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Abstract

We establish the existence and uniqueness of the solution to some inner obstacle problems for a coupling of a multidimensional quasilinear first-order hyperbolic equation set in a region Ω_h with a quasilinear parabolic one set in the complementary $\Omega_p = \Omega \backslash \Omega_h$. We start by providing the definition of a weak solution through an entropy inequality on the whole domain. Since the interface $\partial\Omega_p \cap \partial\Omega_h$ contains the outward characteristics for the first-order operator in Ω_h, the uniqueness proof begins by considering first the hyperbolic zone and then the parabolic one. The existence property uses the vanishing viscosity method and to pass to the limit on the hyperbolic zone, we refer to the notion of process solution.

1 Introduction

1.1 Mathematical setting

Let Ω be a bounded domain of $\mathbb{R}^n$, $n \geq 1$, with a *smooth* boundary Γ and T a finite positive real. This paper is devoted to the mathematical analysis to the unilateral or bilateral inner obstacle problem for the coupling of a quasilinear advection-reaction equation of the form

$$\mathcal{T}_h(u) = \partial_t u - \sum_{i=1}^{n} \partial_{x_i}(K(u)B_i) + g_h(t, x, u) = 0,$$

set in an *hyperbolic zone* $Q_h =]0, T[\times\Omega_h$ with a quasilinear diffusion-advection-reaction equation of the type

$$\mathcal{T}_p(u) = \partial_t u - \sum_{i=1}^{n} \partial_{x_i}(\partial_{x_i}\phi(u) + K(u)B_i) + g_p(t, x, u) = 0,$$

set in a *parabolic area* $Q_p =]0, T[\times(\Omega\backslash\Omega_h)$, complementary to the former, and for suitable conditions across the interface between the two regions Q_p and Q_h. A threshold θ being given, the (bilateral) obstacle problem for $\mathcal{T}_h$ and $\mathcal{T}_p$ may be formally written through the free boundary formulation: find a measurable and bounded function u on $Q \equiv]0,T[\times\Omega$ such that,

$$0 \leq u \leq \theta \text{ on } Q, \tag{1}$$

$$\text{for } i \text{ in } \{h,p\}, \mathcal{T}_i(u) = 0 \text{ on } Q_i \cap [0 < u < \theta], \tag{2}$$

$$\mathcal{T}_i(u) \leq 0 \text{ on } Q_i \cap [0 < u = \theta], \mathcal{T}_i(u) \geq 0 \text{ on } Q_i \cap [0 = u < \theta], \tag{3}$$

$$u = 0 \text{ on }]0,T[\times\Gamma,\ u(0,.) = u_0 \text{ on } \Omega, \tag{4}$$

subject to the transmissions conditions (see Remark 3) along the interface $\Sigma_{hp} =]0,T[\times\Gamma_{hp}$, with $\Gamma_{hp} = \Gamma_h \cap \Gamma_p$ and $\Gamma_i = \partial\Omega_i$, $i \in \{h,p\}$:

$$u_{|Q_h} = u_{|Q_p} \qquad \text{on } \Sigma_{hp} \cap [\mathbf{B}.\nu_h > 0], \tag{5}$$

$$\nabla\phi(u).\nu_p = 0 \qquad \text{on } \Sigma_{hp} \cap [0 < u < \theta], \tag{6}$$

where $\mathbf{B} = (B_1, .., B_n)$, ν_i denoting the outward normal unit vector defined $\mathcal{H}^n$-a.e. on Σ_l and for q in $[0, n+1]$, $\mathcal{H}^q$ being the q-dimensional Hausdorff measure. Lastly $\mathcal{H}^{n-1}(\overline{\Gamma_{hp}} \cap (\overline{\Gamma_l \backslash \Gamma_{hp}})) = 0$.

Remark 1. *Presentation* (1)-(3) *is also valid for the upper unilateral obstacle problem ($u \leq \theta$) (resp. for the lower unilateral obstacle problem ($u \geq \theta$)) by formally replacing the lower bound with "$-\infty$" (resp. the upper bound with "$+\infty$"). Observe that in these situations, for i in $\{h,p\}$, $\mathcal{T}_i(u)$ are non-positive (resp. nonnegative) distributions on Q_i.*

This problem arises from several simplified physical models like infiltration processes in a stratified subsoil viewed as two layers with different geological characteristics and such that in the second layer we can neglect the effects of diffusivity. Indeed when we are interested in the evolution of any effluent c within the flow of substances moving in the subsoil, the first simplified modelling consists in taking into account but one phase saturating the soil, made of two components without any chemical interactions: water and component c. We assume that the distribution of temperatures T and the pressure field P of the fluid phase are determined, sufficiently smooth functions. Then, we refer to P.Bia and M.Combarnous [2] to transcript the mass conservation law for c and we take into account the existence of some saturation thresholds $\theta_{1,c}(T,P)$ and $\theta_{2,c}(T,P)$: beyond those the appearance of a new phase (liquid or solid) for the same number of components changes the thermodynamical nature of the system, which cannot be described through a simplified balance equation. This way, the relations ruling the mass fraction ω_c are formally given by:

$$\mathcal{E}(t,x,\omega_c) = 0 \text{ on } [\theta_1 < \omega_c < \theta_2],$$
$$\mathcal{E}(t,x,\omega_c) \leq 0 \text{ on } [\theta_1 < \omega_c = \theta_2], \mathcal{E}(t,x,\omega_c) \geq 0 \text{ on } [\theta_1 = \omega_c < \theta_2],$$

$$\begin{aligned} \text{where } \mathcal{E}(t,x,\omega_c) \quad &= \quad \rho(T,\omega_c)\left\{\partial_t\omega_c - \frac{k(x)}{\mu(\omega_c)}\nabla\omega_c.(\nabla P - \rho(T,\omega_c)\vec{g})\right\} \\ &\quad -Div\,[\mathcal{A}(x)\rho(T,\omega_c)\nabla\omega_c] \end{aligned} \tag{7}$$

In (7), k denotes the absolute permeability at the point x, μ being the dynamic viscosity of the fluid phase and $\rho(T,\omega_c)$ its density, defined by the composition ω_c at temperature T. Lastly, $\vec{g}$ is the gravity acceleration vector. Furthermore, the molecular diffusion-dispersion effects have been taken into account through the tensor $\mathcal{A}(x)$. But depending on the geological nature of the subsoil these effects may be neglected in favor of the effluent's transport ones. In this situation the evolution of ω_c is ruled by a first-order quasilinear operator.

1.2 Main assumptions on data

For technical reasons (proofs of Theorem 2 and Proposition 3), we assume in this work that the obstacle θ is independent from the time variable. So it will be considered as a measurable function on Ω such that $\theta_{|\Omega_i}$ belongs to $W^{1,+\infty}(\Omega_i)$, i in $\{h,p\}$. In addition $\theta_{|\Omega_p}$ is an element of $H^2(\Omega_p)$. Besides θ is compatible with the boundary condition in the sense $\theta(\bar{\sigma}) \geq 0$ for any $\bar{\sigma}$ of $\partial\Omega$. We set:

$$\forall x \in \Omega,\, \mathcal{C}_\theta(x) = [0, \theta(x)] \text{ and } \mathcal{C}_\theta^\infty = [0, ess \sup_{\bar{\Omega}} \theta]$$

for the bilateral obstacle problem while

$$\mathcal{C}_\theta(x) =]-\infty, \theta(x)] \text{ and } \mathcal{C}_\theta^\infty =]-\infty, ess \sup_{\bar{\Omega}} \theta]$$

for the (upper) unilateral obstacle problem, the reasoning for lower or upper unilateral obstacle problems being similar.

The vector field $\mathbf{B}$ is given in $W^{1,+\infty}(Q)^n$. Especially B_i and $\partial_{x_j} B_i$ are continuous on the whole Ω with respect to the space variable. Moreover

$$\Sigma_{hp} \subset \{\sigma \in \Sigma_h, \mathbf{B}(\sigma).\nu_h \leq 0\}. \tag{8}$$

The initial data u_0 belongs to $L^\infty(\Omega) \cap H_0^1(\Omega)$. In addition, u_0 is an element of $\mathcal{C}_\theta(x)$ for a.e. x in Ω. Besides, for i in $\{h,p\}$, the reaction term g_i is in $W^{1,+\infty}(]0,T[\times\Omega_i \times \mathcal{C}_\theta^\infty)$ and we set

$$M_{g_i} = ess \sup_{(t,x,u)\in]0,T[\times\Omega_i\times\mathcal{C}_\theta^\infty} |\partial_u g_i(t,x,u)|.$$

The transport term K is Lipschitz continuous on $\mathcal{C}_\theta^\infty$ with a constant $\mathcal{K}_K$. Besides K is **nondecreasing**. Thus we may define the nonnegative and nondecreasing time-depending function

$$M_1 : t \in [0,T] \to M_1(t) = ess \sup_{\Omega} \theta\, e^{\mathcal{N}\,t} + \frac{N_3}{\mathcal{N}}(e^{\mathcal{N}\,t} - 1), \tag{9}$$

where $\mathcal{N} = \sum_{i\in\{h,p\}} M_{g_i} + \mathcal{K}_K \|div\mathbf{B}\|_{L^\infty(Q)}$
and $N_3 = \sum_{i\in\{h,p\}} ess \sup_{]0,T[\times\Omega_i} g_i(t,x,0)^- + ess \sup_{]0,T[\times\Omega}(K(0)div\mathbf{B})^-$.
We also introduce the non-positive and non-increasing function

$$M_2 : t \in [0,T] \to M_2(t) = \min(0, ess \inf_{\Omega} u_0)\, e^{\mathcal{N}\,t} + \frac{N_4}{\mathcal{N}}(e^{\mathcal{N}\,t} - 1), \tag{10}$$

with $N_4 = -\sum_{i\in\{h,p\}} ess \sup_{]0,T[\times\Omega_i} g_i(t,x,0)^+ - ess \sup_{]0,T[\times\Omega}(K(0)div\mathbf{B})^+$.

From now, to unify the presentation with the bilateral obstacle problem we set for the unilateral obstacle problem:

$$\mathcal{C}_\theta^\infty = [M_2(T), ess \sup_{\bar{\Omega}} \theta].$$

Lastly ϕ is a nondecreasing function of $W^{1,+\infty}(\mathcal{C}_\theta^\infty)$, with $\phi(0) = 0$, ϕ' is Lipschitz continuous on $\mathcal{C}_\theta^\infty$ and

$$\phi^{-1} \text{ exists and is a continuous function on } \phi(\mathcal{C}_\theta^\infty). \tag{11}$$

We point out that (11) is especially fulfilled when $\mathcal{L}(\{x \in \mathcal{C}_\theta^\infty, \phi'(x) = 0\}) = 0$, where $\mathcal{L}$ refers to the Lebesgue measure on $\mathbb{R}$.

Remark 2. *The monotonicity of K and* (8) *show that the interface Σ_{hp} is included in the set of outward characteristics for the first-order operator in the hyperbolic domain. So on the transmission zone* (5) *is useless since data are leaving the hyperbolic domain. This essential property will guide us for the statement of uniqueness by first considering the behavior of a solution on the hyperbolic area and then on the parabolic one.*

1.3 Notations and functional spaces

In the sequel, σ (resp. $\bar{\sigma}$) is a variable of Σ_i (resp. Γ_i), $i \in \{h, hp, p\}$. This way, $\sigma = (t, \bar{\sigma})$ for any t of $[0, T]$.

We need to consider the Hilbert space

$$V = \{v \in H^1(\Omega_p), v = 0 \ a.e. \ on \ \Gamma_p \backslash \Gamma_{hp}\}.$$

used with the norm $\|v\|_V = \|\nabla v\|_{L^2(\Omega_p)^n}$, equivalent to the classical $H^1(\Omega)$-norm. We denote $\langle\langle ., . \rangle\rangle$ the pairing between V and V' and $\langle ., . \rangle$ the pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. Furthermore, for X and Y two Hilbert spaces, it will be referred to the Hilbert space

$$W(0, T; X; Y) \equiv \{v \in L^2(0, T; X); \partial_t v \in L^2(0, T; Y)\},$$

equipped with the norm $\|v\|_{W(0,T;X;Y)} = \left(\|\partial_t v\|^2_{L^2(0,T;Y)} + \|v\|^2_{L^2(0,T;X)}\right)^{1/2}$. In the sequel, X will be mainly taken equal to $H_0^1(\Omega)$ or V and Y to $H^{-1}(\Omega)$ or V' respectively.

