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# Unilateral contact with adhesion and friction between two hyperelastic bodies

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In the present paper, we consider a thermodynamic model using the contact kinematics developed by A. Curnier, Q.C. He and J.J. Téliéga [C. R. Acad. Sci. Paris Sér. II 314 (1992) 1] involving unilateral contact, adhesion and Coulomb friction between two homogeneous, isotropic and hyperelastic bodies. Adhesion is described by an internal state variable  $\beta^\phi$  introduced by M. Frémond [C. R. Acad. Sci. Paris Sér. II 295 (1982) 913; J. Theor. Appl. Mech. 6 (1987) 383]. Taking the case of contact between a hyperelastic solid and a plane support, we formulate the associated boundary value problem as a minimization problem when no friction is involved. When the intensity of the adhesion obeys a 'static' law, we obtain an existence result for this problem.

## 1. Introduction

In this paper, we propose a thermodynamic model considering both unilateral contact, adhesion and friction between two elastic bodies in the framework of finite deformations [4]. Many interface debonding models have been developed in the framework of small displacements and small deformations. Our model deals with the evolution of the decohesion at the interface between two bodies. Cohesive forces at the interface may act in both the normal and tangential directions. Some of these models take into account unilateral conditions of non-penetration and friction

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between the bodies. This is the case of the model proposed by Cangémi [5] and Raous et al. [20,21]. With our formulation, the interface debonding models are extended to include the case of finite deformations, decohesion, friction and unilateral contact. According to Frémond [10,11], the adhesion is characterized by its intensity  $\beta^\varphi$ , which can vary from 1 (perfect adhesion) to 0 (no adhesion).

The thermodynamic bases of the model are presented in part 2. We begin by giving the contact kinematics in the framework of finite deformations. Using an approach proposed by Curnier et al. [7], we define the gap vector via the orthogonal projection of a point belonging to one of the candidate potential contact surfaces onto the other one. Then we define an objective relative contact velocity of one surface with respect to the other. This local approach is suitable for defining the impenetrability condition via a signed gap vector. Another approach, proposed by Ciarlet and Nečas [6], consists in adopting an overall formulation, assuming a unilateral contact condition under the constraint to stay within a set. Since this does not seem to be very suitable for numerical investigations, we adopted the local approach. The thermodynamic bases of the model are developed and we describe only the contact behavior of one of the two surfaces, as we are able to deduce the behavior of the other one from the extended action–reaction principle [7]. When the free energy density and the pseudo-potential of dissipation are suitably chosen, the complete boundary value problem is obtained.

Finally, in the particular case of contact with a plane rigid support, we propose a variational formulation of the problem with respect to the reference configuration. When the adhesive contact is assumed to be frictionless, the corresponding variational problem can be reduced to a minimization problem, for which an existence result can be obtained if the intensity of the adhesion obeys a ‘static’ law [2,6,7].

## 2. A thermodynamic model for unilateral contact, adhesion and friction

### 2.1. Contact kinematics

Throughout this paper, the variable  $\alpha$  will be assumed to be 1 and 2. We consider two solids  $\mathcal{B}^\alpha$  occupying, in some reference configurations, two domains  $\Omega^\alpha \in \mathbb{R}^3$  with their boundaries denoted by  $\Gamma^\alpha$  and let  $\mathbf{n}^\alpha$  be the corresponding outward normal unit vector. We will identify each particle of  $\mathcal{B}^\alpha$  with its position vector in  $\Omega^\alpha$  denoted by  $\mathbf{X}^\alpha$ . Motions of bodies are defined by the locally invertible and orientation-preserving mappings  $\varphi^\alpha$  such that for all  $t \in [0, T]$

$$\varphi^\alpha(\cdot, t) : \bar{\Omega}^\alpha \rightarrow \mathbb{R}^3.$$

The sets  $\Omega^{\varphi^\alpha} = \varphi^\alpha(\Omega^\alpha, t)$  are the deformed counterparts of  $\Omega^\alpha$ , and  $\mathbf{n}^{\varphi^\alpha}$  is the outward normal unit vector to  $\partial\Omega^{\varphi^\alpha}$ . Let us denote by  $\mathbf{x}^{\varphi^\alpha} = \varphi^\alpha(\mathbf{X}^\alpha, t)$  the position vector in the deformed configuration. Finally, we assume that the boundaries are regular enough, and each of them is decomposed into three disjoint open parts, denoted by  $\Gamma_D^\alpha, \Gamma_T^\alpha, \Gamma_C^\alpha, \Gamma_D^{\varphi^\alpha} = \varphi^\alpha(\Gamma_D^\alpha, t)$ ,  $\Gamma_T^{\varphi^\alpha} = \varphi^\alpha(\Gamma_T^\alpha, t)$  and  $\Gamma_C^{\varphi^\alpha} = \varphi^\alpha(\Gamma_C^\alpha, t)$ , respectively. We assume that the deformations are prescribed on  $\Gamma_D^\alpha$  and  $\Gamma_D^{\varphi^\alpha}$  in both configurations, that the surface density forces  $\mathbf{T}^\alpha, \mathbf{T}^{\varphi^\alpha}$  on  $\Gamma_T^\alpha, \Gamma_T^{\varphi^\alpha}$  are, respectively, applied and that  $\Gamma_C^\alpha, \Gamma_C^{\varphi^\alpha}$  are the potential contact surfaces in the two configurations, where the bodies can

be in unilateral contact. We denote by  $\mathbf{t}^\alpha$ ,  $\mathbf{t}^{\varphi^2}$  the surface density forces on  $\Gamma_C^\alpha$ ,  $\Gamma_C^{\varphi^2}$ , respectively. The bodies are subjected to given body forces denoted by  $\mathbf{f}^\alpha$ ,  $\mathbf{f}^{\varphi^2}$  in  $\Omega^\alpha$ ,  $\Omega^{\varphi^2}$ , respectively. Let us take  $\Omega^{\varphi^1}$  to be the contactor body and  $\Omega^{\varphi^2}$  the target one. Under appropriate smoothness assumptions, we define a one-to-one correspondence via the projection which associates to  $\mathbf{x}^{\varphi^1}$  his proximal point denoted by  $\mathbf{x}^{\Pi^\varphi} = \varphi^2(\mathbf{X}^{\Pi^\varphi})$ . This vector coincides with the orthogonal projection of  $\mathbf{x}^{\varphi^1}$  onto  $\Gamma^{\varphi^2}$ . Let us denote by  $\varphi_p$  this projection (Fig. 1). The point  $\mathbf{X}^{\Pi^\varphi}$  is the solution to the following minimization problem:

$$\|\varphi^1(\mathbf{X}^1, t) - \varphi^2(\mathbf{X}^{\Pi^\varphi}, t)\| = \min_{\mathbf{X}^2 \in \Gamma_C^2} \|\varphi^1(\mathbf{X}^1, t) - \varphi^2(\mathbf{X}^2, t)\|. \quad (1)$$

Each point of  $\Gamma_C^{\varphi^1}$  is thus linked to a point of  $\Gamma_C^{\varphi^2}$ .

In the reference configuration, we define the mappings  $\Theta_p^\varphi : \Gamma_C^1 \rightarrow \Gamma_C^2$  and  $\Pi^\varphi : \Gamma_C^1 \rightarrow \Gamma_C^{\varphi^2}$  by

$$\Theta_p^\varphi = (\varphi^2)^{-1} \circ \varphi_p \circ \varphi^1, \quad \Pi^\varphi = \varphi_p \circ \varphi^1. \quad (2)$$

The problem can therefore be written in both configurations using either the Cauchy stress tensor or the Piola–Kirchhoff stress tensor.

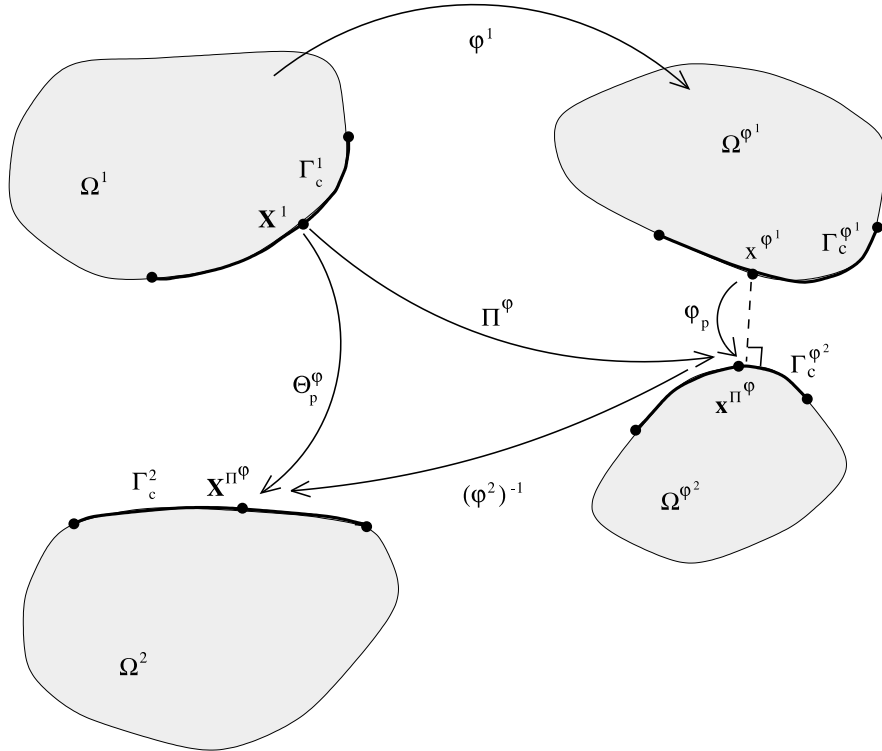


Fig. 1. Contact and kinematics.

