_{xx}u_x \operatorname{sgn}_\eta' u$$

and

$$\Big|\int_{-\infty}^{\infty} u_{xx}u_x \operatorname{sgn}_\eta' u\Big| \le \frac{1}{\eta}\int_{M_\eta}|u_{xx}u_x|.$$

Now we define $\widetilde{u}$ by setting $\widetilde{u} = u$ on M_η and $\widetilde{u} = 0$ elsewhere. Then by Cauchy-Schwartz,

$$\Big|\int_{-\infty}^{\infty} u_{xx}u_x \operatorname{sgn}_\eta' u\Big| \le \frac{1}{\eta}\int_{-\infty}^{\infty}|\widetilde{u}_{xx}\widetilde{u}_x| \le \frac{1}{\eta}\|\widetilde{u}_{xx}\|_{L^2}\|\widetilde{u}_x\|_{L^2}.$$

The second estimate in (3.2) and (4.2) yield

$$\|\widetilde{u}_x\|_{L^2} \leq \|\widetilde{u}\|_{L^2}^{1/2}\|\widetilde{u}_{xx}\|_{L^2}^{1/2} = \Big(\int_{M_\eta} |u|^2\Big)^{1/4}\Big(\int_{M_\eta} |u_{xx}|^2\Big)^{1/4} \leq c\eta^{1/2}\,\mathrm{mes}(M_\eta)^{1/2}$$

and

$$\|\widetilde{u}_{xx}\|_{L^2} \leq \|\widetilde{u}\|_{L^2}^{1/2}\|\widetilde{u}_{xxxx}\|_{L^2}^{1/2} = \Big(\int_{M_\eta} |u|^2\Big)^{1/4}\Big(\int_{M_\eta} |u_{xxxx}|^2\Big)^{1/4} \leq c\eta^{1/2}\,\mathrm{mes}(M_\eta)^{1/2}.$$

Gathering these two last estimates we infer

$$\Big|\int_{-\infty}^{\infty} u_{xx}u_x\,\mathrm{sgn}_\eta'\,u\Big| \leq c\,\mathrm{mes}(M_\eta) \to 0$$

and so $\int_{-\infty}^{\infty} u_{xxx}\,\mathrm{sgn}\,u = 0$. Finally

$$\partial_t\|u(t)\|_{L^1} \leq 0,$$

which complete the proof of Proposition 4.1. $\square$

Corollary 4.1. *Let $u_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and u be a solution of (dKdV) satisfying (2.4). Then,*

$$\forall t > 0, \quad \|u(t)\|_{L^2} \leq c(1+t)^{-1/2\alpha}. \tag{4.3}$$

Proof. If we multiply (dKdV) by u and then integrate the result over $\mathbb{R}$,

$$\partial_t\|u(t)\|_{L^2}^2 = -2\||D|^{\alpha/2}u\|_{L^2}^2 \leq 0.$$

In particular, $\|u(t)\|_{L^2} \leq \|u_0\|_{L^2}$. For all $t > 0$, last equality allow us to write

$$\begin{aligned}
\partial_t\Big[t^{2/\alpha}\|u(t)\|_{L^2}^2\Big] &= \frac{2}{\alpha}t^{2/\alpha-1}\|u(t)\|_{L^2}^2 + t^{2/\alpha}\partial_t\|u(t)\|_{L^2}^2 \\
&= \frac{2}{\alpha}t^{2/\alpha-1}\|u(t)\|_{L^2}^2 - 2t^{2/\alpha}\int_{-\infty}^{\infty}|\xi|^\alpha|\hat{u}(t,\xi)|^2 d\xi \\
&\leq \frac{2}{\alpha}t^{2/\alpha-1}\int_{-\infty}^{\infty}|\hat{u}(t,\xi)|^2 d\xi - 2t^{2/\alpha}\int_{|\xi|>(\alpha t)^{-1/\alpha}}|\xi|^\alpha|\hat{u}(t,\xi)|^2 d\xi \\
&\leq \frac{2}{\alpha}t^{2/\alpha-1}\int_{-\infty}^{\infty}|\hat{u}(t,\xi)|^2 d\xi - \frac{2}{\alpha}t^{2/\alpha-1}\int_{|\xi|>(\alpha t)^{-1/\alpha}}|\hat{u}(t,\xi)|^2 d\xi \\
&= \frac{2}{\alpha}t^{2/\alpha-1}\int_{|\xi|<(\alpha t)^{-1/\alpha}}|\hat{u}(t,\xi)|^2 d\xi \\
&\leq ct^{2/\alpha-1}\|u(t)\|_{L^1}^2\,\mathrm{mes}\{|\xi| < (\alpha t)^{-1/\alpha}\} \\
&\leq ct^{1/\alpha-1}.
\end{aligned}$$

The integration of this inequality over $[0,t]$ provides the desired result. $\square$

Now we show that if $\alpha \geq 1$, solutions of (dKdV) satisfy the maximum principle. The restriction on α is mainly due to the fact that one has $|D|^\alpha 1 = 0$ only if $\alpha \geq 1$.

Lemma 4.2. *If u is a solution to (dKdV) with $\alpha \geq 1$ associated with initial data $u_0 \in L^\infty(\mathbb{R})$, then*

$$\inf u_0 \leq u(t,x) \leq \sup u_0 \tag{4.4}$$

for a.e. $(t,x) \in [0,\infty[\times\mathbb{R}$.

Proof. Let $m = \inf u_0$, $M = \sup u_0$ and $u^+ = \max(0, u - M - \varepsilon)$, $u^- = \min(0, u + m + \varepsilon)$ for some $\varepsilon > 0$. We multiply (dKdV) by u^+ and integrate over $\mathbb{R}$ to get

$$\int_{-\infty}^{\infty} (u_t + u_{xxx} + |D|^\alpha u + uu_x)u^+ = 0. \tag{4.5}$$

On the support of u^+, it is clear that $u_t = u_t^+$, $u_x = u_x^+$ and $|D|^\alpha u = |D|^\alpha u^+$, this last equality follows from the relation $|D|^\alpha 1 = 0$ for $\alpha \geq 1$. We deduce $\int_{-\infty}^{\infty} u_t u^+ = \frac{1}{2}\partial_t \|u^+(t)\|_{L^2}^2$, $\int_{-\infty}^{\infty} u_{xxx}u^+ = \int_{-\infty}^{\infty} u_{xxx}^+ u^+ = 0$ and $\int_{-\infty}^{\infty} |D|^\alpha u u^+ = \int_{-\infty}^{\infty} |D|^\alpha u^+ u^+ = \||D|^{\alpha/2}u^+\|_{L^2}^2$ by Plancherel. On the other hand, one has $\int_{-\infty}^{\infty} uu_x u^+ = \int_{-\infty}^{\infty}(u^+ + M + \varepsilon)u_x^+ u^+ = 0$. Inserting this into (4.5) and integrating over $[0,t]$ we get

$$\|u^+(t)\|_{L^2}^2 + 2\int_0^t \||D|^{\alpha/2}u^+(s)\|_{L^2}^2 ds = \|u^+(0)\|_{L^2}^2 = 0$$

and thus $u^+(t) = 0$ a.e.. Consequently, we have $u(t) \leq M + \varepsilon$ for all $\varepsilon > 0$, and the second part of (4.4) is proved. The same arguments hold with u^+ replaced by u^- and give the first inequality. $\square$

Following [10], we introduce for $\lambda > 1$ the following rescaled solution

$$u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x).$$

Obviously, u_λ satisfies the equation

$$\partial_t u_\lambda + \lambda^{-1}\partial_{xxx}u_\lambda + \lambda^{2-\alpha}|D|^\alpha u_\lambda + u_\lambda \partial_x u_\lambda = 0$$

with initial data $u_{0,\lambda}(x) = \lambda u_0(\lambda x)$.