The function sgn_μ denotes the Lipschitzian and bounded approximation of the function sgn given for any positive μ and any nonnegative real x by:

$$sgn_\mu(x) = \min\left(\frac{x}{\mu}, 1\right) \text{ and } sgn_\mu(-x) = -sgn_\mu(x).$$

Lastly, to simplify the writing, we set for i in $\{h, p\}$:

$$G_i(u, v) = g_i(t, x, u) - div(K(v)\mathbf{B})$$
$$L_i(u, v, w) = -|u - v|\partial_t w - |K(u) - K(v)|\mathbf{B}.\nabla w - sgn(u - v)G_i(u, v)w,$$

and with $\mathbb{I}_{\Omega_i}(x) = 1$ if $x \in \Omega_i$, $\mathbb{I}_{\Omega_i}(x) = 0$ else,

$$L(u, v, w) = L_p(u, v, w)\mathbb{I}_{\Omega_p}(x) + L_h(u, v, w)\mathbb{I}_{\Omega_h}(x),$$
$$g(t, x, u) = g_p(t, x, u)\mathbb{I}_{\Omega_p}(x) + g_h(t, x, u)\mathbb{I}_{\Omega_h}(x),$$
$$\mathcal{F}(u, v, w) = \tfrac{1}{2}\{|K(u) - K(v)| - |K(w) - K(v)| + |K(u) - K(w)|\},$$

2 Statement of Uniqueness

We give the definition of a weak solution to (1)-(6) by firstly keeping in mind that it has to involve an entropy criterion on Q_h and secondly by taking into account the obstacle condition for u. That is why, by considering that (1)-(6) can be viewed as an obstacle problem for an evolutional quasilinear parabolic equation that *strongly degenerates* on a fixed subdomain, we refer to related works ([1],[7]) to propose a weak formulation through a global entropy inequality on the whole Q, the latter giving rise to a variational inequality on the parabolic domain and to an entropy inequality on the hyperbolic one so as to ensure the uniqueness.

2.1 Definition

So it will be said that:

Definition 1. *A measurable function u is a weak solution to* (1)-(6) *if,*

$$\text{for a.e. } t \text{ in }]0,T[,\ u(t,.) \in \mathcal{C}_\theta \text{ a.e. in } \Omega, \quad \phi(u) \in L^2(0,T;V), \tag{12}$$

$\forall \zeta \in \mathcal{D}(Q)$, $\zeta \geq 0$,

$$\int_Q L(u,\kappa,\zeta)dxdt - \int_{Q_p} \nabla|\phi(u)-\phi(\kappa)|.\nabla\zeta dxdt + \int_{Q_p} \Delta\phi(\kappa) sgn(u-\kappa)\zeta dxdt$$
$$+ \int_{\Sigma_{hp}} \nabla\phi(\kappa).\nu_h sgn(\phi(u)-\phi(\kappa))\zeta d\mathcal{H}^n \geq 0, \tag{13}$$

where $\kappa = k\theta$, $k \in [0,1]$ for the bilateral obstacle problem and $\kappa = k+\theta$, $k \in [M_2(T) - ess\sup_{\bar{\Omega}} \theta, 0]$ for the unilateral one,

$\forall \zeta \in L^1(\Sigma_h \backslash \Sigma_{hp})$, $\zeta \geq 0$,

$$ess \lim_{\tau \to 0^-} \int_{\Sigma_h \backslash \Sigma_{hp}} \mathcal{F}(u(\sigma+\tau\nu_h), 0, \kappa(\bar{\sigma}))\mathbf{B}(\sigma).\nu_h \zeta d\mathcal{H}^n \leq 0, \tag{14}$$

$$ess \lim_{t \to 0^+} \int_\Omega |u(t,x) - u_0(x)| dx = 0. \tag{15}$$

Remark 3.
(i) When θ is nonnegative on Q, the formulation for the (upper) unilateral obstacle problem is a special situation of (13) *for the bilateral obstacle problem by considering $k \leq 1$ only.*

(ii) The link between (2),(3),(6) *and* (13) *can be achieved through two inequalities resulting form* (13) *and that will be useful in the sequel. In* (13)*, we take $\kappa(x) = \theta(x)$, that means $k=1$ in the case of a bilateral obstacle condition and $k=0$ in the case of an unilateral one. It comes (with $\mathcal{T} = \mathcal{T}_h \mathbb{I}_{\Omega_h} + \mathcal{T}_p \mathbb{I}_{\Omega_p}$),*

$$\begin{aligned} \int_Q u\partial_t\zeta dxdt \quad &\leq \int_{Q_p} (\nabla\phi(u) + K(u)\mathbf{B}).\nabla\zeta dxdt + \int_Q g(t,x,u)\zeta dxdt \\ &+ \int_{\Sigma_{hp}} \nabla\phi(\theta).\nu_h(1+sgn(\phi(u)-\phi(\theta)))\zeta d\mathcal{H}^n \\ &- \int_Q (1+sgn(u-\theta))\mathcal{T}(\theta)\zeta dxdt. \end{aligned} \tag{16}$$

In (13)*, we take $\kappa(x) = 0$ for the bilateral obstacle problem and $\kappa(x) = M_2(T) -$*

ess $\sup_{\bar\Omega}\theta + \theta(x)$ *for the unilateral one (so that* $u-\kappa \geq 0$ *a.e.). One has,*

$$\begin{aligned}\int_Q u\partial_t\zeta dxdt \quad &\geq \quad \int_{Q_p} \left(\nabla\phi(u) + K(u)\mathbf{B}\right).\nabla\zeta dxdt + \int_Q g(t,x,u)\zeta dxdt \\ &+ \quad \int_{\Sigma_{hp}} \nabla\phi(\kappa).\nu_h(1 - sgn(\phi(u) - \phi(\kappa)))\zeta d\mathcal{H}^n \\ &- \quad \int_Q (1 - sgn(u-\kappa))\mathcal{T}(\kappa)\zeta dxdt. \end{aligned} \tag{17}$$

Let $\mathbf{V} = (u, -\mathbb{I}_{\Omega_p}\nabla\phi(u) - K(u)\mathbf{B})$ *in* $L^2(Q)^{n+1}$*. For any* ζ *in* $\mathcal{D}(Q_i)$*,* i *in* $\{h,p\}$*, we take* ζ^+ *and* ζ^- *as test-functions in* (16) *and* (17)*. By writing that* $\zeta = \zeta^+ - \zeta^-$*, we deduce the existence of a constant* C *such that*

$$\left|\int_{Q_i} \mathbf{V}.(\partial_t\zeta, \nabla\zeta)dxdt\right| \leq C\|\zeta\|_{L^2(Q_i)}.$$

That means $\mathbf{V}_{|Q_i}$ *belongs to* $H_{div_{(t,x)}}(Q_i) = \{\mathbf{v} \in L^2(Q_i)^{n+1}, div_{(t,x)}\mathbf{v} \in L^2(Q_i)\}$*. We deduce that* $\mathbf{V}_{|Q_i}.\nu_i$ *belongs to* $H_{00}^{-1/2}(\Sigma_{hp})$*, the topological dual of* $H_{00}^{1/2}(\Sigma_{hp})$*. In addition, we derive from* (16) *and* (17) *that a.e. on* Q_i*,*

$$\begin{aligned}-div_{(t,x)}\mathbf{V}_{|Q_i} &\leq g_i(t,x,u) - (1 + sgn(u-\theta))\mathcal{T}_i(\theta), \\ -div_{(t,x)}\mathbf{V}_{|Q_i} &\geq g_i(t,x,u) - (1 - sgn(u-\kappa))\mathcal{T}_i(\kappa).\end{aligned}$$

We multiply each inequality with ζ *in* $\mathcal{D}(Q)$*,* $\zeta \geq 0$ *and add up with respect to* i*. By denoting* $\lfloor .,. \rfloor$ *the pairing between* $H_{00}^{1/2}(\Sigma_{hp})$ *and* $H_{00}^{-1/2}(\Sigma_{hp})$ *it comes:*

$$\begin{aligned} & \int_Q \mathbf{V}.(\partial_t\zeta, \nabla\zeta)dxdt - \lfloor \mathbf{V}_{|Q_h}.\nu_h + \mathbf{V}_{|Q_p}.\nu_p, \zeta\rfloor \\ \leq \quad & \int_Q (g(t,x,u) - (1 + sgn(u-\theta))\mathcal{T}(\theta))\zeta dxdt, \\ \textit{and} \quad & \int_Q \mathbf{V}.(\partial_t\zeta, \nabla\zeta)dxdt - \lfloor \mathbf{V}_{|Q_h}.\nu_h + \mathbf{V}_{|Q_p}.\nu_p, \zeta\rfloor \\ \geq \quad & \int_Q (g(t,x,u) - (1 - sgn(u-\kappa))\mathcal{T}(\kappa))\zeta dxdt. \end{aligned}$$

Now what follows is formal. We are interested with the bilateral obstacle problem (the reasoning for the unilateral one being similar) and assume that $[0 < u < \theta]$ *is an open subset of* Q*,* $\mathcal{H}^n$*-measurable. We consider in* (16) *and* (17) *that* ζ *has a compact support in* $Q_i \cap [0 < u < \theta]$*. Since* $(1 + sgn(u-\theta))\zeta = (1 - sgn(u-\kappa))\zeta = 0$ *a.e. and* $(1 + sgn(\phi(u) - \phi(\theta)))\zeta = (1 - sgn(\phi(u) - \phi(\kappa)))\zeta = 0$ $\mathcal{H}^n$*-a.e., we deduce that for* i *in* $\{h,p\}$*,*

$$div_{(t,x)}(\mathbf{V}_{|Q_i}) = 0 \textit{ on } Q_i \cap [0 < u < \theta],$$

that is namely (2). *Then, for ζ with a compact support in $Q \cap [0 < u < \theta]$, by comparing* (16) *and* (17) *with above inequality, we may say*

$$\lfloor \mathbf{V}_{|Q_h}.\nu_h + \mathbf{V}_{|Q_p}.\nu_p, \zeta \rfloor = 0,$$

that is (6) *in a certain sense. Furthermore if we take ζ with a support in $Q_i \cap [0 < u = \theta]$ and in $Q_i \cap [0 = u < \theta]$ - if it has a meaning - we find* (3). *Besides, for any nonnegative ζ a compact support in $Q \cap [0 < u = \theta]$,*

$$\lfloor \mathbf{V}_{|Q_h}.\nu_h + \mathbf{V}_{|Q_p}.\nu_p, \zeta \rfloor = \int_{\Sigma_{hp}} \nabla\phi(\theta).\nu_h \zeta d\mathcal{H}^n,$$

$$\textit{and } \lfloor \mathbf{V}_{|Q_h}.\nu_h + \mathbf{V}_{|Q_p}.\nu_p, \zeta \rfloor \geq \int_Q \mathcal{T}(\theta)\zeta dxdt.$$

This way,

$$\nabla\phi(\theta).\nu_h \geq \mathcal{T}(\theta)\ \mathcal{H}^n\text{- a.e. on } \Sigma_{hp} \cap [0 < u = \theta].$$

On $\Sigma_{hp} \cap [0 = u < \theta]$, since $K(0) = \phi(0) = 0$, for any nonnegative ζ a compact support in $Q \cap [0 = u < \theta]$

$$\lfloor \mathbf{V}_{|Q_h}.\nu_h + \mathbf{V}_{|Q_p}.\nu_p, \zeta \rfloor = 0 \leq \int_Q g(t, x, 0)\zeta dxdt.$$

2.2 Study on the hyperbolic zone

We derive from (13) and (14) an entropy inequality on the hyperbolic domain that will be the starting point to establish a time-Lipschitzian dependence in $L^1(\Omega_h)$ of a weak solution to (1)-(6) with respect to the corresponding initial data. To do so we claim a first lemma proved as in [1]:

Lemma 1. *Let u be a measurable and bounded function on Q satisfying* (13) *and* (14). *Then for any κ as in Definition 1 and any φ of $\mathcal{D}(]0, T[\times\mathbb{R}^n)$, $\varphi \geq 0$,*

$$\begin{aligned} -\int_{Q_h} L_h(u, \kappa, \zeta)dxdt \quad &\leq \quad -ess \lim_{\tau\to 0^-} \int_{\Sigma_h\setminus\Sigma_{hp}} |K(u(\sigma + \tau\nu_h))|\mathbf{B}(\sigma).\nu_h\varphi(\sigma)d\mathcal{H}^n \\ &\quad + \int_{\Sigma_h\setminus\Sigma_{hp}} |K(\kappa)(\sigma)|\mathbf{B}(\sigma).\nu_h\varphi(\sigma)d\mathcal{H}^n \end{aligned} \tag{18}$$

In order to use the method of doubling variables, we now need a technical result based on properties of mollifiers and already pointed out in [10],[11]. From (18) we argue that for any open subset Σ_{loc} of Σ_h and κ as in Definition 1,

$$ess \lim_{\tau\to 0^-} \int_{\Sigma_{loc}} |K(u(\sigma + \tau\nu_h)) - K(\kappa(\sigma))|\mathbf{B}(\sigma).\nu_h\beta(\sigma)d\mathcal{H}^n \text{ exists} \tag{19}$$

and, $\exists\gamma \in L^\infty(\Sigma_{loc})$ such that

$$ess \lim_{\tau\to 0^-} \int_{\Sigma_{loc}} K(u(\sigma + \tau\nu_h))\mathbf{B}(\sigma).\nu_h\beta(\sigma)d\mathcal{H}^n = \int_{\Sigma_{loc}} \gamma_\kappa(\sigma)\beta(\sigma)d\mathcal{H}^n, \tag{20}$$

for any β in $L^1(\Sigma_{loc})$. In the sequel (19) and (20) will be used with $\Sigma_{loc} = \Sigma_{hp}$ or $\Sigma_{loc} = \Sigma_h \backslash \Sigma_{hp}$. We define the sequence $(\mathcal{W}_\delta)_{\delta>0}$ on $\mathbb{R}^{n+1}$

$$\forall \delta > 0,\ \forall p = (t, x) \in \mathbb{R}^{n+1},\ \mathcal{W}_\delta(p) = \rho_\delta(t) \prod_{i=1}^{n} \rho_\delta(x_i),$$

where $(\rho_\delta)_{\delta>0}$ is a standard sequence of mollifiers on $\mathbb{R}$. We focus on $\Sigma_h \backslash \Sigma_{hp}$ the proof developed in [11] on the whole boundary to state:

Lemma 2. *Let u be a measurable and bounded function on Q_h such that* (19) *holds. Then for any continuous function φ on $Q_h \cup \Sigma_h$*

$$\lim_{\delta \to 0^+} \int_{Q_h} \int_{\Sigma_h \backslash \Sigma_{hp}} |K(u(p))| \mathbf{B}(\tilde{\sigma}).\nu_h \varphi(\frac{\tilde{\sigma} + p}{2}) \mathcal{W}_\delta(\tilde{\sigma} - p) d\mathcal{H}^n_{\tilde{\sigma}} dp$$
$$= \frac{1}{2} ess \lim_{\tau \to 0^-} \int_{\Sigma_h \backslash \Sigma_{hp}} |K(u(\sigma + \tau\nu_h))| \mathbf{B}(\sigma)\nu_h \varphi(\sigma) d\mathcal{H}^n$$

and,

$$\lim_{\delta \to 0^+} \int_{Q_h} ess \lim_{\tau \to 0^-} \int_{\Sigma_h \backslash \Sigma_{hp}} |K(u(\sigma + \tau\nu_h))| \mathbf{B}(\sigma).\nu_h \varphi(\frac{\sigma + \tilde{p}}{2}) \mathcal{W}_\delta(\sigma - \tilde{p}) d\mathcal{H}^n_\sigma d\tilde{p}$$
$$= \frac{1}{2} ess \lim_{\tau \to 0^-} \int_{\Sigma_h \backslash \Sigma_{hp}} |K(u(\sigma + \tau\nu_h))| \mathbf{B}(\sigma).\nu_h \varphi(\sigma) d\mathcal{H}^n.$$

From Lemma 1 and Lemma 2 we derive:

Theorem 1. *Let u_1 and u_2 be two bounded and measurable functions on Q_h, for a.e. t in $]0, T[$, $u(t, .)$ and $v(t, .)$ belong to $\mathcal{C}_\theta$ a.e. on Q_h, satisfying* (18) *and* (15) *respectively for initial data $u_{0,1}$ and $u_{0,2}$. Then*

$$\textit{for a.e. } t \textit{ in }]0, T[, \quad \int_{\Omega_h} |u_1(t, .) - u_2(t, .)| dx \leq e^{M_{g_h} t} \int_{\Omega_h} |u_{0,1} - u_{0,2}| dx.$$

Proof. We choose in (18) for u_1 written in variables $p = (t, x)$,

$$\kappa(x) = u_2(\tilde{t}, \tilde{x}) - \theta(\tilde{x}) + \theta(x),$$

in the case of an (upper) unilateral constraint while

$$\kappa(x) = \begin{cases} \dfrac{u_2(\tilde{t}, \tilde{x})}{\theta(\tilde{x})} \theta(x) \text{ if } \theta(\tilde{x}) \neq 0, \\ 0 \text{ else,} \end{cases}$$

for a bilateral obstacle condition, and similarly in (18) for u_2 written in variables $\tilde{p} = (\tilde{t}, \tilde{x})$. Furthermore in (18) for u_1,

$$\varphi(p) = \zeta(\frac{p + \tilde{p}}{2}) \mathcal{W}_\delta(p - \tilde{p}),$$

where δ is positive and large enough, and ζ belongs to $\mathcal{D}(]0, T[\times \mathbb{R}^n)$, $\zeta \geq 0$. Similarly in (18) for u_2. We integrate over Q_h on the $\tilde{p}$ variables for u_1 and on the p variables for u_2. We add up. Through techniques developed in [7] we pass to the limit with δ on the left-hand side. The right-hand side goes to 0 with δ, thanks to Lemma 2 for u_1 and u_2. It comes:

$$\begin{aligned} - & \int_{Q_h} \{|u_1 - u_2|\partial_t\zeta - |K(u_1) - K(u_2)|\mathbf{B}.\nabla\zeta\}dxdt \\ \leq & \ - \int_{Q_h} sgn(u_1 - u_2)(g_h(t,x,u_1) - g_h(t,x,u_2))\zeta dxdt. \end{aligned}$$

For $\zeta \equiv \alpha\psi$ where α belongs to $\mathcal{D}(]0,T[)$, $\alpha \geq 0$ and ψ to $\mathcal{D}(\mathbb{R}^n)$, $\psi \geq 0$, $\psi \equiv 1$ on Q_h, the Lipschitz condition for g_h provides:

$$-\int_{Q_h} |u_1 - u_2|\alpha'(t)dxdt \leq M'_{g_h} \int_{Q_h} |u_1 - u_2|\alpha(t)dxdt.$$

When α is the element of a sequence approximating $\mathbb{I}_{[0,t]}$, t being given outside a set of measure zero, the desired inequality is obtained thanks to the initial condition (15) for u_1 and u_2 and to the Gronwall's Lemma. □

2.3 Study in the parabolic zone

We consider now the behavior of a weak solution u to (1)-(6) on the parabolic domain. With this view, we characterize u on Q_p through a *strong* variational inequality (in the sense of J.L.Lions in [9]) including the contribution of entering data from the hyperbolic zone. Indeed:

Proposition 1. *Let u be a measurable and bounded function on Q such that $\nabla\phi(u)$ belongs to $L^2(Q_p)^n$ and satisfying* (13)*. Then $\partial_t u$ belongs to $L^2(0,T;V')$. Furthermore, for any v in $L^2(0,T;V)$ such that for a.e. t in $]0,T[$, $\phi^{-1}(v(t,.))$ is an element of $\mathcal{C}_\theta$ a.e. on Ω_p,*

$$\begin{aligned} \int_0^T \langle\langle \partial_t u, v - \phi(u)\rangle\rangle dt + \int_{Q_p} \left(\nabla\phi(u) + K(u)\mathbf{B}\right).\nabla(v - \phi(u))dxdt \\ + \int_{Q_p} g_p(t,x,u)(v - \phi(u))dxdt \\ +ess \lim_{\tau\to 0^-} \int_{\Sigma_{hp}} K(u(\sigma + \tau\nu_h))\mathbf{B}(\sigma).\nu_h(v - \phi(u))d\mathcal{H}^n \geq 0. \end{aligned} \tag{21}$$

Proof. Thanks to a density argument (16) and (17) still hold for any nonnegative ζ in $\mathcal{D}(0,T;H^1_0(\Omega))$. Now let φ be given in $\mathcal{D}(0,T;V)$. We consider $\hat{\varphi}$ an extension of φ to $\mathcal{D}(0,T;H^1_0(\Omega))$ and we take $\zeta = \hat{\varphi}\xi_\varrho$ in (16) and (17), where ξ_ϱ belongs to $W^{1,+\infty}(\Omega)$, $0 \leq \xi_\varrho \leq 1$, and fulfills for any positive ϱ:

$$\xi_\varrho(x) = \begin{cases} 1 \text{ if } x \in \bar{\Omega}_p, \\ 0 \text{ if } x \in \Omega_h,\ dist(x,\Gamma_{hp}) \geq \varrho,\ \|\nabla\xi_\varrho\|_\infty \leq C/\varrho. \end{cases}$$

To pass to the limit when ϱ goes to 0^+, we claim that due to (19) (see [1]),

$$\lim_{\varrho\to 0^+}\int_{Q_h} K(u)\hat{\varphi}\mathbf{B}.\nabla\xi_\varrho dxdt = ess\lim_{\tau\to 0^-}\int_{\Sigma_{hp}} K(u(\sigma+\tau\nu_h))\varphi(\sigma)\mathbf{B}.\nu_h d\mathcal{H}^n.$$

This way, for any φ in $\mathcal{D}(0,T;V)$, $\varphi\geq 0$,

$$\begin{aligned}
\int_{Q_p} u\partial_t\varphi dxdt \quad &\leq \quad \int_{Q_p}(\nabla\phi(u)+K(u)\mathbf{B}).\nabla\varphi dxdt+\int_{Q_p} g_p(t,x,u)\varphi dxdt\\
&+\quad \int_{\Sigma_{hp}}\nabla\phi(\theta).\nu_h(1+sgn(u-\theta))\varphi d\mathcal{H}^n\\
&-\quad \int_{Q_p}(1+sgn(u-\theta))\mathcal{T}_p(\theta)\varphi dxdt\\
&+\quad ess\lim_{\tau\to 0^-}\int_{\Sigma_{hp}} K(u(\sigma+\tau\nu_h))\varphi(\sigma)\mathbf{B}.\nu_h d\mathcal{H}^n. \qquad (22)
\end{aligned}$$

and

$$\begin{aligned}
\int_{Q_p} u\partial_t\varphi dxdt \quad &\geq \quad \int_{Q_p}(\nabla\phi(u)+K(u)\mathbf{B}).\nabla\varphi dxdt+\int_{Q_p} g_p(t,x,u)\varphi dxdt\\
&+\quad \int_{\Sigma_{hp}}\nabla\phi(\kappa).\nu_h(1-sgn(u_\epsilon-\kappa))\varphi d\mathcal{H}^n\\
&-\quad \int_{Q_p}(1-sgn(u-\kappa))\mathcal{T}_p(\kappa)\varphi dxdt\\
&+\quad ess\lim_{\tau\to 0^-}\int_{\Sigma_{hp}} K(u(\sigma+\tau\nu_h))\varphi(\sigma)\mathbf{B}.\nu_h d\mathcal{H}^n.. \qquad (23)
\end{aligned}$$