Let us now consider a parametrization of  $\Gamma_C^{\varphi^2}$  which will help to construct a local frame for the solid  $\Omega^{\varphi^2}$  [8,13,14]. We define therefore a differentiable and invertible mapping  $\gamma^2$  such that

$$\gamma^2 : A^2 \rightarrow \Gamma_C^2, \quad (3)$$

where  $A^2$  is an open subset of  $\mathbb{R}^2$ . Let  $\gamma^{\varphi^2} : A^2 \rightarrow \Gamma_C^{\varphi^2}$  be such that

$$\gamma^{\varphi^2} = \varphi^2 \circ \gamma^2. \quad (4)$$

Each element of  $A^2$  is denoted by

$$\xi^{\varphi^2} = \begin{pmatrix} \xi_1^{\varphi^2} \\ \xi_2^{\varphi^2} \end{pmatrix} = \left( \gamma^{\varphi^2} \right)^{-1} \left( \mathbf{x}^{\varphi^2} \right).$$

For the position vector  $\mathbf{X}^2 \in \Gamma_C^2$  and the corresponding vector  $\mathbf{x}^{\varphi^2} \in \Gamma_C^{\varphi^2}$ , we have the relation (Fig. 2)

$$\mathbf{X}^2 = \gamma^2 \left( \xi^{\varphi^2} \right), \quad \mathbf{x}^{\varphi^2} = \gamma^{\varphi^2} \left( \xi^{\varphi^2} \right) = \varphi^2 \left( \gamma^2 \left( \xi^{\varphi^2} \right), t \right).$$

The minimization problem (1) can then be written as follows:

$$\left\| \varphi^1 \left( \mathbf{X}^1, t \right) - \varphi^2 \left( \gamma^2 \left( \xi_p^{\varphi^2} \right), t \right) \right\| = \min_{\xi \in A^2} \left\| \varphi^1 \left( \mathbf{X}^1, t \right) - \varphi^2 \left( \gamma^2 \left( \xi \right), t \right) \right\|, \quad (5)$$

where  $\xi_p^{\varphi^2}$  denotes the minimizer.

We suppose that these parametrizations are sufficiently smooth, so that we can define a local frame for the surface of the second solid at  $\mathbf{x}^{I\varphi^2}$ . In the deformed configuration, we define the local frame  $(\boldsymbol{\tau}_1^{\varphi^2}, \boldsymbol{\tau}_2^{\varphi^2}, \mathbf{n}^{\varphi^2})$  by

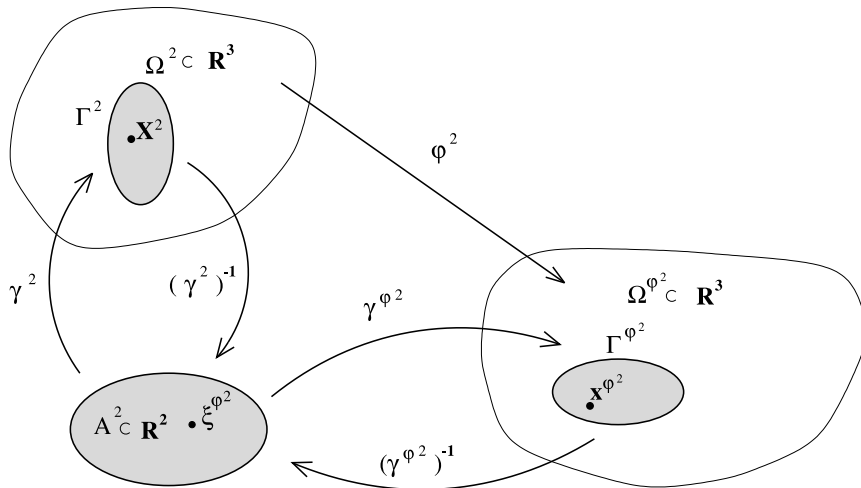


Fig. 2. Parametrization of the target's potential contact zone.

$$\begin{cases} \boldsymbol{\tau}_\alpha^{\varphi^2}(\mathbf{x}^{\varphi^2}) = \tilde{\boldsymbol{\tau}}_\alpha^{\varphi^2}(\mathbf{X}^2) & \hat{\boldsymbol{\tau}}_\alpha^{\varphi^2}(\boldsymbol{\xi}^{\varphi^2}) & \frac{\partial \gamma^{\varphi^2}(\boldsymbol{\xi}^{\varphi^2})}{\partial \xi_\alpha^{\varphi^2}}, \\ \mathbf{n}^{\varphi^2}(\mathbf{x}^{\varphi^2}) = \tilde{\mathbf{n}}^{\varphi^2}(\mathbf{X}^2) & \hat{\mathbf{n}}^{\varphi^2}(\boldsymbol{\xi}^{\varphi^2}) & \frac{\hat{\boldsymbol{\tau}}_1^{\varphi^2}(\boldsymbol{\xi}^{\varphi^2}) \wedge \hat{\boldsymbol{\tau}}_2^{\varphi^2}(\boldsymbol{\xi}^{\varphi^2})}{\|\hat{\boldsymbol{\tau}}_1^{\varphi^2}(\boldsymbol{\xi}^{\varphi^2}) \wedge \hat{\boldsymbol{\tau}}_2^{\varphi^2}(\boldsymbol{\xi}^{\varphi^2})\|} \end{cases}$$

and in the reference configuration we define  $(\boldsymbol{\tau}_1^2, \boldsymbol{\tau}_2^2, \mathbf{n}^2)$  by

$$\begin{cases} \boldsymbol{\tau}_\alpha^2(\mathbf{X}^2) & \hat{\boldsymbol{\tau}}_\alpha^2(\boldsymbol{\xi}^{\varphi^2}) & \frac{\partial \gamma^2(\boldsymbol{\xi}^{\varphi^2})}{\partial \xi_\alpha^{\varphi^2}} = (\nabla \varphi^2)^{-1}(\gamma^2(\boldsymbol{\xi}^{\varphi^2})) \hat{\boldsymbol{\tau}}_\alpha^{\varphi^2}(\boldsymbol{\xi}^{\varphi^2}), \\ \mathbf{n}^2(\mathbf{X}^2) = \hat{\mathbf{n}}^2(\boldsymbol{\xi}^{\varphi^2}) & \frac{\hat{\boldsymbol{\tau}}_1^2(\boldsymbol{\xi}^{\varphi^2}) \wedge \hat{\boldsymbol{\tau}}_2^2(\boldsymbol{\xi}^{\varphi^2})}{\|\hat{\boldsymbol{\tau}}_1^2(\boldsymbol{\xi}^{\varphi^2}) \wedge \hat{\boldsymbol{\tau}}_2^2(\boldsymbol{\xi}^{\varphi^2})\|}. \end{cases}$$

We will denote by  $(\boldsymbol{\tau}_1^\varphi, \boldsymbol{\tau}_2^\varphi, \mathbf{n}^\varphi)$ ,  $(\tilde{\boldsymbol{\tau}}_1^\varphi, \tilde{\boldsymbol{\tau}}_2^\varphi, \tilde{\mathbf{n}}^\varphi)$ ,  $(\hat{\boldsymbol{\tau}}_1^\varphi, \hat{\boldsymbol{\tau}}_2^\varphi, \hat{\mathbf{n}}^\varphi)$  these local frames constructed at  $\mathbf{x}^{I\varphi}$ ,  $I\varphi(\mathbf{X}^1)$  and  $\gamma^{\varphi^2}(\boldsymbol{\xi}_p^\varphi)$ , respectively, and by  $(\cdot)_n$ ,  $(\cdot)_{\tilde{n}}$ ,  $(\cdot)_{\hat{n}}$ ,  $(\cdot)_\tau$ ,  $(\cdot)_{\tilde{\tau}}$ ,  $(\cdot)_{\hat{\tau}}$  the normal and tangential components.

