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Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org

pp. **X–XX**

RELAXATION RATE, DIFFUSION APPROXIMATION AND FICK'S LAW FOR INELASTIC SCATTERING BOLTZMANN MODELS

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(Communicated by the associate editor name)

ABSTRACT. We consider the linear dissipative Boltzmann equation describing inelastic interactions of particles with a fixed background. For the simplified model of Maxwell molecules first, we give a complete spectral analysis, and deduce from it the optimal rate of exponential convergence to equilibrium. Moreover we show the convergence to the heat equation in the diffusive limit and compute explicitely the diffusivity. Then for the physical model of hard spheres we use a suitable entropy functional for which we prove explicit inequality between the relative entropy and the production of entropy to get exponential convergence to equilibrium with explicit rate. The proof is based on inequalities between the entropy production functional for hard spheres and Maxwell molecules. Mathematical proof of the convergence to some heat equation in the diffusive limit is also given. From the last two points we deduce the first explicit estimates on the diffusive coefficient in the Fick's law for (inelastic hard-spheres) dissipative gases.

1. **Introduction.** The linear Boltzmann equation for *granular particles* models the dynamics of dilute particles (test particles with negligible mutual interactions) immersed in a fluid at thermal equilibrium that undergo *inelastic collisions* characterized by the fact that the total kinetic energy of the system is dissipated during collision. Such an equation introduced in [16, 21, 15] provides an efficient description of the dynamics of a mixture of impurities in a gas [13, 10]. Assuming the fluid at thermal equilibrium and neglecting the mutual interactions of the particles, the evolution of the distribution of the particles phase is modelled by the linear Boltzmann equation which reads

$$\partial_t f + v \cdot \nabla_x f = Q(f), \tag{1.1}$$

with suitable initial condition $f(x,v,0) = f_0(x,v)$, $(x,v) \in \mathbb{R}^3_x \times \mathbb{R}^3_v$. Here above, the collision operator $Q(\cdot)$ is a *linear scattering operator* given by

$$Q(f) = \mathcal{B}(f, \mathcal{M}_1),$$

2000 *Mathematics Subject Classification.* Primary: 35B40; Secondary: 82C40.

Key words and phrases. Granular gas dynamics; linear Boltzmann equation; entropy production; spectral gap; diffusion approximation; Fick's law; diffusive coefficient.

G.T. acknowledges the support of the Italian MIUR project "Kinetic and hydrodynamic equations of complex collisional systems".

where $\mathcal{B}(\cdot,\cdot)$ denotes the usual quadratic Boltzmann collision operator for granular gases (cf. [9] for instance) and $\mathcal{M}_1$ stands for the distribution function of the host fluid which is assumed to be a given Maxwellian with bulk velocity u_1 and temperature Θ_1 (see Section 2 for details). Notice that we shall deal in this paper with the collision operator Q corresponding to ***hard-spheres interactions*** as well as with the one associated to ***Maxwell molecules interactions***. The inelasticity is modeled by a ***constant normal restitution coefficient***. The main goals and results of this paper are the following:

(1) First, we explicit the exponential rate of convergence towards equilibrium for the solution to the space-homogeneous version of (1.1) (for both Maxwellian molecules and hard-spheres interactions) ***through a quantitative estimate of the spectral gap of*** Q. It is computed in an explicit way for Maxwell molecules, and estimated in an explicit way for hard-spheres, see Theorem 3.8 (together with its Corollary 3.9 for its consequence on the asymptotic behavior of space-homogeneous solutions), where μ_{max} is defined in Theorem 3.2, and the constant C^* is detailed in Remark 3.6.

(2) Second, we investigate the problem of the ***diffusion approximation for*** (1.1). Precisely, we show that the macroscopic limit ϱ of Eq. (1.1) in the diffusive scaling is the solution to some (parabolic) heat equation (see Proposition 4.3 together with Theorems 4.4 and 4.9). ***When dealing with Maxwell molecules interactions, the diffusivity of this heat equation can then be computed explicitly*** (see Theorem 4.4 together with the computation of Remark 4.5 for the diffusivity). ***This is no more the case for the equation corresponding to hard-spheres interactions but we provide some new quantitative estimates on it*** (see Theorem 4.9 together with the estimate of Proposition 4.10).

Concerning point (1), it is known from [16, 21, 15] that the linear collision operator admits a unique steady state given by a (normalized) Maxwellian distribution function $\mathcal{M}$ with bulk velocity u_1 and temperature $\Theta^{\#} \leqslant \Theta_1$. Moreover, thanks to the spectral analysis of Q performed in [1] (in the hard-spheres case), the solution to the space-homogeneous version of (1.1) is known to converge exponentially (in some pertinent L^2 norm) towards this equilibrium $\mathcal{M}$ as times goes to infinity. This exponential convergence result is based upon the existence of a positive spectral gap for the collision operator Q and relies on compactness arguments, via Weyl's Theorem. ***Because of this non constructive approach, at least for hard-spheres interactions, no explicit estimate on the relaxation rate were available by now***. It is one of the objectives of this paper to fill this blank. It is well-known that the kinetic description of gases through the Boltzmann equation is relevant *only on some suitable time scale* [9, 11]. Providing explicit estimates of the relaxation rate is the only way to make sure that the time scale for the equilibration process is smaller than the one on which the kinetic modeling is relevant. Another motivation to look for an explicit relaxation rate relies more on methodological aspects. Compactness methods do not rely on any physical argument and it seems to us more natural to look for a method which relies as much as possible on physical mechanisms, e.g. dissipation of entropy.