Lemma 4.3. *Let $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and u be a solution of (dKdV) with $1 < \alpha < 2$ satisfying (2.4). For $j \geq 0$, $T > 0$ and $0 < t < T$, there exists $c = c(t,T)$ such that for all $\lambda > 1$, one has $\|\partial_x^j u_\lambda(t)\|_{L^2} \leq c$.*

Proof. The method of proof is based on an induction on j. If $j = 0$, one easily deduce from Corollary 4.1 and Lemma 4.2 that $\|u(t)\|_{L^p} \leq ct^{-1/\alpha p}$ for $2 \leq p \leq \infty$ and thus

$$\|u_\lambda(t)\|_{L^p} \leq c\lambda^{1-(1+2/\alpha)/p} t^{-1/\alpha p}. \tag{4.6}$$

In particular for $p = 2$ and $\lambda > 1$, $\|u_\lambda(t)\|_{L^2} \leq c(t)$. Suppose now that the result is true for all $k < j$. Consider $S^\lambda_\alpha(t)$ (resp. $G^\lambda_\alpha(t)$), the semigroup generated by $\lambda^{-1}\partial_{xxx} + \lambda^{2-\alpha}|D|^\alpha$ (resp. $\lambda^{2-\alpha}|D|^\alpha$) so that we have for $0 < t, t' < T$

$$u_\lambda(t+t') = S^\lambda_\alpha(t) * u_\lambda(t') - \frac{1}{2}\int_0^t S^\lambda_\alpha(t-s) * \partial_x u^2_\lambda(s+t')ds. \tag{4.7}$$

It is worth noticing that $\|S^\lambda_\alpha(t) * f\|_{L^2} = \|G^\lambda_\alpha(t) * f\|_{L^2}$ and

$$\|\partial^j_x G^\lambda_\alpha(t)\|_{L^p} = \|\partial^j_x G_\alpha(\lambda^{2-\alpha}t)\|_{L^p} \tag{4.8}$$

for all $1 \leq p \leq \infty$. Application of ∂^j_x to (4.7) and computing the L^2-norm lead to

$$\begin{aligned} \|\partial^j_x u_\lambda(t+t')\|_{L^2} &\leq \|\partial^j_x G^\lambda_\alpha(t) * u_\lambda(t')\|_{L^2} \\ &\quad + c\sum_{k=0}^{j}\int_0^t \|\partial_x G^\lambda_\alpha(t-s) * \partial^k_x u_\lambda(s+t')\partial^{j-k}_x u_\lambda(s+t')\|_{L^2}ds. \end{aligned} \tag{4.9}$$

By the inductive hypothesis, the first term in the right-hand side of (4.9) is bounded by

$$\|\partial_x G^\lambda_\alpha(t) * \partial^{j-1}_x u_\lambda(t')\|_{L^2} \leq ct^{-1/\alpha}\|\partial^{j-1}_x u_\lambda(t')\|_{L^2} \leq c(t')t^{-1/\alpha}.$$

By symmetry, it is sufficient in the sum $\sum_{k=0}^{j}$ in (4.9) to consider the indexes $k = 0, ...E(j/2)$. The case $k = 0$ is a special case and has to be treated separately. Using Young and Hölder inequalities and next estimates (4.8) and (4.6), we obtain

$$\begin{aligned} &\|\partial_x G^\lambda_\alpha(t-s) * u_\lambda(s+t')\partial^j_x u_\lambda(s+t')\|_{L^2} \\ &\quad \leq \|\partial_x G^\lambda_\alpha(t-s)\|_{L^{2/(3-\alpha)}}\|u_\lambda(s+t')\|_{L^{2/(\alpha-1)}}\|\partial^j_x u_\lambda(s+t')\|_{L^2} \\ &\quad \leq [\lambda^{2-\alpha}(t-s)]^{-(\alpha+1)/2\alpha}\lambda^{1-(\alpha-1)(1+2/\alpha)/2}(s+t')^{(1-\alpha)/2\alpha}\|\partial^j_x u_\lambda(s+t')\|_{L^2} \\ &\quad \leq c(s+t')(t-s)^{-(\alpha+1)/2\alpha}\|\partial^j_x u_\lambda(s+t')\|_{L^2} \end{aligned} \tag{4.10}$$

since $-(2-\alpha)(\alpha+1)/2\alpha+1-(\alpha-1)(1+2/\alpha)/2=0$. When $k\geq 1$, we use the inductive hypothesis combined with (3.2) to get

$$\begin{aligned}&\|\partial_x G_\alpha^\lambda(t-s)*\partial_x^k u_\lambda(s+t')\partial_x^{j-k}u_\lambda(s+t')\|_{L^2}\\&\quad\leq \|\partial_x G_\alpha^\lambda(t-s)\|_{L^1}\|\partial_x^k u_\lambda(s+t')\|_{L^\infty}\|\partial_x^{j-k}u_\lambda(s+t')\|_{L^2}\\&\quad\leq c(s+t')(t-s)^{-1/\alpha}. \qquad (4.11)\end{aligned}$$

Bounding $c(s+t')$ in (4.10)-(4.11) by $\sup_{0\leq s\leq T} c(s+t')$ and inserting these inequalities into (4.9) let us conclude that

$$\begin{aligned}\|\partial_x^j u_\lambda(t+t')\|_{L^2}&\leq c(t')t^{-1/\alpha}+c(t',T)\\&\qquad+c(t')\int_0^t (t-s)^{-(\alpha+1)/2\alpha}\|\partial_x^j u_\lambda(s+t')\|_{L^2}ds.\end{aligned}$$

This implies by the generalized Gronwall lemma [5] that for $t'=t$,

$$\|\partial_x^j u_\lambda(2t)\|_{L^2}\leq c(t,T)$$

where $c(t,T)$ is independent of $\lambda>1$. □

As noticed in Section 2, these uniform estimates (in λ) imply uniform estimates in time of the solution.

Corollary 4.2. *Let $u_0\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})\cap H^j(\mathbb{R})$ for some $j\geq 0$. Assume that u is a solution of (dKdV) with $1<\alpha<2$ satisfying (2.4). Then assumption (2.5) is satisfied, i.e.*

$$\sup_{t\geq 0}\|\partial_x^j u(t)\|_{L^2}<\infty.$$

Proof. First since $u_0\in H^j(\mathbb{R})$, we have $u\in C([0,\infty[;H^j(\mathbb{R}))$ and thus $\sup_{0\leq t\leq 1}\|\partial_x^j u(t)\|_{L^2}<\infty$. On the other hand, one easily verifies that $\|\partial_x^j u_\lambda(t)\|_{L^2}=\lambda^{j+1/2}\|\partial_x^j u(\lambda^2 t)\|_{L^2}$. Taking $t=1$ and $\lambda=t^{1/2}$ in this equality we deduce

$$t^{j/2+1/4}\|\partial_x^j u(t)\|_{L^2}=\|\partial_x^j u_\lambda(1)\|_{L^2}\leq c$$

by Lemma 4.3. This implies for $t\geq 1$ that $\|\partial_x^j u(t)\|_{L^2}\leq c$ as desired. □

5 Decay of solutions to (dKdV)

In this section we prove Theorem 2.2 which has already been shown in the special cases $(p,j)=(1,0)$ and $(p,j)=(2,0)$ in the previous section.

Lemma 5.1. *Let $u_0 \in H^j(\mathbb{R})\cap L^1(\mathbb{R})$ and u be a solution satisfying (2.4)-(2.5). Then, for all $t>1$ and $N\geq 1$,*

$$\Big\|\int_0^t S_\alpha(t-s)*\partial_x u^2(s)ds\Big\|_{\dot{H}^j} \leq ct^{-1/2\alpha-j/\alpha}+ct^{-\gamma}\sup_{t/2\leq s\leq t}\|\partial_x^j u(s)\|_{L^2}^{1-1/N}+t^{-\gamma}\sup_{t/2\leq s\leq t}\|\partial_x^j u(s)\|_{L^2} \quad (5.1)$$

with $\gamma=\gamma(\alpha)>0$.

Corollary 5.1. *If $u_0 \in H^j(\mathbb{R})\cap L^1(\mathbb{R})$ and if (2.4)-(2.5) hold true,*

$$\|u(t)\|_{\dot{H}^j}\leq c(1+t)^{-1/2\alpha-j/\alpha}$$

for any $t>0$.