We write $\varphi=\varphi^+-\varphi^-$ and use (22)-(23) with φ^+ and φ^-. Since u is bounded and $\phi(u)$ belongs to $L^2(0,T;V)$ we argue that there exists a constant C such as

$$\forall\varphi\in\mathcal{D}(0,T;V),\ \left|\int_0^T\int_{\Omega_p} u\partial_t\varphi dxdt\right|\leq C\|\varphi\|_{L^2(0,T;V)},$$

which ensures that $\partial_t u$ belongs to $L^2(0,T;V')$ (see Appendix of [3]). Thus,

$$\forall\varphi\in\mathcal{D}(0,T;V),\ -\int_0^T\int_{\Omega_p} u\partial_t\varphi dxdt=\int_0^T\langle\langle\partial_t u,\varphi\rangle\rangle dt,$$

Thus by density, we may rewrite (22) and (23) with φ in $L^2(0,T;V)$. Then we consider $\varphi=(v-\phi(u))^+$ and $\varphi=(v-\phi(u))^-$ respectively, with v as in the statement of Proposition 1 so that, due to the obstacle condition for u, $(1+sgn(u-\theta))(v-\phi(u))^+=0$ and respectively $(1-sgn(u-\kappa))(v-\phi(u))^-=0$ a.e. on Q_p and $\mathcal{H}^n$-a.e. on Σ_{hp}. By adding up Inequality (21) follows, that completes the proof of Proposition 1. □

2.4 The uniqueness theorem

Theorem 1 ensures a uniqueness property on the hyperbolic zone. On the parabolic one, the lack of regularity of the time partial derivative of a weak solution to (1)-(6) requires a doubling the time variable and uses a suitable time-integration by parts formula. Furthermore, to deal with the convective terms, we assume that

$$\zeta \to K \circ \phi^{-1} \text{ is Lipschitz continuous on } \phi(\mathcal{C}_\theta^\infty). \tag{24}$$

Then we have:

Theorem 2. *Under* (24) *Problem* (1)-(6) *admits at most one weak solution.*

Proof. On the parabolic zone, it refers to that developed in [8]. Indeed, let u_1 and u_2 be two weak solutions to (1)-(6). Thanks to Lemma 1 and Theorem 1, we know that $u_1 = u_2$ a.e. on Q_h. In addition, θ being independent from the time variable on Q_p we may choose in (21) for u_1 written in variables (t, x)

$$v_1(t,x) = \phi(u_1)(t,x) - \frac{\mu\alpha_\delta}{\|\alpha_\delta\|_\infty} sgn_\mu(\phi(u_1)(t,x) - \phi(u_2)(\tilde{t},x)),$$

and in (21) for u_2 written in variables $(\tilde{t}, x)$

$$v_2(\tilde{t},x) = \phi(u_2)(\tilde{t},x) + \frac{\mu\alpha_\delta}{\|\alpha_\delta\|_\infty} sgn_\mu(\phi(u_1)(t,x) - \phi(u_2)(\tilde{t},x).$$

For any positive δ, $\alpha_\delta(t,\tilde{t}) = \gamma\left(\frac{t+\tilde{t}}{2}\right)\rho_\delta\left(\frac{t-\tilde{t}}{2}\right)$, where γ is an element of $\mathcal{D}(]0,T[)$, $\gamma \geq 0$, and δ is small enough for α_δ to belong to $\mathcal{D}(]0,T[\times]0,T[)$. To simplify the writing we add a "tilde" superscript to any function in the $\tilde{t}$ variable; q stands for (t,x) while $\tilde{q}$ stands for $(\tilde{t},x)$. By adding up (and denoting $w_{\mu,\delta}(u_1,\tilde{u}_2) = sgn_\mu(\phi(u_1) - \phi(\tilde{u}_2))\alpha_\delta)$, it comes:

$$\begin{aligned}
&\int_{]0,T[\times Q_p} \langle\langle \partial_t u_1 - \partial_{\tilde{t}} u_2, w_{\mu,\delta}(u_1,\tilde{u}_2)\rangle\rangle dt d\tilde{t} \\
&+ \int_{]0,T[\times Q_p} \nabla\{\phi(u_1) - \phi(\tilde{u}_2)\}.\nabla w_{\mu,\delta}(u_1,\tilde{u}_2) dq d\tilde{t} \\
&\leq - \int_{]0,T[\times Q_p} \{K(u_1) - K(\tilde{u}_2)\}\mathbf{B}.\nabla w_{\mu,\delta}(u_1,\tilde{u}_2) dq d\tilde{t} \\
&+ \int_{]0,T[\times Q_p} K(\tilde{u}_2)(\mathbf{B} - \tilde{\mathbf{B}}).\nabla w_{\mu,\delta}(u_1,\tilde{u}_2) dq d\tilde{t} \\
&- \int_{]0,T[\times Q_p} \{g_p(q,u_1) - g_p(\tilde{q},\tilde{u}_2)\} w_{\mu,\delta}(u_1,\tilde{u}_2) dq d\tilde{t} \\
&- \int_0^T ess \lim_{\tau\to 0^-} \int_{\Sigma_{hp}} K(u_1(\sigma + \tau\nu_h))\mathbf{B}.\nu_h w_{\mu,\delta}(u_1,\tilde{u}_2) d\mathcal{H}_\sigma^n d\tilde{t} \\
&+ \int_0^T ess \lim_{\tau\to 0^-} \int_{\Sigma_{hp}} K(u_2(\tilde{\sigma} + \tau\nu_h))\tilde{\mathbf{B}}.\nu_h w_{\mu,\delta}(u_1,\tilde{u}_2) d\mathcal{H}_{\tilde{\sigma}}^n dt,
\end{aligned}$$

To deal with the first term in the left-hand side, we use a time-integration by parts formula in the same spirit as in ([5], the Mignot-Bamberger Lemma). For the second integral in the right-hand side, a Green formula is used since (24) ensures that $K(\tilde{u}_2) = (K \circ \phi^{-1})(\phi(\tilde{u}_2))$ belongs to $L^2(0,T;H^1(\Omega_p))$. For the boundary integrals we argue that due to the uniqueness property on the hyperbolic zone, $u_2(\tilde{\sigma} + \tau\nu_h) = u_1(\tilde{\sigma} + \tau\nu_h)$ for a.e. $(\tilde{\sigma}, \tau)$. This way, as a consequence of (20),

$$\begin{aligned}
& \int_0^T ess \lim_{\tau \to 0^-} \int_{\Sigma_{hp}} K(u_2(\tilde{\sigma} + \tau\nu_h))\tilde{\mathbf{B}}.\nu_h w_{\mu,\delta}(u_1, \tilde{u}_2) d\mathcal{H}^n_{\tilde{\sigma}} dt \\
= & \int_0^T ess \lim_{\tau \to 0^-} \int_{\Sigma_{hp}} K(u_1(\tilde{\sigma} + \tau\nu_h))\tilde{\mathbf{B}}.\nu_h w_{\mu,\delta}(u_1, \tilde{u}_2) d\mathcal{H}^n_{\tilde{\sigma}} dt \\
= & \int_0^T \int_{\Sigma_{hp}} \gamma(\tilde{\sigma}) w_{\mu,\delta}(u_1, \tilde{u}_2) d\mathcal{H}^n_{\tilde{\sigma}} dt,
\end{aligned}$$

where γ belongs to $L^\infty(\Sigma_{hp})$. It follows that

$$\begin{aligned}
- & \int_{]0,T[\times Q_p} \left(\int_{\tilde{u}_2}^{u_1} sgn_\mu(\phi(r) - \phi(\tilde{u}_2)) dr \right) \partial_t \alpha_\delta dq d\tilde{t} \\
- & \int_{]0,T[\times Q_p} \left(\int_{\tilde{u}_2}^{u_1} sgn_\mu(\phi(u_1) - \phi(r)) dr \right) \partial_{\tilde{t}} \alpha_\delta dq d\tilde{t} \\
\leq & \ \|\mathbf{B}\|_{L^\infty(Q_p)} \int_{]0,T[\times Q_p} |K(u_1) - K(\tilde{u}_2)| |\nabla w_{\mu,\delta}(u_1, \tilde{u}_2)|_n dq d\tilde{t} \\
+ & \int_{]0,T[\times Q_p} div(K(\tilde{u}_2)(\mathbf{B} - \tilde{\mathbf{B}})) w_{\mu,\delta}(u_1, \tilde{u}_2) dq d\tilde{t} \\
- & \int_{]0,T[\times \Sigma_{hp}} K(\tilde{u}_2)(\mathbf{B} - \tilde{\mathbf{B}}).\nu_h w_{\mu,\delta}(u_1, \tilde{u}_2) dq d\tilde{t} \\
- & \int_{]0,T[\times Q_p} \{g(q, u_1) - g(\tilde{q}, \tilde{u}_2)\} w_{\mu,\delta}(u_1, \tilde{u}_2) dq d\tilde{t} \\
- & \int_0^T \int_{\Sigma_{hp}} \gamma(\sigma) w_{\mu,\delta}(u_1, \tilde{u}_2) d\mathcal{H}^n_\sigma d\tilde{t} + \int_0^T \int_{\Sigma_{hp}} \gamma(\tilde{\sigma}) w_{\mu,\delta}(u_1, \tilde{u}_2) d\mathcal{H}^n_{\tilde{\sigma}} dt.
\end{aligned}$$

We take the limit with respect to μ. For the first integral in the right-hand side we refer to (24) and use the Sacks Lemma; so that it goes to 0. Thus one has:

$$
\begin{aligned}
- & \int\limits_{]0,1[\times Q_p} |u_1 - \tilde{u}_2|(\partial_t \alpha_\delta + \partial_{\tilde{t}} \alpha_\delta) dq d\tilde{t} \\
\leq & \int\limits_{]0,T[\times Q_p} |div(K(\tilde{u}_2)(\mathbf{B} - \tilde{\mathbf{B}}))| \alpha_\delta dq d\tilde{t} \\
+ & \int\limits_{]0,T[\times \Sigma_{hp}} |K(\tilde{u}_2)(\mathbf{B} - \tilde{\mathbf{B}}).\nu_h| \alpha_\delta dq d\tilde{t} \\
+ & \quad M_{g_p} \int\limits_{]0,T[\times Q_p} |u_1 - \tilde{u}_2| \alpha_\delta dq d\tilde{t} + \int\limits_0^T \int\limits_0^T \int\limits_{\Gamma_{hp}} |\gamma(t,s) - \gamma(\tilde{t},s)| \alpha_\delta ds dt d\tilde{t},
\end{aligned}
$$

We come back to the definition of α_δ to express its partial derivatives with respect to t and $\tilde{t}$. This way we may pass to the limit with δ through the classical argument of the Lebesgue points for an integrable function on $]0,T[$: all the terms in the right-hand side tend to 0 ($\mathbf{B}$ being smooth) except the first integral in the third line. The end is classical: it uses a piecewise linear approximation of $\mathbb{I}_{]0,t[}$, t given outside of a set of measure zero. Thanks to (15) and to the Gronwall's Lemma we complete the proof of Theorem 2. □