Precisely, the strategy we adopt to treat the above point (1) is based upon an explicit estimate of the spectral gap of Q. For Maxwell molecules interactions, we use the Fourier-based approach introduced by Bobylev [6] for the study of the linearized (elastic) Boltzmann equation and then used for the study of the spectrum of the *linearized* inelastic collision operator in [7], and we provide an explicit description of the whole spectrum of this linear *scattering* collision operator . Then, to treat the case of hard-spheres interactions, our method is based upon the *entropy-entropy-production method*. Precisely, we show that the entropy production functional (naturally associated to the $L^2(\mathcal{M}^{-1})$ norm) corresponding to the hard-spheres model can be bounded from below (up to some explicit constant)

by the one associated to the Maxwell molecules model (Proposition 3.3). Note that such a comparison between entropy production rates for hard-spheres and Maxwell molecules interactions is inspired by the approach of [2] which deals with the linearized (elastic) Boltzmann equation. In the present case, the method of proof is different and simpler, being based upon the careful study of a convolution integral. Such an entropy production estimate allows us to prove, via a suitable coercivity estimate of Q (Theorem 3.8), that any space-homogenous solution to (1.1) converges exponentially towards equilibrium with an explicit rate that depends on the model under investigation.

Concerning now point (2), various attempts to derive hydrodynamic equations from the dissipative nonlinear Boltzmann equation exist in the literature, mostly based upon suitable moment closure methods [23, 4, 5] or on the study of the linearized version of the Boltzmann equation around self-similar solutions (homogeneous cooling state) [3, 8] in some weak inelastic regime. Dealing with the linear Boltzmann equation (1.1), hydrodynamic models describing the evolution of the momentum and the temperature of the gas have been obtained in [10] as a closed set of *dissipative Euler equations* for some pseudo-Maxwellian approximation of Q. Similar results have been obtained in [13] where numerical methods are proposed for the resolution of both the kinetic and hydrodynamic models. The work [4] proposes two closure methods, based upon a maximum entropy principle, of the moment equations for the density, macroscopic velocity and temperature. These closure methods lead to a single diffusion equation for the hydrodynamical variable. In the present paper, we shall discuss the diffusion approximation of the linear Boltzmann equation (1.1) with the main objective of providing a ***rigorous derivation of the Fick's law for dissipative gases and an estimate on the diffusive coefficient***. Recall that the diffusion approximation for the linear Boltzmann equation consists in looking for the limit, as the small parameter ε goes to 0, of the solution to the following re-scaled kinetic equation:

$$\varepsilon \partial_t f_\varepsilon(t,x,v) + v \cdot \nabla_x f_\varepsilon(t,x,v) = \frac{1}{\varepsilon} Q(f_\varepsilon)(t,x,v), \tag{1.2}$$

with suitable initial condition. We consider indeed here the *Navier-Stokes scaling*, namely, we assume the mean free path to be a small parameter $\lambda = \varepsilon \ll 1$ and, at the same time, we rescale time as $t \to t/\varepsilon$ in order to see emerging the diffusive hydrodynamical regime (and not the Euler hydrodynamical description, which would be a trivial transport equation in our case). Performing a formal Hilbert asymptotic expansion of the solution allows us to expect the solution f_ε to converges towards a limit f with $Q(f) = 0$. Therefore, the expected limit of f_ε is of the form $f(t,x,v) = \varrho(t,x)\mathcal{M}(v)$, and the diffusion approximation problem consists in expressing $\varrho(t,x)$ as the solution to some suitable diffusion equation. Actually, standard approach consists in using the continuity relation

$$\partial_t \varrho(t,x) + \mathrm{div}_x j(t,x) = 0,$$

between the density ϱ and the current vector $j(t,x)$ together with a suitable *Fick's law* that links the current j to the gradient of ϱ:

$$j(t,x) = -\mathsf{D}\, \nabla_x \varrho(t,x)$$

for some suitable diffusion coefficient (diffusivity) $\mathsf{D} > 0$ which depends on the kind of interactions we are dealing with. For Maxwell molecules interactions, the expression of the diffusivity can be made explicit while this is no more the case when dealing with hard-spheres model. The method we adopt for the proof of the diffusive limit follows very closely the work of P. Degond, T. Goudon and F. Poupaud [12]. Though more general than ours since it deals with models without detailed balance law, the study of [12] is restricted

to the case of a collision operator for which the collision frequency is controlled from above in a way that excludes the case of physical hard-spheres interactions (recall that, for hard-spheres, the collision frequency behaves asymptotically like $(1 + |v|)$ [1]). Actually, the analysis of [12] can be make valid under the only hypothesis that the coercivity estimate obtained in Theorem 3.8 (and strenghten in Theorem 3.10) holds true. Precisely, such a coercivity estimate of Q allows to obtain satisfactory *a priori bounds* for the solution to the re-scaled equation (1.2). We are then able to prove the weak convergence of the density ϱ_ε and current j_ε of the solution f_ε towards suitable limit density ϱ and limit current j. It is also possible, via compensated-compactness arguments from [14, 17, 22] to prove strong convergence result in L^2 norm.

The organization of the paper is as follows. In Section 2 we present the models we shall deal with as well as some related known results we shall use in the sequel. In Section 3 we perform the computations of the spectrum in the Maxwell molecules case and then prove the crucial entropy production estimates in the hard-spheres case (from which we deduces the explicit convergence rate to equilibrium for the space-homogenous version of (1.1)). Section 4 is dealing with the above point (2). We first prove *a priori* estimates valid for both models of hard-spheres and Maxwell molecules and based upon the coercivity estimates obtained in Section 3. Then, we deal separately with the cases of Maxwell molecules and hard-spheres proving for both models the convergence towards suitable macroscopic equations, providing for Maxwell molecules an explicit expression of the diffusivity, and for hard-spheres explicit estimates on it.