In this framework, we define the gap vector  $\mathbf{d}^\varphi$  for any deformation  $\varphi$  by

$$\mathbf{d}^\varphi(\mathbf{x}^{\varphi^1}, t) = \mathbf{x}^{\varphi^1} - \varphi_p(\mathbf{x}^{\varphi^1}, t) = \mathbf{x}^{\varphi^1} - \gamma^{\varphi^2}(\boldsymbol{\xi}_p^\varphi), \quad (6)$$

with the nominal counterpart, which can be written as follows:

$$\tilde{\mathbf{d}}^\varphi(\mathbf{X}^1, t) = \varphi^1(\mathbf{X}^1, t) - \varphi^2(\boldsymbol{\Theta}_p^\varphi(\mathbf{X}^1), t) = \varphi^1(\mathbf{X}^1, t) - \varphi^2(\gamma^2(\boldsymbol{\xi}_p^\varphi), t). \quad (7)$$

These quantities can be expressed in the previous local frames as follows:

$$\mathbf{d}^\varphi(\mathbf{x}^{\varphi^1}, t) = d_{\hat{\mathbf{n}}^\varphi}^\varphi(\mathbf{x}^{\varphi^1}, t) \hat{\mathbf{n}}^\varphi = d_{\tilde{\mathbf{n}}^\varphi}^\varphi(\mathbf{x}^{\varphi^1}, t) \tilde{\mathbf{n}}^\varphi \quad (8)$$

and also

$$\tilde{\mathbf{d}}^\varphi(\mathbf{X}^1, t) = \tilde{d}_{\tilde{\mathbf{n}}^\varphi}^\varphi(\mathbf{X}^1, t) \tilde{\mathbf{n}}^\varphi = \tilde{d}_{\hat{\mathbf{n}}^\varphi}^\varphi(\mathbf{X}^1, t) \hat{\mathbf{n}}^\varphi. \quad (9)$$

Let us calculate the material derivative of this gap vector:

$$\begin{aligned} \frac{d}{dt} \left\{ \varphi^1(\mathbf{X}^1, t) - \varphi^2(\gamma^2(\boldsymbol{\xi}_p^\varphi), t) \right\} &= \partial_t \varphi^1(\mathbf{X}^1, t) - \partial_t \varphi^2(\gamma^2(\boldsymbol{\xi}_p^\varphi), t) - \sum_{k=1}^2 \dot{\xi}_{pk}^\varphi \hat{\boldsymbol{\tau}}_k^\varphi \\ &= \partial_t \varphi^1(\mathbf{X}^1, t) - \partial_t \varphi^2(\gamma^2(\boldsymbol{\xi}_p^\varphi), t) - \dot{\boldsymbol{\xi}}_p^\varphi. \end{aligned}$$

From (9) we can write another expression for the material derivative of the gap vector as

$$\dot{\mathbf{d}}^\varphi(\mathbf{x}^{\varphi^1}, t) = \dot{\mathbf{d}}^\varphi(\mathbf{X}^1, t) = \widehat{d}_{\hat{\mathbf{n}}^\varphi}^\varphi \hat{\mathbf{n}}^\varphi + d_{\hat{\mathbf{n}}^\varphi}^\varphi \dot{\hat{\mathbf{n}}^\varphi} = \widehat{\tilde{d}}_{\hat{\mathbf{n}}^\varphi}^\varphi \hat{\mathbf{n}}^\varphi + \tilde{d}_{\hat{\mathbf{n}}^\varphi}^\varphi \dot{\hat{\mathbf{n}}^\varphi}. \quad (10)$$

We define a contact relative velocity as follows:

$$\mathbf{V}_{\text{re}}^\varphi(\mathbf{x}^{\varphi^1}, t) = \partial_t \mathbf{x}^{\varphi^1} - \partial_t \gamma^{\varphi^2}(\boldsymbol{\xi}_p^\varphi) - d_{\hat{\mathbf{n}}^\varphi}^\varphi(\mathbf{x}^{\varphi^1}, t) \dot{\hat{\mathbf{n}}^\varphi}. \quad (11)$$

This is an objective quantity and it is reduced to the classical relative velocity, that is,  $\mathbf{V}_{\text{re}}^\varphi(\mathbf{x}^{\varphi^1}, t) = \partial_t \mathbf{x}^{\varphi^1}(t) - \partial_t \gamma^{\varphi^2}(\boldsymbol{\xi}_p^\varphi)$  in the case of established contact [8,18,19].

From (11) and (10), the contact velocity can be written:

$$\mathbf{V}_{\text{re}}^\varphi(\mathbf{x}^{\varphi^1}, t) = \widehat{d}_{\hat{\mathbf{n}}^\varphi}^\varphi(\mathbf{x}^{\varphi^1}, t) \hat{\mathbf{n}}^\varphi + \dot{\boldsymbol{\xi}}_p^\varphi = \widehat{\tilde{d}}_{\hat{\mathbf{n}}^\varphi}^\varphi(\mathbf{X}^1, t) \hat{\mathbf{n}}^\varphi + \dot{\boldsymbol{\xi}}_p^\varphi. \quad (12)$$

## 2.2. The two principles of thermodynamics

Let us denote by  $\epsilon^\varphi$  the internal energy in an arbitrary domain  $\mathcal{D}^{\varphi^x} \subset \Omega^{\varphi^x} \cup \Gamma_C^{\varphi^x}$  such that

$$\epsilon^\varphi = \int_{\mathcal{D}^{\varphi^x}} \rho^{\varphi^x} e^{\varphi^x} \, d\mathbf{x}^{\varphi^x} + \int_{\partial \mathcal{D}^{\varphi^x} \cap \Gamma_C^{\varphi^1}} \rho_S^{\varphi^1} e_S^{\varphi^1} \, da^{\varphi^1},$$

where  $\rho^{\varphi^x}$  is the density and  $\rho_S^{\varphi^1}$  is the surface density of the first solid. In this paper, we study the quasi-static case and using the principle of virtual powers, the first principle reads

$$\dot{\epsilon}^\varphi = \mathcal{P}_e + \mathcal{P}_{\text{cal}} = -\mathcal{P}_i + \mathcal{P}_{\text{cal}},$$

where

$$\begin{aligned} \mathcal{P}_e &= \sum_{\alpha=1}^2 \left[ \int_{\mathcal{D}^{\varphi^x}} \mathbf{f}^{\varphi^x} \cdot \mathbf{v}^{\varphi^x} \, d\mathbf{x}^{\varphi^x} + \int_{\partial \mathcal{D}^{\varphi^x} \setminus \Gamma_C^{\varphi^x}} \tilde{\mathbf{T}}^{\varphi^x} \cdot \mathbf{v}^{\varphi^x} \, da^{\varphi^x} \right], \\ \mathcal{P}_i &= - \sum_{\alpha=1}^2 \left[ \int_{\mathcal{D}^{\varphi^x}} \boldsymbol{\sigma}^{\varphi^x} : \mathbb{D}^{\varphi^x} \, d\mathbf{x}^{\varphi^x} - \int_{\partial \mathcal{D}^{\varphi^x} \cap \Gamma_C^{\varphi^x}} \mathbf{t}^{\varphi^x} \cdot \mathbf{v}^{\varphi^x} \, da^{\varphi^x} \right], \\ \mathcal{P}_{\text{cal}} &= - \sum_{\alpha=1}^2 \int_{\partial \mathcal{D}^{\varphi^x}} \mathbf{q}^{\varphi^x} \cdot \mathbf{n}^{\varphi^x} \, da^{\varphi^x} \end{aligned} \quad (13)$$

and  $\mathbf{v}^{\varphi^x}$  is the virtual admissible velocity,  $\tilde{\mathbf{T}}^{\varphi^x}$  is the Cauchy stress vector,  $\boldsymbol{\sigma}^{\varphi^x}$  is the Cauchy stress tensor,  $\mathbb{D}^{\varphi^x}$  is the rate of deformation tensor,  $\mathbf{t}^{\varphi^x} = \boldsymbol{\sigma}^{\varphi^x} \mathbf{n}^{\varphi^x}$  is the Cauchy stress vector, which depends on the adhesive state, and  $\mathbf{q}^{\varphi^x}$  is the surface heat flux vector.

Using the following extended action–reaction principle of contact forces:

$$\int_{\Gamma_C^{\varphi^1}} \left( \mathbf{t}^{\varphi^1}(\mathbf{x}^{\varphi^1}) + j_p(\mathbf{x}^{\varphi^1}) \mathbf{t}^{\varphi^2}(\varphi_p(\mathbf{x}^{\varphi^1})) \right) d\mathbf{a}^{\varphi^1} = 0 \quad \forall \mathbf{x}^{\varphi^1} \in \Gamma_C^{\varphi^1}, \quad (14)$$

where  $j_p$  satisfies  $da^{\varphi^2} = j_p((\mathbf{x}^{\varphi^1}) da^{\varphi^1}$  (see [12]), the internal power becomes

$$\mathcal{P}_i = - \sum_{\alpha=1}^2 \int_{\mathcal{D}^{\varphi^\alpha}} \boldsymbol{\sigma}^{\varphi^\alpha} : \mathbb{D}^{\varphi^\alpha} d\mathbf{x}^{\varphi^\alpha} + \int_{\partial\mathcal{D}^{\varphi^1} \cap \Gamma_C^{\varphi^1}} \mathbf{t}^{\varphi^1} \cdot [\mathbf{v}^{\varphi^1} - \mathbf{v}^{\varphi^2} \circ \varphi_p] d\mathbf{a}^{\varphi^1}. \quad (15)$$

Assuming that there is no flux of material between surface and volume, the volume and surface mass conservation equations reads

$$\begin{cases} \dot{\rho}^{\varphi^\alpha} + \rho^{\varphi^\alpha} \operatorname{div}^{\varphi^\alpha} \mathbf{v}^{\varphi^\alpha} = 0 & \text{in } \Omega^{\varphi^\alpha}, \\ \dot{\rho}_S^{\varphi^1} + \rho_S^{\varphi^1} \operatorname{div}_S^{\varphi^1} \mathbf{v}^{\varphi^1} = 0 & \text{on } \Gamma_C^{\varphi^1}, \end{cases} \quad (16)$$

where  $\operatorname{div}^{\varphi^1}$  and  $\operatorname{div}_S^{\varphi^1}$  are the volume and surface divergence operators [9,16].