Proof of Lemma 5.1. One proceeds by induction on j. For $j=0$ we use the integral formulation (1.6) and estimates (3.5) and (4.3) :

$$\Big\|\int_0^t S_\alpha(t-s)*\partial_x u^2(s)ds\Big\|_{L^2}=2\|u(t)-S_\alpha(t)*u_0\|_{L^2} \leq 2\|u(t)\|_{L^2}+2\|S_\alpha(t)\|_{L^2}\|u_0\|_{L^1}\leq ct^{-1/2\alpha}.$$

Now assume the statement (and thus Corollary 5.1) is true for the $k<j$. We split the left-hand side of (5.1) into

$$\Big\|\int_0^t S_\alpha(t-s)*\partial_x u^2(s)ds\Big\|_{\dot{H}^j}\leq \int_0^{t/2}\dots ds+\int_{t/2}^t \dots ds := I+II.$$

By the Young inequality and estimates (3.5), (4.3), we have

$$\begin{aligned} I &\leq \int_0^{t/2}\|\partial_x^{j+1}S_\alpha(t-s)\|_{L^2}\|u(s)\|_{L^2}^2 ds\\ &\leq c\int_0^{t/2}(t-s)^{-1/2\alpha-(j+1)/\alpha}(1+s)^{-1/\alpha}ds\\ &\leq ct^{-1/2\alpha-j/\alpha}\Big(t^{-1/\alpha}\int_0^t(1+s)^{-1/\alpha}ds\Big)\end{aligned}$$

and for $t > 1$,

$$t^{-1/\alpha}\int_0^t (1+s)^{-1/\alpha}ds \le c\begin{cases} t^{-1/\alpha} & \text{if} \quad \alpha < 1 \\ t^{-1}\log t & \text{if} \quad \alpha = 1 \\ t^{1-2/\alpha} & \text{if} \quad \alpha > 1 \end{cases} \le c.$$

To estimate II, we use Plancherel and we split low and high frequencies,

$$\begin{aligned} II &= c\int_{t/2}^t \Big(\int_{-\infty}^{\infty} e^{-2(t-s)|\xi|^\alpha}|\xi|^{2(j+1)}|\widehat{u^2}(s,\xi)|^2 d\xi\Big)^{1/2} ds \\ &\le c\int_{t/2}^t \Big(\int_{|\xi|<1} \dots d\xi\Big)^{1/2} ds + c\int_{t/2}^t \Big(\int_{|\xi|>1} \dots d\xi\Big)^{1/2} ds := II_1 + II_2. \end{aligned}$$

If $|\xi| < 1$, then $e^{-2|\xi|^\alpha} \ge e^{-2}$, hence

$$\begin{aligned} II_1 &\le c\int_{t/2}^t \Big(\int_{-\infty}^{\infty} e^{-2(1+t-s)|\xi|^\alpha}|\xi|^{2(j+1)}|\widehat{u^2}(s,\xi)|^2 d\xi\Big)^{1/2} \\ &= c\int_{t/2}^t \|\partial_x S_\alpha(1+t-s) * \partial_x^j u^2(s)\|_{L^2} ds \\ &\le c\int_{t/2}^t \|\partial_x S_\alpha(1+t-s)\|_{L^2}\|\partial_x^j u^2(s)\|_{L^1} ds \\ &\le c\int_{t/2}^t (1+t-s)^{-3/2\alpha}\sum_{k=0}^j \|\partial_x^k u(s)\|_{L^2}\|\partial_x^{j-k}u(s)\|_{L^2} ds. \end{aligned}$$

Corollary 5.1 with $k < j$ implies that

$$\begin{aligned} \sum_{k=0}^j \|\partial_x^k u(s)\|_{L^2}\|\partial_x^{j-k}u(s)\|_{L^2} &\le c\sum_{k=1}^{j-1}(1+s)^{-1/2\alpha-k/\alpha}(1+s)^{-1/2\alpha-(j-k)/\alpha} \\ &\quad + c\|u(s)\|_{L^2}\|\partial_x^j u(s)\|_{L^2} \\ &\le c(1+s)^{-1/\alpha-j/\alpha} + (1+s)^{-1/2\alpha}\|\partial_x^j u(s)\|_{L^2}. \end{aligned} \tag{5.2}$$

For the contribution of the first term in (5.2), we have

$$\int_{t/2}^t (1+t-s)^{-3/2\alpha}(1+s)^{-1/\alpha-j/\alpha}ds \le ct^{-1/2\alpha-j/\alpha}\Big(t^{-1/2\alpha}\int_0^t (1+s)^{-3/2\alpha}ds\Big)$$

and for $t > 1$,

$$t^{-1/2\alpha}\int_0^t (1+s)^{-3/2\alpha}ds \le c\begin{cases} t^{-1/2\alpha} & \text{if} \quad \alpha < 3/2 \\ t^{-1/3}\log t & \text{if} \quad \alpha = 3/2 \\ t^{1-2/\alpha} & \text{if} \quad \alpha > 3/2 \end{cases} \le c. \tag{5.3}$$

For the second one, one can write

$$\int_{t/2}^{t}(1+t-s)^{-3/2\alpha}(1+s)^{-1/2\alpha}\|\partial_x^j u(s)\|_{L^2}ds$$
$$\le c\Big(t^{-1/2\alpha}\int_0^t(1+s)^{-3/2\alpha}ds\Big)\sup_{t/2\le s\le t}\|\partial_x^j u(s)\|_{L^2}\le ct^{-\gamma}\sup_{t/2\le s\le t}\|u(s)\|_{\dot H^j}$$

in view of (5.3). Term II_2 is bounded by

$$\begin{aligned}II_2&\le\int_{t/2}^{t}\Big(\int_{-\infty}^{\infty}e^{-2(t-s)}|\xi|^{2(j+1)}|\widehat{u^2}(s,\xi)|^2d\xi\Big)^{1/2}ds\\&=c\int_{t/2}^{t}e^{-(t-s)}\|\partial_x^{j+1}u^2(s)\|_{L^2}ds\\&\le c\int_{t/2}^{t}e^{-(t-s)}\sum_{k=0}^{j+1}\|\partial_x^k u(s)\partial_x^{j+1-k}u(s)\|_{L^2}ds.\end{aligned}$$

By symmetry, it suffices in the previous sum to consider the values $k=0,1,\dots,E((j+1)/2)$. When $k=0$, assumption (2.5) and Lemma 3.2 provide

$$\begin{aligned}\|u(s)\partial_x^{j+1}u(s)\|_{L^2}&\le\|u(s)\|_{L^\infty}\|\partial_x^{j+1}u(s)\|_{L^2}\\&\le c\|u(s)\|_{L^2}^{1/2}\|u_x(s)\|_{L^2}^{1/2}\|\partial_x^j u(s)\|_{L^2}^{1-1/N}\|\partial_x^{j+N}u(s)\|_{L^2}^{1/N}\\&\le c(1+s)^{-1/4\alpha}\|\partial_x^j u(s)\|_{L^2}^{1-1/N}\end{aligned}$$

for any $N\ge1$. For $k=1$, we have by similar calculations

$$\begin{aligned}\|u_x(s)\partial_x^j u(s)\|_{L^2}&\le\|u_x(s)\|_{L^\infty}\|\partial_x^j u(s)\|_{L^2}\\&\le c\|u(s)\|_{L^2}^{1/4}\|u_{xx}(s)\|_{L^2}^{3/4}\|\partial_x^j u(s)\|_{L^2}\\&\le c(1+s)^{-1/8\alpha}\|\partial_x^j u(s)\|_{L^2}.\end{aligned}$$

Note that if $k=2$, we must have $j\ge3$. If $j\ge4$, one has by the inductive hypothesis

$$\begin{aligned}\|\partial_x^2u(s)\partial_x^{j-1}u(s)\|_{L^2}&\le c\|\partial_x^2u(s)\|_{L^\infty}\|\partial_x^{j-1}u(s)\|_{L^2}\\&\le c\|\partial_x^2u(s)\|_{L^2}^{1/2}\|\partial_x^3u(s)\|_{L^2}^{1/2}(1+s)^{-1/2\alpha-(j-1)/\alpha}\\&\le c(1+s)^{-1/2\alpha-j/\alpha}.\end{aligned}$$