2. **Preliminaries.**

2.1. **The model.**

As explained in the introduction, we shall deal with a *linear scattering operator* Q given by

$$Q(f) = \frac{1}{2\pi\lambda} \int_{\mathbb{R}^3\times\mathbb{S}^2} B(q,n)\left[J_B f(x,v_\star,t)\mathcal{M}_1(w_\star) - f(x,v,t)\mathcal{M}_1(w)\right] \mathrm{d}w\,\mathrm{d}n \qquad (2.1)$$

where λ is the mean free path, $q = v - w$ is the relative velocity, $v_\star$ and $w_\star$ are the pre-collisional velocities which result, respectively, in v and w after collision. The main feature of the *binary dissipative collisions* is that part of the normal relative velocity is lost in the interaction, so that

$$(v^\star - w^\star)\cdot n = -e(v-w)\cdot n, \qquad (2.2)$$

where $n \in \mathbb{S}^2$ is the unit vector in the direction of impact and $0 < e < 1$ is the so-called normal restitution coefficient. Generally, such a coefficient should depend on (v, w) but, for simplicity, we shall only deal with a ***constant normal restitution coefficient*** e. The collision kernel $B(q, n)$ depends on the microscopic interaction (see below) while the term J_B corresponds to the product of the Jacobian of the transformation $(v_\star, w_\star) \to (v, w)$ with the ratio of the lengths of the collision cylinders [9]. Note that in such a scattering model, the microscopic masses of the dilute particles m and that of the host particles m_1 can be different. We will assume throughout this paper that the distribution function $\mathcal{M}_1$ of the host fluid is a given normalized Maxwellian function:

$$\mathcal{M}_1(v) = \left(\frac{m_1}{2\pi\Theta_1}\right)^{3/2} \exp\left\{-\frac{m_1(v-u_1)^2}{2\Theta_1}\right\}, \qquad v \in \mathbb{R}^3, \qquad (2.3)$$

where $u_1 \in \mathbb{R}^3$ is the given bulk velocity and $\Theta_1 > 0$ is the given effective temperature of the host fluid. For particles of masses m colliding inelastically with particles of mass

m_1, the restitution coefficient being constant, the expression of the pre-collisional velocities $(v_\star, w_\star)$ are given by [9, 21]

$$v_\star = v - 2\alpha\frac{1-\beta}{1-2\beta}\,(q\cdot n)\,n, \qquad w_\star = w + 2(1-\alpha)\frac{1-\beta}{1-2\beta}\,(q\cdot n)\,n; \tag{2.4}$$

where $q = v - w$, α is the mass ratio and β denotes the inelasticity parameter

$$\alpha = \frac{m_1}{m+m_1} \in (0,1), \quad \beta = \frac{1-e}{2} \in [0, 1/2).$$

We shall investigate in this paper several collision operators corresponding to various interactions collision kernels. Namely, we will deal with

- the linear Boltzmann operator for ***hard-sphere interactions*** $Q = Q_{\mathsf{hs}}$ for which

$$B(q,n) = B_{\mathrm{hs}}(q,n) = |q\cdot n|, \qquad \text{and} \qquad J_{B_{\mathrm{hs}}} =: J_{\mathsf{hs}} = \frac{1}{e^2};$$

- the scattering operator $Q = Q_{\mathsf{max}}$ corresponding to the ***Maxwell molecules approximation*** for which

$$B(q,n) = B_{\mathsf{max}}(q,n) = |\widetilde{q}\cdot n|, \quad \widetilde{q} = q/|q|, \qquad \text{and} \qquad J_{B_{\mathsf{max}}} = J_{\mathsf{max}} = \frac{1}{e^2}\frac{|v-w|}{|v^\star - w^\star|}.$$

It will be sometimes convenient to express the collision operator Q in the following weak form:

$$< \psi, Q(f) >= \frac{1}{2\pi\lambda}\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B(q,n) f(v)\mathcal{M}_1(w)\left[\psi(v^\star) - \psi(v)\right]\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}n \tag{2.5}$$

for any regular ψ, where $(v^\star, w^\star)$ denote the post-collisional velocities given by

$$v^\star = v - 2\alpha(1-\beta)\,(q\cdot n)\,n, \qquad w^\star = w + 2(1-\alpha)(1-\beta)\,(q\cdot n)\,n. \tag{2.6}$$

In particular, one sees that the dissipative feature of microscopic collision is measured, at the macroscopic level, only through the parameter:

$$\kappa = \alpha(1-\beta) = \frac{\alpha}{2}(1+e) \in (0,1)$$

appearing in the expression of $v^\star$. Accordingly, the macroscopic properties of Q are those of the classical linear Boltzmann gas whenever $\kappa = 1/2$ (which is equivalent to $v^\star = v$). It can be shown for both cases that the number density of the dilute gas is the unique conserved macroscopic quantity (as in the elastic case). In contrast with the nonlinear Boltzmann equation for granular gases, the temperature, though not conserved, remains bounded away from zero, which prevents the solution to the linear Boltzmann equation to converge towards a Dirac mass.

Moreover let us remark that from the dual form we see that the collision operator in fact depends only on two real parameters m_1 and κ (plus u_1 of course) and not u_1 plus three parameters α, β, m_1 as a first guess would suggest.

2.2. **Universal equilibrium and H-Theorem.** A very important feature of these inelastic scattering models is the existence (and uniqueness) of a *universal equilibrium*, that is independent of range of the microscopic-interactions (that is of the collision kernel B) and depending only on the parameters m_1, κ and u_1. Precisely, the background forces the system to adopt a Maxellian steady state (with density equal to 1):

Theorem 2.1. *The Maxwellian velocity distribution:*

$$\mathcal{M}(v) = \left(\frac{m}{2\pi\Theta^{\#}}\right)^{3/2} \exp\left\{-\frac{m(v-u_1)^2}{2\Theta^{\#}}\right\}, \qquad v \in \mathbb{R}^3, \tag{2.7}$$

with

$$\Theta^{\#} = \frac{(1-\alpha)(1-\beta)}{1-\alpha(1-\beta)}\Theta_1 \tag{2.8}$$

is the unique equilibrium state of Q with unit mass.