Using (14) and (16), one can show that

$$\sum_{\alpha=1}^2 \int_{\mathcal{D}^{\varphi^\alpha}} \rho^{\varphi^\alpha} \dot{\epsilon}^{\varphi^\alpha} d\mathbf{x}^{\varphi^\alpha} + \int_{\partial\mathcal{D}^{\varphi^1} \cap \Gamma_C^{\varphi^1}} \rho_S^{\varphi^1} \dot{\epsilon}_S^{\varphi^1} d\mathbf{a}^{\varphi^1} = -\mathcal{P}_i + \mathcal{P}_{\text{cal}}. \quad (17)$$

Using the decomposition with respect to the local frame, the definition of the contact velocity (11) and the local moment balance [7], the internal power can be written as follows:

$$\mathcal{P}_i = - \sum_{\alpha=1}^2 \int_{\mathcal{D}^{\varphi^\alpha}} \boldsymbol{\sigma}^{\varphi^\alpha} : \mathbb{D}^{\varphi^\alpha} d\mathbf{x}^{\varphi^\alpha} + \int_{\partial\mathcal{D}^{\varphi^1} \cap \Gamma_C^{\varphi^1}} \left( t_{\hat{n}^{\varphi^1}}^{\varphi^1} \dot{\hat{d}}_{\hat{n}^{\varphi^1}} + \mathbf{t}_{\hat{\tau}^{\varphi^1}}^{\varphi^1} \cdot \dot{\hat{\xi}}_p^{\varphi^1} \right) d\mathbf{a}^{\varphi^1}.$$

We therefore obtain the local form of the first principle, as  $\mathcal{D}^{\varphi^\alpha}$  is arbitrary in  $\Omega^{\varphi^\alpha}$ :

$$\begin{cases} \rho^{\varphi^\alpha} \dot{\epsilon}^{\varphi^\alpha} = \boldsymbol{\sigma}^{\varphi^\alpha} : \mathbb{D}^{\varphi^\alpha} - \operatorname{div}^{\varphi^\alpha} \mathbf{q}^{\varphi^\alpha} & \text{in } \Omega^{\varphi^\alpha}, \quad \alpha = 1, 2, \\ \rho_S^{\varphi^1} \dot{\epsilon}_S^{\varphi^1} = -t_{\hat{n}^{\varphi^1}}^{\varphi^1} \dot{\hat{d}}_{\hat{n}^{\varphi^1}} - \mathbf{t}_{\hat{\tau}^{\varphi^1}}^{\varphi^1} \cdot \dot{\hat{\xi}}_p^{\varphi^1} & \text{on } \Gamma_C^{\varphi^1}. \end{cases} \quad (18)$$

Let us now denote by  $H^\varphi$  the entropy of  $\mathcal{D}^{\varphi^\alpha}$  given by

$$H^\varphi = \int_{\mathcal{D}^{\varphi^\alpha}} \rho^{\varphi^\alpha} h^{\varphi^\alpha} d\mathbf{x}^{\varphi^\alpha} + \int_{\partial\mathcal{D}^{\varphi^1} \cap \Gamma_C^{\varphi^1}} \rho_S^{\varphi^1} h_S^{\varphi^1} d\mathbf{a}^{\varphi^1}.$$

The second principle of thermodynamics reads:

$$\dot{H}^\varphi \geq \int_{\partial\mathcal{D}^{\varphi^\alpha}} \frac{\mathbf{q}^{\varphi^\alpha} \cdot \mathbf{n}^{\varphi^\alpha}}{T} d\mathbf{a}^{\varphi^\alpha},$$



which yields to the following local form for the second principle, as  $\mathcal{D}^{\varphi^\alpha}$  is arbitrary in  $\Omega^{\varphi^\alpha}$ :

$$\begin{cases} \rho_S^{\varphi^1} \dot{h}^{\varphi^\alpha} \geq -\operatorname{div}^{\varphi^\alpha} \left( \frac{\mathbf{q}}{T} \right) & \text{in } \Omega^{\varphi^\alpha}, \quad \alpha = 1, 2, \\ \dot{h}_S^{\varphi^1} \geq 0 & \text{on } \Gamma_C^{\varphi^1}, \end{cases} \quad (19)$$

where  $T$  is the absolute temperature.

We classically introduce the Helmholtz free energies  $\psi^{\varphi^\alpha}$  and  $\Psi^{\varphi^1}$  such that

$$\psi^{\varphi^\alpha} = e^{\varphi^\alpha} - h^{\varphi^\alpha} T, \quad \Psi^{\varphi^1} = e_S^{\varphi^1} - h_S^{\varphi^1} T.$$

In any isothermal process, relations (18)–(22) lead to the following Clausius–Duhem inequalities which must be satisfied for any actual evolution of the bodies:

$$\begin{cases} \rho^{\varphi^\alpha} \dot{\psi}^{\varphi^\alpha} \leq \boldsymbol{\sigma}^{\varphi^\alpha} : \mathbb{D}^{\varphi^\alpha} & \text{in } \Omega^{\varphi^\alpha}, \quad \alpha = 1, 2, \\ \rho_S^{\varphi^1} \dot{\Psi}^{\varphi^1} \leq -t_{n^\varphi}^{\varphi^1} \hat{d}_{n^\varphi}^{\varphi^1} - \mathbf{t}_{\tau^\varphi}^{\varphi^1} \cdot \dot{\boldsymbol{\zeta}}_p^{\varphi^1} & \text{on } \Gamma_C^{\varphi^1}. \end{cases} \quad (20)$$

### 2.3. State and complementary laws

Since the previous relations have been established quite generally, we will now present our specific choice for expressions of the free energies.

As contact situations can occur either with or without adhesion, the surface variables  $d_{n^\varphi}^{\varphi^1}$  and  $\boldsymbol{\zeta}_p^{\varphi^1}$  (which correspond to the normal and the tangential components of the displacement jump across the interface under the assumption of small displacements) does not describe completely the adhesion state. According to Frémond [10,11], we will characterize the adhesion by the internal state variable  $\beta^\varphi \in [0, 1]$ , which represents the intensity of the adhesion between the two surfaces, such that

$$\begin{cases} 0 < \beta^\varphi < 1, & \text{the adhesion is partial,} \\ \beta^\varphi = 1, & \text{the adhesion is total,} \\ \beta^\varphi = 0, & \text{there is no adhesion.} \end{cases} \quad (21)$$

We consider a homogeneous, isotropic and hyperelastic material characterized by its stored energy function  $\hat{\mathbf{W}}^\alpha$  such that

$$\mathbb{P}^\alpha = \frac{\partial \hat{\mathbf{W}}^\alpha}{\partial \nabla \varphi^\alpha}, \quad \alpha = 1, 2, \quad (22)$$

where  $\mathbb{P}^\alpha$  is the first Piola Kirchhoff stress tensor.

To describe the deformation of the bodies in the deformed configuration, we choose the Hencky strain tensor [17] because it is conjugated to the Cauchy stress tensor when the solids are hyperelastic and isotropic, we have the following relations (see [23]):

$$\boldsymbol{\sigma}^{\varphi^z} : \mathbb{D}^{\varphi^z} = \boldsymbol{\sigma}^{\varphi^z} : \mathbb{H}^{\varphi^z} \quad \text{and} \quad \boldsymbol{\sigma}^{\varphi^z} = \frac{\rho^{\varphi^z}}{\rho^z} \frac{\partial \mathbf{W}_0^{\varphi^z}(\mathbb{H}^{\varphi^z})}{\partial \mathbb{H}^{\varphi^z}}, \quad (23)$$

where  $\mathbf{W}_0^{\varphi^z}$  is the strain energy density of the hyperelastic material with respect to the configuration  $\Omega^{\varphi^z}$ .

The superscripts  $(\cdot)^{\text{R}}$ ,  $(\cdot)^{\text{IR}}$  will denote, respectively, the reversible and irreversible parts. We take  $(\mathbf{t}_{\hat{n}^\varphi}^{\varphi^1})^{\text{R}}$ ,  $(\mathbf{t}_{\hat{z}^\varphi}^{\varphi^1})^{\text{R}}$  to denote the thermodynamic forces associated with  $d_{\hat{n}^\varphi}^\varphi$  and  $\xi_p^\varphi$  (the expressions for the irreversible components will be given later by the complementary laws). The thermodynamics forces associated to the state variables are presented in Table 1.