If $j=3$, then

$$\begin{aligned}\|u_{xx}(s)u_{xx}(s)\|_{L^2} &\leq \|u_{xx}(s)\|_{L^\infty}\|u_{xx}(s)\|_{L^2}\\ &\leq c\|u_{xx}(s)\|_{L^2}^{1/2}\|u_{xxx}(s)\|_{L^2}^{1/2}\|u_x(s)\|_{L^2}^{1/2}\|u_{xxx}(s)\|_{L^2}^{1/2}\\ &\leq c(1+s)^{-2/\alpha}\|\partial_x^j u(s)\|_{L^2}.\end{aligned}$$

In the end for $k\geq 3$ (and thus $j\geq 5$),

$$\begin{aligned}\|\partial_x^k u(s)\partial_x^{j+1-k}u(s)\|_{L^2} &\leq \|\partial_x^k u(s)\|_{L^2}\|\partial_x^{j+1-k}u(s)\|_{L^\infty}\\ &\leq \|\partial_x^k u(s)\|_{L^2}\|\partial_x^{j+1-k}u(s)\|_{L^2}^{1/2}\|\partial_x^{j+2-k}u(s)\|_{L^2}^{1/2}\\ &\leq c(1+s)^{-1/2\alpha-k/\alpha+(-1/2\alpha-(j+1-k)/\alpha)/2+(-1/2\alpha-(j+2-k)/\alpha)/2}\\ &\leq c(1+s)^{-5/2\alpha-j/\alpha}\leq c(1+s)^{-1/2\alpha-j/\alpha}\end{aligned}$$

This allows us to conclude that

$$\begin{aligned}II_2 &\leq c\int_{t/2}^t e^{-(t-s)}[(1+s)^{-1/2\alpha-j/\alpha}+s^{-\gamma}\|\partial_x^j u(s)\|_{L^2}+(1+s)^{-\gamma}\|\partial_x^j u(s)\|_{L^2}^{1-1/N}]ds\\ &\leq c[t^{-1/2\alpha-j/\alpha}+t^{-\gamma}\sup_{t/2\leq s\leq t}\|\partial_x^j u(s)\|_{L^2}+t^{-\gamma}\sup_{t/2\leq s\leq t}\|\partial_x^j u(s)\|_{L^2}^{1-1/N}]\int_0^t e^{-(t-s)}ds\\ &\leq ct^{-1/2\alpha-j/\alpha}+t^{-\gamma}\sup_{t/2\leq s\leq t}\|\partial_x^j u(s)\|_{L^2}+ct^{-\gamma}\sup_{t/2\leq s\leq t}\|\partial_x^j u(s)\|_{L^2}^{1-1/N}.\end{aligned}$$

□

In order to prove Corollary 5.1, we need the following elementary result.

Lemma 5.2. *Let $f:\mathbb{R}^+\to\mathbb{R}^+$ bounded, and $0<\gamma<\beta$ and $N\geq 1$. We assume*

$$\forall t\geq 1,\quad f(t)\leq ct^{-\beta}+ct^{-\gamma}\sup_{s\sim t}f(s)^{1-1/N}.$$

Then for t and N large enough, $f(t)\leq ct^{-\beta}$.

Proof. We show by induction that for all $n\geq 0$, $f(t)\leq ct^{-\min(\beta,\gamma(1-N)(1-\frac{1}{N})^n+\gamma N)}$. Thus for n large enough, one obtains $f(t)\leq ct^{-\min(\beta,\gamma N+1)}$ and it suffices to choose N so that $\beta\leq\gamma N+1$. □

Proof of Corollary 5.1. By (2.5), we only need to consider t large enough. Using (3.5) and Lemma 5.1, it follows that

$$\begin{aligned}\|\partial_x^j u(t)\|_{L^2} &\leq \|\partial_x^j S_\alpha(t)*u_0\|_{L^2}+\left\|\frac{1}{2}\int_0^t\partial_x^j S_\alpha(t-s)*\partial_x u^2(s)ds\right\|_{L^2}\\ &\leq ct^{-1/2\alpha-j/\alpha}+ct^{-\gamma}\sup_{t/2\leq s\leq t}\|\partial_x^j u(s)\|_{L^2}+ct^{-\gamma}\sup_{t/2\leq s\leq t}\|\partial_x^j u(s)\|_{L^2}^{1-1/N}.\end{aligned}$$

Letting $t \to \infty$, we deduce $\|\partial_x^j u(t)\|_{L^2} \to 0$. For $t \gg 1$, we thus have $\|\partial_x^j u(t)\|_{L^2} \leq 1$ and

$$\|\partial_x^j u(t)\|_{L^2} \leq ct^{-1/2\alpha - j/\alpha} + ct^{-\gamma} \sup_{t/2 \leq s \leq t} \|\partial_x^j u(s)\|_{L^2}^{1-1/N}.$$

Applying Lemma 5.2 with $f(t) = \|\partial_x^j u(t)\|_{L^2}$ and $\beta = 1/2\alpha + j/\alpha$, we obtain the desired result. $\square$

Proof of Theorem 2.2. The result is already proved in the case $p = 2$. When $p = \infty$, we use (3.2) and Corollary 5.1 to get

$$\|u(t)\|_{\dot{H}^{\infty,j}} \leq c\|u(t)\|_{\dot{H}^j}^{1/2} \|u(t)\|_{\dot{H}^{j+1}}^{1/2} \leq c(1+t)^{-1/\alpha - j/\alpha}.$$

The other cases follow by an interpolation argument. $\square$

6 Asymptotic expansion

6.1 First order

In this subsection we prove Theorem 2.3. As previously, it suffices to show the result when $p = 2$ and $u_0 \in H^{j+2}(\mathbb{R}) \cap L^1(\mathbb{R})$.

First, since $u \in C_b(\mathbb{R}^+, H^j(\mathbb{R}))$,

$$\|u(t) - S_\alpha(t) * u_0\|_{\dot{H}^j} \leq \|u(t)\|_{\dot{H}^j} + \|G_\alpha(t)\|_{L^1} \|u_0\|_{\dot{H}^j} \leq c$$

and we reduce to consider the case $t \geq 1$. Using the integral formulation of (dKdV), we have

$$\begin{aligned}
\|u(t) - S_\alpha(t) * u_0\|_{\dot{H}^j} &\leq \frac{1}{2} \int_0^t \|\partial_x^j S_\alpha(t-s) * \partial_x u^2\|_{L^2} ds \\
&= \int_0^{t/2} \ldots ds + \int_{t/2}^t \ldots ds := I + II.
\end{aligned}$$

Term I is bounded by

$$
\begin{aligned}
I &\leq c\int_0^{t/2} \|\partial_x^{j+1} S_\alpha(t-s)\|_{L^2}\|u(s)\|_{L^2}^2 ds \\
&\leq c\int_0^{t/2}(t-s)^{-1/2\alpha-(j+1)/\alpha}(1+s)^{-1/\alpha}ds \\
&\leq ct^{-1/2\alpha-(j+1)/\alpha}\int_0^t (1+s)^{-1/\alpha}ds \\
&\leq c\begin{cases} t^{(-1/2\alpha-j/\alpha)-1/\alpha} & \text{if} \quad \alpha<1, \\ t^{(-1/2-j)-1}\log(t) & \text{if} \quad \alpha=1, \\ t^{(-1/2\alpha-j/\alpha)-(2/\alpha-1)} & \text{if} \quad \alpha>1. \end{cases}
\end{aligned}
$$

To estimate II we use Plancherel and we split low and high frequencies,

$$
\begin{aligned}
II &= c\int_{t/2}^t \Big(\int_{-\infty}^\infty e^{-2(t-s)|\xi|^\alpha}|\xi|^{2(j+1)}|\widehat{u^2}(s,\xi)|^2 d\xi\Big)^{1/2}ds \\
&\leq c\int_{t/2}^t \Big(\int_{|\xi|<1}\dots d\xi\Big)^{1/2}ds + c\int_{t/2}^t \Big(\int_{|\xi|>1}\dots d\xi\Big)^{1/2}ds := II_1 + II_2.
\end{aligned}
$$

