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# NECESSARY AND SUFFICIENT CONDITIONS TO BE AN EIGENVALUE FOR LINEARLY RECURRENT DYNAMICAL CANTOR SYSTEMS

XAVIER BRESSAUD, FABIEN DURAND, AND ALEJANDRO MAASS

Abstract. We give necessary and sufficient conditions to have measurable and continuous eigenfunctions for linearly recurrent Cantor dynamical systems. We also construct explicitly an example of linearly recurrent system with nontrivial Kronecker factor and a trivial maximal equicontinuous factor.

## 1. Introduction

Let $(X,T)$ be a topological dynamical system, that is, $X$ is a compact metric space and $T : X \to X$ is a homeomorphism. Let $\mu$ be a $T$-invariant probability measure on $X$. In the classification of dynamical systems in ergodic theory and topological dynamics rotation factors play a central role. In the measure theoretical context this is reflected by the existence of a $T$-invariant sub $\sigma$-algebra $\mathcal{K}_\mu$ of the Borel $\sigma$-algebra of $X$, $\mathcal{B}_X$, such that

$$L^2(X,\mathcal{K}_\mu,\mu) = \overline{< \{f \in L^2(X,\mathcal{B}_X,\mu) \setminus \{0\}; \exists \lambda \in \mathbb{C}, f \circ T = \lambda f\} >}.$$

It is the subspace spanned by the eigenfunctions which determines the Kronecker factor. From a purely topological point of view the role of the Kronecker factor is played by the maximal equicontinuous factor. It can be defined in several ways. When $(X,T)$ is minimal (all orbits are dense), it is determined by the continuous eigenfunctions. So it is relevant to ask whether there exist continuous eigenfunctions; or even under which conditions measure theoretical eigenvalues can be associated to continuous eigenfunctions.

In [CDHM] these questions are considered for *linearly recurrent systems*. These systems are characterized by the existence of a nested sequence of clopen (for closed and open) Kakutani-Rohlin (CKR) partitions of the system $(\mathcal{P}(n); n \in \mathbb{N})$ verifying some technical conditions we call **(KR1)**, **(KR2)**,..., **(KR6)** (see below), and such that the height of the towers of each partition increases "linearly" from one level to the other. A partial answer to the former question is given in terms of the sequence of matrices $(M(n); n \geq 1)$ relating towers from different levels in [CDHM]. A complete answer to this question is given in the next theorem.

We need some extra notations. For each real number $x$ we write $\|x\|$ for the distance of $x$ to the nearest integer. For a vector $V = (v_1, \ldots, v_m)^T \in \mathbb{R}^m$, we write

$$\|V\| = \max_{1\leq j\leq m} |v_j| \text{ and } \|V\| = \max_{1\leq j\leq m} \|v_j\| .$$

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For $n \geq 2$ we put $P(n) = M(n) \cdots M(2)$ and $H(1) = M(1)$.

**Theorem 1.** *Let $(X, T)$ be a linearly recurrent Cantor system given by an increasing sequence of CKR partitions with associated matrices $(M(n); n \geq 1)$, and let $\mu$ be the unique invariant measure. Let $\lambda = \exp(2i\pi\alpha)$.*

(1) *$\lambda$ is an eigenvalue of $(X, T)$ with respect to $\mu$ if and only if*

$$\sum_{n\geq 2} \|\alpha P(n) H(1)\|^2 < \infty \ .$$

(2) *$\lambda$ is a continuous eigenvalue of $(X, T)$ if and only if*

$$\sum_{n\geq 2} \|\alpha P(n) H(1)\| < \infty \ .$$

In [CDHM] the authors prove the necessary condition in the statement (1) and the sufficient condition in the statement (2). One of the most relevant facts is that both conditions do not depend on the order of levels in the towers defining the system but just on the matrices.

## 2. Definitions and background

2.1. **Dynamical systems.** By a *topological dynamical system* we mean a couple $(X, T)$ where $X$ is a compact metric space and $T : X \to X$ is a homeomorphism. We say that it is a *Cantor system* if $X$ is a Cantor space; that is, $X$ has a countable basis of its topology which consists of closed and open sets (clopen sets) and does not have isolated points. We only deal here with minimal Cantor systems.
A complex number $\lambda$ is a *continuous eigenvalue* of $(X, T)$ if there exists a continuous function $f : X \to \mathbb{C}$, $f \neq 0$, such that $f \circ T = \lambda f$; $f$ is called a *continuous eigenfunction* (associated to $\lambda$). Let $\mu$ be a $T$-invariant probability measure, i.e., $T\mu = \mu$, defined on the Borel $\sigma$-algebra $\mathcal{B}_X$ of $X$. A complex number $\lambda$ is an *eigenvalue* of the dynamical system $(X, T)$ with respect to $\mu$ if there exists $f \in L^2(X, \mathcal{B}_X, \mu)$, $f \neq 0$, such that $f \circ T = \lambda f$; $f$ is called an *eigenfunction* (associated to $\lambda$). If the system is ergodic, then every eigenvalue is of modulus 1, and every eigenfunction has a constant modulus. Of course continuous eigenvalues are eigenvalues.
In this paper we mainly consider topological dynamical systems $(X, T)$ which are uniquely ergodic and minimal. That is, systems that admit a unique invariant probability measure which is ergodic, and such that the unique $T$-invariant sets are $X$ and $\emptyset$.

2.2. **Partitions and towers.** Sequences of partitions associated to minimal Cantor systems were used in [HPS] to build representations of such systems as adic transformations on ordered Bratteli diagrams. Here we do not introduce the whole formalism of Bratteli diagrams since we will only use the language describing the tower structure. Both languages are very close. We recall some definitions and fix some notations.
Let $(X, T)$ be a minimal Cantor system. A *clopen Kakutani-Rokhlin partition* (CKR partition) is a partition $\mathcal{P}$ of $X$ given by

$$\mathcal{P} = \{T^{-j} B_k; 1 \leq k \leq C,\ 0 \leq j < h_k\} \tag{2.1}$$

where $C$ is a positive integer, $B_1, \ldots, B_C$ are clopen subsets of $X$ and $h_1, \ldots, h_k$ are positive integers. For $1 \leq k \leq C$, the $k$-th *tower* of $\mathcal{P}$ is

$$\mathcal{T}_k = \bigcup_{j=0}^{h_k-1} T^{-j} B_k$$

and its height is $h_k$; the *roof* of $\mathcal{P}$ is the set $B = \bigcup_{1\le k\le C} B_k$. Let

$$\big(\mathcal{P}(n) = \{T^{-j}B_k(n); 1 \le k \le C(n),\ 0 \le j < h_k(n)\}\ ;\ n \in \mathbb{N}\big) \tag{2.2}$$

be a sequence of CKR partitions. For every $n \in \mathbb{N}$ and $1 \le k \le C(n)$, $B(n)$ is the roof of $\mathcal{P}(n)$ and $\mathcal{T}_k(n)$ is the $k$-th tower of $\mathcal{P}(n)$. We assume that $\mathcal{P}(0)$ is the trivial partition, that is, $B(0) = X$, $C(0) = 1$ and $h_1(0) = 1$.
We say that $(\mathcal{P}(n); n \in \mathbb{N})$ is *nested* if for every $n \in \mathbb{N}$ it satisfies:

**(KR1)** $B(n+1) \subseteq B(n)$;

**(KR2)** $\mathcal{P}(n+1) \succeq \mathcal{P}(n)$; i.e., for all $A \in \mathcal{P}(n+1)$ there exists $A' \in \mathcal{P}(n)$ such that $A \subseteq A'$;

**(KR3)** $\bigcap_{n\in\mathbb{N}} B(n)$ consists of a unique point;

**(KR4)** the sequence of partitions spans the topology of $X$;

In [HPS] it is proven that given a minimal Cantor system $(X, T)$ there exists a nested sequence of CKR partitions fulfilling **(KR1)**–**(KR4)** (**(KR1)**, **(KR2)**, **(KR3)** and **(KR4)**) and the following additional technical conditions:

**(KR5)** for all $n \ge 1$, $1 \le k \le C(n-1)$, $1 \le l \le C(n)$, there exists $0 \le j < h_l(n)$ such that $T^{-j}B_l(n) \subseteq B_k(n-1)$;

**(KR6)** for all $n \ge 1$, $B(n) \subseteq B_1(n-1)$.

We associate to $(\mathcal{P}(n); n \in \mathbb{N})$ the sequence of matrices $(M(n); n \ge 1)$, where $M(n) = (m_{l,k}(n); 1 \le l \le C(n), 1 \le k \le C(n-1))$ is given by

$$m_{l,k}(n) = \#\{0 \le j < h_l(n); T^{-j}B_l(n) \subseteq B_k(n-1)\}.$$

Notice that **(KR5)** is equivalent to: for all $n \ge 1$, $M(n)$ has strictly positive entries. For $n \ge 0$ set $H(n) = (h_l(n); 1 \le l \le C(n))^T$. As the sequence of partitions is nested $H(n) = M(n)H(n-1)$ for $n \ge 1$. Notice that $H(1) = M(1)$. For $n > m \ge 0$ we define

$$P(n, m) = M(n)M(n-1)\dots M(m+1) \text{ and } P(n) = P(n, 1)\ .$$

Clearly

$$P_{l,k}(n, m) = \#\big\{0 \le j < h_l(n);\ T^{-j}B_l(n) \subseteq B_k(m)\big\},$$

for $1 \le l \le C(n),\ 1 \le k \le C(m)$, and

$$P(n, m)H(m) = H(n) = P(n)H(1)\ .$$

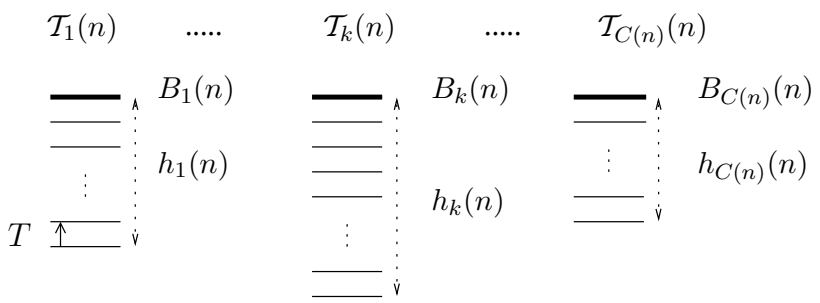


FIGURE 1. CKR partition of level $n$: (a) $X$ is partitioned in $C(n)$ towers. Each tower $\mathcal{T}_k(n)$, $1 \leq k \leq C(n)$, is composed by $h_k(n)$ disjoint sets, called *stages* of the tower. The top of a tower is the roof $B_k(n)$. (b) The dynamics of $T$ consists in going up from one stage to the other of a tower until the roof. Points in a roof are sent to the bottom of the towers; two points in the same roof can be send to different towers.

2.3. **Linearly recurrent systems.** The notion of linearly recurrent minimal Cantor system (also called linearly recurrent system) in the generality we present below was stated in [CDHM]. It is an extension of the concept of linearly recurrent subshift introduced in [DHS].

**Definition 2.** *A minimal Cantor system $(X,T)$ is linearly recurrent (with constant $L$) if there exists a nested sequence of CKR partitions $(\mathcal{P}(n) = \{T^{-j}B_k(n); 1 \leq k \leq C(n), 0 \leq j < h_k(n)\}; n \in \mathbb{N})$ satisfying* **(KR1)**–**(KR6)** *and*

**(LR)** *there exists $L$ such that for all $n \geq 1$, $l \in \{1, \dots, C(n)\}$ and $k \in \{1, \dots, C(n-1)\}$*

$$h_l(n) \leq L\ h_k(n-1)\ .$$

Most of the basic dynamical properties of linearly recurrent minimal Cantor systems are described in [CDHM]. In particular, they are uniquely ergodic and the unique invariant measure is never strongly mixing. In addition, $C(n) \leq L$ for any $n \in \mathbb{N}$ and the set of matrices $\{M(n); n \geq 1\}$ is finite.
To prove Theorem 1 we will need to consider property

**(KR5')** for all $n \geq 2$, $1 \leq k \leq C(n-1)$, $1 \leq l \leq C(n)$, there exist $0 \leq j < j' < h_l(n)$ such that $T^{-j}B_l(n) \subseteq B_k(n-1)$ and $T^{-j'}B_l(n) \subseteq B_k(n-1)$,

instead of **(KR5)**. This condition is equivalent to say that the coefficients of $M(n)$ are strictly larger than 1 for $n \geq 2$.
Let $(X,T)$ be a linearly recurrent system given by a nested sequence of CKR partitions $(\mathcal{P}(n); n \in \mathbb{N})$ which verifies **(KR1)**-**(KR6)** and **(LR)**. Then the sequence of partitions defined by $\mathcal{P}'(0) = \mathcal{P}(0)$ and $\mathcal{P}'(n) = \mathcal{P}(2n-1)$ for $n \geq 1$, is a sequence of nested CKR partitions of the system which verifies **(KR1)**–**(KR4)**, **(KR5')**, **(KR6)** and **(LR)** (with another constant). It follows that $M'(1) = M(1)$ and $M'(n) = M(2n-1)M(2n-2)$ for $n \geq 2$, where $(M(n); n \geq 1)$ and $(M'(n); n \geq 1)$ are the sequence of matrices associated to the partitions

$(\mathcal{P}(n); n \in \mathbb{N})$ and $(\mathcal{P}'(n); n \in \mathbb{N})$ respectively. Moreover,

$$\sum_{n\geq 2} \|\alpha P(n)H(1)\|^p < \infty \;\Leftrightarrow\; \sum_{n\geq 2} \|\alpha P'(n)H(1)\|^p < \infty \tag{2.3}$$

where $\alpha \in \mathbb{R}$ and $p \in \{1, 2\}$.

## 3. Markov chain associated to a linearly recurrent system

Let $(X, T)$ be a linearly recurrent system and let $\mu$ be its unique invariant measure. Consider a sequence $(\mathcal{P}(n); n \geq 0)$ of CKR partitions which satisfies **(KR1)**-**(KR6)** and **(LR)** with constant $L$ and let $(M(n); n \geq 1)$ be the sequence of matrices associated. The purpose of this section is to formalize the fact that there exists a Markovian measurable structure behind the tower structure.
The following relation will be of constant use in the paper. For $n \geq 1$ put $\mu(n) = (\mu(B_t(n)); 1 \leq t \leq C(n))$ (the vector of measures of the roofs at level $n$). It follows directly from the structure of towers that for $1 \leq k < n$

$$\mu(n-k) = M^T(n-k+1)\cdots M^T(n)\mu(n)\ . \tag{3.1}$$

3.1. **First entrance times and combinatorial structure of the towers.** In this subsection we define several concepts that will be extensively used later. An illustration of them is given in Figure 2.
Define the first entrance time map to the roof $B(n)$, $r_n : X \to \mathbb{N}$, by

$$r_n(x) = \min\{j \geq 0; T^j(x) \in B(n)\}\ .$$

Since $(X, T)$ is minimal and $B(n)$ is a clopen set, then $r_n$ is finite and continuous. Define the tower of level $n$ map $\tau_n : X \to \mathbb{N}$ by

$$\tau_n(x) = k \text{ if and only if } x \in \mathcal{T}_k(n) \text{ for some } 1 \leq k \leq C(n)\ .$$

Remark that

$$r_n(T(x)) - r_n(x) = \begin{cases} -1 & \text{if } x \notin B(n), \\ h_k(n) - 1 & \text{if } x \in B(n) \text{ and } \tau_n(T(x)) = k. \end{cases} \tag{3.2}$$