Note that this universal equilibrium is coherent with the remark that the collision only depends on m_1 and κ (and u_1) from the dual form, since

$$\frac{m}{\Theta^{\#}} = \frac{m_1}{\Theta_1}\frac{1-\kappa}{\kappa}.$$

This explicit Maxwellian equilibrium state allows to develop entropy, spectral and hydrodynamical analysis on both the models in the same way. First, from [18, 19] and [15], the existence and uniqueness of such an equilibrium state allows to establish a linear version of the famous H–Theorem. Precisely, for any *convex* C^1–function $\Phi : \mathbb{R}^+ \to \mathbb{R}$, the associated so-called H–functional (relatively to the equilibrium $\mathcal{M}$)

$$H_\Phi(f|\mathcal{M}) = \int_{\mathbb{R}^3} \mathcal{M}(v)\,\Phi\left(\frac{f(v)}{\mathcal{M}(v)}\right) \mathrm{d}v\,, \tag{2.9}$$

is decreasing along the flow of the equation (1.1) (this is the opposite of a physical entropy), with its associated dissipation functional vanishing only when f is co-linear to the equilibrium $\mathcal{M}$:

Theorem 2.2 (***Formal* H*–Theorem***). *Let $f(t,v) \geqslant 0$ be a space homogeneous solution then we have formally*

$$\frac{\mathrm{d}}{\mathrm{d}t}H_\Phi(f(t)|\mathcal{M}) = \int_{\mathbb{R}^3_v} Q(f)(t,v)\,\Phi'\left(\frac{f(t,v)}{M(v)}\right) \mathrm{d}v \leqslant 0 \qquad (t \geqslant 0). \tag{2.10}$$

The application of the H-Theorem with $\Phi(x) = (x-1)^2$ suggests the following Hilbert space setting: the unknown distribution f has to belong to the weighted Hilbert space $L^2(\mathcal{M}^{-1}) = L^2(\mathbb{R}^3_v\,;\,\mathcal{M}^{-1}(v)\,\mathrm{d}v)$. Consequently, one defines the Maxwell molecules and hard spheres collision operators, *associated to the mean-free path* $\lambda = 1$, with their suitable domains, as follows:

$$\begin{cases} \mathscr{D}(\mathcal{L}_{\mathsf{hs}}) \subset L^2(\mathcal{M}^{-1}), \qquad \mathrm{Range}(\mathcal{L}_{\mathsf{hs}}) \subset L^2(\mathcal{M}^{-1}), \\ \mathcal{L}_{\mathsf{hs}} f = Q_{\mathsf{hs}} f \qquad \text{for any } f \in \mathscr{D}(\mathcal{L}_{\mathsf{hs}}), \end{cases}$$

and

$$\begin{cases} \mathscr{D}(\mathcal{L}_{\mathsf{max}}) = L^2(\mathcal{M}^{-1}), \qquad \mathrm{Range}(\mathcal{L}_{\mathsf{max}}) \subset L^2(\mathcal{M}^{-1}), \\ \mathcal{L}_{\mathsf{max}} f = Q_{\mathsf{max}} f \qquad \text{for any } f \in \mathscr{D}(\mathcal{L}_{\mathsf{max}}). \end{cases}$$

Precisely,

$$\mathscr{D}(\mathcal{L}_{\mathsf{hs}}) = \left\{ f \in L^2(\mathcal{M}^{-1})\,;\, \sigma_{\mathsf{hs}} f \in L^2(\mathcal{M}^{-1}) \right\}$$

where σ_{hs} is the collision frequency associated to the hard-spheres collision kernel:

$$\sigma_{\mathsf{hs}}(v) = \frac{1}{2\pi}\int_{\mathbb{R}^3\times\mathbb{S}^2} |q\cdot n|\mathcal{M}_1(w)\,\mathrm{d}w\,\mathrm{d}n, \qquad v \in \mathbb{R}^3.$$

Note that σ_{hs} is unbounded [1]: there exist positive constants ν_0, ν_1 such that

$$\nu_0(1+|v-u_1|) \leqslant \sigma_{\mathsf{hs}}(v) \leqslant \nu_1(1+|v-u_1|), \qquad \forall v \in \mathbb{R}^3.$$

For this reason, $\mathscr{D}(\mathcal{L}_{\mathsf{hs}}) \neq L^2(\mathcal{M}^{-1})$. On the contrary, the collision frequency σ_{max} associated to the Maxwell molecules collision kernel,

$$\sigma_{\mathsf{max}}(v) = \frac{1}{2\pi} \int_{\mathbb{R}^3 \times \mathbb{S}^2} |\widetilde{q} \cdot n| \mathcal{M}_1(w) \, \mathrm{d}w \, \mathrm{d}n = 1$$

is independent of the velocity v and $\mathcal{L}_{\mathsf{max}}$ is a bounded operator in $L^2(\mathcal{M}^{-1})$. We recall (see [1]) that $\mathcal{L}_{\mathsf{hs}}$ is a negative self–adjoint operator of $L^2(\mathcal{M}^{-1})$. Moreover, let us introduce the dissipation entropy functionals associated to $\mathcal{L}_{\mathsf{max}}$ and $\mathcal{L}_{\mathsf{hs}}$:

$$\mathcal{D}_{\mathsf{max}}(f) := -\int_{\mathbb{R}^3} \mathcal{L}_{\mathsf{max}}(f)(v) \, f(v) \, \mathcal{M}^{-1}(v) \, \mathrm{d}v, \qquad f \in L^2(\mathcal{M}^{-1})$$

and

$$\mathcal{D}_{\mathsf{hs}}(f) := -\int_{\mathbb{R}^3} \mathcal{L}_{\mathsf{hs}}(f)(v) \, f(v) \, \mathcal{M}^{-1}(v) \, \mathrm{d}v, \qquad f \in \mathscr{D}(\mathcal{L}_{\mathsf{hs}}).$$

Note that, by virtue of (2.10), if $f(t)$ denotes the (unique) solution to (1.1) in $L^2(\mathcal{M}^{-1})$ for hard-spheres interactions, then, with the choice $\Phi(x) = (x-1)^2$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \| f(t) - \mathcal{M} \|^2_{L^2(\mathcal{M}^{-1})} = \frac{\mathrm{d}}{\mathrm{d}t} H_\Phi(f(t) | \mathcal{M}) = -2 \, \mathcal{D}_{\mathsf{hs}}(f(t)). \tag{2.11}$$

The same occurs for $\mathcal{D}_{\mathsf{max}}(f(t))$. This is the reason why we are looking for a control estimate for both $\mathcal{D}_{\mathsf{hs}}(f)$ and $\mathcal{D}_{\mathsf{max}}(f)$ with respect to the $L^2(\mathcal{M}^{-1})$ norm of f. It will be useful to derive an alternative expression for both $\mathcal{D}_{\mathsf{max}}$ and $\mathcal{D}_{\mathsf{hs}}$:

Proposition 2.3. *For any $f \in \mathscr{D}(\mathcal{L}_{\mathsf{hs}})$,*

$$\mathcal{D}_{\mathsf{hs}}(f) = \frac{1}{4\pi} \int_{\mathbb{R}^3_v \times \mathbb{R}^3_w \times \mathbb{S}^2} |q \cdot n| \left[\frac{f(v^\star)}{\mathcal{M}(v^\star)} - \frac{f(v)}{\mathcal{M}(v)} \right]^2 \mathcal{M}_1(w) \mathcal{M}(v) \, \mathrm{d}w \, \mathrm{d}v \, \mathrm{d}n \geqslant 0.$$

In the same way,

$$\mathcal{D}_{\mathsf{max}}(f) = \frac{1}{4\pi} \int_{\mathbb{R}^3_v \times \mathbb{R}^3_w \times \mathbb{S}^2} |\widetilde{q} \cdot n| \left[\frac{f(v^\star)}{\mathcal{M}(v^\star)} - \frac{f(v)}{\mathcal{M}(v)} \right]^2 \mathcal{M}_1(w) \mathcal{M}(v) \, \mathrm{d}w \, \mathrm{d}v \, \mathrm{d}n \geqslant 0.$$

for any $f \in L^2(\mathcal{M}^{-1})$.

Proof. The proof is a straightforward particular case of the above H-Theorem. Precisely, let us fix $f \in \mathscr{D}(\mathcal{L}_{\mathsf{hs}})$, and set $f = g\mathcal{M}$ then one has

$$\int_{\mathbb{R}^3_v} \mathcal{L}_{\mathsf{hs}}(f) \, f \, \mathcal{M}^{-1} \, \mathrm{d}v = \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} \frac{|q \cdot n|}{2\pi\sigma} \mathcal{M}(v) \mathcal{M}_1(w) g(v) \left[g(v^\star) - g(v) \right] \, \mathrm{d}w \, \mathrm{d}v \, \mathrm{d}n.$$

Moreover, for any $\varphi \in \mathscr{D}(\mathcal{L}_{\mathsf{hs}})$, since $\mathcal{M}$ is an equilibrium state of Q_{hs},

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} \frac{|q \cdot n|}{2\pi\sigma} \mathcal{M}(v) \mathcal{M}_1(w) \left[\varphi(v^\star) - \varphi(v) \right] \, \mathrm{d}w \, \mathrm{d}v \, \mathrm{d}n = \langle \mathcal{L}_{\mathsf{hs}}(\mathcal{M}), \varphi \rangle_{L^2(\mathbb{R}, \, \mathrm{d}v)} = 0.$$

It is easy to deduce then that

$$\int_{\mathbb{R}^3_v} \mathcal{L}_{\mathsf{hs}}(f) \, f \, \mathcal{M}^{-1} \, \mathrm{d}v = \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} \frac{|q \cdot n|}{2\pi\sigma} \mathcal{M}(v) \mathcal{M}_1(w) g(v^\star) \left[g(v) - g(v^\star) \right] \, \mathrm{d}w \, \mathrm{d}v \, \mathrm{d}n.$$

Finally taking the mean of these two quantities leads to the desired result. The same reasoning also holds for $\mathcal{D}_{\mathsf{max}}$. □

3. **Quantitative estimates of the spectral gap.** In this section, we strengthen the above result in providing a quantitative lower bound for both $\mathcal{D}_{\mathsf{max}}(f)$ and $\mathcal{D}_{\mathsf{hs}}(f)$. The estimate for $\mathcal{D}_{\mathsf{max}}(f)$ is related to the spectral properties of the collision operator $\mathcal{L}_{\mathsf{max}}$ while that of $\mathcal{D}_{\mathsf{hs}}(f)$ relies on a suitable comparison with $\mathcal{D}_{\mathsf{max}}(f)$. From now on, $\langle\cdot,\cdot\rangle$ denotes the scalar product of $L^2(\mathcal{M}^{-1})$ and $\mathfrak{S}(\mathcal{L})$, $\mathfrak{S}_p(\mathcal{L})$ shall denote respectively the spectrum and the point spectrum of a given (non necessarily bounded) operator in $L^2(\mathcal{M}^{-1})$.

3.1. **Spectral study for Maxwell molecules.** We already saw that the operator $\mathcal{L}_{\mathsf{max}} : L^2(\mathcal{M}^{-1}) \to L^2(\mathcal{M}^{-1})$ is bounded, and it is easily seen that $\mathcal{L}_{\mathsf{max}}$ splits as

$$\mathcal{L}_{\mathsf{max}} f = \mathcal{L}^+_{\mathsf{max}} f - f(v)$$

where $\mathcal{L}^+_{\mathsf{max}}$ is compact and self-adjoint (the proof can be done similarly as in [1]) and we used that the collision frequency σ_{max} associated to the Maxwell molecules is constant. Moreover, since

$$\langle \mathcal{L}_{\mathsf{max}} f, f\rangle \leqslant 0, \qquad \forall f \in L^2(\mathcal{M}^{-1}),$$

the operator is easily seen to generate a C^0-semigroup of contractions, and it is known that $\mathfrak{S}(\mathcal{L}_{\mathsf{max}}) \subset (-\infty, 0]$. Finally, since $\mathrm{Id} + \mathcal{L}_{\mathsf{max}} = \mathcal{L}^+_{\mathsf{max}}$ is a positive, self-adjoint compact operator, one sees that the spectrum of $\mathcal{L}_{\mathsf{max}}$ is made of a discrete set of eigenvalues with finite algebraic multiplicities plus possibly $\{-1\}$ in the essential spectrum, with

$$\mathfrak{S}_p(\mathcal{L}_{\mathsf{max}}) \subset (-1, 0] \tag{3.1}$$

and where the only possible accumulation point is $\{-1\}$. Clearly, since $\mathcal{L}_{\mathsf{max}}(\mathcal{M}) = 0$, $\lambda_{0,0} := 0$ is an eigenvalue of $\mathcal{L}_{\mathsf{max}}$ with eigenspace given by $\mathrm{Span}(\mathcal{M})$. There are other eigenvalues of $\mathcal{L}_{\mathsf{max}}$ of peculiar interest. Namely, for any $f \in L^2(\mathcal{M}^{-1})$, the weak formulation (2.5) yields