The free energy depends on the Hencky strain tensor according to the following relation:

$$\psi^{\varphi^z}(\mathbb{H}^{\varphi^z}) = \frac{1}{\rho^z} \mathbf{W}_0^{\varphi^z}(\mathbb{H}^{\varphi^z}). \quad (24)$$

The surface free energy  $\Psi^{\varphi^1}$  is chosen onto  $\Gamma_C^{\varphi^1}$  as follows:

$$\Psi^{\varphi^1} = \Psi_0^{\varphi^1}(d_{\hat{n}^\varphi}^\varphi, \xi_p^\varphi, \beta^\varphi) = \frac{1}{\rho_S^{\varphi^1}} (Id_K(d_{\hat{n}^\varphi}^\varphi) + Id_P(\beta^\varphi)) + \Psi_0^{\varphi^1}(d_{\hat{n}^\varphi}^\varphi, \xi_p^\varphi, \beta^\varphi), \quad (25)$$

where  $K = \{d_n; d_n \geq 0\}$ ,  $P = \{\gamma; 0 \leq \gamma \leq 1\}$ ,  $Id_K$ ,  $Id_P$  are the indicator functions of  $K$  and  $P$  and  $\Psi_0^{\varphi^1}$  is assumed to be objective. Moreover,  $\Psi_0^{\varphi^1}$  is assumed to be equal to 0 when  $\beta^\varphi = 0$  and differentiable at  $(d_{\hat{n}^\varphi}^\varphi, \xi_p^\varphi, \beta^\varphi)$ .

We choose the following specific form for  $\Psi_0^{\varphi^1}(d_{\hat{n}^\varphi}^\varphi, \xi_p^\varphi, \beta^\varphi)$ :

$$\Psi_0^{\varphi^1} = c_N \beta^\varphi (d_{\hat{n}^\varphi}^\varphi)^2 + c_T \beta^\varphi \left\| \xi_p^\varphi \right\|^2,$$

where  $c_N \geq 0$ ,  $c_T \geq 0$  and

$$\begin{cases} \beta^\varphi = 1 & \text{if } 0 \leq \delta \leq \delta_0, \\ 0 < \beta^\varphi < 1 & \text{if } \delta_0 \leq \delta \leq \delta_c, \\ \beta^\varphi = 0 & \text{if } \delta_c \leq \delta, \end{cases}$$

with  $\delta_0$  a mechanical parameter for which the decohesion begins and  $\delta_c$  the one for which decohesion is complete. Here  $\delta$  denotes the norm of the gap between the two bodies:

Table 1  
The thermodynamic variables

State variables	$\mathbb{H}^{\varphi^z}$	$d_{\hat{n}^\varphi}^\varphi$	$\xi_p^\varphi$	$\beta^\varphi$
Thermodynamic forces	$-(\boldsymbol{\sigma}^\varphi)^{\text{R}}$	$(\mathbf{t}_{\hat{n}^\varphi}^{\varphi^1})^{\text{R}}$	$(\mathbf{t}_{\hat{z}^\varphi}^{\varphi^1})^{\text{R}}$	$G_\beta$

$$\delta = \sqrt{(d_{\hat{n}^\varphi}^\varphi)^2 + \|\xi_p^\varphi\|^2}.$$

For example, one can take

- $\beta(\delta) = \left(\frac{\delta_c - \delta}{\delta_c}\right)^2, \quad \delta_0 = 0, \quad \delta_c < \infty,$
- $\beta(\delta) = \frac{\delta_c - \delta}{\delta_c - \delta_0}, \quad \delta_0 < \delta_c < \infty,$
- $\beta(\delta) = \frac{\delta_c - \delta}{\delta_c - 2\delta_0 + \delta}, \quad \delta_0 < \delta_c < \infty,$
- $\beta(\delta) = \left(\frac{\delta_0}{\delta}\right)^2, \quad \delta_c = \infty.$

This formulation is similar to the classical continuum damage mechanics formulation, in the case of surface damage, and generalizes some formulas used in small displacements (see [15]).

We consider the following state laws:

$$\begin{aligned} (\boldsymbol{\sigma}^{\varphi^\alpha})^{\mathbf{R}} &= \rho^{\varphi^\alpha} \frac{\partial \Psi^{\varphi^\alpha}}{\partial \mathbb{H}^{\varphi^\alpha}} \text{ in } \Omega^{\varphi^\alpha}, \quad \alpha = 1, 2, \\ -(\mathbf{t}_{\hat{n}^\varphi}^{\varphi^1})^{\mathbf{R}} &\in \rho_S^{\varphi^1} \partial_{d_{\hat{n}^\varphi}^\varphi} \Psi^{\varphi^1}, \quad -(\mathbf{t}_{\xi_p^\varphi}^{\varphi^1})^{\mathbf{R}} \in \rho_S^{\varphi^1} \partial_{\xi_p^\varphi} \Psi^{\varphi^1}, \quad -G_\beta \in \rho_S^{\varphi^1} \partial_{\beta^\varphi} \Psi^{\varphi^1} \text{ on } \Gamma_C^{\varphi^1}, \end{aligned} \quad (26)$$

where  $\hat{\partial}_y$  is the sub-differential with respect to  $y$ .

We now choose the following expression for the pseudo-potential of dissipation:

$$\Phi^{\varphi^1} = \Phi^{\varphi^1}(\xi_p^\varphi, \dot{\beta}^\varphi; \chi) = \frac{\mu}{\rho_S^{\varphi^1}} \left| t_{\hat{n}^\varphi}^{\varphi^1} + \rho_S^{\varphi^1} \frac{\partial \Psi_0^{\varphi^1}}{\partial d_{\hat{n}^\varphi}^\varphi} \right| \|\xi_p^\varphi\| + \Phi_0^{\varphi^1}(\dot{\beta}^\varphi, t_{\hat{n}^\varphi}^{\varphi^1}, \beta^\varphi), \quad \chi = (t_{\hat{n}^\varphi}^{\varphi^1}, d_{\hat{n}^\varphi}^\varphi, \xi_p^\varphi, \beta^\varphi), \quad (27)$$

where  $\mu$  is the friction coefficient,  $\Phi_0^{\varphi^1}$  is a given, frame indifferent and differentiable mapping that is equal to 0 when  $\beta^\varphi = 0$ , finite and convex with respect to  $\dot{\beta}^\varphi$ .

The complementary laws are as follows:

$$\begin{aligned} (\boldsymbol{\sigma}^{\varphi^\alpha})^{\mathbf{IR}} &= 0 \text{ in } \Omega^{\varphi^\alpha}, \quad \alpha = 1, 2, \\ -(\mathbf{t}_{\hat{n}^\varphi}^{\varphi^1})^{\mathbf{IR}} &= 0, \quad -(\mathbf{t}_{\xi_p^\varphi}^{\varphi^1})^{\mathbf{IR}} \in \rho_S^{\varphi^1} \partial_{\xi_p^\varphi} \Phi^{\varphi^1}, \quad G_\beta \in \rho_S^{\varphi^1} \hat{\partial}_{\beta^\varphi} \Phi^{\varphi^1} \text{ on } \Gamma_C^{\varphi^1}. \end{aligned} \quad (28)$$

Let us define the dissipation of the system as follows:

$$\begin{aligned} d^{\varphi^x} &= -\rho^{\varphi^x} \dot{\psi}^{\varphi^x} + \boldsymbol{\sigma}^{\varphi^x} : \mathbb{H}^{\varphi^x} \quad \text{in } \Omega^{\varphi^x}, \\ D^{\varphi^1} &= -\rho_S^{\varphi^1} \dot{\Psi}^{\varphi^1} - t_{\hat{n}^{\varphi^1}}^{\varphi^1} \hat{d}_{\hat{n}^{\varphi^1}}^{\varphi^1} - \mathbf{t}_{\hat{\tau}^{\varphi^1}}^{\varphi^1} \cdot \dot{\boldsymbol{\xi}}_p^{\varphi^1} \quad \text{on } \Gamma_C^{\varphi^1}, \end{aligned}$$

where  $\dot{\Psi}^{\varphi^1}$  is the right derivative of  $\Psi^{\varphi^1}$ . Using (26) one obtains the positivity of  $D^{\varphi^1}$  on  $\Gamma_C^{\varphi^1}$  if

$$-\left(t_{\hat{n}^{\varphi^1}}^{\varphi^1}\right)^{\text{IR}} \hat{d}_{\hat{n}^{\varphi^1}}^{\varphi^1} - \left(\mathbf{t}_{\hat{\tau}^{\varphi^1}}^{\varphi^1}\right)^{\text{IR}} \cdot \dot{\boldsymbol{\xi}}_p^{\varphi^1} + G_\beta \dot{\beta}^{\varphi^1} \geq 0. \quad (29)$$

The previous inequality will be satisfied, for example, in the case of a positive pseudo-potential of dissipation which is zero when  $\dot{\boldsymbol{\xi}}_p^{\varphi^1} = \mathbf{0}$ ,  $\dot{\beta}^{\varphi^1} = 0$ .