3 The Existence Property

3.1 The obstacle problem to the second order

We propose to approximate the weak solution to (1)-(6) through a sequence of solutions to viscous problems deduced from (1)-(6) by adding a diffusion term only in the hyperbolic area. This is in accordance with the proposed physical modelling of two layers in the subsoil with different geological characteristics. So for any positive ϵ, we introduce

$$\mathcal{T}_{\epsilon,h}(u) = \partial_t u - \sum_{i=1}^{n} \partial_{x_i}(\epsilon \partial_{x_i} \phi(u) + K(u) B_i) + g_h(t,x,u),$$

and we consider the free boundary problem: find a measurable and bounded function u_ϵ on Q such that formally (for the bilateral obstacle problem),

$$0 \leq u_\epsilon \leq \theta \text{ on } Q, \tag{25}$$

$$\mathcal{T}_p(u_\epsilon) = 0 \text{ on } Q_p \cap [0 < u_\epsilon < \theta], \mathcal{T}_{\epsilon,h}(u_\epsilon) = 0 \text{ on } Q_h \cap [0 < u_\epsilon < \theta], \tag{26}$$

$$\mathcal{T}_{\epsilon,h}(u_\epsilon) \leq 0 \text{ on } Q_h \cap [0 < u_\epsilon = \theta], \mathcal{T}_{\epsilon,h}(u_\epsilon) \geq 0 \text{ on } Q_h \cap [0 = u_\epsilon < \theta], \tag{27}$$

$$\mathcal{T}_p(u_\epsilon) \leq 0 \text{ on } Q_p \cap [0 < u_\epsilon = \theta], \mathcal{T}_p(u_\epsilon) \geq 0 \text{ on } Q_p \cap [0 = u_\epsilon < \theta], \tag{28}$$

$$u_\epsilon = 0 \text{ on } \Sigma,\ u_\epsilon(0,.) = u_0 \text{ on } \Omega, \tag{29}$$

and to have a well-posed problem, we express the transmission conditions across the interface (that will be discussed in Remark 6)

$$-\epsilon \nabla \phi(u_\epsilon).\nu_h = \nabla \phi(u_\epsilon).\nu_p \text{ on } \Sigma_{hp} \cap [0 < u_\epsilon < \theta], \tag{30}$$

$$u_{\epsilon|Q_h} = u_{\epsilon|Q_p} \text{ on } \Sigma_{hp}. \tag{31}$$

Our aim is to prove first that (25)-(31) has a unique weak solution and secondly to establish some estimates proper to study the behavior of the sequence $(u_\epsilon)_{\epsilon>0}$ when ϵ goes to 0^+. We obtain an existence result to (25)-(31) by using the artificial viscosity method - to regularize ϕ - and by relaxing the obstacle condition. That is why we start by introducing a Lipschitz bounded extension $K^\star$ and $g_i^\star$, for i in $\{h,p\}$, of K and g_i outside $\mathcal{C}_\theta^\infty$ through (for a generic function f):

$$f^\star(z) = \left\{ \begin{array}{l} f(z) \text{ if } z \in \mathcal{C}_\theta^\infty, \\ f(l_{\mathcal{C}_\theta^\infty}) \text{ if } z \leq l_{\mathcal{C}_\theta^\infty}, f(ess\sup_{\bar\Omega}\theta) \text{ if } z \geq ess\sup_{\bar\Omega}\theta, \end{array} \right.$$

where $l_{\mathcal{C}_\theta^\infty} = \min \mathcal{C}_\theta^\infty$ depending on the unilateral or bilateral case. For ϕ we choose an increasing Lipschitz extension $\phi^\star$ outside $\mathcal{C}_\theta^\infty$, so that due to (11), $(\phi^\star)^{-1}$ exists and is a continuous function on $\phi^\star(\mathcal{C}_\theta^\infty)$.

Then, for any positive parameter η, we set $\phi_\eta^\star = \phi^\star + \eta \mathbb{I}_\mathbb{R}$ and $\beta(x,u) = (-u^- + (u-\theta(x))^+)$ for a bilateral constraint (while for a unilateral obstacle condition β is reduced to $\beta(x,u) = (u-\theta(x))^+)$) and $\lambda_{\epsilon,\eta}$, a $\mathcal{C}^1(\bar\Omega)$-class approximation of $\lambda_\epsilon = \mathbb{I}_{\Omega_p}(x) + \epsilon \mathbb{I}_{\Omega_h}$ such that

$$\exists N > 0, \forall \epsilon > 0, \forall \eta > 0,\ 0 < \lambda_{\epsilon,\eta} \leq N \text{ a.e. in } \Omega,$$
$$\|\nabla \lambda_{\epsilon,\eta}\|_\infty \leq \frac{C(\epsilon)}{\eta} \text{ and when } \eta \text{ goes to } 0^+,\ \lambda_{\epsilon,\eta} \to \lambda_\epsilon \text{ a.e. on } \Omega.$$

This way (see *e.g.* [6])

Theorem 3. *There exists a unique solution*

$$u_{\epsilon,\eta} \in W(0,T; H_0^1(\Omega) \cap H^2(\Omega); L^2(\Omega)) \cap L^\infty(Q)$$

to the nondegenerate-penalized problem:

$$\begin{aligned} \partial_t u_{\epsilon,\eta} - div(\lambda_{\epsilon,\eta}(x)\nabla\phi_\eta^\star(u_{\epsilon,\eta}) &+ K^\star(u_{\epsilon,\eta})\mathbf{B}) + g^\star(t,x,u_{\epsilon,\eta}) \\ &= -\frac{1}{\eta}\beta(x,u_{\epsilon,\eta}) \text{ a.e. on } Q, \qquad (32) \\ u_{\epsilon,\eta}(0,.) &= u_0 \text{ a.e. in } \Omega. \qquad (33) \end{aligned}$$

Now we state some *a priori* estimates for $(u_{\epsilon,\eta})_{\eta>0}$ that are sufficient to study its limit when η goes to 0^+. Indeed (with $Q_s =]0,s[\times\Omega$, s in $]0,T]$):

Proposition 2. *There exists a constant C independent from ϵ and η such that:*

$$\forall t \in [0,T],\ M_2(t) \leq u_{\epsilon,\eta}(t,.) \leq M_1(t) \text{ a.e. in } \Omega, \qquad (34)$$
$$\|\beta(x,u_{\epsilon,\eta})\|_{L^1(Q)} \leq C\eta, \qquad (35)$$
$$\|\lambda_{\epsilon,\eta}^{1/2}\nabla\widehat{\phi^\star}_\eta(u_{\epsilon,\eta})\|_{L^2(Q)^n} \leq C, \qquad (36)$$
$$\forall s \in]0,T], \epsilon\|\partial_t\widehat{\phi^\star}_\eta(u_{\epsilon,\eta})\|_{L^2(Q_s)}^2 + \frac{\epsilon}{2}\|\lambda_{\epsilon,\eta}^{1/2}\nabla\phi_\eta^\star(u_{\epsilon,\eta})(s,.)\|_{L^2(\Omega)^n}^2 \leq C, \qquad (37)$$

where M_1 and M_2 are defined in (9) *and* (10) *and* $\widehat{\phi^\star}_\eta(x) = \int_0^x \sqrt{(\phi^\star)'_\eta(\tau)}d\tau.$

Proof. For (34) we use a cut-off method in L^1 by considering the $L^2(Q_s)$-scalar product between (32) and $sgn^+_\mu(u_{\epsilon,\eta} - M_1(t))$ for the majoration by M_1, and $-sgn^-_\mu(u_{\epsilon,\eta} - M_2(t))$ for the minoration by M_2. It is also a cut-off method in L^1 that provides (35). From the energy equality fulfilled by $u_{\epsilon,\eta}$ we derive (36). To conclude we take the $L^2(Q_s)$-scalar product between (32) and $\epsilon\partial_t\phi^\star_\eta(u_{\epsilon,\eta})$. Concerning the penalized term,

$$\begin{aligned}
& \frac{\epsilon}{\eta}\int_{]0,s[\times\Omega} (u_{\epsilon,\eta}-\theta)^+\partial_t\phi^\star_\eta(u_{\epsilon,\eta}) \\
= & \frac{\epsilon}{\eta}\int_{]0,s[\times\Omega} (u_{\epsilon,\eta}-\theta)^+(\phi^\star_\eta)'((u_{\epsilon,\eta}-\theta)^+ + \theta)\partial_t(u_{\epsilon,\eta}-\theta)^+ dxdt \\
= & \frac{\epsilon}{\eta}\int_{]0,s[\times\Omega} \partial_t\left(\int_0^{(u_{\epsilon,\eta}-\theta)^+} \tau(\phi^\star_\eta)'(\tau+\theta)d\tau\right) dxdt \\
= & \frac{\epsilon}{\eta}\int_\Omega \left(\int_0^{(u_{\epsilon,\eta}(s,.)-\theta)^+} \tau(\phi^\star_\eta)'(\tau+\theta)d\tau\right) dx \geq 0.
\end{aligned}$$

The same reasoning and the same sign condition hold for $-(u_{\epsilon,\eta})^-$. We bound the convective and reactive terms by using (34),(36) and the Young inequality (see [8]). Thanks to the density of $\mathcal{D}(0,T;H^1_0(\Omega)\cap H^2(\Omega))$ into $W(0,T;H^1_0(\Omega)\cap H^2(\Omega);L^2(\Omega))$, the diffusive term is integrated by parts and then with respect to t. So that the constant C in (36) depends on $\|\phi(u_0)\|_{H^1_0(\Omega)}$ and $\|u_0\|_{H^1_0(\Omega)}$. $\square$

The parameter ϵ being fixed, $(\phi^\star_\eta(u_{\epsilon,\eta}))_{\eta>0}$ remains at least in a bounded set of $H^1(Q)$. As a result, the compactness embedding of the latter space into $L^2(Q)$ and the continuity of $(\phi^\star)^{-1}$ provide the existence of a measurable function u_ϵ and a subsequence - still denoted $(u_{\epsilon,\eta})_{\eta>0}$ - such that when η goes to 0^+, $(u_{\epsilon,\eta})_{\eta>0}$ goes to u_ϵ in $L^q(Q)$, $1 \leq q < +\infty$ and $(\phi^\star_\eta(u_{\epsilon,\eta}))_{\eta>0}$ goes to $\phi(u_\epsilon)$ weakly in $H^1(Q)$ and strongly in $\mathcal{C}^0([0,T];L^2(\Omega))$. This leads to:

Theorem 4. *Problem* (25)-(31) *has at least a weak solution u_ϵ such that*

$$\forall t \in]0,T[, u_\epsilon(t,.) \in \mathcal{C}_\theta \ \text{a.e.} \ \ \text{in } \Omega, \tag{38}$$

$$\phi(u_\epsilon) \in W(0,T,H^1_0(\Omega),L^2(\Omega)), \tag{39}$$

$$u_\epsilon(0,.) = u_0 \ \text{a.e. in } \Omega, \tag{40}$$

and for any v in $W(0,T;H^1_0(\Omega),L^2(\Omega))$, $v(T,.) = u_\epsilon(T,.)$ a.e. in Ω, such that *for any t in $[0,T]$, $\phi^{-1}(v(t,.)) \in \mathcal{C}_\theta$ a.e. on Ω,*

$$\int_Q \partial_t v(v-\phi(u_\epsilon))dxdt + \int_Q \lambda_\epsilon(x)\nabla\phi(u_\epsilon).\nabla(v-\phi(u_\epsilon))dxdt$$
$$+\int_Q K(u_\epsilon)\mathbf{B}.\nabla(v-\phi(u_\epsilon))dxdt + \int_Q g(t,x,u_\epsilon)(v-\phi(u_\epsilon))dxdt$$
$$-\int_Q (u_\epsilon - v)\partial_t(v-\phi(u_\epsilon))dxdt$$
$$+\int_\Omega (u_0 - v(0,.))(\phi(u_0)-v(0,.))dx \geq 0. \tag{41}$$

Remark 4. *In* (38),(40),(41) *the trace of u_ϵ with respect to the time variable has to be understood, for any t in $[0,T]$, in the sense $u_\epsilon(t,.) = \phi^{-1}(\phi(u_\epsilon)(t,.))$.*

Proof. Obstacle Condition (38) follows from (35) while (40) comes from (33) and from the strong convergence of $(\phi^\star_\eta(u_{\epsilon,\eta}))_{\eta>0}$ toward $\phi(u_\epsilon)$ in $\mathcal{C}^0([0,T];L^2(\Omega))$. To obtain (41) we take the $L^2(Q)$-scalar product between (32) and $v-\phi^\star_\eta(u_{\epsilon,\eta})$. To study the penalized term, we write:

$$-\frac{1}{\eta}\int_Q (u_{\epsilon,\eta}-\theta)^+(v-\phi^\star_\eta(u_{\epsilon,\eta}))dxdt = -\frac{1}{\eta}\int_Q (u_{\epsilon,\eta}-\theta)^+(v-\phi^\star(u_{\epsilon,\eta}))dxdt + \int_Q (u_{\epsilon,\eta}-\theta)^+u_{\epsilon,\eta}dxdt,$$

where in the right-hand side the first term is nonnegative and the second one goes to 0 (due to (35)). The same reasoning is still true for the negative part in $\beta(x,.)$. For the evolution term, we artificially introduce the quantity $\partial_t\phi(v)(\phi(v) - \phi^\star(u_{\epsilon,\eta}))$. Then we integrate by parts in time and use the definition of v. This allows us to take the η-limit. Just note that in the diffusive term we take in fact the " $\liminf$ " and the weak convergence of gradients in $L^2(Q)$. $\square$