$$\begin{aligned}\int_{\mathbb{R}^3_v} (v-u_1)\mathcal{L}_{\mathsf{max}}(f)\,\mathrm{d}v &= \frac{1}{2\pi}\int_{\mathbb{R}^3_v\times\mathbb{R}^3_w\times\mathbb{S}^2} |\widetilde{q}\cdot n| f(v)\mathcal{M}_1(w)\left[v^\star - v\right]\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}n\\ &= -\frac{\alpha(1-\beta)}{\pi}\int_{\mathbb{R}^3_v} f(v)\,\mathrm{d}v\int_{\mathbb{R}^3_w}\mathcal{M}_1(w)\,\mathrm{d}w\int_{\mathbb{S}^2}|\widetilde{q}\cdot n|(q\cdot n)\,n\,\mathrm{d}n.\end{aligned}$$

Using the fact that $\displaystyle\int_{\mathbb{S}^2}|\widetilde{q}\cdot n|(q\cdot n)\,n\,\mathrm{d}n = \pi\, q$, one has

$$\begin{aligned}\int_{\mathbb{R}^3_v}(v-u_1)\mathcal{L}_{\mathsf{max}}(f)\,\mathrm{d}v &= -\alpha(1-\beta)\int_{\mathbb{R}^3_v} f(v)\,\mathrm{d}v\int_{\mathbb{R}^3_w}(v-w)\mathcal{M}_1(w)\,\mathrm{d}w\\ &= -\alpha(1-\beta)\int_{\mathbb{R}^3_v}(v-u_1)f(v)\,\mathrm{d}v.\end{aligned} \tag{3.2}$$

The operator $\mathcal{L}_{\mathsf{max}}$ being self-adjoint in $L^2(\mathcal{M}^{-1})$, one obtains the identity

$$\langle f, \mathcal{L}_{\mathsf{max}}((v_i-u_{1,i})\mathcal{M})\rangle = -\alpha(1-\beta)\,\langle f, (v_i-u_{1,i})\mathcal{M}\rangle\,, \qquad i=1,2,3, \quad \forall f\in L^2(\mathcal{M}^{-1}).$$

If we denote

$$\lambda_{0,1} = \alpha(1-\beta) = \kappa \in (0,1),$$

this means that $-\lambda_{0,1}$ is an eigenvalue of $\mathcal{L}_{\mathsf{max}}$ associated to the ***momentum eigenvectors*** $(v_i-u_{1,i})\mathcal{M}(v)$ for any $i=1,2,3$. In the same way, technical calculations show that, for any $f\in L^2(\mathcal{M}^{-1})$,

$$\begin{aligned}\langle \mathcal{L}_{\mathsf{max}}(f)\,, |v-u_1|^2\mathcal{M}\rangle &= \int_{\mathbb{R}^3}\mathcal{L}_{\mathsf{max}}(f)|v-u_1|^2\,\mathrm{d}v\\ &= -2\alpha(1-\beta)(1-\alpha(1-\beta))\langle f, |v-u_1|^2\mathcal{M}\rangle + \frac{6\Theta_1}{m_1}\alpha^2(1-\beta)^2\langle f,\mathcal{M}\rangle,\end{aligned}$$

and, since $\mathcal{L}_{\mathsf{max}}$ is self-adjoint, setting

$$\lambda_{1,0} = 2\alpha(1-\beta)(1-\alpha(1-\beta)) = 2\kappa(1-\kappa),$$

one has $\langle \mathcal{L}_{\mathsf{max}}(|v-u_1|^2\mathcal{M}), f\rangle = -\lambda_{1,0}\langle |v-u_1|^2\mathcal{M}, f\rangle$, for any $f \perp \mathrm{span}(\mathcal{M})$. Equivalently,

$$\mathcal{L}_{\mathsf{max}}(\mathcal{E}) = -\lambda_{1,0}\mathcal{E},$$

where $\mathcal{E}(v) = \left(|v-u_1|^2 - \frac{3\Theta^{\#}}{m}\right)\mathcal{M}$, is the ***energy eigenfunction***, associated to $-\lambda_{1,0}$. To summarize, we obtained three particular eigenvalues $\lambda_{0,0}, \lambda_{0,1}$ and $\lambda_{1,0}$ of $\mathcal{L}_{\mathsf{max}}$ associated respectively to the equilibrium, momentum and energy eigenfunctions.

To provide a full picture of the spectrum of $\mathcal{L}_{\mathsf{max}}$, we adopt the strategy of Bobylev [6] based on the application of the Fourier transform to the elastic Boltzmann equation. Namely, we are looking for $\lambda > 0$ such that the equation

$$\mathcal{L}_{\mathsf{max}} f = -\lambda f, \qquad f \in L^2(\mathcal{M}^{-1}),\ f \neq 0, \tag{3.3}$$

admits a solution. Applying the Fourier transform $\mathscr{F}$ to the both sides of the above equation, we are lead to:

$$\mathscr{F}\left[\mathcal{L}_{\mathsf{max}} f\right](\xi) = -\lambda \widehat{f}(\xi), \qquad \xi \in \mathbb{R}^3,$$

where $\widehat{f} = \mathscr{F}(f)$. One deduces immediately from the calculations performed in [21], that

$$\mathscr{F}\left[\mathcal{L}_{\mathsf{max}} f\right](\xi) = \frac{1}{2\pi}\int_{\mathbb{S}^2} |\widetilde{\xi}\cdot n|\left[\widehat{f}(\xi^+)\widehat{\mathcal{M}_1}(\xi^-) - \widehat{f}(\xi)\widehat{\mathcal{M}_1}(0)\right]\,\mathrm{d}n$$

with $\widetilde{\xi} = \xi/|\xi|$ and

$$\xi^+ = \xi - 2\alpha(1-\beta)(\xi\cdot n)n, \qquad \xi^- = 2\alpha(1-\beta)(\xi\cdot n)n$$

while $\widehat{\mathcal{M}_1}$ is the Fourier transform of the background Maxwellian distribution, given by

$$\widehat{\mathcal{M}_1}(\xi) = \exp\left\{-iu_1\cdot\xi - \frac{\Theta_1|\xi|^2}{2m_1}\right\}.$$