Let $n \geq 1$ and $1 \leq t \leq C(n)$. By hypothesis **(KR5)**, several stages in the tower $\mathcal{T}_t(n)$ are included in the roof $B(n-1)$, in particular stage $B_t(n)$. The number of such stages is

$$m_t(n) = \sum_{k=1}^{C(n-1)} m_{t,k}(n) = \#\{0 \leq j < h_t(n); T^{-j}B_t(n) \subseteq B(n-1)\}\ .$$

Let $\{e_1, e_2, \ldots, e_{m_t(n)}\} = \{0 \leq j < h_t(n); T^{-j}B_t(n) \subseteq B(n-1)\}$ with $h_t(n) > e_1 > e_2 > ... > e_{m_t(n)} = 0$. The integers $e_1, ..., e_{m_t(n)}$ are the first entrance times of points belonging to $\mathcal{T}_t(n) \cap B(n-1)$ into $B_t(n)$. Moreover, for all $1 \leq l \leq m_t(n)$ there is a unique $k \in \{1, ..., C(n-1)\}$ such that

$$T^{-e_l}B_t(n) \subseteq B_k(n-1).$$

Denote this $k$ by $\theta_l^t(n-1)$. From **(KR6)** we have

$$\theta^t_{m_t(n)}(n-1) = 1. \tag{3.3}$$

We set

$$\theta^t(n-1) = \theta_1^t(n-1)\cdots\theta_{m_t(n)}^t(n-1) \in \{1,...,C(n-1)\}^*. \tag{3.4}$$

Remark that $e_l - e_{l+1}$ is the height of the $\theta_{l+1}^t(n-1)$-th tower of $\mathcal{P}(n-1)$ for $1 \le l < m_t(n)$. Thus,

$$e_l = \sum_{k=l+1}^{m_t(n)} h_{\theta_k^t(n-1)}(n-1).$$

Now, the tower $\mathcal{T}_t(n)$ can be decomposed as a disjoint union of the towers of $\mathcal{P}(n-1)$ it intersects. More precisely, $\mathcal{T}_t(n) = \bigcup_{l=1}^{m_t(n)} \mathcal{E}_{l,t}(n-1)$, where

$$\mathcal{E}_{l,t}(n-1) = \bigcup_{j=e_{l-1}-1}^{e_l} T^{-j}B_t(n) = \bigcup_{j=0}^{h_{\theta_l^t(n-1)}(n-1)-1} T^{-j-e_l}B_t(n).$$

By definition,

$$\mathcal{E}_{l,t}(n-1) \subseteq \bigcup_{j=0}^{h_{\theta_l^t(n-1)}(n-1)-1} T^{-j}B_{\theta_l^t(n-1)}(n-1).$$

For $x \in X$ denote by $l_n(x)$ the unique integer in $\{1,...,m_{\tau_n(x)}(n)\}$ such that $x \in \mathcal{E}_{l_n(x),\tau_n(x)}(n-1)$. The following lemma follows from the construction. The proof is left to the reader.

**Lemma 3.** *For all $x \in X$ we have*

$$\bigcap_{k=1}^{n} \mathcal{E}_{l_k(x),\tau_k(x)}(k-1) = T^{-r_n(x)}B_{\tau_n(x)}(n); \tag{3.5}$$

$$\{x\} = \bigcap_{n\ge 1} \mathcal{E}_{l_n(x),\tau_n(x)}(n-1). \tag{3.6}$$

*Moreover, given*

$$(t_n; n\ge 0) \in \prod_{n\ge 0}\{1,...,C(n)\}, \quad (j_n; n\ge 1) \in \prod_{n\ge 1}\{1,...,m_{t_n}(n)\}$$

*such that $\theta_{j_n}^{t_n}(n-1) = t_{n-1}$ for $n \ge 1$, then there exists a unique $x \in X$ such that*

$$((l_n(x),\tau_n(x)); n\ge 1) = ((j_n,t_n); n\ge 1)\ . \tag{3.7}$$

Remark that the set in (3.5) is the atom of the partition $P(n)$ containing $x$.

For all $n \ge 1$ and $x \in X$ define $s_{n-1}(x) = (s_{n-1,t}(x); 1 \le t \le C(n-1))$ by

$$s_{n-1,t}(x) = \#\{j; r_{n-1}(x) < j \le r_n(x),\ T^j x \in B_t(n-1)\}.$$

It also holds that,

$$s_{n-1,t}(x) = \#\{j; l_n(x) < j \le m_{\tau_n(x)}(n),\ \theta_j^{\tau_n(x)}(n-1) = t\}.$$

In other words, the vector $s_{n-1}(x)$ counts, in each coordinate $1 \le t \le C(n-1)$, the number of times the tower $\mathcal{T}_t(n-1)$ is crossed by a point $x$, after its first return to the roof of level $n-1$, and before reaching the roof of the tower of level $n$ it

belongs to. Notice that $s_{n-1}$ does not consider the order in which the towers are visited. In the following figure we illustrate the notations introduced previously.

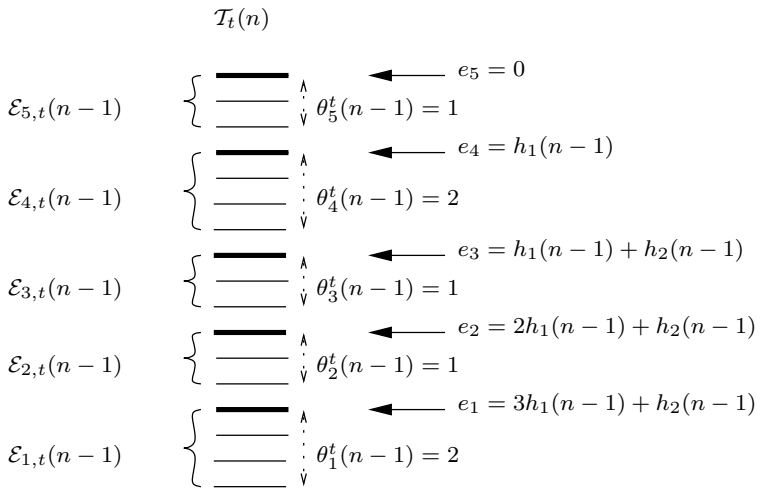


FIGURE 2. In the figure we present tower $t$ of $\mathcal{P}(n)$ in a particular example. We assume that in $\mathcal{P}(n-1)$ there are only two towers and that $m_t(n) = 5$. If $x \in \mathcal{E}_{1,t}(n-1)$ then $s_{n-1}(x) = (3,1)^T$ and $l_n(x) = 1$. If $x \in \mathcal{E}_{4,t}(n-1)$ then $s_{n-1}(x) = (1,0)^T$ and $l_n(x) = 4$.

A direct computation yields to the following lemma. It will be extensively used in the sequel. Denote by $< \cdot, \cdot >$ the usual scalar product.

**Lemma 4.** *For all $x \in X$ and all $n \geq 2$ it holds,*

$$r_1(x) = s_0(x);\ r_n(x) = r_{n-1}(x) + < s_{n-1}(x), H(n-1) >;$$
$$r_n(x) = \sum_{j=2}^{n-1} < s_j(x), P(j)H(1) > + < s_1(x), H(1) > + s_0(x).$$

3.2. **Markov property for the towers.** Now we prove the sequence of random variables $(\tau_n; n \in \mathbb{N})$ is a non-stationary Markov chain. We need some preliminary computations. Let $n \geq 1$. From Lemma 3 we have

$$\mu(B_{\tau_n(x)}(n)) = \mu\left(\bigcap_{k=1}^{n} \mathcal{E}_{l_k(x),\tau_k(x)}(k-1)\right).$$