#### 2.4. The quasi-static problem

Using the expression for  $\Psi^{\varphi^1}$  given by (25), we can write the state laws (26) via a partial sub-differentiation of  $\Psi^{\varphi^1}$ . The first inclusions in (26) become

$$\left(-\left(t_{\hat{n}^{\varphi^1}}^{\varphi^1}\right)^{\text{R}}, -\left(\mathbf{t}_{\hat{\tau}^{\varphi^1}}^{\varphi^1}\right)^{\text{R}}\right) \in \rho_S^{\varphi^1} \nabla_{(d_{\hat{n}^{\varphi^1}}^{\varphi^1}, \boldsymbol{\xi}_p^{\varphi^1})} \Psi_0^{\varphi^1} \left(d_{\hat{n}^{\varphi^1}}^{\varphi^1}, \boldsymbol{\xi}_p^{\varphi^1}\right) + \partial \text{Id}_K(d_{\hat{n}^{\varphi^1}}^{\varphi^1}),$$

and by making appropriate variational choices, we obtain [3]

$$-\left(t_{\hat{n}^{\varphi^1}}^{\varphi^1}\right)^{\text{R}} \in \rho_S^{\varphi^1} \frac{\partial \Psi_0^{\varphi^1}}{\partial d_{\hat{n}^{\varphi^1}}^{\varphi^1}} + \partial \text{Id}_K(d_{\hat{n}^{\varphi^1}}^{\varphi^1}), \quad (30)$$

$$\left(\mathbf{t}_{\hat{\tau}^{\varphi^1}}^{\varphi^1}\right)^{\text{R}} = -\rho_S^{\varphi^1} \frac{\partial \Psi_0^{\varphi^1}}{\partial \boldsymbol{\xi}_p^{\varphi^1}} \quad \text{on } \Gamma_C^{\varphi^1}. \quad (31)$$

We can now write (30) as follows:

$$\forall \psi, \quad \text{Id}_K(d_{\hat{n}^\psi}^\psi) - \text{Id}_K(d_{\hat{n}^{\varphi^1}}^{\varphi^1}) \geq -\left(\left(t_{\hat{n}^{\varphi^1}}^{\varphi^1}\right)^{\text{R}} + \rho_S^{\varphi^1} \frac{\partial \Psi_0^{\varphi^1}}{\partial d_{\hat{n}^{\varphi^1}}^{\varphi^1}}\right) (d_{\hat{n}^\psi}^\psi - d_{\hat{n}^{\varphi^1}}^{\varphi^1}),$$

which is equivalent to

$$\forall \psi \in K, \quad \left(\left(t_{\hat{n}^{\varphi^1}}^{\varphi^1}\right)^{\text{R}} + \rho_S^{\varphi^1} \frac{\partial \Psi_0^{\varphi^1}}{\partial d_{\hat{n}^{\varphi^1}}^{\varphi^1}}\right) (d_{\hat{n}^\psi}^\psi - d_{\hat{n}^{\varphi^1}}^{\varphi^1}) \geq 0.$$

By making suitable variational choices, we obtain

$$\begin{aligned} \left(t_{\hat{n}^{\varphi^1}}^{\varphi^1}\right)^{\text{R}} + \rho_S^{\varphi^1} \frac{\partial \Psi_0^{\varphi^1}}{\partial d_{\hat{n}^{\varphi^1}}^{\varphi^1}} &\geq 0, \quad d_{\hat{n}^{\varphi^1}}^{\varphi^1} \geq 0, \\ \left(\left(t_{\hat{n}^{\varphi^1}}^{\varphi^1}\right)^{\text{R}} + \rho_S^{\varphi^1} \frac{\partial \Psi_0^{\varphi^1}}{\partial d_{\hat{n}^{\varphi^1}}^{\varphi^1}}\right) d_{\hat{n}^{\varphi^1}}^{\varphi^1} &= 0 \quad \text{on } \Gamma_C^{\varphi^1}, \end{aligned} \quad (32)$$

which correspond to the unilateral contact conditions involved in finite deformations.

Let us now study the contribution of the friction. According to the complementary laws (28), taking a partial sub-differentiation of  $\Phi^{\varphi^1}$  given by (27), we can obtain

$$\mu \left| t_{\hat{n}^\varphi}^{\varphi^1} + \rho_S^{\varphi^1} \frac{\partial \Psi_0^{\varphi^1}}{\partial d_{\hat{n}^\varphi}^\varphi} \right| \left( \left\| \dot{\xi}_p^\psi \right\| - \left\| \dot{\xi}_p^\varphi \right\| \right) \geq - \left( \mathbf{t}_{\hat{\tau}^\varphi}^{\varphi^1} \right)^{\text{IR}} \cdot \left( \dot{\xi}_p^\psi - \dot{\xi}_p^\varphi \right) \quad \forall \psi,$$

and from (31)

$$\mu \left| t_{\hat{n}^\varphi}^{\varphi^1} + \rho_S^{\varphi^1} \frac{\partial \Psi_0^{\varphi^1}}{\partial d_{\hat{n}^\varphi}^\varphi} \right| \left( \left\| \dot{\xi}_p^\psi \right\| - \left\| \dot{\xi}_p^\varphi \right\| \right) \geq - \left( \mathbf{t}_{\hat{\tau}^\varphi}^{\varphi^1} + \rho_S^{\varphi^1} \frac{\partial \Psi_0^{\varphi^1}}{\partial \dot{\xi}_p^\varphi} \right) \cdot \left( \dot{\xi}_p^\psi - \dot{\xi}_p^\varphi \right) \quad \forall \psi. \quad (33)$$

The above relations are equivalent to the friction law [5]:

$$\left\| \left( \mathbf{t}_{\hat{\tau}^\varphi}^{\varphi^1} \right)^{\text{IR}} \right\| \leq \mu \left| t_{\hat{n}^\varphi}^{\varphi^1} + \rho_S^{\varphi^1} \frac{\partial \Psi_0^{\varphi^1}}{\partial d_{\hat{n}^\varphi}^\varphi} \right| \quad \text{on } \Gamma_C^{\varphi^1} \quad (34)$$

and

$$\begin{cases} \left\| \left( \mathbf{t}_{\hat{\tau}^\varphi}^{\varphi^1} \right)^{\text{IR}} \right\| < \mu \left| t_{\hat{n}^\varphi}^{\varphi^1} + \rho_S^{\varphi^1} \frac{\partial \Psi_0^{\varphi^1}}{\partial d_{\hat{n}^\varphi}^\varphi} \right| & \Rightarrow \quad \dot{\xi}_p^\varphi = \mathbf{0}, \\ \left\| \left( \mathbf{t}_{\hat{\tau}^\varphi}^{\varphi^1} \right)^{\text{IR}} \right\| = \mu \left| t_{\hat{n}^\varphi}^{\varphi^1} + \rho_S^{\varphi^1} \frac{\partial \Psi_0^{\varphi^1}}{\partial d_{\hat{n}^\varphi}^\varphi} \right| & \Rightarrow \quad \exists \lambda \geq 0, \quad \dot{\xi}_p^\varphi = -\lambda \left( \mathbf{t}_{\hat{\tau}^\varphi}^{\varphi^1} \right)^{\text{IR}}. \end{cases}$$

To express the evolution of the adhesion, we use the Legendre–Fenchel transformation, denoted by  $(\cdot)^*$ , which, if the functionals are smooth enough, gives

$$G_\beta \in \rho_S^{\varphi^1} \partial_{\beta^\varphi} \Phi^{\varphi^1} \iff \dot{\beta}^\varphi \in \partial_{G_\beta} \left( \rho_S^{\varphi^1} \Phi^{\varphi^1} \right)^*. \quad (35)$$

Inequations (32), (34) and (35), together with the equilibrium conditions, the boundary conditions and the generalized action–reaction principle (14), yield to the following boundary value problem:

**Problem  $\mathcal{P}_1$ .** Find  $\varphi^\alpha : \Omega^{\varphi^\alpha} \times ]0, T[ \rightarrow \mathbb{R}^3$ ,  $\beta^\varphi : \Gamma_C^{\varphi^1} \rightarrow \mathbb{R}$  with  $\varphi^\alpha(0) = \varphi_0^\alpha$  in  $\Omega^{\varphi^\alpha}$ ,  $\beta^\varphi(0) = \beta_0^\varphi$  on  $\Gamma_C^{\varphi^1}$ , such that for all  $t \in ]0, T[$  and  $\alpha = 1, 2$ ,

$$\operatorname{div}^{\varphi^\alpha} \boldsymbol{\sigma}^{\varphi^\alpha} + \mathbf{f}^{\varphi^\alpha} = \mathbf{0}, \quad \boldsymbol{\sigma}^{\varphi^\alpha} = \rho^{\varphi^\alpha} \frac{\partial \psi^{\varphi^\alpha}}{\partial \mathbb{H}^{\varphi^\alpha}} \quad \text{in } \Omega^{\varphi^\alpha},$$

$$\varphi^\alpha = \varphi_D^\alpha \quad \text{on } \Gamma_D^{\varphi^\alpha}, \quad \boldsymbol{\sigma}^{\varphi^\alpha} \mathbf{n}^{\varphi^\alpha} = \mathbf{T}^{\varphi^\alpha} \quad \text{on } \Gamma_T^{\varphi^\alpha},$$

$$\boldsymbol{\sigma}^{\varphi^2} \mathbf{n}^{\varphi^2} = \mathbf{0} \quad \text{on } \Gamma_C^{\varphi^2} \setminus \varphi_p \left( \Gamma_C^{\varphi^1} \right),$$

$$j_p(\mathbf{x}^{\varphi^1}) \left( \boldsymbol{\sigma}^{\varphi^2} \mathbf{n}^{\varphi^2} \right) \left( \varphi_p(\mathbf{x}^{\varphi^1}) \right) + \mathbf{t}^{\varphi^1}(\mathbf{x}^{\varphi^1}) = \mathbf{0} \quad \text{on } \Gamma_C^{\varphi^1},$$