Now, we observe that

Proposition 3. *If u_1 and u_2 are two weak solutions to* (38)-(41) *for initial data $u_{0,1}$ and $u_{0,2}$ respectively, then (with $M_g = M_{g_h} + M_{g_p}$),*

$$\text{for a.e. } t \text{ in }]0,T[, \int_\Omega |u_1(t,x)-u_2(t,x)|dx \leq \int_\Omega |u_{0,1}-u_{0,2}|dx\, e^{M_g t}.$$

Proof. We develop the same reasoning (on the whole Q) as in Theorem 2 (on Q_p) by doubling the time variable and using the same test-functions (remind that θ is independent from the time variable on the whole Q). Observe that there are no here boundary integrals. Besides, to deal with the evolution terms, we perform first an integration by parts with respect to the time variable by considering that α_δ has a compact support in $]0,T[\times]0,T[$. Then we refer to the integration formula proved in [8] through some convexity inequalities:

Lemma 3. *Let u be a measurable and bounded function (by a constant M) on Q and f a function defined on $\Omega \times [-M, M]$ such that for any x in Ω, $\lambda \to f(x,\lambda)$ is nondecreasing and continuous and for all λ in $[-M, M]$, $x \to f(x,\lambda)$ is measurable and bounded on Ω and $\partial_t f(.,u)$ belongs to $L^1(Q)$. Then, for any α of $\mathcal{C}^1([0,T])$, $\alpha \geq 0$, such that $\alpha(T) = \alpha(0) = 0$,*

$$\int_Q u\partial_t(f(x,u)\alpha)dxdt = \int_Q \left(\int_v^u f(x,r)dr\right)\partial_t \alpha dxdt,$$

for any measurable function v bounded by M on Ω.

This way,

$$\begin{aligned}
&\int_{]0,T[\times Q} u_1\partial_t(sgn_\mu(\phi(u_1)-\phi(\tilde{u}_2))\alpha_\delta)dqd\tilde{t} \\
-&\int_{]0,T[\times Q} \tilde{u}_2\partial_{\tilde{t}}(sgn_\mu(\phi(u_1)-\phi(\tilde{u}_2))\alpha_\delta)dqd\tilde{t} \\
=&\int_{]0,T[\times Q} \left(\int_{\tilde{u}_2}^{u_1} sgn_\mu(\phi(r)-\phi(\tilde{u}_2))dr\right)\partial_t\alpha_\delta dqd\tilde{t} \\
-&\int_{]0,T[\times Q} \left(\int_{u_1}^{\tilde{u}_2} sgn_\mu(\phi(u_1)-\phi(r))dr\right)\partial_{\tilde{t}}\alpha_\delta dqd\tilde{t}
\end{aligned}$$

The conclusion follows. □

3.2 The viscous limit

As a consequence of the uniqueness property stated in Proposition 3, we make sure that the whole sequence $(u_{\epsilon,\eta})_{\eta>0}$ converges toward u_ϵ when η goes to 0^+. Thus, by considering the *a priori* estimates of Proposition 2 for $(u_{\epsilon,\eta})_{\eta>0}$, we may derive some estimates for $(u_\epsilon)_{\epsilon>0}$. Indeed

Proposition 4.

$$(u_\epsilon)_{\epsilon>0} \text{ is a bounded sequence in } L^\infty(Q), \tag{42}$$

and there exists a constant C independent from ϵ such that

$$\epsilon^{1/2}\|\nabla\widehat{\phi}(u_\epsilon)\|_{L^2(Q_h)^n} + \|\nabla\widehat{\phi}(u_\epsilon)\|_{L^2(Q_p)^n} \leq C. \tag{43}$$

Relations (42) and (43) are not sufficient to study the behavior of the sequence $(u_\epsilon)_{\epsilon>0}$ when ϵ goes to 0^+: we also need an estimate of $\partial_t u_\epsilon$ in a suitable space. To this purpose, we prove that u_ϵ fulfills an entropy inequality on Q that also be used as a starting point to establish (13) for the corresponding ϵ-limit.

Proposition 5. *Assume that*

$$\theta_{|\Omega_h} \textit{ belongs to } H^2(\Omega_h). \tag{44}$$

Then there exists a constant C, independent from ϵ such that,

$$\|\partial_t u_\epsilon\|_{L^2(0,T;H^{-1}(\Omega))} \leq C. \tag{45}$$

Proof. We set $\kappa(x) = k\theta(x)$, $k \in [0,1]$, for a bilateral constraint or $\kappa(x) = k + \theta(x)$, $M_2(T) - ess\sup_{\bar{\Omega}} \theta \leq k \leq 0$ for a unilateral obstacle condition. We consider the $L^2(Q)$-scalar product between (32) and $w_\mu^{\epsilon,\eta} \equiv sgn_\mu(\phi_\eta^\star(u_{\epsilon,\eta}) - \phi_\eta(\kappa))\zeta$, where ζ belongs to $\mathcal{D}(]-\infty,T[\times\Omega)$, $\zeta \geq 0$. We observe first that the penalized term is nonnegative. The other integrals are performed through the following transformations:
For the evolution term, with $I_\mu(u_{\epsilon,\eta},\kappa) = \int_\kappa^{u_{\epsilon,\eta}} sgn_\mu(\phi_\eta^\star(\tau) - \phi_\eta(\kappa))d\tau$

$$\begin{aligned}\int_Q \partial_t u_{\epsilon,\eta} w_\mu^{\epsilon,\eta} dxdt &= \int_Q \partial_t I_\mu(u_{\epsilon,\eta},\kappa)\zeta dxdt \\ &= -\int_Q I_\mu(u_{\epsilon,\eta},\kappa)\partial_t\zeta dxdt - \int_\Omega I_\mu(u_0,\kappa)\zeta(0,.)dx.\end{aligned}$$

For the diffusion term,

$$\begin{aligned}\int_Q \lambda_{\epsilon,\eta}\nabla\phi_\eta^\star(u_{\epsilon,\eta}).\nabla w_\mu^{\epsilon,\eta} dxdt &= \int_Q \lambda_{\epsilon,\eta}\nabla(\phi_\eta^\star(u_{\epsilon,\eta}) - \phi_\eta(\kappa)).\nabla w_\mu^{\epsilon,\eta} dxdt \\ &+ \int_Q \lambda_{\epsilon,\eta}\nabla\phi_\eta(\kappa).\nabla w_\mu^{\epsilon,\eta} dxdt\end{aligned}$$

We develop the partial derivatives in the first term in the right-hand side and we use the fact that $sgn_\mu(.)$ is nondecreasing. To take the limit with η, we remind that due to (34) and (37), $(w_\mu(u_{\epsilon,\eta},\kappa))_{\eta>0}$ is a bounded sequence of $H^1(Q)\cap L^\infty(Q)$, uniformly with respect to η and so, thanks to the convergence properties of $(u_{\epsilon,\eta})_{\eta>0}$ toward u_ϵ, converges toward $w_\mu^\epsilon \equiv sgn_\mu(\phi(u_\epsilon) - \phi(\kappa))\zeta$ strongly in $L^q(Q)$, $1 \leq q < +\infty$, weakly in $H^1(Q)$. Then, the η-limit being taken, we use the Green formula in the second of the diffusion term by sharing the integration field into Q_h (where $\lambda_\epsilon = \epsilon$) and Q_p (where $\lambda_\epsilon = 1$). It comes:

$$\begin{aligned}&-\int_Q I_\mu(u_\epsilon,\kappa)\partial_t\zeta dxdt - \int_\Omega I_\mu(u_0,\kappa)\zeta(0,.)dx \\ &+ \int_Q \lambda_\epsilon sgn_\mu(\phi(u_\epsilon) - \phi(\kappa))\nabla(\phi(u_\epsilon) - \phi(\kappa)).\nabla\zeta dxdt \\ &+ \int_Q (K(u_\epsilon) - K(\kappa))\mathbf{B}.\nabla w_\mu^\epsilon dxdt + \int_{\Sigma_{hp}} (\epsilon - 1)\nabla\phi(\kappa).\nu_h w_\mu^\epsilon d\mathcal{H}^n \\ &- \int_Q (\lambda_\epsilon\Delta\phi(\kappa) + div(K(\kappa)\mathbf{B}) - g(t,x,u_\epsilon))\, w_\mu^\epsilon dxdt \leq 0.\end{aligned} \tag{46}$$

We pass to the limit with μ through the Lebesgue dominated convergence Theorem and the Sacks Lemma to deal with the first term in the third line (remind that (24) holds). It follows:

$$
\begin{aligned}
&-\int_Q L(u_\epsilon,\kappa,\zeta)dxdt-\int_\Omega |u_0-\kappa|\zeta(0,.)dx\\
&+\int_Q \lambda_\epsilon \nabla|\phi(u_\epsilon)-\phi(\kappa)|.\nabla\zeta dxdt-\int_Q \lambda_\epsilon\Delta\phi(\kappa)sgn(u_\epsilon-\kappa)\zeta dxdt\\
&+\int_{\Sigma_{hp}} (\epsilon-1)\nabla\phi(\kappa).\nu_h sgn(\phi(u_\epsilon)-\phi(\kappa))\zeta d\mathcal{H}^n\leq 0. \qquad (47)
\end{aligned}
$$

Now the arguments are similar to those developed in Proposition 1 to prove that $\partial_t u$ is in $L^2(0,T;V')$. In (47) we consider that ζ is a nonnegative element of $\mathcal{D}(Q)$ and so, thanks to a density argument, we may choose ζ in $\mathcal{D}(0,T;H_0^1(\Omega))$, $\zeta\geq 0$. Thus for $k=1$ in the case of a bilateral obstacle and $k=0$ in the case of a unilateral one (so that $\kappa(x)=\theta(x)$), one has (with $\mathcal{T}_\epsilon=\mathcal{T}_{\epsilon,h}\mathbb{I}_{\Omega_h}+\mathcal{T}_p\mathbb{I}_{\Omega_p}$):

$$
\begin{aligned}
&\int_Q u_\epsilon\partial_t\zeta dxdt\\
\leq\ &\int_Q \lambda_\epsilon\nabla\phi(u_\epsilon).\nabla\zeta dxdt+\int_Q K(u_\epsilon)\mathbf{B}.\nabla\zeta dxdt+\int_Q g(t,x,u_\epsilon)\zeta dxdt\\
-\ &(\epsilon-1)\int_{\Sigma_{hp}} \nabla\phi(\theta).\nu_h(1+sgn(\phi(u_\epsilon)-\phi(\theta)))\zeta d\mathcal{H}^n\\
-\ &\int_Q (1+sgn(u_\epsilon-\theta))\mathcal{T}_\epsilon(\theta)\zeta dxdt,
\end{aligned}
$$

and for $\kappa(x)=0$ in the case of a bilateral constraint and $\kappa(x)=M_2(T)-ess\sup_{\bar\Omega}\theta+\theta(x)$ for an unilateral one (thus $u_\epsilon-\kappa\geq 0$ a.e.) it comes:

$$
\begin{aligned}
&\int_Q u_\epsilon\partial_t\zeta dxdt\\
\geq\ &\int_Q \lambda_\epsilon\nabla\phi(u_\epsilon).\nabla\zeta dxdt+\int_Q K(x,u_\epsilon)\mathbf{B}.\nabla\zeta dxdt+\int_Q g(t,x,u_\epsilon)\zeta dxdt\\
-\ &(\epsilon-1)\int_{\Sigma_{hp}} \nabla\phi(\kappa).\nu_h(1-sgn(\phi(u_\epsilon)-\phi(\kappa)))\zeta d\mathcal{H}^n\\
-\ &\int_Q (1-sgn(u_\epsilon-\kappa))\mathcal{T}_\epsilon(\kappa)\zeta dxdt.
\end{aligned}
$$

For any ζ in $\mathcal{D}(0,T,H_0^1(\Omega))$, we write $\zeta=\zeta^+-\zeta^-$ and use the two previous inequalities with ζ^+ and ζ^-. Thanks to estimates of Proposition 4, and to the continuity of the trace operator from V into $L^2(\Gamma_{hp})$ (so from $H_0^1(\Omega)$ into

$L^2(\Gamma_{hp})$), we prove the existence of a constant C (independent from ϵ) such as:

$$\forall \zeta \in \mathcal{D}(0,T;H_0^1(\Omega)),\ \left| \int_Q u_\epsilon \partial_t \zeta dxdt \right| \leq C\|\zeta\|_{L^2(0,T;H_0^1(\Omega))}.$$

Thus $\partial_t u_\epsilon$ belongs to $L^2(0,T;H^{-1}(\Omega))$ and for any ζ of $\mathcal{D}(0,T;H_0^1(\Omega))$,

$$-\int_Q u_\epsilon \partial_t \zeta dxdt = \int_0^T \langle \partial_t u_\epsilon, \zeta \rangle dt.$$

Estimate (45) follows that completes the proof of Proposition 5.