Let $(t_i \in \{1, ..., C(i)\}; 0 \leq i \leq n)$. The set $[\tau_n = t_n]$ is the tower $\mathcal{T}_{t_n}(n)$. For $0 \leq k < n$, $\tau_k(x)$ is constant on each level of $\mathcal{T}_{t_n}(n)$. By a simple induction, the number of levels of this tower where $\tau_0(x) = t_0, \ldots, \tau_{n-1}(x) = t_{n-1}$ is equal to $m_{t_1,t_0}(1) \cdots m_{t_n,t_{n-1}}(n)$. In other words, the set $[\tau_0 = t_0, ..., \tau_n = t_n]$ is the union of $m_{t_1,t_0}(1) \cdots m_{t_n,t_{n-1}}(n)$ levels of the tower $\mathcal{T}_{t_n}(n)$ and

$$\mu[\tau_0 = t_0, ..., \tau_n = t_n] = m_{t_1,t_0}(1) \cdots m_{t_n,t_{n-1}}(n)\mu(B_{t_n}(n)) \ . \tag{3.8}$$

In particular, from the last equality and the definition of the matrices $(M(n); n \geq 1)$ we deduce

$$\mu[\tau_n = t_n | \tau_{n-1} = t_{n-1}] = \frac{m_{t_n,t_{n-1}}(n)\mu(B_{t_n}(n))}{\mu(B_{t_{n-1}}(n-1))}.$$

Now, given the sequence $(\mathcal{P}(n); n \in \mathbb{N})$ we can prove $(\tau_n; n \in \mathbb{N})$ is a Markov chain on the probability space $(X, \mathcal{B}_X, \mu)$. Therefore, by (3.1), the matrix $Q(n) = (q_{t,\bar{t}}(n); 1 \leq \bar{t} \leq C(n), 1 \leq t \leq C(n-1))$ with

$$q_{t,\bar{t}}(n) = \frac{m_{\bar{t},t}(n)\mu(B_{\bar{t}}(n))}{\mu(B_t(n-1))}$$

is a stochastic matrix.

**Lemma 5.** *The sequence of random variables $(\tau_n; n \in \mathbb{N})$ is a non-stationary Markov chain with associated stochastic matrices $(Q(n); n \geq 1)$.*

*Proof.* From (3.8) we get

$$\begin{aligned}
&\mu[\tau_n = \bar{t} | \tau_{n-1} = t, \tau_{n-2} = t_{n-2}, \ldots, \tau_0 = t_0] \\
&= \frac{m_{t_1,t_0}(1) \cdots m_{t,t_{n-2}}(n-1) m_{\bar{t},t}(n) \mu(B_{\bar{t}}(n))}{m_{t_1,t_0}(1) \cdots m_{t,t_{n-2}}(n-1) \mu(B_t(n-1))} \\
&= \frac{m_{\bar{t},t}(n)\mu(B_{\bar{t}}(n))}{\mu(B_t(n-1))} \\
&= \mu[\tau_n = \bar{t} | \tau_{n-1} = t] \\
&= q_{t,\bar{t}}(n).
\end{aligned}$$

□

The following lemma provides an exponential mixing property for non-stationary ergodic Markov chains. It is a standard result. The proof can be adapted from that of Corollary 2 page 141 of [Se]. That is, this corollary can be generalized to the case of a non-stationary Markov chain where the stochastic matrices have not necessarily the same dimension. Alternatively, a direct proof follows from inequality (3.3) Theorem 3.1 page 81 of [Se] in the case of our particular matrices.

**Lemma 6.** *Let $(\tau_n; n \in \mathbb{N})$ be the non-stationary Markov chain defined in the previous subsection. There exist $c \in \mathbb{R}_+$ and $\beta \in [0, 1[$ such that for all $n, k \in \mathbb{N}$, with $k \leq n$,*

$$\sup_{1 \leq t \leq C(n-k), 1 \leq \bar{t} \leq C(n)} |\mu[\tau_n = \bar{t} | \tau_{n-k} = t] - \mu[\tau_n = \bar{t}]| \leq c\beta^k \ .$$

## 4. Measurable eigenvalues

The main purpose of this section is to prove Statement (1) of Theorem 1 (this is done in Subsection 4.2). In the first subsection we give a general necessary and sufficient condition to be a measurable eigenfunction of a minimal Cantor system.

### 4.1. **A necessary and sufficient condition to be an eigenvalue.**

We give a general necessary and sufficient condition to be an eigenvalue. We do not use it directly to prove our result, but we think it gives an idea of the classical way to tackle the problem and shows that the difficulty relies in understanding the stochastic behavior of the sequence $(r_n; n \in \mathbb{N})$. We would like to stress the fact

that we still do not have a convincing interpretation of the sequence of functions $\rho_n$ which appears in the next theorem.

**Theorem 7.** *Let $(X,T)$ be a minimal Cantor system and let $\mu$ be an invariant measure. Let $(\mathcal{P}(n); n \in \mathbb{N})$ be a sequence of CKR partitions verifying* **(KR1)-(KR4)**. *A complex number $\lambda = \exp(2i\pi\alpha)$ is an eigenvalue of $(X,T)$ with respect to $\mu$ if and only if there exist real functions $\rho_n : \{1,...,C(n)\} \to \mathbb{R}$, $n \in \mathbb{N}$, such that*

$$\alpha\left(r_n(x) + \rho_n \circ \tau_n(x)\right) \text{ converges } (\mathrm{mod}\ \mathbb{Z}) \tag{4.1}$$

*for $\mu$-almost every $x \in X$ when $n$ tends to infinity.*

*Proof.* Let $\lambda = \exp(2i\pi\alpha)$ be a complex number of modulus 1 such that (4.1) holds and let $g$ be the corresponding limit function. Consider $x \notin \cap_{n\in\mathbb{N}} B(n)$, so $x$ does not belong to $B(n)$ for all large enough $n \in \mathbb{N}$. Then, from (3.2) we get

$$\frac{\exp(2i\pi g(Tx))}{\exp(2i\pi g(x))} = \lim_{n\to\infty} \lambda^{r_n(Tx)-r_n(x)} = \lambda^{-1}.$$

This implies $\lambda$ is an eigenvalue of $(X,T)$ with respect to $\mu$.

Now, assume $\lambda$ is an eigenvalue of $(X,T)$ with respect to $\mu$ and let $g \in L^2(X,\mathcal{B}_X,\mu)$ be an associated eigenfunction. For all $n \in \mathbb{N}$ let $\phi_n = \lambda^{-r_n}$ and $\psi_n = g/\phi_n$. The map $\phi_n$ is $\mathcal{P}(n)$-measurable and bounded, then

$$\phi_n \mathbb{E}_\mu(\psi_n|\mathcal{P}(n)) = \mathbb{E}_\mu(\phi_n\psi_n|\mathcal{P}(n)) = \mathbb{E}_\mu(g|\mathcal{P}(n)) \xrightarrow[n\to\infty]{} g$$

$\mu$-almost everywhere. Since $\psi_n \circ T^{-j}/\psi_n = \lambda^{r_n\circ T^{-j} - r_n - j}$, the restriction of $\psi_n$ to each tower of level $n$ is invariant under $T$. Thus $\mathbb{E}_\mu(\psi_n|\mathcal{P}(n))$ is constant on each of these towers and is therefore equal to the average of $\psi_n$ on each tower.
To finish, for $1 \le i \le C(n)$ we define $\rho_n(i)$ such that

$$\mathrm{Arg}\lambda^{-\rho_n(i)} = \mathrm{Arg}\left(\frac{1}{\mu(B_i(n))}\int_{B_i(n)} \psi_n \mathrm{d}\mu\right).$$

This ends the proof. □

*Remark* 8. The same proof works if we remove the Cantor and clopen hypotheses.

4.2. **Eigenvalues of linearly recurrent systems.** In this subsection we prove Statement (1) of Theorem 1. Recall $(X,T)$ is linearly recurrent and $\mu$ is the unique invariant measure. Let $(\mathcal{P}(n); n \ge 0)$ be a sequence of CKR partitions such that **(KR1)-(KR6)** and **(LR)** with constant $L$ are satisfied. Let $(M(n); n \ge 1)$ be the associated sequence of matrices.
We will need the following lemma. Its proof can be found in [CDHM].