II_1 is treated as follows

$$
\begin{aligned}
II_1 &\leq c\int_{t/2}^t \|S_\alpha(1+t-s)\|_{L^2}\|\partial_x^{j+1}u^2(s)\|_{L^1}ds \\
&\leq c\int_{t/2}^t (1+t-s)^{-1/2\alpha}(1+s)^{-2/\alpha-j/\alpha}ds \\
&\leq ct^{-2/\alpha-j/\alpha}\int_0^t (1+s)^{-1/2\alpha}ds \\
&\leq c\begin{cases} t^{-2/\alpha-j/\alpha} & \text{if} \quad \alpha<1/2, \\ t^{-4-2j}\log t & \text{if} \quad \alpha=1/2, \\ t^{(-1/2\alpha-j/\alpha)+1-2/\alpha} & \text{if} \quad \alpha>1/2, \end{cases} \\
&\leq c\begin{cases} t^{(-1/2\alpha-j/\alpha)-1/\alpha} & \text{if} \quad \alpha<1, \\ t^{(-1/2-j)-1}\log(t) & \text{if} \quad \alpha=1, \\ t^{(-1/2\alpha-j/\alpha)-(2/\alpha-1)} & \text{if} \quad \alpha>1. \end{cases}
\end{aligned}
$$

For the last term, we have

$$\begin{aligned}
II_2 &\le c\int_{t/2}^{t} e^{-(t-s)}\|\partial_x^{j+1}u^2(s)\|_{L^2}ds \\
&\le c\int_{t/2}^{t} e^{-(t-s)}(1+s)^{-1/2\alpha-j/\alpha-2/\alpha} \\
&\le ct^{(-1/2\alpha-j/\alpha)-2/\alpha},
\end{aligned}$$

which is acceptable.

6.2 Higher orders

Here we find higher orders terms in the asymptotic expansion of the solution to (dKdV), i.e. we give a demonstration of Theorems 2.4 and 2.5.

6.2.1 The case $0<\alpha<1$

First consider the case $0<\alpha<1$, our proof follows Karch's one [9] (see also [4]).

Proof of Theorem 2.4 (i). By interpolation, we only need to consider the case $p=2$ and $u_0\in H^{j+2}(\mathbb{R})$. Split the quantity

$$\begin{aligned}
&\Big\|u(t)-S_\alpha(t)*u_0+\frac{1}{2}\Big(\int_0^\infty\int_{-\infty}^\infty u^2(s,y)dyds\Big)\partial_x G_\alpha(t)\Big\|_{\dot H^j} \\
&\le \frac{1}{2}\Big\|\int_0^t \partial_x[S_\alpha(t-s)-G_\alpha(t-s)]*u^2(s)ds\Big\|_{\dot H^j} \\
&\quad+\frac{1}{2}\Big\|\int_0^t \partial_x G_\alpha(t-s)*u^2(s)ds-\Big(\int_0^\infty\int_{-\infty}^\infty u^2(s,y)dyds\Big)\partial_x G_\alpha(t)\Big\|_{\dot H^j} \\
&:= I+II.
\end{aligned}$$

To estimate I, we write

$$\begin{aligned}
I &\le c\int_0^t \|\partial_x^{j+1}[S_\alpha(t-s)-G_\alpha(t-s)]*u^2(s)\|_{L^2}ds \\
&=\int_0^{t/2}\dots ds+\int_{t/2}^t\dots ds := I_1+I_2.
\end{aligned}$$

Concerning I_1, we use (3.4) with $N=0$,

$$\begin{aligned} I_1 &\le c\int_0^{t/2} \|\partial_x^{j+1}[S_\alpha(t-s)-G_\alpha(t-s)]\|_{L^2}\|u(s)\|_{L^2}^2 ds \\ &\le c\int_0^{t/2}(t-s)^{-1/2\alpha-(j+1)/\alpha+1-3/\alpha}(1+s)^{-1/\alpha}ds \\ &\le ct^{-1/2\alpha-j/\alpha-1/\alpha}t^{1-3/\alpha}, \end{aligned}$$

which shows that $t^{1/2\alpha+j/\alpha+1/\alpha}I_1 \to 0$. To deal with the integrand over $[t/2,t]$, we note that $\|[S_\alpha(t-s)-G_\alpha(t-s)]*u^2(s)\|_{\dot H^{j+1}} \le c\|u^2(s)\|_{\dot H^{j+1}}$, hence

$$\begin{aligned} I_2 &\le c\int_{t/2}^t \|\partial_x^{j+1}u^2(s)\|_{L^2}ds \\ &\le c\int_{t/2}^t (1+s)^{-1/2\alpha-j/\alpha-2/\alpha}ds \\ &\le ct^{-1/2\alpha-j/\alpha-1/\alpha}t^{1-1/\alpha}, \end{aligned}$$

which is acceptable. Now we estimate term II by

$$\begin{aligned} II &\le \frac{1}{2}\Big\|\Big(\int_t^\infty\int_{-\infty}^\infty u^2(s,y)dyds\Big)\partial_x G_\alpha(t)\Big\|_{\dot H^j} \\ &\quad+\frac{1}{2}\Big\|\int_0^t\Big[\partial_x G_\alpha(t-s)*u^2(s)-\Big(\int_{-\infty}^\infty u^2(s,y)dy\Big)\partial_x G_\alpha(t)\Big]ds\Big\|_{\dot H^j} \\ &:= II_1+II_2. \end{aligned}$$

Obviously,

$$II_1 \le c\int_t^\infty \|u(s)\|_{L^2}^2 ds\|\partial_x^{j+1}G_\alpha(t)\|_{L^2} \le ct^{(-1/2\alpha-j/\alpha)-1/\alpha}\int_t^\infty (1+s)^{-1/\alpha}ds$$

and it is clear that $\int_t^\infty(1+s)^{-1/\alpha}ds \to 0$ as $t\to\infty$. To estimate II_2 one fixes $\delta>0$ and we bound it by

$$\begin{aligned} II_2 &\le c\Big\|\int_0^t\Big(\int_{-\infty}^\infty \partial_x[G_\alpha(t-s,\cdot-y)-G_\alpha(t,\cdot)]u^2(s,y)dy\Big)ds\Big\|_{\dot H^j} \\ &\le c\int_0^t\Big\|\int_{-\infty}^\infty \partial_x^{j+1}[G_\alpha(t-s,\cdot-y)-G_\alpha(t,\cdot)]u^2(s,y)dy\Big\|_{L^2}ds \\ &= \int_0^{\delta t}\ldots ds+\int_{\delta t}^t \ldots ds \\ &= II_{21}+II_{22}. \end{aligned}$$

Then we split II_{21} in two parts,

$$\begin{aligned}
II_{21} &\leq c\int_{[0,\delta t]\times\mathbb{R}} \|\partial_x^{j+1}[G_\alpha(t-s,\cdot-y)-G_\alpha(t,\cdot)]u^2(s,y)\|_{L^2}dsdy \\
&= c\int_{\Omega_1}\dots dsdy + c\int_{\Omega_2}\dots dyds \\
&= II_{211}+II_{212},
\end{aligned}$$

where

$$\begin{aligned}
\Omega_1 &= [0,\delta t]\times[-\delta t^{1/\alpha},+\delta t^{1/\alpha}], \\
\Omega_2 &= [0,\delta t]\times(]-\infty,-\delta t^{1/\alpha}[\cup]+\delta t^{1/\alpha},\infty[).
\end{aligned}$$

For all $(s,y)\in\Omega_1$, a straightforward calculation provides

$$\begin{aligned}
&\|\partial_x^{j+1}[G_\alpha(t-s,\cdot-y)-G_\alpha(t,\cdot)]\|_{L^2} \\
&\qquad = t^{-1/2\alpha-j/\alpha-1/\alpha}\|\partial_x^{j+1}[G_\alpha(1-s/t,\cdot-yt^{-1/\alpha})-G_\alpha(1,\cdot)]\|_{L^2}.
\end{aligned}$$

Hence, using the continuity of the translation on L^2, for all $\varepsilon>0$, we can find a $\delta>0$ such that

$$\begin{aligned}
&t^{(1/2\alpha+j/\alpha)+1/\alpha}\sup_{(s,y)\in\Omega_1}\|\partial_x^{j+1}[G_\alpha(t-s,\cdot-y)-G_\alpha(t,\cdot)]\|_{L^2} \\
&\qquad\qquad \leq \sup_{\substack{0\leq\tau\leq\delta\\|z|\leq\delta}}\|\partial_x^{j+1}[G_\alpha(1-\tau,\cdot-z)-G_\alpha(1,\cdot)]\|_{L^2}\leq\varepsilon.
\end{aligned}$$