□

Remark 5. *By referring to [8], we may assert that as soon as $\partial_t u_\epsilon$ belongs to $L^2(0,T;H^{-1}(\Omega))$ we can perform a time-integration by parts in* (41) *so that the weak solution to* (25)-(29) *fulfills the "strong" variational inequality for any measurable function v, v in $\mathcal{C}_\theta$ a.e. on Ω with $\phi(v)$ in $H_0^1(\Omega)$:*

$$\langle \partial_t u_\epsilon, \phi(v) - \phi(u_\epsilon)\rangle + \int_\Omega (\lambda_\epsilon \nabla\phi(u_\epsilon) + K(u_\epsilon)\mathbf{B}).\nabla(\phi(v) - \phi(u_\epsilon))dx$$
$$+ \int_\Omega g(t,x,u_\epsilon)(\phi(v) - \phi(u_\epsilon))dx \geq 0 \textit{ for a.e. } t \textit{ in }]0,T[.$$

Remark 6. *By reasoning as in Remark 3 and denoting $\mathbf{V}_\epsilon = (u_\epsilon, -\lambda_\epsilon \nabla\phi(u_\epsilon) - K(u_\epsilon)\mathbf{B})$ we make sure that $\mathbf{V}_{\epsilon|Q_i}$ belongs to $H_{div_{(t,x)}}(Q_i)$ and a.e. on Q_i,*

$$-div_{(t,x)}\mathbf{V}_{\epsilon|Q_i} \leq g_i(t,x,u_\epsilon) - (1 + sgn(u_\epsilon - \theta))\mathcal{T}_i(\theta),$$
$$-div_{(t,x)}\mathbf{V}_{\epsilon|Q_i} \geq g_i(t,x,u_\epsilon) - (1 - sgn(u_\epsilon - \kappa))\mathcal{T}_i(\kappa).$$

So that if $[0 < u_\epsilon < \theta]$ is an open subset of Q, $\mathcal{H}^n$-measurable, we argue that

$$div_{(t,x)}(\mathbf{V}_{\epsilon,|Q_i}) = 0 \textit{ on } Q_i \cap [0 < u_\epsilon < \theta],$$

that is (26) *and for any nonnegative ζ with a support in $Q \cap [0 < u_\epsilon < \theta]$*

$$\lfloor \mathbf{V}_{\epsilon|Q_h}.\nu_h + \mathbf{V}_{\epsilon|Q_p}.\nu_p, \zeta \rfloor = 0,$$

that corresponds to (30)*. Observe that* (31) *holds since $\phi(u_\epsilon)$ belongs to $H^1(Q)$. Eventually $\nabla\phi(\theta).\nu_h \geq \mathcal{T}_\epsilon(\theta)$ $\mathcal{H}^n$- a.e. on $\Sigma_{hp} \cap [0 < u_\epsilon = \theta]$.*

To study the behavior of the sequence $(u_\epsilon)_{\epsilon>0}$ and characterize the corresponding limit we need and additional assumption on ϕ:

$$\phi^{-1} \text{ is Hölder continuous on } \phi(\mathcal{C}_\theta^\infty) \text{ with an exponent } \tau \text{ in }]0,1[. \tag{48}$$

In this framework,

Proposition 6. *If* (48) *holds, there exists a measurable function u in $L^\infty(Q)$, with for a.e. t in $]0,T[$, $u(t,.) \in \mathcal{C}_\theta$ a.e. on Ω, $\phi(u)$ in $L^2(0,T;V)$ and such that up to a subsequence when ϵ goes to 0^+,*

$$u_\epsilon \to u \text{ in } L^\infty(Q) \text{ weak} - \star, \text{ and in } L^q(Q_p),\ 1 \le q < +\infty,$$
$$\nabla\phi(u_\epsilon) \rightharpoonup \nabla\phi(u) \text{ weakly in } L^2(Q_p)^n,\ \epsilon\nabla\phi(u_\epsilon) \to 0^+ \text{ strongly in } L^2(Q_h)^n.$$

Proof. The strong convergence in $L^q(Q_p)$ for $(u_\epsilon)_{\epsilon>0}$ refers to the arguments put forward in [5], chapter 2. From (45) the sequence $(\partial_t u_\epsilon)_{\epsilon>0}$ remains fixed in a bounded subset of $L^2(0,T;H^{-1}(\Omega_p))$ and due to (43), the sequence $(\phi(u_\epsilon))_{\epsilon>0}$ is bounded in $L^2(0,T;V)$ uniformly with respect to ϵ. Using that

$$\forall s \in]0,1[,\ L^2(0,T;V) \hookrightarrow L^2(0,T;H^1(\Omega_p)) \hookrightarrow L^2(0,T;W^{s,2}(\Omega_p)),$$

we argue that $u_\epsilon \equiv \phi^{-1}(\phi(u_\epsilon))$ is bounded in $L^{2/\tau}(0,T;W^{\tau\, s,2/\tau}(\Omega_p))$. The compactness embedding of $W^{\tau\, s,2/\tau}(\Omega_p)$ into $L^{2/\tau}(\Omega_p)$ and the J.L.Lions compactness Theorem ([9], p. 57) ensure that $\mathcal{W} \equiv \{v \in L^{2/\tau}(0,T;W^{\tau\, s,2/\tau}(\Omega_p));\ \partial_t v \in L^2(0,T;H^{-1}(\Omega_p))\}$ is compactly embedded in $L^{2/\tau}(0,T;L^{2/\tau}(\Omega_p))$.

□

The previous convergence properties for $(u_\epsilon)_{\epsilon>0}$ are sufficient To characterize the function u. On the hyperbolic zone we take advantage of (42) and of:

Claim 1. *(see [4]) - If $\mathcal{O}$ be an open bounded subset of $\mathbb{R}^q$ $(q \ge 1)$ and $(u_n)_{n>0}$ a sequence of measurable functions on $\mathcal{O}$ such that,*

$$\exists M > 0, \forall n > 0, \|u_n\|_{L^\infty(\mathcal{O})} \le M,$$

there exist a subsequence $(u_{\varphi(n)})_{n>0}$ and a measurable π in $L^\infty(]0,1[\times\mathcal{O})$ such that for all continuous and bounded functions f on $\mathcal{O}\times]-M,M[$,

$$\forall \xi \in L^1(\mathcal{O}),\ \lim_{n\to+\infty} \int_{\mathcal{O}} f(x,u_{\varphi(n)})\xi dx = \int_{]0,1[\times\mathcal{O}} f(x,\pi(\alpha,w))d\alpha\xi dx.$$

Such a result has first been applied to the approximation through the artificial viscosity method of the Cauchy problem in $\mathbb{R}^p$ for conservation laws, as one can establish a uniform L^∞-control of approximate solutions. It has also been applied to the numerical analysis of transport equations since "Finite-Volume" schemes only give an L^∞-estimate uniformly with respect to the mesh length of the numerical solution (see [4]). Here the approximating sequence is the sequence of solutions to viscous problems (25)-(29) and we state:

Theorem 5. *If*

$$(K \circ \phi^{-1})' \text{ is continuous on } \phi(\mathcal{C}_\theta^\infty), \tag{49}$$

then (1)-(6) *has a weak solution that is the limit in $L^q(Q)$, $1 \le q < +\infty$ of the whole sequence of solutions to viscous Problems* (25)-(31) *when ϵ goes to 0^+.*

Proof. We consider the function u highlighted in Proposition 6. Since $(u_{\epsilon|\Omega_h})_{\epsilon>0}$ is uniformly bounded, there exist a subsequence - still labelled $(u_{\epsilon|\Omega_h})_{\epsilon>0}$ - and a measurable and bounded function π - called a *process* - on $]0,1[\times Q_h$ such that for any continuous bounded function ψ on $Q_h \times \mathcal{C}_\theta^\infty$ and for any ξ of $L^1(Q_h)$

$$\lim_{\epsilon\to 0^+} \int_{Q_h} \psi(t,x,u_\epsilon)\xi dxdt = \int_{]0,1[\times Q_h} \psi(t,x,\pi(\alpha,t,x))\xi d\alpha dxdt. \tag{50}$$

We first establish that on Q_h, the *process* π is reduced to $u_{|\Omega_h}$ and secondly we prove that u is a weak solution to (1)-(6) for initial data u_0. To do so, we come back to (46) in order to take the ϵ-limit and then the μ-limit separately on the parabolic zone and on the hyperbolic one. Thanks to the convergence properties of $(u_\epsilon)_{\epsilon>0}$ toward u there are no difficulties to pass to these limits in the integrals over Q_p. Especially for the convective term we refer to the Sacks Lemma. For the boundary integrals we use the fact that $(sgn_\mu(\phi(u_\epsilon) - \phi(\kappa))\zeta)_{\epsilon>0}$ is a bounded sequence of $L^2(0,T;V)$ that weakly converges toward $sgn_\mu(\phi(u) - \phi(\kappa))\zeta$ in $L^2(0,T;V)$ up to a subsequence. Then we refer to the continuity and to the linearity of the trace operator from V into $L^2(\Gamma_{hp})$. On the hyperbolic zone, we take the ϵ-limit thanks to (50) since all the nonlinearities are continuous with respect to u_ϵ. However the flux term

$$I_{\epsilon,\mu} = \int_{Q_h} K(u_\epsilon) sgn'_\mu(\phi(u_\epsilon) - \phi(\kappa))\nabla(\phi(u_\epsilon) - \phi(\kappa)).\mathbf{B}\zeta dxdt$$

has to be carefully studied since we only have weak convergences for $(u_\epsilon)_{\epsilon>0}$ and for $(\nabla\phi(u_\epsilon))_{\epsilon>0}$. That is why we introduce

$$H_\mu(v,w) = \int_w^v (K\circ\phi^{-1})(\tau) sgn'_\mu(\tau - w)d\tau.$$

So that after an integration by parts with respect to τ,

$$\begin{aligned} I_{\epsilon,\mu} &= \int_{Q_h} \nabla(H_\mu(\phi(u_\epsilon),\phi(\kappa))).\mathbf{B}\zeta dxdt \\ &- \int_{Q_h} \left(\int_{\phi(\kappa)}^{\phi(u_\epsilon)} (K\circ\phi^{-1})'(\tau) sgn'_\mu(\tau - \phi(\kappa))d\tau \right) \nabla\phi(\kappa).\mathbf{B}\zeta dxdt. \end{aligned}$$

Thanks to the Green formula,

$$\begin{aligned} I_{\epsilon,\mu} &= -\int_{Q_h} H_\mu(\phi(u_\epsilon),\phi(\kappa))(\zeta div\mathbf{B} + \nabla\zeta.\mathbf{B})dxdt \\ &+ \int_{\Sigma_{hp}} H_\mu(\phi(u_\epsilon),\phi(\kappa))\mathbf{B}.\nu_h\zeta d\mathcal{H}^n \\ &- \int_{Q_h} \left(\int_{\phi(\kappa)}^{\phi(u_\epsilon)} (K\circ\phi^{-1})'(\tau) sgn'_\mu(\tau - \phi(\kappa))d\tau \right) \nabla\phi(\kappa).\mathbf{B}\zeta dxdt. \end{aligned}$$