**Lemma 9.** *Let $u \in \mathbb{R}^{C(1)}$ be a real vector such that $\|\!|P(n)u|\!\| \to 0$ as $n \to \infty$. Then, there exist $m \ge 2$, an integer vector $w \in \mathbb{Z}^{C(m)}$ and a real vector $v \in \mathbb{R}^{C(m)}$ with*

$$P(m)u = v + w \text{ and } \|P(n,m)v\| \to 0 \text{ as } n \to \infty\ .$$

Assume the following condition holds:

$$\sum_{n\ge 2} \|\!|\alpha P(n)H(1)|\!\|^2 < \infty\ . \tag{4.2}$$

Then, $\|\!| P(n)(\alpha H(1)) |\!\| \to 0$ as $n \to \infty$. From Lemma 9 there exist an integer $n_0 \geq 2$, a real vector $v \in \mathbb{R}^{C(n_0)}$ and an integer vector $w \in \mathbb{Z}^{C(n_0)}$ such that, $P(n_0)(\alpha H(1)) = v + w$ and $P(n, n_0)v \to 0$ as $n \to \infty$. By modifying a finite number of towers, if needed, we can assume without loss of generality that $n_0 = 1$ and that $H(1) = (1, ..., 1)^T$. So condition (4.2) implies

$$\sum_{n \geq 2} \|P(n)v\|^2 < \infty \ . \tag{4.3}$$

From (2.3), we can also assume without loss of generality that **(KR5')** holds. That is, entries of matrices $M(n)$ are larger than 2 for all $n \geq 2$.
For $n \geq 1$ we define $g_n : X \to \mathbb{R}$ by

$$g_n(x) = s_0(x) + < s_1(x), v > + \sum_{j=2}^{n-1} < s_j(x), P(j)v > \ .$$

Since we are assuming $H(1) = (1, ..., 1)^T$, then $s_0 = 0$ and

$$g_n(x) = \sum_{j=1}^{n-1} < s_j(x), P(j)v > \ ,$$

where we set $P(1) = Id$.

**Lemma 10.** *If (4.2) holds, then the sequence* $(f_n = g_n - \mathbb{E}_\mu(g_n); n \geq 1)$ *converges in* $L^2(X, \mathcal{B}_X, \mu)$.

*Proof.* Let $n \geq 1$. Recall that $\mathcal{P}(n)$ is the partition of level $n$ and let $\mathcal{T}(n)$ be the coarser partition $\{\mathcal{T}_j(n); 1 \leq j \leq C(n)\}$. As usual we identify the finite partitions with the $\sigma$-algebras they span and we use the same notation. Thus $\mathcal{T}(n)$ is the $\sigma$-algebra spanned by the random variable $\tau_n$.
Let $X_n$ be the random variable given by

$$X_n = < s_n, P(n)v > - \mathbb{E}_\mu(< s_n, P(n)v >) \ .$$

We decompose it as $X_n = Y_n + Z_n$ where

$$Y_n = \mathbb{E}_\mu(X_n | \mathcal{P}(n)) \text{ and } Z_n = < s_n, P(n)v > - \mathbb{E}_\mu(< s_n, P(n)v > | \mathcal{P}(n)) \ .$$

We write $\kappa_n = \|P(n)v\|$. Observe that for some positive constant $K$ and all $n \geq 1$ we have $|X_n| \leq K\kappa_n$, $|Y_n| \leq K\kappa_n$ and $|Z_n| \leq K\kappa_n$.
First we show that the series $\sum Z_n$ converges. Let $m$ and $n$ be positive integers with $m < n$. The random variable $Z_m$ is measurable with respect to $\mathcal{P}(m+1)$, thus also with respect to $\mathcal{P}(n)$. Since $\mathbb{E}_\mu(Z_n | \mathcal{P}(n)) = 0$ we get $\mathbb{E}_\mu(Z_m \cdot Z_n) = 0$. As $|Z_n| \leq K\kappa_n$ for every $n \geq 1$, the series $\sum \mathbb{E}_\mu(Z_n^2)$ converges, and thus the orthogonal series $\sum Z_n$ converges in $L^2(X, \mathcal{B}_X, \mu)$.
Now we prove that the series $\sum Y_n$ converges in $L^2(X, \mathcal{B}_X, \mu)$. Fix $j \geq 1$ and $1 \leq \bar{t} \leq C(n+1)$. The set $\mathcal{E}_{j,\bar{t}}(n)$ is included in the tower $\mathcal{T}_t(n)$ where $t = \theta_j^{\bar{t}}(n)$. Moreover, the intersection of all levels of $\mathcal{T}_t(n)$ with $\mathcal{E}_{j,\bar{t}}(n)$ are levels of $\mathcal{T}_{\bar{t}}(n+1)$ (see Figure 2) and thus have the same measure $\mu(B_{\bar{t}}(n+1))$. As each level of the tower $\mathcal{T}_t(n)$ has measure $\mu(B_t(n))$ we have

$$\mu(\mathcal{E}_{j,\bar{t}}(n) | \mathcal{P}(n))(x) = \begin{cases} \frac{\mu(B_{\bar{t}}(n+1))}{\mu(B_t(n))} & \text{if } x \in \mathcal{T}_t(n) \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

Observe that this conditional probability is constant in each atom of $\mathcal{T}(n)$ and thus

$$\mu(\mathcal{E}_{j,\bar{t}}(n)|\mathcal{P}(n)) = \mu(\mathcal{E}_{j,\bar{t}}(n)|\mathcal{T}(n)) \ .$$

As $s_n$ is constant on each set $\mathcal{E}_{j,\bar{t}}(n)$, the same property holds for $X_n$ and thus

$$Y_n = \mathbb{E}_\mu(X_n|\mathcal{P}(n)) = \mathbb{E}_\mu(X_n|\mathcal{T}(n)) \ . \tag{4.5}$$

In particular $Y_n$ is equal to a constant on each set $[\tau_n = \bar{t}\,]$ and we write $y_{\bar{t}}$ for this constant. Fix $k$ with $0 \leq k \leq n$. If $\tau_{n-k}(x) = t$ we have

$$\mathbb{E}_\mu(Y_n|\mathcal{T}(n-k))(x) = \sum_{\bar{t}=1}^{C(n)} \mu[\tau_n = \bar{t}\ |\tau_{n-k} = t]y_{\bar{t}} \ .$$

We have

$$\sum_{\bar{t}=1}^{C(n)} \mu[\tau_n = \bar{t}\,]y_{\bar{t}} = \mathbb{E}_\mu(Y_n) = 0 \ .$$

We deduce from Lemma (6), the fact that $C(n)$ is bounded independently of $n$ and $|Y_n| \leq K\kappa_n$ that for some positive constant $C$

$$|\mathbb{E}_\mu(Y_n|\mathcal{T}(n-k))(x)| \leq \sum_{\bar{t}=1}^{C(n)} |\mu[\tau_n = \bar{t}|\tau_{n-k} = t] - \mu[\tau_n = \bar{t}]|y_{\bar{t}} \leq C\beta^k\kappa_n \ .$$

As $Y_{n-k}$ is measurable with respect to $\mathcal{T}(n-k)$ we have

$$|\mathbb{E}_\mu(Y_n \cdot Y_{n-k})| \leq C\beta^k\kappa_n\mathbb{E}_\mu(|Y_{n-k}|) \leq C\beta^k\kappa_n\kappa_{n-k} \ .$$

For $1 \leq m < n$ we compute

$$\begin{aligned}
\mathbb{E}_\mu\left(\left(\sum_{k=m}^{n} Y_k\right)^2\right) &= \sum_{m\leq j,l\leq n} \mathbb{E}_\mu(Y_j \cdot Y_l) \leq C \sum_{m\leq j,l\leq n} \beta^{j-l}\kappa_j\kappa_l \\
&= C\sum_{r=0}^{n-m}\beta^r\sum_{l=m}^{n-r}\kappa_l\kappa_{l+r} \leq C\sum_{r=0}^{n-m}\beta^r\sum_{l=m}^{n}\kappa_l^2 \\
&\leq \frac{C}{1-\beta}\sum_{l=m}^{n}\kappa_l^2 \ .
\end{aligned}$$

Since the series $\sum \kappa_j^2$ converges, the partial sums of the series $\sum Y_n$ form a Cauchy sequence in $L^2(X, \mathcal{B}_X, \mu)$. □

The following lemma completes the proof of Theorem 1 (1).