$$t_{\hat{n}^{\varphi}}^{\varphi^1} + \rho_S^{\varphi^1} \frac{\partial \Psi_0^{\varphi^1}}{\partial d_{\hat{n}^{\varphi}}^{\varphi}} \geq 0, \quad d_{\hat{n}^{\varphi}}^{\varphi} \geq 0, \quad \left( t_{\hat{n}^{\varphi}}^{\varphi^1} + \rho_S^{\varphi^1} \frac{\partial \Psi_0^{\varphi^1}}{\partial d_{\hat{n}^{\varphi}}^{\varphi}} \right) d_{\hat{n}^{\varphi}}^{\varphi} = 0 \quad \text{on } \Gamma_C^{\varphi^1},$$

$$\left\| \mathbf{t}_{\hat{\boldsymbol{\zeta}}_p^{\varphi}}^{\varphi^1} + \rho_S^{\varphi^1} \frac{\partial \Psi_0^{\varphi^1}}{\partial \boldsymbol{\zeta}_p^{\varphi}} \right\| \leq \mu \left| t_{\hat{n}^{\varphi}}^{\varphi^1} + \rho_S^{\varphi^1} \frac{\partial \Psi_0^{\varphi^1}}{\partial d_{\hat{n}^{\varphi}}^{\varphi}} \right| \quad \text{on } \Gamma_C^{\varphi^1}$$

and

$$\begin{cases} \left\| \mathbf{t}_{\hat{\boldsymbol{\zeta}}_p^{\varphi}}^{\varphi^1} + \rho_S^{\varphi^1} \frac{\partial \Psi_0^{\varphi^1}}{\partial \boldsymbol{\zeta}_p^{\varphi}} \right\| < \mu \left| t_{\hat{n}^{\varphi}}^{\varphi^1} + \rho_S^{\varphi^1} \frac{\partial \Psi_0^{\varphi^1}}{\partial d_{\hat{n}^{\varphi}}^{\varphi}} \right| & \Rightarrow \quad \dot{\boldsymbol{\zeta}}_p^{\varphi} = \mathbf{0}, \\ \left\| \mathbf{t}_{\hat{\boldsymbol{\zeta}}_p^{\varphi}}^{\varphi^1} + \rho_S^{\varphi^1} \frac{\partial \Psi_0^{\varphi^1}}{\partial \boldsymbol{\zeta}_p^{\varphi}} \right\| = \mu \left| t_{\hat{n}^{\varphi}}^{\varphi^1} + \rho_S^{\varphi^1} \frac{\partial \Psi_0^{\varphi^1}}{\partial d_{\hat{n}^{\varphi}}^{\varphi}} \right| & \Rightarrow \quad \exists \lambda \geq 0, \quad \dot{\boldsymbol{\zeta}}_p^{\varphi} = -\lambda \left( \mathbf{t}_{\hat{\boldsymbol{\zeta}}_p^{\varphi}}^{\varphi^1} + \rho_S^{\varphi^1} \frac{\partial \Psi_0^{\varphi^1}}{\partial \boldsymbol{\zeta}_p^{\varphi}} \right), \end{cases}$$

$$\dot{\beta}^{\varphi} \in \partial_{G_{\beta}} \left( \rho_S^{\varphi^1} \Phi^{\varphi^1} \right)^*, \quad -G_{\beta} \in \rho_S^{\varphi^1} \partial_{\beta^{\varphi}} \Psi^{\varphi^1} \quad \text{on } \Gamma_C^{\varphi^1}.$$

### 3. Application to the adhesive contact between a hyperelastic body and a plane rigid support

#### 3.1. The quasi-static problem

Let us now study the contact between a hyperelastic body and a rigid fixed body (Fig. 3). We shall write  $\mathcal{P}_1$  using the Piola–Kirchhoff stress tensor and we neglect the superscript 1 in the following. The local frame is reduced to  $(\mathbf{n}, \boldsymbol{\tau})$ , the previous state variables become  $u_n^{\varphi}, \mathbf{u}_{\boldsymbol{\tau}}^{\varphi}, \beta^{\varphi}$ .

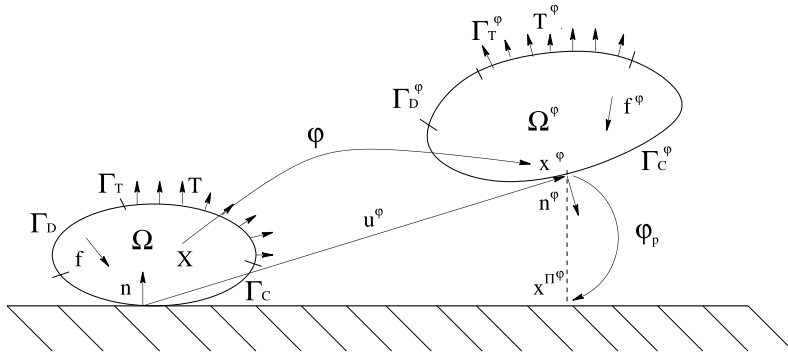


Fig. 3. Contact between an hyperelastic body and a rigid plane support.

The hyperelastic constitutive equation can be likewise written in a nominal formulation (22). In this particular case, the boundary value problem is reduced to:

**Problem  $\mathcal{P}_2$ .** Find  $\varphi : \Omega \times ]0, T[ \rightarrow \mathbb{R}^3$ ,  $\beta^\varphi : \Gamma_C \rightarrow \mathbb{R}$  with  $\varphi(0) = \varphi_0$  in  $\Omega$ ,  $\beta^\varphi(0) = \beta_0^\varphi$  on  $\Gamma_C$ , such that  $\forall t \in ]0, T[$ :

$$\operatorname{div} \hat{\mathbb{P}} + \mathbf{f} = 0 \text{ in } \Omega, \quad \hat{\mathbb{P}} = \frac{\partial \hat{\mathbf{W}}}{\partial \nabla \varphi} \text{ in } \Omega,$$

$$\varphi = \varphi_D \text{ on } \Gamma_D, \quad \hat{\mathbb{P}} \mathbf{n} = \mathbf{T} \text{ on } \Gamma_T,$$

$$t_n + \rho_S \frac{\partial \Psi_0^\varphi}{\partial u_n^\varphi} \geq 0, \quad u_n^\varphi \geq 0, \quad \left( t_n + \rho_S \frac{\partial \Psi_0^\varphi}{\partial u_n^\varphi} \right) u_n^\varphi = 0 \text{ on } \Gamma_C,$$

$$\left\| \mathbf{t}_\tau + \rho_S \frac{\partial \Psi_0^\varphi}{\partial \mathbf{u}_\tau^\varphi} \right\| \leq \mu \left| t_n + \rho_S \frac{\partial \Psi_0^\varphi}{\partial u_n^\varphi} \right| \text{ on } \Gamma_C,$$

and

$$\begin{cases} \left\| \mathbf{t}_\tau + \rho_S \frac{\partial \Psi_0^\varphi}{\partial \mathbf{u}_\tau^\varphi} \right\| < \mu \left| t_n + \rho_S \frac{\partial \Psi_0^\varphi}{\partial u_n^\varphi} \right| \Rightarrow \dot{\mathbf{u}}_\tau^\varphi = \mathbf{0}, \\ \left\| \mathbf{t}_\tau + \rho_S \frac{\partial \Psi_0^\varphi}{\partial \mathbf{u}_\tau^\varphi} \right\| = \mu \left| t_n + \rho_S \frac{\partial \Psi_0^\varphi}{\partial u_n^\varphi} \right| \Rightarrow \exists \lambda \geq 0, \quad \dot{\mathbf{u}}_\tau^\varphi = -\lambda \left( \mathbf{t}_\tau + \rho_S \frac{\partial \Psi_0^\varphi}{\partial \mathbf{u}_\tau^\varphi} \right), \end{cases}$$

$$\dot{\beta}^\varphi \in \partial_{G_\beta} (\rho_S \Phi^\varphi)^*, \quad \text{where } -G_\beta \in \rho_S \partial_{\beta^\varphi} \Psi^\varphi \text{ on } \Gamma_C,$$

where  $\operatorname{div}$  denotes the volume divergence operator with respect to the reference configuration and  $\mathbf{t} = \hat{\mathbb{P}} \mathbf{n}$ .