Since $\phi(u_\epsilon)$ is an element of $L^2(0,T;H^1(\Omega))$, for a.e. t of $]0,T[$, $(\phi(u_\epsilon)_{|\Omega_h})_{|\Gamma_{hp}} = (\phi(u_\epsilon)_{|\Omega_p})_{|\Gamma_{hp}}$. We take now the ϵ-limit through (50). For the boundary integral, we argue as previously by considering that $(H_\mu(\phi(u_\epsilon),\phi(\kappa))\zeta)_{\epsilon>0}$ is a bounded sequence in $L^2(0,T;V)$. It comes $\lim_{\epsilon\to 0^+} I_{\epsilon,\mu} = I_\mu$ where

$$\begin{aligned} I_\mu &= -\int_{Q_h\times]0,1[} H_\mu(\phi(\pi),\phi(\kappa))(\zeta div\mathbf{B} + \nabla\zeta.\mathbf{B})d\alpha dxdt \\ &+ \int_{\Sigma_{hp}} H_\mu(\phi(u),\phi(\kappa))\mathbf{B}.\nu_h\zeta d\mathcal{H}^n \\ &- \int_{Q_h\times]0,1[} \left(\int_{\phi(\kappa)}^{\phi(\pi)} (K\circ\phi^{-1})'(\tau)sgn'_\mu(\tau-\phi(\kappa))d\tau\right)\nabla\phi(\kappa).\mathbf{B}\zeta d\alpha dxdt. \end{aligned}$$

To pass to the limit with μ, we come back to the definition of $sign'_\mu$ and we use the fact that since $K\circ\phi^{-1}$ is continuous on $\phi(\mathcal{C}_\theta^\infty)$, $(H_\mu(v,w))_{\mu>0}$ converges toward $sgn(v-w)K(w)$ a.e. on $Q_h\times]0,1[$ and $d\mathcal{H}^n$-a.e. on Σ_{hp}. In the same way, due to (49), $\left(\int_w^v (K\circ\phi^{-1})'(\tau)sgn'_\mu(\tau-w)d\tau\right)_{\mu>0}$ converges toward $sgn(v-w)(K\circ\phi^{-1})'(w)$ a.e. on $Q_h\times]0,1[$. From the Lebesgue dominated convergence Theorem, it follows that $\lim_{\mu\to 0^+} I_\mu = I$ where

$$\begin{aligned} I &= -\int_{Q_h\times]0,1[} sgn(\pi-\kappa)(K(\kappa)\mathbf{B}.\nabla\zeta + \zeta div(K(\kappa)\mathbf{B}))d\alpha dxdt \\ &+ \int_{\Sigma_{hp}} sgn(\phi(u)-\phi(\kappa))K(\kappa)\mathbf{B}.\nu_h\zeta d\mathcal{H}^n, \end{aligned}$$

since ϕ is strictly increasing. Eventually,

$$\begin{aligned} &- \int_{Q_p} L_p(u,\kappa,\zeta)dxdt - \int_{Q_h\times]0,1[} L_h(\pi,\kappa,\zeta)d\alpha dxdt - \int_\Omega |u_0-\kappa|\zeta(0,.)dx \\ &+ \int_{Q_p} \nabla|\phi(u)-\phi(\kappa)|.\nabla\zeta dxdt - \int_{Q_p} \Delta\phi(\kappa)sgn(u-\kappa)\zeta dxdt \\ &- \int_{\Sigma_{hp}} \nabla\phi(\kappa).\nu_h sgn(\phi(u)-\phi(\kappa))\zeta d\mathcal{H}^n \leq 0. \end{aligned} \tag{51}$$

For ζ in $\mathcal{D}(]-\infty,T]\times\Omega_h)$, we deduce that

$$-\int_{Q_h\times]0,1[} L_h(\pi,\kappa,\zeta)d\alpha dxdt \leq \int_{\Omega_h} |u_0-\kappa|\zeta(0,.)dx.$$

Therefore, by following F.Otto's ideas in [10], but here in the context of a process solution, we may be sure that,

$$ess\lim_{t\to 0^+} \int_{]0,1[\times\Omega_h} |\pi(\alpha,t,x)-\Lambda(x)|d\alpha dx \leq \int_{\Omega_h} |u_0-\Lambda(x)|dx, \tag{52}$$

where $\Lambda(x) = k(x)\theta(x)$, $k(.)$ being a measurable function on Ω_h, $0\leq k\leq 1$ a.e. in Ω_h for the bilateral obstacle problem and $\Lambda(x) = k(x)+\theta(x)$, $M_2(T)-$

$ess\sup_{\bar{\Omega}} \theta \leq k \leq 0$ a.e. in Ω_h for the unilateral one. Initial Condition (15) on Ω_h for π is obtained by choosing:

$$\Lambda(x) = \begin{cases} \dfrac{u_0(x)}{\theta(x)} \text{ if } \theta(x) \neq 0, \\ 0 \text{ else,} \end{cases} \quad \text{in the case of the bilateral constraint and}$$

$$\Lambda(x) = u_0(x) - \theta(x), \text{ for the unilateral one.}$$

Now to establish (14) for π, we take advantage of the approximation properties of u through $(u_\epsilon)_{\epsilon>0}$ and of u_ϵ through $(u_{\epsilon,\eta})_{\eta>0}$ to come back to (32)-(33) and consider the $L^2(Q)$-scalar product between (32) and $\partial_1 H_l(u_{\epsilon,\eta}, \kappa)\zeta$, where ζ belongs to $\mathcal{D}(]0,T[\times\overline{\Omega_h})$, $\zeta \geq 0$, $\zeta(t,.) = 0$ on Γ_{hp} for any t of $[0,T]$, and

$$\begin{aligned} \forall l \in \mathbb{N}^\star,\ H_l(z,w) &= \left((dist(z,[0,w]))^2 + \left(\frac{1}{l}\right)^2 \right)^{1/2} - \frac{1}{l}, \\ \mathcal{Q}_l(z,w) &= \int_w^z \partial_1 H_l(\tau,w)(K^\star)'(\tau)d\tau, \end{aligned}$$

is the family of boundary entropy-entropy flux pair introduced by F.Otto [10]. We emphasize that $\partial_1 H_l(u_{\epsilon,\eta}, \kappa)\zeta$ is an element of $W(0,T;H_0^1(\Omega_h);L^2(\Omega_h))$ so that calculations may be performed as if we were in a single domain. Especially the Green formula does not give rise to integrals along the interface. By arguing that $0 \leq \partial_1 H_l(u_{\epsilon,\eta}, \kappa)\beta(x, u_{\epsilon,\eta})$ a.e. on Q_h, it comes:

$$\begin{aligned} &- \int_{Q_h} \left(H_l(u_{\epsilon,\eta},\kappa)\partial_t\zeta - \mathcal{Q}_l(u_{\epsilon,\eta},\kappa)\mathbf{B}.\nabla\zeta - G_{h,l}(u_{\epsilon,\eta},\kappa)\zeta \right) dxdt \\ \leq\ & - \int_{Q_h} \lambda_{\epsilon,\eta}(\partial_1 H_l(u_{\epsilon,\eta},\kappa)\nabla\zeta + \zeta\partial^2_{21}H_l(u_{\epsilon,\eta},\kappa)\nabla\kappa).\nabla\phi^\star_\eta(u_{\epsilon,\eta})dxdt, \end{aligned}$$

the convexity of the function $\xi \to H_l(\xi,.)$ being taken into account and

$$\begin{aligned} G_{h,l}(u_{\epsilon,\eta},\kappa) &= \int_\kappa^{u_{\epsilon,\eta}} \left((K^\star)'(\tau)\mathbf{B}.\nabla\kappa + K^\star(\tau)div\mathbf{B}\right)\partial^2_{11}H_l(\tau,\kappa)d\tau \\ &+ g^\star_h(t,x,u_{\epsilon,\eta})\partial_1 H_l(u_{\epsilon,\eta},\kappa). \end{aligned}$$

On account of the convergences properties of $(u_{\epsilon,\eta})_{\eta>0}$ toward u_ϵ, we take the η-limit. Then, as previously, we take the ϵ-limit thanks to (50). It follows that:

$$- \int_{]0,1[\times Q_h} \left(H_l(\pi,\kappa)\partial_t\zeta - \mathcal{Q}_l(\pi,\kappa)\mathbf{B}.\nabla\zeta - G_{h,l}(\pi,\kappa)\zeta \right) d\alpha dxdt \leq 0.$$

At this point, we adapt F.Otto's works providing that:

$$ess \lim_{\tau\to 0^-} \int_{]0,1[\times\Sigma_h\setminus\Sigma_{hp}} \mathcal{Q}_l(\pi(\alpha,\sigma+\tau\nu),\kappa)\mathbf{B}(\sigma).\nu_h\zeta d\alpha d\mathcal{H}^n \leq 0,$$

for any ζ of $L^1(\Sigma_h \backslash \Sigma_{hp})$, $\zeta \geq 0$. Boundary condition (14) for π follows by observing that $(\mathcal{Q}_l)_{l\in\mathbb{N}^*}$ converges uniformly toward $\mathcal{F}(z,0,\kappa)$ as l goes to $+\infty$.

So π fulfills (14),(15) and (18) with similar integrations fields. This way, by reasoning as in Theorem 1, if $\pi_1(\alpha,.,.)$ and $\pi_2(\beta,.,.)$ are two process solutions for initial data $u_{0,1}$ and $u_{0,2}$, then for a.e. t in $]0,T[$,

$$\int_{]0,1[\times\Omega_h} |\pi_1(\alpha,t,x) - \pi_2(\beta,t,x)| d\alpha d\beta dx dt \leq \int_{\Omega_h} |u_{0,1} - u_{0,2}| dx\, e^{M_{g_h} t}.$$

When $u_{0,1} = u_{0,2}$ on Ω_h, there exists a measurable function u_h on Q_h such that a.e. on Q_h, $u_h = \pi_1(\alpha,.) = \pi_2(\beta,.)$ for a.e. α and β in $]0,1[$. Besides the uniqueness property warrants that the whole sequence $(u_\epsilon)_{\epsilon>0}$ strongly converges to u_h in $L^q(Q_h)$, $1 \leq q < +\infty$. Thus $u_h = u_{|\Omega_h}$ a.e. on Q_h and u satisfies (12)-(14). To complete the proof of Theorem 5 we only need to ensure that u fulfills (15). Owing to (52) we just have to concentrate on Ω_p. We consider (51) for $\zeta(t,x) = \psi(t)\zeta(x)$ with ψ in $\mathcal{D}(]-\infty,T[)$, $\psi \geq 0$, and ζ in $\mathcal{D}(\Omega_p)$, $\zeta \geq 0$:

$$-\int_0^T \left(\int_{\Omega_p} |u-k|\zeta dx + f(t) \right) \psi'(t) dt \leq \int_{\Omega_p} |u_0 - k|\zeta\psi(0) dx,$$

$$\begin{aligned} \text{with } f(t) \quad &= \quad \int_{\Omega_p} (\int_0^t [-|K(u(\tau,x)) - K(\kappa)|\mathbf{B}.\nabla\zeta \\ &+ \quad g_p(\tau,x,u(\tau,x)) sgn(u(\tau,x)-k)\zeta - |\phi(u(\tau,x)) - \phi(\kappa)|\Delta\zeta] d\tau) dx. \end{aligned}$$

So the time-depending function $t \to \int_{\Omega_p} |u-\kappa|\zeta dx + f(t)$ is identified a.e. with a non-increasing and bounded function, so it has an essential limit when t goes to 0^+, t in $]0,T[\backslash\mathcal{O}$, where $\mathcal{L}(\mathcal{O}) = 0$. As f goes to 0 with t, it comes

$$ess \lim_{t\to 0^+} \int_{\Omega_p} |u-\kappa|\zeta dx \leq \int_{\Omega_p} |u_0 - \kappa|\zeta dx,$$

for any function ζ of $\mathcal{D}(\Omega_p)$, $\zeta \geq 0$. As a consequence, thanks to F.Otto's reasoning in [10] we may announce (with Λ as in (52)):

$$ess \lim_{t\to 0^+} \int_{\Omega_p} |u(t,x) - \Lambda(x)| dx \leq \int_{\Omega_p} |u_0 - \Lambda(x)| dx$$

and we argue as for (52), which concludes the proof of Theorem 5. □

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