The fundamental property is that even if the equilibrium distribution $\mathcal{M}$ does not make the integrand of the collision operator vanish pointwise (as it is the case in the elastic case), surprisingly it still satisfies a pointwise relation in Fourier variables as was noticed in [21]. A simple computation yields

$$\widehat{\mathcal{M}}(\xi) = \exp\left\{-iu_1\cdot\xi - \frac{\Theta^{\#}|\xi|^2}{2m}\right\}$$

and thus one checks easily that, $\widehat{\mathcal{M}}(\xi^+)\,\widehat{\mathcal{M}_1}(\xi^-) = \widehat{\mathcal{M}}(\xi)$, for any $n \in \mathbb{S}^2$, and any $\xi \in \mathbb{R}^3$. Thus, following the method of Bobylev [6], we rescale $\widehat{f}(\xi) = \widehat{\mathcal{M}}(\xi)\,\varphi(\xi)$ and define the corresponding re-scaled operator $\mathscr{L}$

$$\mathscr{L}\varphi(\xi) = \frac{1}{2\pi}\int_{\mathbb{S}^2} |\widetilde{\xi}\cdot n|\left[\varphi(\xi^+) - \varphi(\xi)\right]\,\mathrm{d}n.$$

Then, Eq. (3.3) amounts to find $\lambda > 0$ such that the equation $\mathscr{L}\varphi(\xi) = -\lambda\varphi(\xi)$ admits a non zero solution φ with

$$f = \mathscr{F}^{-1}(\varphi\widehat{\mathcal{M}}^{-1}) \in L^2(\mathcal{M}^{-1}) \tag{3.4}$$

where $\mathscr{F}^{-1}$ stands for the inverse Fourier transform. Using, as in the elastic case [6, p. 136], the symmetry properties of the operator $\mathscr{L}$ together with condition (3.4), one obtains that functions of the form

$$\varphi_{n,\ell,m}(\xi) = |\xi|^{2n+\ell}\,\mathbf{Y}_{\ell,m}(\widetilde{\xi}), \qquad n \geqslant 0,$$

$\mathbf{Y}_{\ell,m}$ being a spherical harmonic ($\ell \in \mathbb{N}$, $m = -\ell, \dots, \ell$), are eigenvectors of $\mathscr{L}$, associated to the eigenvalues $-\lambda_{n,\ell} = -\lambda_{n,\ell}(\kappa)$ where, according to [7],

$$\lambda_{n,\ell} = 1 - \frac{1}{2\kappa(1-\kappa)} \int_{1-2\kappa}^{1} s^{2n+\ell+1}\, \mathbf{P}_\ell\left(\frac{1-2\kappa+s^2}{(2-2\kappa)s}\right) \mathrm{d}s \tag{3.5}$$

where $\mathbf{P}_\ell$ is the ℓ-th Legendre polynomial, $n \in \mathbb{N}$, $\ell \in \mathbb{N}$. Note that, according to (3.1), $\lambda_{n,\ell} \in (0,1]$. Technical calculations prove that the eigenvalues we already found out, namely,

$$\lambda_{0,0} = 0, \quad \lambda_{0,1} = \kappa, \quad \text{and} \quad \lambda_{1,0} = 2\kappa(1-\kappa),$$

do actually correspond to the couples $(n,\ell) = (0,0)\,;(0,1)$ and $(1,0)$ respectively. Moreover, from the well-known Legendre polynomials property:

$$(\ell+1)\mathbf{P}_{\ell+1}(x) = (2\ell+1)x\mathbf{P}_\ell(x) - \ell\,\mathbf{P}_{\ell-1}(x), \qquad x \in \mathbb{R},\ \ell \geqslant 1,$$

one obtains the recurrence formula

$$\lambda_{n,\ell+1} = \frac{2\ell+1}{\ell+1}\frac{\nu}{1+\nu}\lambda_{n,\ell} + \frac{2\ell+1}{(\ell+1)(1+\nu)}\lambda_{n+1,\ell} - \frac{\ell}{\ell+1}\lambda_{n+1,\ell-1}, \qquad n,\ell \geqslant 0 \tag{3.6}$$

where $\nu = 1-2\kappa \in (-1,1)$. Such a recurrence formula together with the relation

$$\lambda_{n,0} = 1 - \frac{1}{n+1}\frac{1-\nu^{2n+2}}{1-\nu^2},$$

allow to prove by induction over $\ell \in \mathbb{N}$ that

$$\lambda_{n+1,\ell} \geqslant \lambda_{n,\ell} \quad \text{and} \quad \lambda_{n,\ell+1} \geqslant \lambda_{n,\ell} \quad \text{for any} \quad n \in \mathbb{N}.$$

Consequently, one sees that $\min\{\lambda_{n,\ell}\,;\, n,\ell \geqslant 0\} \setminus \{0\} = \min\{\lambda_{1,0};\lambda_{0,1}\}$ which means that spectral gap of $\mathcal{L}_{\mathsf{max}}$ is given by

$$\mu_{\mathsf{max}} = \min\{\lambda_{1,0};\lambda_{0,1}\} = \min\{\kappa; 2\kappa(1-\kappa)\}.$$

Remark 3.1. *Note that,*

$$\lambda_{0,1} \leqslant \lambda_{1,0} \Longleftrightarrow \kappa \leqslant 1/2 \Longleftrightarrow \alpha(1+e) \leqslant 1.$$

*In particular, if $m_1 \leqslant m$ then $\lambda_{0,1} \leqslant \lambda_{1,0}$. Assuming for a while that we are dealing with species of gases with same masses $m = m_1$, then in the true inelastic case (*i.e., *$e < 1$) one also has $\lambda_{1,0} > \lambda_{0,1}$. This situation is very particular to inelastic scattering and means that the cooling process of the temperature happens more rapidly than the forcing of the momentum by the background, whereas when $\kappa = 1/2$ these two processes happen at exactly the same speed ($\lambda_{0,1} = \lambda_{1,0}$). Note also that, whenever $\lambda_{0,1} > \lambda_{1,0}$ (due to the ratio of mass different from* 1*), the smallest eigenvalue corresponds to the momentum relaxation and not the energy relaxation anymore. This contrasts very much with the linearized case (see* [7]*). Note also that the first eigenvalues are ordered as illustrated in Fig. 1.*

The above result can be summarized in the following where the last statement follows from the fact that $\mathcal{L}_{\mathsf{max}} + \mathrm{Id}$ is a self-adjoint compact operator of $L^2(\mathcal{M}^{-1})$.