Let us now establish a variational formulation for the previous boundary value problem. In what follows, we will denote by  $Q$  and  $K'$  the following sets:

$$\begin{aligned} Q &= \{ \psi : \Omega \rightarrow \mathbb{R}^3; \psi = \varphi_D \text{ on } \Gamma_D \}, \\ K' &= \{ \psi \in Q; u_n^\psi \geq 0 \text{ on } \Gamma_C \}. \end{aligned} \tag{36}$$

We shall adopt the notations:

$$(A\varphi, \eta) = \int_\Omega \hat{\mathbb{P}} : \nabla \eta \, d\mathbf{X}, \quad L(\eta) = \int_{\Gamma_T} \mathbf{T} \cdot \eta \, da + \int_\Omega \mathbf{f} \cdot \eta \, d\mathbf{X}, \tag{37}$$

$$j(\beta^\varphi, \varphi, \eta) = \int_{\Gamma_C} \mu \left| t_n + \rho_S \frac{\partial \Psi_0^\varphi}{\partial u_n^\varphi} \right| \|\mathbf{u}_\tau^\eta\| \, da, \tag{38}$$

$$C_T(\beta^\varphi, \varphi, \eta) = \int_{\Gamma_C} \rho_S \frac{\partial \Psi_0^\varphi}{\partial \mathbf{u}_t^\varphi} \cdot \mathbf{u}_t^\eta \, da, \quad C_N(\beta^\varphi, \varphi, \eta) = \int_{\Gamma_C} \rho_S \frac{\partial \Psi_0^\varphi}{\partial u_n^\varphi} u_n^\eta \, da. \quad (39)$$

The problem  $\mathcal{P}_2$  is formally equivalent to the following variational formulation [3]:

**Problem  $\mathcal{P}_2^L$ .** Find  $\varphi, \beta^\varphi$  such that  $\varphi(0) = \varphi_0$  in  $\Omega$  and  $\beta^\varphi(0) = \beta_0^\varphi$  on  $\Gamma_C$  such that  $\forall t \in ]0, T[, \varphi(t) \in K'$  and

$$\begin{aligned} & (A\varphi, \partial_t \eta - \partial_t \varphi) + j(\beta^\varphi, \varphi, \partial_t \eta) - j(\beta^\varphi, \varphi, \partial_t \varphi) + C_T(\beta^\varphi, \varphi, \partial_t \eta) \\ & \geq L(\partial_t \eta - \partial_t \varphi) + \int_{\Gamma_C} t_n(\partial_t \eta - \partial_t \varphi)_n \quad \forall \eta \in Q, \end{aligned} \quad (40)$$

$$\int_{\Gamma_C} t_n(u_n^w - u_n^\varphi) + C_N(\beta^\varphi, \varphi, w - \varphi) \geq 0 \quad \forall w \in K', \quad (41)$$

$$\dot{\beta}^\varphi \in \partial_{G_\beta}(\rho_S \Phi^\varphi)^*, \quad -G_\beta \in \rho_S \partial_{\beta^\varphi} \Psi^\varphi. \quad (42)$$

We note that we can simplify this expression by coupling (40) and (41) and obtain

$$\begin{aligned} & (A\varphi, \eta - \varphi) + j(\beta^\varphi, \varphi, \partial_t \varphi + \eta - \varphi) - j(\beta^\varphi, \varphi, \partial_t \eta - \eta + \varphi) + C(\beta^\varphi, \varphi, \eta - \varphi) \\ & \geq L(\eta - \varphi) \quad \forall \eta \in K', \end{aligned} \quad (43)$$

$$\dot{\beta}^\varphi \in \partial_{G_\beta}(\rho_S \Phi^\varphi)^*, \quad -G_\beta \in \rho_S \partial_{\beta^\varphi} \Psi^\varphi, \quad (44)$$

with  $C = C_T + C_N$ .

### 3.2. An existence result for the frictionless contact problem

Let us consider the previous system (44) in a static form

$$\beta^\varphi \in \partial_{G_\beta} A^\varphi, \quad -G_\beta \in \rho_S \partial_{\beta^\varphi} \Psi^\varphi, \quad (45)$$

where  $A^\varphi$  is deduced by a discretization of the pseudo-potential of dissipation (see [22]).

By assuming  $G_\beta$  in (45) does not depend explicitly on  $\beta^\varphi$ , problem (43), (44) can be reduced to the following minimization:

$$\inf_{\eta \in \mathcal{Q}} \{I(\eta) + J(\eta) - L(\eta)\},$$

where

$$I(\eta) = \int_{\Omega} \hat{\mathbf{W}}(\nabla \eta) \, d\mathbf{X}, \quad J(\eta) = \int_{\Gamma_C} \rho_S \hat{\Psi}_0(\eta) \, da.$$



Let us study this minimization problem over an appropriate set  $\mathcal{Q}$  of admissible deformations:

$$\mathcal{Q} = \{\eta \in W^{1,p}(\Omega; \mathbb{R}^3); \text{ cof } \nabla \eta \in L^q(\Omega; \mathbb{R}^9), \det \nabla \eta \in L^r(\Omega), \\ \det \nabla \eta > 0 \text{ p.p. in } \Omega, \eta = \varphi_D \text{ p.p. on } \Gamma_D, \eta_n \geq 0 \text{ p.p. on } \Gamma_C\}.$$

We assume that:

- (a)  $\hat{\Psi}_0(\cdot)$  is lower semi-continuous on  $\mathbb{R}^3$  and  $\hat{\Psi}_0(\eta) \geq 0 \forall \eta \in \mathcal{Q}$ ;
- (b)  $\hat{\mathbf{W}}$  is polyconvex [2];
- (c) there exist constants  $\alpha > 0$ ,  $\beta$ ,  $p \geq 2$ ,  $q \geq p/(p-1)$ ,  $r > 1$ , such that

$$\hat{\mathbf{W}}(\mathbb{F}) \geq \alpha(\|\mathbb{F}\|^p + \|\text{cof } \mathbb{F}\|^q + (\det \mathbb{F})^r) + \beta,$$

for all  $\mathbb{F} \in \mathbf{M}_+^3$  (where  $\mathbf{M}_+^3$  is the set of matrices with positive determinants);

- (d)  $\lim_{\det \mathbb{F} \rightarrow 0^+} \hat{\mathbf{W}}(\mathbb{F}) = +\infty$  for all  $\mathbb{F} \in \mathbf{M}_+^3$ .

We have the following result:

**Theorem 3.1.** *Let  $\Omega$  be a domain in  $\mathbb{R}^3$  and  $\hat{\mathbf{W}} : \mathbf{M}_+^3 \rightarrow \mathbb{R}$  be a strain energy density with properties (b), (c), (d) and let  $L : W^{1,p}(\Omega, \mathbb{R}^3) \rightarrow \mathbb{R}$  be continuous.*

*Let  $\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_T \cup \bar{\Gamma}_C$  be a da-measurable partition of the boundary of  $\Omega$  with area  $\Gamma_D > 0$ , and let  $\varphi_D : \Gamma_D \rightarrow \mathbb{R}^3$  be a measurable function such that the set  $\mathcal{Q}$  is non-empty. We assume that the integrand of  $J$  has the property (a). Let us assume that*

$$\inf_{\eta \in \mathcal{Q}} [I(\eta) + J(\eta)] < \infty. \quad (46)$$

*Then there exists at least one function  $\varphi$  such that:*

$$\varphi \in \mathcal{Q} \quad \text{and} \quad I(\varphi) + J(\varphi) = \inf_{\eta \in \mathcal{Q}} [I(\eta) + J(\eta)]. \quad (47)$$

**Proof.** Let  $(\varphi^k)$  an infimizing sequence for the functional  $I + J$ , i.e., a sequence that satisfies  $\varphi^k \in \mathcal{Q} \forall k \in \mathbb{N}^*$  and

$$\lim_{k \rightarrow \infty} (I(\varphi^k) + J(\varphi^k)) = \inf_{\eta \in \mathcal{Q}} [I(\eta) + J(\eta)],$$

and let us consider a sub-sequence of the infimizing sequence  $(\varphi^k)$  such that  $\varphi^k \rightharpoonup \varphi$  in  $W^{1,p}(\Omega; \mathbb{R}^3)$ .

From assumptions (a) and (46),  $\inf_{\eta \in \mathcal{Q}} I(\eta) < \infty$ ,  $\varphi \in \mathcal{Q}$  and [2,6]

$$I(\varphi) \leq \liminf_{k \rightarrow \infty} I(\varphi^k).$$

We now show that  $J$  is sequentially weakly lower semi-continuous. For this purpose, we consider a sequence  $(\eta^k) \in \mathcal{Q}$  such that  $\eta^k \rightharpoonup \eta$  in  $W^{1,p}(\Omega; \mathbb{R}^3)$ . Then from a compactness result (see, e.g.,

[1]),  $\eta^k \rightarrow \eta$  in  $L^p(\Gamma_C; \mathbb{R}^3)$ . This means that there exists a subsequence of  $(\eta^k)$  that converges almost everywhere towards  $\eta$ . Under the hypothesis (a), this sub-sequence satisfies

$$\hat{\Psi}_0(\eta) \leq \liminf_{k \rightarrow \infty} \hat{\Psi}_0(\eta^k).$$

Using this relation and Fatou's lemma, we obtain

$$J(\eta) \leq \liminf_{k \rightarrow \infty} \int_{\Gamma_C} \rho \hat{\Psi}_0(\eta^k) \, da = \liminf_{k \rightarrow \infty} J(\eta^k).$$

It can therefore be concluded that  $J$  is sequentially weakly lower semi-continuous. Then the minimization problem has the solution  $\varphi$  since,

$$\inf_{\eta \in \mathcal{Z}} [I(\eta) + J(\eta)] \leq I(\varphi) + J(\varphi) \leq \liminf_{k \rightarrow \infty} [I(\varphi^k) + J(\varphi^k)] = \inf_{\eta \in \mathcal{Z}} [I(\eta) + J(\eta)]. \quad \square$$

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