We deduce

$$t^{(1/2\alpha+j/\alpha)+1/\alpha}II_{211}\leq c\varepsilon\int_0^{\delta t}\|u(s)\|_{L^2}^2ds\leq c\varepsilon\int_0^{\delta t}(1+s)^{-1/\alpha}ds\leq c\varepsilon.$$

Now for any $(s,y)\in\Omega_2$, we have

$$\begin{aligned}
\|\partial_x^{j+1}[G_\alpha(t-s,\cdot-y)-G_\alpha(t,\cdot)]\|_{L^2} &\leq \|\partial_x^{j+1}G_\alpha(t-s)\|_{L^2}+\|\partial_x^{j+1}G_\alpha(t)\|_{L^2} \\
&\leq ct^{-1/2\alpha-(j+1)/\alpha},
\end{aligned}$$

which yields

$$t^{(1/2\alpha+j/\alpha)+1/\alpha}II_{212}\leq c\int_0^\infty\int_{|y|\geq\delta t^{1/\alpha}}u^2(s,y)dyds\to 0$$

by the dominated convergence theorem.
It remains to estimate II_{22}, we have

$$
\begin{aligned}
II_{22} &= c\int_{\delta t}^{t} \|\partial_x^{j+1} G_\alpha(t-s) * u^2(s) - \|u(s)\|_{L^2}^2 \partial_x^{j+1} G_\alpha(t)\|_{L^2} ds \\
&\leq c\int_{\delta t}^{t} \|\partial_x^{j+1} G_\alpha(t-s) * u^2(s)\|_{L^2} ds + c\int_{\delta t}^{t} (1+s)^{-1/\alpha} ds \|\partial_x^{j+1} G_\alpha(t)\|_{L^2} \\
&= II_{221} + II_{222}.
\end{aligned}
$$

The first term is bounded by

$$
\begin{aligned}
II_{221} &\leq c\int_{\delta t}^{t} \Big(\int_{-\infty}^{\infty} |\xi|^{2(j+1)} e^{-2(t-s)|\xi|^\alpha} |\widehat{u^2}(s,\xi)|^2 d\xi\Big)^{1/2} ds \\
&\leq c\int_{\delta t}^{t} [\|G_\alpha(1+t-s)\|_{L^2} \|\partial_x^{j+1} u^2(s)\|_{L^1} + e^{-(t-s)} \|\partial_x^{j+1} u^2(s)\|_{L^2}] ds \\
&\leq c\int_{\delta t}^{t} [(1+t-s)^{-1/2\alpha} (1+s)^{-2/\alpha - j/\alpha} + e^{-(t-s)} (1+s)^{-5/2\alpha - j/\alpha}] ds \\
&\leq ct^{-1/2\alpha - j/\alpha - 1/\alpha} \Big(t^{-1/2\alpha} \int_0^t (1+s)^{-1/2\alpha} ds\Big) + ct^{-5/2\alpha - j/\alpha}
\end{aligned}
$$

and thus $t^{1/2\alpha + j/\alpha + 1/\alpha} II_{221} \to 0$. On the other hand, we have immediately

$$II_{222} \leq ct^{-1/2\alpha - j/\alpha - 1/\alpha} t^{1-1/\alpha},$$

which achieves the proof of (2.7). $\square$

6.2.2 The case $\alpha = 1$

The proof of (2.8) uses the same arguments together with the following result.

Lemma 6.1. *Under the assumptions of Theorem 2.4 (ii),*

$$\lim_{t\to\infty} \frac{1}{\log t} \int_0^t \int_{-\infty}^{\infty} u^2(s,y) dy ds = \frac{M^2}{2\pi}.$$

Proof. First note that

$$\frac{1}{\log t} \int_0^1 \int_{-\infty}^{\infty} u^2(s,y) dy ds \leq \frac{c}{\log t} \int_0^1 (1+s)^{-1} ds \leq \frac{c}{\log t} \to 0$$

and it remains to calculate the limit as $t \to \infty$ of

$$\frac{1}{\log t}\int_1^t \int_{-\infty}^{\infty} u^2(s,y)dyds = \frac{1}{\log t}\int_1^t \int_{-\infty}^{\infty} (u^2(s,y) - (MG_1(s,y))^2)dyds$$
$$+ \frac{1}{\log t}\int_1^t \int_{-\infty}^{\infty} (MG_1(s,y))^2 dyds. \tag{6.1}$$

Using Theorem 2.3 as well as estimate (2.2), we get for all $s > 1$

$$\begin{aligned}\int_{-\infty}^{\infty} |u^2(s,y) - (MG_1(s,y))^2|dy &\le \|u(s) + MG_1(s)\|_{L^2}\|u(s) - MG_1(s)\|_{L^2} \\ &\le cs^{-1/2}\big(\|u(s) - S_1(s) * u_0\|_{L^2} \\ &\quad + \|S_1(s) * u_0 - MG_1(s)\|_{L^2}\big) \\ &\le cs^{-1/2}(s^{-3/2}\log s + s^{-3/2}) \\ &\le cs^{-2}\log s.\end{aligned}$$

It follows that

$$\frac{1}{\log t}\int_1^t \int_{-\infty}^{\infty} |u^2(s,y) - (MG_1(s,y))^2|dyds \le \frac{c}{\log t}\int_1^t s^{-2}\log s ds \to 0$$

by dominated convergence. The last term in (6.1) is equal to

$$\begin{aligned}\frac{1}{\log t}\int_1^t \int_{-\infty}^{\infty} (MG_1(s,y))^2 dyds &= \frac{M^2}{\log t}\int_1^t \int_{-\infty}^{\infty} s^{-2}(G_1(1,y/s))^2 dyds \\ &= \frac{M^2}{\log t}\int_1^t \frac{ds}{s}\int_{-\infty}^{\infty} (G_1(1,x))^2 dx \\ &= M^2\|G_1(1)\|_{L^2}^2 \\ &= \frac{M^2}{2\pi}.\end{aligned}$$

□

Proof of Theorem 2.4 (ii). It is sufficient to show that

$$\frac{t^{3/2+j}}{\log t}\Big\|\int_0^t \partial_x S_1(t-s) * u^2(s)ds - \frac{M^2}{2\pi}(\log t)\partial_x G_1(t)\Big\|_{\dot{H}^j} \to 0.$$

for all $j \ge 0$. As in Theorem 2.4 (i), we can replace $S_\alpha(t-s)$ by $G_\alpha(t-s)$

by writing

$$
\begin{aligned}
&\Big\| \int_0^t \partial_x S_1(t-s) * u^2(s)ds - \frac{M^2}{2\pi}(\log t)\partial_x G_1(t)\Big\|_{\dot{H}^j} \\
&\leq \Big\| \int_0^t \partial_x [S_1(t-s) - G_1(-t-s)] * u^2(s)ds \Big\|_{\dot{H}^j} \\
&\quad + \Big\| \int_0^t \partial_x G_1(t-s) * u^2(s)ds - \frac{M^2}{2\pi}(\log t)\partial_x G_1(t)\Big\|_{\dot{H}^j}
\end{aligned}
$$

and using (3.4). Last term in the previous inequality is bounded by

$$
\begin{aligned}
&\leq \Big\| \int_0^t \partial_x G_1(t-s) * u^2(s)ds - \Big(\int_0^t \int_{-\infty}^{\infty} u^2(s,y)dyds\Big)\partial_x G_1(t)\Big\|_{\dot{H}^j} \\
&\quad + \Big\| \Big(\int_0^t \int_{-\infty}^{\infty} u^2(s,y)dyds\Big)\partial_x G_1(t) - \frac{M^2}{2\pi}(\log t)\partial_x G_1(t)\Big\|_{\dot{H}^j}.
\end{aligned}
$$

The first term is estimated exactly in the same way that II_2 in Theorem 2.4 (i) and for the second one, Lemma 6.1 provides

$$
\begin{aligned}
&\frac{t^{3/2+j}}{\log t}\Big\| \Big(\int_0^t \int_{-\infty}^{\infty} u^2(s,y)dyds\Big)\partial_x G_1(t) - \frac{M^2}{2\pi}(\log t)\partial_x G_1(t)\Big\|_{\dot{H}^j} \\
&\leq t^{3/2+j}\Big| \frac{1}{\log t}\int_0^t \int_{-\infty}^{\infty} u^2(s,y)dyds - \frac{M^2}{2\pi}\Big| \|\partial_x^{j+1} G_1(t)\|_{L^2} \\
&\leq c\Big| \frac{1}{\log t}\int_0^t \int_{-\infty}^{\infty} u^2(s,y)dyds - \frac{M^2}{2\pi}\Big| \to 0.
\end{aligned}
$$

□

6.2.3 The case $1 < \alpha < 2$

Finally we consider the case $1 < \alpha < 2$.