**Lemma 11.** *Let $f \in L^2(X, \mathcal{B}_X, \mu)$ be the limit of sequence $(f_n; n \geq 1)$. The function $\exp(2i\pi f)$ is an eigenfunction of $(X, T)$ with respect to $\mu$ associated to the eigenvalue $\exp(2i\pi\alpha)$.*

*Proof.* Remark that $g_n(x) = \alpha r_{n-1}(x) \pmod{\mathbb{Z}}$. From relation (3.2) we get

$$f_n(Tx) = f_n(x) - \alpha \pmod{\mathbb{Z}}$$

holds outside of the roof $B(n)$ and $\mu(B(n)) \to 0$ as $n \to \infty$. We conclude using Lemma 10. □

## 5. Continuous Eigenvalues of Linearly Recurrent Systems

Let $(X,T)$ be a linearly recurrent dynamical system with constant $L$. The main purpose of this section is to prove the necessary condition in the statement (2) of Theorem 1. We recall the sufficient condition was proven in [CDHM].

5.1. **A necessary and sufficient condition to be a continuous eigenvalue.** In this subsection we only assume that $(\mathcal{P}(n); n \in \mathbb{N})$ is a sequence of CKR partitions describing the system $(X,T)$ which satisfies **(KR1)**-**(KR6)**. We give a general necessary and sufficient condition to be a continuous eigenvalue.

**Proposition 12.** *Let $\lambda = \exp(2i\pi\alpha)$ be a complex number of modulus 1. The following conditions are equivalent,*

1. *$\lambda$ is a continuous eigenvalue of the minimal Cantor system $(X,T)$;*
2. *$(\lambda^{r_n(x)}; n \geq 1)$ converges uniformly in $x$, i.e., the sequence $(\alpha r_n(x); n \geq 1)$ converges* (mod $\mathbb{Z}$) *uniformly in $x$.*

*Proof.* We start proving that (1) implies (2). Let $g$ be a continuous eigenfunction associated to $\lambda$. For all $n \geq 1$ and all $x \in X$ we have $T^{r_n(x)}(x) \in B(n) \subseteq B_1(n-1)$ (the last inclusion is due to **(KR6)**). Hence, using **(KR3)**, we deduce that $\lim_{n\to\infty} T^{r_n(x)}(x) = u$ uniformly in $x$, where $u$ is the unique element of $\cap_{n\geq 0} B(n)$. The eigenfunction $g$ being uniformly continuous we have that $\lambda^{r_n(x)} = g(T^{r_n(x)}(x))/g(x)$ tends to $g(u)/g(x)$ uniformly in $x$.
Now we prove (2) implies (1). We set $\phi(x) = \lim_{n\to\infty} \lambda^{r_n(x)}$. Since the convergence is uniform and $r_n$ is continuous, then $\phi$ is continuous.
Let $x$ be such that $x \notin B(n)$ for infinitely many $n$. Then, from (3.2), we obtain $\phi(T(x)) = \lambda^{-1}\phi(x)$. Using the minimality of $(X,T)$ and the continuity of $\phi$, we obtain that $\phi(T(y)) = \lambda^{-1}\phi(y)$ for all $y \in X$. Consequently $\lambda$ is a continuous eigenvalue. □

**Corollary 13.** *Let $\lambda$ be a complex number of modulus* 1*.*

1. *If $\lambda$ is a continuous eigenvalue of $(X,T)$ then*
$$\lim_{n\to\infty} \lambda^{h_{j_n}(n)} = 1$$
*uniformly in $(j_n; n \in \mathbb{N}) \in \prod_{n\in\mathbb{N}} \{1, \ldots, C(n)\}$.*
2. *If*
$$\sum_{m\geq 1} \left( \frac{\sup_{k\in\{1,\ldots,C(m+1)\}} h_k(m+1)}{\inf_{k\in\{1,\ldots,C(m)\}} h_k(m)} \right) \sup_{k\in\{1,\ldots,C(m)\}} \mid \lambda^{h_k(m)} - 1 \mid < \infty$$
*then $\lambda$ is a continuous eigenvalue of $(X,T)$.*

*Proof.* Let $g$ be a continuous eigenfunction of $\lambda$. Then, it is uniformly continuous. Let $\epsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that $|g(y) - g(u)| < \epsilon/2$ for all $y \in B_1(n_0)$, where $\{u\} = \cap_{n\in\mathbb{N}} B(n)$.
Let $(j_n; n \in \mathbb{N}) \in \prod_{n\in\mathbb{N}} \{1, \ldots, C(n)\}$. For all $n \in \mathbb{N}$ we take $x(n) \in B_{j_n}(n)$ and we set $y(n) = T^{-h_{j_n}(n)}(x(n)) \in B(n)$. Hence, using **(KR6)**, for all $n \geq n_0 + 1$ the points $x(n)$ and $y(n)$ belong to $B(n) \subseteq B_1(n_0)$. Consequently

$$\begin{aligned} |\lambda^{h_{j_n}(n)} - 1| &= |g(T^{h_{j_n}(n)} y(n)) - g(y(n))| \\ &\leq |g(x(n)) - g(u)| + |g(u) - g(y(n))| < \epsilon. \end{aligned}$$

Now we prove (2). It suffices to remark, by Lemma 4, that for all $x \in X$ and all $0 < n < m$,

$$| \lambda^{r_m(x)} - \lambda^{r_n(x)} | = | 1 - \lambda^{r_m(x)-r_n(x)} |$$

$$\leq \sum_{l=n}^{m-1} \frac{\sup_{k\in\{1,\dots,C(l+1)\}} h_k(l+1)}{\inf_{k\in\{1,\dots,C(l)\}} h_k(l)} \sup_{k\in\{1,\dots,C(l)\}} | 1 - \lambda^{h_k(l)} | .$$

Hence, from Proposition 12, $\lambda$ is a continuous eigenvalue. □

Remark that for linearly recurrent systems statement (2) gives the sufficient condition for $\lambda$ to be a continuous eigenvalue. This was proved in [CDHM].

5.2. **The linearly recurrent case.** Now we assume $(X, T)$ is linearly recurrent and we prove Theorem 1 part (2). We also assume without loss of generality that the sequence of partitions verifies **(KR5')**, that is entries of $M(n)$ are bigger than 2 for any $n \geq 2$ (see discussion in subsection 2.3). To prove the result we introduce an intermediate statement which gives a more precise interpretation to the necessary condition.

**Proposition 14.** *Let $\lambda = \exp(2i\pi\alpha)$ be a complex number of modulus 1. The following properties are equivalent.*

(1) *$\lambda$ is a continuous eigenvalue of the minimal Cantor system $(X, T)$.*
(2) *There exist $n_0 \in \mathbb{N}$, $v \in \mathbb{R}^{C(n_0)}$, $z \in \mathbb{Z}^{C(n_0)}$, such that $\alpha P(n_0)H(1) = v + z$, $P(n, n_0)v \to 0$ as $n \to \infty$ and the series*

$$\sum_{j\geq n_0+1} < s_j(x), P(j, n_0)v >$$

*converges for every $x \in X$.*
(3) *$\sum_{n\geq 2} \|\alpha P(n)H(1)\| < \infty$*

*Proof.* In [CDHM] it is proven that (3) implies (1).