Proof of Theorem 2.5. We prove the result when $p = 2$ and $u_0 \in H^{j+2}(\mathbb{R})$.
Step 1. $\|F^n(t)\|_{\dot{H}^j}$ decays like $\|u(t)\|_{\dot{H}^j}$.
If $n = 0$, then for all $j \geq 0$, $\|F^0(t)\|_{\dot{H}^j} = \|\partial_x^j S_\alpha(t) * u_0\|_{L^2} \leq c(1+t)^{-1/2\alpha - j/\alpha}$.
Let $n \geq 0$ such that for all $j \geq 0$, $\|F^n(t)\|_{\dot{H}^j} \leq c(1+t)^{-1/2\alpha - j/\alpha}$. Then, for

any $t \leq 1$,

$$\begin{aligned}
\|F^{n+1}(t)\|_{\dot{H}^j} &\leq \|S_\alpha(t) * u_0\|_{\dot{H}^j} + \int_0^t \|S_\alpha(t-s) * \partial_x (F^n(s))^2\|_{\dot{H}^j} ds \\
&\leq \|G_\alpha(t)\|_{L^1} \|u_0\|_{\dot{H}^j} + \int_0^1 \|G_\alpha(t-s)\|_{L^1} \|\partial_x^{j+1}(F^n(s))\|_{L^2} ds \\
&\leq c.
\end{aligned}$$

Now assume $t > 1$. We have

$$\begin{aligned}
\|F^{n+1}(t)\|_{\dot{H}^j} &\leq \|S_\alpha(t) * u_0\|_{\dot{H}^j} + \int_0^t \|S_\alpha(t-s) * \partial_x (F^n(s))^2\|_{\dot{H}^j} ds \\
&\leq c(1+t)^{-1/2\alpha - j/\alpha} + \int_0^{t/2} \dots ds + \int_{t/2}^t \dots ds.
\end{aligned}$$

The integrand over $[0, t/2]$ is estimated as follows

$$\begin{aligned}
\int_0^{t/2} \dots ds &\leq \int_{t/2}^t \|\partial_x^{j+1} S_\alpha(t-s)\|_{L^2} \|F^n(s)\|_{L^2}^2 ds \\
&\leq c \int_0^{t/2} (t-s)^{-1/2\alpha - (j+1)/\alpha} (1+s)^{-1/\alpha} ds \\
&\leq c t^{-1/2\alpha - j/\alpha} \Big(t^{-1/\alpha} \int_0^t (1+s)^{-1/\alpha} ds \Big) \\
&\leq c t^{-1/2\alpha - j/\alpha}.
\end{aligned}$$

For the second one, one splits

$$\begin{aligned}
\int_{t/2}^t \dots ds &= c \int_{t/2}^t \Big(\int_{-\infty}^\infty |\xi|^{2(j+1)} e^{-2(t-s)|\xi|^\alpha} |\widehat{(F^n(s))^2}(\xi)|^2 d\xi \Big)^{1/2} ds \\
&\leq c \int_{t/2}^t \Big(\int_{|\xi|<1} \dots d\xi \Big)^{1/2} ds + \int_{t/2}^t \Big(\int_{|\xi|>1} \dots d\xi \Big)^{1/2} ds \\
&:= I + II.
\end{aligned}$$

Term I is bounded by

$$
\begin{aligned}
I &\le c\int_{t/2}^{t} \|\partial_x^{j+1} S_\alpha(1+t-s) * F^n(s)\|_{L^2} ds \\
&\le c\int_{t/2}^{t} \|S_\alpha(1+t-s)\|_{L^2} \|\partial_x^{j+1}(F^n(s)^2\|_{L^1} ds \\
&\le c\int_{t/2}^{t} (1+t-s)^{-1/2\alpha} \sum_{k=0}^{j+1} \|\partial_x^k F^n(s)\|_{L^2} \|\partial_x^{j+1-k} F^n(s)\|_{L^2} ds \\
&\le c\int_{t/2}^{t} (1+t-s)^{-1/2\alpha} (1+s)^{-2/\alpha - j/\alpha} ds \\
&\le ct^{-1/2\alpha - j/\alpha}\Big(t^{-3/2\alpha}\int_0^t (1+s)^{-1/2\alpha} ds\Big) \\
&\le ct^{-1/2\alpha - j/\alpha}
\end{aligned}
$$

and II is estimated by

$$
\begin{aligned}
\int_{t/2}^{t} e^{-(t-s)} \|\partial_x^{j+1}(F^n(s)^2\|_{L^2} ds &\le c\int_{t/2}^{t} e^{-(t-s)} \sum_{k=0}^{j+1} \|\partial_x^k F^n(s)\|_{L^2} \|\partial_x^{j+1-k} F^n(s)\|_{L^\infty} ds \\
&\le c\int_{t/2}^{t} e^{-(t-s)} \sum_{k=0}^{j+1} \|\partial_x^k F^n(s)\|_{L^2} \|\partial_x^{j+1-k} F^n(s)\|_{L^2}^{1/2} \\
&\quad \times \|\partial_x^{j+2-k} F^n(s)\|_{L^2}^{1/2} ds \\
&\le c\int_{t/2}^{t} e^{-(t-s)} (1+s)^{-5/2\alpha - j/\alpha} ds \\
&\le ct^{-5/2\alpha - j/\alpha} \le ct^{-1/2\alpha - j/\alpha}.
\end{aligned}
$$

We have showed that $\|F^{n+1}(t)\|_{\dot H^j} \le c(1+t)^{-1/2\alpha - j/\alpha}$ and by induction, this estimate becomes true for any $n \ge 0$.

Step 2. We claim that if for all $j \ge 0$, $\|u(t) - F^n(t)\|_{\dot H^j} \le c(1+t)^{-r_j(n)}$ and $r_j(n) = \frac{j}{\alpha} + r_0(n)$, then

$$
\|u(t) - F^{n+1}(t)\|_{\dot H^j} \le c \begin{cases} (1+t)^{-1/2\alpha - j/\alpha - 1/\alpha} & \text{if } \ 1 - \frac{1}{2\alpha} - r_0(n) < 0, \\ (1+t)^{-1/2\alpha - j/\alpha - 1/\alpha} \log(1+t) & \text{if } \ 1 - \frac{1}{2\alpha} - r_0(n) = 0, \\ (1+t)^{-1/2\alpha - j/\alpha - 1/\alpha + 1 - 1/2\alpha - r_0(n)} & \text{if } \ 1 - \frac{1}{2\alpha} - r_0(n) > 0. \end{cases}
$$

Indeed, first for $t \le 1$ it is clear that $\|u(t) - F^{n+1}(t)\|_{\dot H^j}$ is bounded. If $t > 1$

we have by definition of F^n,

$$\|u(t)-F^{n+1}(t)\|_{\dot H^j} \le \frac{1}{2}\int_0^t \|\partial_x^{j+1} S_\alpha(t-s) * [u^2(s)-(F^n(s))^2]\|_{L^2} ds$$
$$= \int_0^{t/2} \dots ds + \int_{t/2}^t \dots ds := III + IV.$$