We prove that (1) implies (2): assume $\lambda$ is a continuous eigenvalue of $(X, T)$. We deduce from statement (1) of Corollary 13 that $\|\alpha P(n)H(1)\|$ converges to 0 as $n$ tends to $\infty$. By Lemma 9, there are $n_0 \in \mathbb{N}$, $v \in \mathbb{R}^{C(n_0)}$ and $z \in \mathbb{Z}^{C(n_0)}$ such that $\alpha P(n_0)H(1) = v + z$ and $P(n, n_0)v \to 0$ as $n \to \infty$. By modifying a finite number of towers we can assume without loss of generality that $n_0 = 1$.
By Lemma 4, for $n \geq 1$ and $x \in X$, $r_n(x) = \sum_{j=1}^{n-1} < s_j(x), P(j)H(1) > +s_0(x)$, where we put $P(1) = I$. Then,

$$\alpha r_n(x) = \sum_{j=1}^{n-1} < s_j(x), P(j)v > + \sum_{j=1}^{n-1} < s_j(x), P(j)z > +\alpha s_0(x).$$

From Proposition 12, $\sum_{j=1}^{n-1} < s_j(x), P(j)v > +\alpha s_0(x) \to v(x) \pmod{\mathbb{Z}}$ as $n \to \infty$. We distinguish two cases: if $v(x) \in (0, 1)$, we write $\sum_{j=1}^{n-1} < s_j(x), P(j)v > +\alpha s_0(x) = V_n(x) + v_n(x)$ with $V_n(x) \in \mathbb{Z}$ and $v_n(x) \in [0, 1)$; if $v(x) = 0$ we consider $v_n(x) \in [-1/2, 1/2)$. Then, in both cases, $(v_n(x); n \in \mathbb{N})$ converges and a fortiori $(v_{n+1}(x) - v_n(x); n \geq 1) \to 0$ as $n \to \infty$. Moreover,

$$\sum_{j=1}^{n} < s_j(x), P(j)v > - \sum_{j=1}^{n-1} < s_j(x), P(j)v >$$
$$=< s_n(x), P(n)v >= V_{n+1}(x) - V_n(x) + v_{n+1}(x) - v_n(x).$$

Since, for a linearly recurrent system $\{s_n(x); x \in X, n \in \mathbb{N}\}$ is bounded, $P(n)v \to 0$ and $(v_{n+1}(x) - v_n(x)) \to 0$ as $n \to \infty$. We conclude $V_n(x)$ is a constant integer for all large enough $n \in \mathbb{N}$. Consequently the series $\sum_{j\geq 2} < s_j(x), P(j)v >$ converges.

Now we prove that (2) implies (3): we assume, without loss of generality, that $n_0 = 1$ and that for any $x \in X$ the series

$$\sum_{j\geq 2} < s_j(x), P(j)v > \in \mathbb{R}$$

converges. It suffices to prove that $\sum_{j\geq 2} \|P(j)v\| < \infty$.
For $n \geq 2$ define $i(n) \in \{1, ..., C(n)\}$ such that

$$| < e_{i(n)}, P(n)v > | = \max_{i\in\{1,...,C(n)\}} | < e_i, P(n)v > |$$

where $e_i$ is the $i$-th canonical vector of $\mathbb{R}^{C(n)}$. Let

$$I^+ = \{n \geq 2; < e_{i(n)}, P(n)v >\geq 0\},\ I^- = \{n \geq 2; < e_{i(n)}, P(n)v >< 0\}.$$

To prove $\sum_{j\geq 2} \|P(j)v\| < \infty$ we only need to show

$$\sum_{j\in I^+} < e_{i(j)}, P(j)v >< \infty \text{ and } -\sum_{j\in I^-} < e_{i(j)}, P(j)v >< \infty.$$

Since arguments we will use are similar in both cases we only prove the first fact. To prove $\sum_{j\in I^+} < e_{i(j)}, P(j)v >< \infty$ we only show

$$\sum_{j\in I^+\cap 2\mathbb{N}} < e_{i(j)}, P(j)v >< \infty, \tag{5.1}$$

and analogously it can be proven

$$\sum_{j\in I^+\cap(2\mathbb{N}+1)} < e_{i(j)}, P(j)v >< \infty.$$

We construct two points $x, y \in X$ such that $s_n(x) - s_n(y) = e_{i(n)}$ if $n \in I^+ \cap 2\mathbb{N}$ and $s_n(x) - s_n(y) = 0$ elsewhere. By hypothesis, from this fact we conclude (5.1). To construct $x$ and $y$, according to Lemma 3, we only need to produce sequences

$$(t_n; n \in \mathbb{N}) \in \Pi_{n\in\mathbb{N}}\{1, ..., C(n)\},\quad (j_n; n \geq 1) \in \Pi_{n\geq 1}\{1, ..., m_{t_n}(n)\}$$

and

$$(\bar{t}_n; n \in \mathbb{N}) \in \Pi_{n\in\mathbb{N}}\{1, ..., C(n)\},\quad (\bar{j}_n; n \geq 1) \in \Pi_{n\geq 1}\{1, ..., m_{\bar{t}_n}(n)\}$$

such that

$$\theta_{j_n}^{t_n}(n-1) = t_{n-1} \text{ and } \theta_{\bar{j}_n}^{\bar{t}_n}(n-1) = \bar{t}_{n-1} \text{ for all } n \geq 1. \tag{5.2}$$

The point $x$ is the unique one such that $\tau_n(x) = t_n$ and $l_n(x) = j_n$. Point $y$ is defined analogously with respect to $\bar{t}_n$ and $\bar{j}_n$. Given $n \in (I^+ \cap 2\mathbb{N})^c$ put $t_n = \bar{t}_n = 1$.

For $n \in I^+ \cap 2\mathbb{N}$, by property **(KR5')**, there exist $k \in \{1, ..., m_1(n+1)-1\}$ such that $\theta^1_{k+1}(n) = i(n)$. Put

$$\bar{t}_n = i(n),\ \bar{j}_n = m_{\bar{t}_n}(n),\ t_n = \theta^1_k(n) \text{ and } j_n = m_{t_n}(n).$$

Using (3.3) we obtain $\theta^{t_n}_{j_n}(n-1) = 1$ and $\theta^{\bar{t}_n}_{\bar{j}_n}(n-1) = 1$. Then, we set $\bar{t}_{n+1} = t_{n+1} = 1$, $\bar{j}_{n+1} = k+1$ and $j_{n+1} = k$. Consequently, the relations (5.2) are satisfied for $n$ and $n+1$.
Now we treat the remaining case: $n \in (I^+)^c \cap 2\mathbb{N}$. We recall that $t_n = \bar{t}_n = t_{n+1} = \bar{t}_{n+1} = 1$. It suffices to set

$$j_n = \bar{j}_n = m_1(n) \text{ and } j_{n+1} = \bar{j}_{n+1} = m_1(n+1)$$

to fulfill the relations (5.2).
For each $n \in I^+ \cap 2\mathbb{N}$, the towers of level $n$ visited by $x$ and $y$ after their first entrance time to $B(n)$ and before their first entrance time to $B(n+1)$ are

$$\mathcal{S}_n(x) = \{\theta^1_{k+1}(n), \ldots, \theta^1_{m_1(n+1)}(n)\} \text{ and } \mathcal{S}_n(y) = \{\theta^1_{k+2}(n), \ldots, \theta^1_{m_1(n+1)}(n)\}$$

respectively. Therefore, $s_n(x) - s_n(y) = e_{i(n)}$.
On the other hand, if $n \notin I^+ \cap 2\mathbb{N}$ then $\mathcal{S}_n(x)$ and $\mathcal{S}_n(y)$ are the empty set. Hence $s_n(x) = s_n(y) = 0$. □

## 6. Example: measurable and non continuous eigenvalues

We construct explicitly a system with a nontrivial Kronecker factor but having a trivial equicontinuous factor. Let us consider the commuting matrices

$$A = \begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

We set $\varphi = \frac{1+\sqrt{5}}{2}$. Let $e = (\varphi, 1)^T$, $f = (-1, \varphi)^T$, $\alpha_A = 3 + 2\varphi$, $\beta_A = 5 - 2\varphi$, $\alpha_B = 1 + \varphi$ and $\beta_B = 2 - \varphi$. Observe that $\alpha_A > \alpha_B > \beta_A > 1 > \beta_B > 0$ and $\{e, f\}$ is a base of $\mathbb{R}^2$ made of the common eigenvectors associated to eigenvalues $\alpha_A, \beta_A$ of $A$ and $\alpha_B, \beta_B$ of $B$ respectively.
We define recursively the sequence $(v_n; n \geq 1)$ of real numbers by: $v_1 = 1$ and for all $n > 1$

$$v_{n+1} = \begin{cases} \beta_A v_n & \text{if } nv_n \leq 1 \\ \beta_B v_n & \text{if } nv_n > 1 \end{cases}$$

Notice that the sequence $(nv_n; n \geq 1)$ is uniformly bounded and uniformly bounded away from 0. Now let $H(1) = M(1) = (1, 1)^T$ and for $n \geq 1$

$$M(n+1) = \begin{cases} A & \text{if } nv_n \leq 1 \\ B & \text{if } nv_n > 1 \end{cases}$$

Remark $M(n) = A$ for infinitely many values of $n$.
Define the words in $\{1, 2\}^*$, $\theta^1(A) = 2211111$, $\theta^2(A) = 22211$, $\theta^1(B) = 211$ and $\theta^2(B) = 21$. Let$(X, T)$ be a minimal Cantor system such that there is a sequence of CKR partitions $(\mathcal{P}(n); n \in \mathbb{N})$ verifying **(KR1)-(KR6)** with associated sequence of matrices $(M(n); n \geq 1)$. Moreover, we require that for $n \geq 1$ and $t \in \{1, 2\}$, $\theta^t(n) = \theta^t(M(n+1))$ holds (see (3.4) for the definition of $\theta^t(n)$). This is possible by [HPS] using Bratteli diagrams. It is clear that $(X, T)$ is linearly recurrent. We call $\mu$ its unique ergodic measure.

A symbolic way to see this system is by considering the substitutions $\sigma_A : \{1,2\} \to \{1,2\}^*$, $\sigma_A(1) = 2211111$, $\sigma_A(2) = 22211$, and $\sigma_B : \{1,2\} \to \{1,2\}^*$, $\sigma_B(1) = 211$, $\sigma_B(2) = 21$. Define a sequence of substitutions $(\sigma_n; n \geq 1)$ by $\sigma_1 = Id$ and, for all $n > 1$, $\sigma_{n+1} = \sigma_n \circ \sigma_{M(n)}$. It follows that $...111\sigma_n(1).\sigma_n(2)222...$ converges to some $\omega \in \{1,2\}^{\mathbb{Z}}$, where the dot indicates the position to the left of 0 coordinate. We set $X = \overline{\{T^n(\omega), n \in \mathbb{Z}\}}$, where $T$ is the shift map.
Before to study the system $(X,T)$ defined by this sequence of matrices we need a general property. We keep notations of previous sections.

**Lemma 15.** *Let $v \in \mathbb{R}^{C(1)}$. If $\lim_{n\to\infty} \|P(n)v\| = 0$, then $v$ is orthogonal to the vector $\mu(1) = (\mu(B_k(1)); 1 \leq k \leq C(1))^T$.*

*Proof.* Let $v \in \mathbb{R}^{C(1)}$ be such that $\lim_{n\to\infty} \|P(n)v\| = 0$. Then, for $n > 1$

$$\begin{aligned} | < \mu(1), v > | &= | < P^T(n)\mu(n), v > | \\ &= | < \mu(n), P(n)v > | \\ &\leq \|P(n)v\|, \end{aligned}$$

and the last term converges to 0 as $n \to \infty$. Thus $v$ is orthogonal to $\mu(1)$. □

**Proposition 16.** *Let $(X,T)$ be the linearly recurrent system defined above. The set of eigenvalues of $(X,T)$ is*

$$E_\mu = \left\{\exp(2i\pi\alpha) \in \mathbb{C}; \alpha = (\varphi - 1, 2 - \varphi) A^{-l} w,\ l \geq 0, w \in \mathbb{Z}^2\right\}$$

*None of these eigenvalues is continuous except the trivial one.*

*Proof.* Let $v = -(\varphi-2)f = (\varphi-2, \varphi-1)^T$ and $n \geq 2$. Hence, $P(n)v = P(n)(\varphi H(1)) \pmod{\mathbb{Z}^2}$. Also, since $v$ is an eigenvector of $A$ and $B$, from the definition of $v_n$ we get

$$P(n)v = \beta_{M(n)} \cdots \beta_{M(2)} v = v_n v.$$

The sequence $(v_n)_{n\geq 1}$ was constructed so that $nv_n$ is uniformly bounded and uniformly bounded away from 0. It follows that

$$\sum_{n\geq 2} v_n = \infty \text{ and } \sum_{n\geq 2} v_n^2 < \infty\ , \tag{6.1}$$

and

$$\sum_{n\geq 2} \|P(n)v\| = \infty \text{ and } \sum_{n\geq 2} \|P(n)v\|^2 < \infty. \tag{6.2}$$

In particular, $\lim_{n\to\infty} \|P(n)v\| = 0$ and, by Lemma 15, $v$ is orthogonal to $\mu(1) = (\mu(B_1(1)), \mu(B_2(1))^T$.

**Claim:** *Let $\alpha \in \mathbb{R}$ and $\lambda = \exp(2i\pi\alpha)$: $|||P(n)(\alpha H(1))||| \to 0$ as $n \to \infty$ holds if and only if $\lambda \in E_\mu$. Moreover, if $\lambda \in E_\mu$ then $|||P(n)(\alpha H(1))||| = c\|P(n)v\|$, for some positive constant $c$.*

*Proof of the claim.* First assume $|||P(n)(\alpha H(1))||| \to 0$ as $n \to \infty$ holds. By Lemma 9, there exist $m \geq 2$, an integer vector $w \in \mathbb{Z}^{C(m)}$ and a real vector $v' \in \mathbb{R}^{C(m)}$ with $P(m)(\alpha H(1)) = v' + w$ and $\|P(n)P(m)^{-1}v'\| \to 0$ as $n \to \infty$. From Lemma 15, vector $P(m)^{-1}v'$ is orthogonal to $\mu(1)$. Hence, there exists $k \in \mathbb{R}$ such that $P(m)^{-1}v' = kv$ and

$$P(m)(\alpha H(1)) = kP(m)v + w. \tag{6.3}$$

Suppose $k = 0$. It is not difficult to show by induction that $\gcd(h_1(m), h_2(m)) = 1$. Then, since $w$ is an integer vector, $\alpha \in \mathbb{Z}$ and $\lambda = 1$ which belongs to $E_\mu$.
Suppose $k \neq 0$. Then, $k = W_1 - W_2$ where $P(m)^{-1}w = (W_1, W_2)^T$. This gives,

$$\alpha H(1) = \begin{pmatrix} \varphi - 1 & 2 - \varphi \\ \varphi - 1 & 2 - \varphi \end{pmatrix} P(m)^{-1} w. \tag{6.4}$$

The determinants of the matrices $A$ and $B$ are respectively equal to 11 and 1. Therefore, since $P(m) = A^{l_m} B^{k_m}$ for some $l_m, k_m \geq 0$,

$$\alpha = (\varphi - 1, 2 - \varphi) A^{-l_m} w',$$

with $w' \in \mathbb{Z}^2$. So $\lambda \in E_\mu$.
Conversely, let $\lambda \in E_\mu$. Then, since $M(n) = A$ for infinitely many $n \geq 2$, for $n$ large enough we get

$$P(n)(\alpha H(1)) = P(n) \begin{bmatrix} \varphi - 2 & -(\varphi - 2) \\ \varphi - 1 & -(\varphi - 1) \end{bmatrix} + w,$$

where $w \in \mathbb{Z}^2$. Therefore, $\|\!| P(n)(\alpha H(1)) |\!\| = c \|P(n)v\|$, for some positive constant $c$, which proves the claim.

Finally, the proposition follows from Theorem 1 and property (6.2). □

This proposition gives an example of a minimal dynamical system with a nontrivial Kronecker factor and a trivial maximal equicontinuous factor.

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