We bound the contribution of III by

$$\begin{aligned}
III &\le c\int_0^{t/2} \|\partial_x^{j+1} S_\alpha(t-s)\|_{L^2} \|u^2(s)-(F^n(s))^2\|_{L^1} ds \\
&\le c\int_0^{t/2} (t-s)^{-1/2\alpha-(j+1)/\alpha} \|u(s)-F^n(s)\|_{L^2}(\|u(s)\|_{L^2}+\|F^n(s)\|_{L^2}) ds \\
&\le c\int_0^{t/2} (t-s)^{-1/2\alpha-(j+1)/\alpha}(1+s)^{-1/2\alpha-r_0(n)} ds \\
&\le c\begin{cases} (1+t)^{-1/2\alpha-j/\alpha-1/\alpha} & \text{if } \ 1-\frac{1}{2\alpha}-r_0(n)<0, \\ (1+t)^{-1/2\alpha-j/\alpha-1/\alpha}\log(1+t) & \text{if } \ 1-\frac{1}{2\alpha}-r_0(n)=0, \\ (1+t)^{-1/2\alpha-j/\alpha-1/\alpha+1-1/2\alpha-r_0(n)} & \text{if } \ 1-\frac{1}{2\alpha}-r_0(n)>0. \end{cases}
\end{aligned}$$

Then we decompose IV as

$$\begin{aligned}
IV &= c\int_{t/2}^t \Big(\int_{-\infty}^{\infty} |\xi|^{2(j+1)} e^{-2(t-s)|\xi|^\alpha} |\mathcal{F}[u^2(s)-(F^n(s))^2](\xi)|^2 d\xi\Big)^{1/2} ds \\
&\le c\int_{t/2}^t \Big(\int_{|\xi|<1} \dots d\xi\Big)^{1/2} ds + \int_{t/2}^t \Big(\int_{|\xi|>1} \dots d\xi\Big)^{1/2} ds \\
&:= IV_1 + IV_2.
\end{aligned}$$

Low frequencies are treated as follows,

$$\begin{aligned}
IV_1 &\le \int_{t/2}^t \|\partial_x^{j+1} S_\alpha(1+t-s) * [u^2(s)-(F^n(s))^2]\|_{L^2} ds \\
&\le c\int_{t/2}^t \|S_\alpha(1+t-s)\|_{L^2} \|\partial_x^{j+1}[u^2(s)-(F^n(s))^2]\|_{L^1} ds \\
&\le c\int_{t/2}^t (1+t-s)^{-1/2\alpha} \sum_{k=0}^{j+1} \|\partial_x^k[u(s)-F^n(s)]\|_{L^2}(\|\partial_x^{j+1-k}u(s)\|_{L^2}+\|\partial_x^{j+1-k}F^n(s)\|_{L^2}) ds \\
&\le c\int_{t/2}^t (1+t-s)^{-1/2\alpha} \sum_{k=0}^{j+1} (1+s)^{-r_k(n)-1/2\alpha-(j+1-k)/\alpha} ds \\
&\le c\sum_{k=0}^{j+1} t^{-r_k(n)+k/\alpha-j/\alpha+1-2/\alpha} \qquad (6.2)
\end{aligned}$$

and since $r_k(n) = \frac{k}{\alpha} + r_0(n)$, we infer $IV_1 \le ct^{-r_0(n)-j/\alpha+1-2/\alpha}$. In the same way,

$$\begin{aligned}
IV_2 &\le c\int_{t/2}^{t} e^{-(t-s)}\|\partial_x^{j+1}[u^2(s)-(F^n(s))^2]\|_{L^2}ds \\
&\le c\int_{t/2}^{t} e^{-(t-s)}\sum_{k=0}^{j+1}(1+s)^{-r_k(n)-1/\alpha-(j+1-k)/\alpha}ds \\
&\le c\sum_{k=0}^{j+1} t^{-r_k(n)+k/\alpha-j/\alpha-2/\alpha} \\
&\le ct^{-r_0(n)-j/\alpha-2/\alpha}. \qquad (6.3)
\end{aligned}$$

Combining (6.2) and (6.3), we deduce

$$IV \le c\begin{cases} (1+t)^{-1/2\alpha-j/\alpha-1/\alpha} & \text{if } 1-\frac{1}{2\alpha}-r_0(n)<0, \\ (1+t)^{-1/2\alpha-j/\alpha-1/\alpha}\log(1+t) & \text{if } 1-\frac{1}{2\alpha}-r_0(n)=0, \\ (1+t)^{-1/2\alpha-j/\alpha-1/\alpha+1-1/2\alpha-r_0(n)} & \text{if } 1-\frac{1}{2\alpha}-r_0(n)>0. \end{cases}$$

Step 3. Construction of $r_j(n)$ and conclusion.
We define the sequence $r_j(n)$ by iteration. Set $r_j(0) = \frac{1}{2\alpha} + \frac{j}{\alpha} + \frac{2}{\alpha} - 1$ for all $j \ge 0$. We have $\|u(t) - F^0(t)\|_{\dot{H}^j} \le c(1+t)^{-r_j(0)}$ by Theorem 2.3. If $r_j(n)$ is constructed for all j, then we set

$$r_j(n+1) = \begin{cases} \frac{1}{2\alpha}+\frac{j}{\alpha}+\frac{1}{\alpha} & \text{if } 1-\frac{1}{2\alpha}-r_0(n)\le 0, \\ r_0(n)+\frac{j}{\alpha}+\frac{2}{\alpha}-1 & \text{if } 1-\frac{1}{2\alpha}-r_0(n)>0. \end{cases} \qquad (6.4)$$

We easily see that $r_j(n) = \frac{j}{\alpha} + r_0(n)$ for all j, thus Step 2 shows that for any $n \ge 0$ satisfying $1 - \frac{1}{2\alpha} - r_0(n) \le 0$,

$$\begin{aligned}
&\|u(t) - F^{n+1}(t)\|_{\dot{H}^j} \\
&\quad \le c\begin{cases} (1+t)^{-1/2\alpha-j/\alpha-1/\alpha} & \text{if } 1-\frac{1}{2\alpha}-r_0(n)<0, \\ (1+t)^{-1/2\alpha-j/\alpha-1/\alpha}\log(1+t) & \text{if } 1-\frac{1}{2\alpha}-r_0(n)=0. \end{cases} \qquad (6.5)
\end{aligned}$$

Let us prove that the sequence $n \mapsto r_j(n)$ is eventually constant. Suppose that $1 - \frac{1}{2\alpha} - r_0(n) > 0$ for all $n \ge 0$. Then by (6.4) we obtain $r_j(n+1) = r_0(n) + \frac{j}{\alpha} + \frac{2}{\alpha} - 1$ $(\forall n)$. In particular $r_0(n+1) = r_0(n) + \frac{2}{\alpha} - 1$ and thus $r_0(n) = n(\frac{2}{\alpha} - 1) + r_0(0) = (n+1)(\frac{2}{\alpha} - 1) + \frac{1}{2\alpha}$. Since $\frac{2}{\alpha} - 1 > 0$, this contradicts the assumption $r_0(n) < 1 - \frac{1}{2\alpha}$ for n large enough. Hence there exists $n \ge 0$ such that $1 - \frac{1}{2\alpha} - r_0(n) \le 0$ and we can set

$$N = \min\big\{n \ge 0 : 1 - \frac{1}{2\alpha} - r_0(n) \le 0\big\}.$$

For this value of N, it is not too difficult to see that

$$r_j(n) = \begin{cases} (n+1)(\frac{2}{\alpha}-1)+\frac{1}{2\alpha}+\frac{j}{\alpha} & \text{if} \quad n \le N, \\ \frac{1}{2\alpha}+\frac{j}{\alpha}+\frac{1}{\alpha} & \text{if} \quad n > N. \end{cases}$$

It follows that $N = \min\{n \ge 0 : 1 - \frac{1}{\alpha} - (n+1)(\frac{2}{\alpha}-1) \le 0\} = \min\{n \ge 0 : \alpha \le \frac{2n+3}{n+2}\}$. From this and (6.5) we infer

$$\|u(t) - F^{N+1}(t)\|_{\dot{H}^j} \le c \begin{cases} (1+t)^{-1/2\alpha - j/\alpha - 1/\alpha} & \text{if} \quad \alpha < \frac{2N+3}{N+2}, \\ (1+t)^{-1/2\alpha - j/\alpha - 1/\alpha} \log(1+t) & \text{if} \quad \alpha = \frac{2N+3}{N+2}. \end{cases}$$

□

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