Theorem 3.2. *The operator $-\mathcal{L}_{\mathsf{max}}$ is a bounded self-adjoint positive operator of $L^2(\mathcal{M}^{-1})$ whose spectrum is composed of an essential part $\{+1\}$ plus the following discrete part:*

$$\mathfrak{S}_p(-\mathcal{L}_{\mathsf{max}}) = \{\lambda_{n,\ell}\,;\, n,\ell \in \mathbb{N}\} \subset [0,1)$$

where $\lambda_{n,\ell}$ is given by (3.5)*. Moreover, $\lambda_{0,0} = 0$ is a simple eigenvalue of $\mathcal{L}_{\mathsf{max}}$ associated to the eigenvector $\mathcal{M}$ and $-\mathcal{L}_{\mathsf{max}}$ admits a positive spectral gap*

$$\mu_{\mathsf{max}} = \min\{\lambda_{1,0};\lambda_{0,1}\} = \min\{\kappa; 2\kappa(1-\kappa)\}.$$

Finally, there exists a Hilbert basis of $L^2(\mathcal{M}^{-1})$ made of eigenvectors of $-\mathcal{L}_{\mathsf{max}}$.

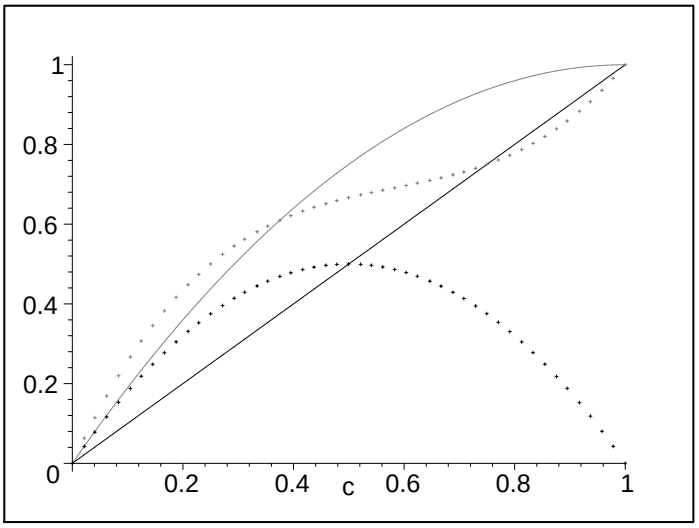

FIGURE 1. Evolution of $\lambda_{0,1}$ (black solid line), $\lambda_{1,0}$ (black dotted line), $\lambda_{1,1}$ (grey dotted line) and $\lambda_{0,2}$ (grey solid line) as functions of $c \in [0,1]$

3.2. **Entropy estimate for Maxwell molecules.** The result of the above section allows to provide a quantitative version of the H-Theorem. Precisely, for any $f \in L^2(\mathcal{M}^{-1})$ orthogonal to $\mathcal{M}$, using the decomposition of both $\mathcal{L}_{\max} f$ and f on the Hilbert basis of $L^2(\mathcal{M}^{-1})$ made of eigenvectors of $-\mathcal{L}_{\max}$ (see Theorem 3.2), it is easily proved that:

$$\mathcal{D}_{\max}(f) := -\int_{\mathbb{R}^3_v} \mathcal{L}_{\max}(f)\, f\, \mathcal{M}^{-1}\, \mathrm{d}v \geqslant \mu_{\max}\, \|f\|^2_{L^2(\mathcal{M}^{-1})}, \qquad \forall f \perp \mathrm{Span}(\mathcal{M}). \tag{3.7}$$

It is well-known that such a *coercivity estimate* allows to obtain an exponential relaxation rate to equilibrium for the solution to the space homogeneous Boltzmann equation. Namely, given $f_0(v) \in L^2(\mathbb{R}^3, \mathcal{M}^{-1}(v)\,\mathrm{d}v)$ with unit mass

$$\int_{\mathbb{R}^3} f_0(v)\,\mathrm{d}v = 1,$$

let f_t be the unique solution of (1.1) with initial condition $f_{t=0} = f_0$. According to the conservation of mass, it is clear that $(f_t - \mathcal{M})$ is orthogonal to $\mathcal{M}$ (for the $L^2(\mathbb{R}^3_v, \mathcal{M}^{-1}(v)\,\mathrm{d}v)$ scalar product) and, within the entropy language:

$$\frac{\mathrm{d}}{\mathrm{d}t} H_\Phi(f_t|\mathcal{M}) = -2\mathcal{D}_{\max}(f_t) \leqslant -2\mu_{\max}\, H_\Phi(f_t|\mathcal{M})$$

for $\Phi(x) = (x-1)^2$ or equivalently,

$$\left(\int_{\mathbb{R}^3} (f_t - \mathcal{M})^2\, \mathcal{M}^{-1}\, \mathrm{d}v\right)^{1/2} \leqslant \left(\int_{\mathbb{R}^3} (f_0 - \mathcal{M})^2\, \mathcal{M}^{-1}\, \mathrm{d}v\right)^{1/2} \exp\left(-\mu_{\max} t\right), \qquad \forall t \geqslant 0.$$

We obtain in this way an *explicit* exponential relaxation rate towards equilibrium for the solution to the space homogeneous linear Boltzmann equation which is valid for granular gases of Maxwell molecules and generalizes a well-known result for classical gases [11]. More interesting is the fact that the knowledge of the spectral gap of $\mathcal{L}_{\max}$ allows to recover an explicit estimate of the spectral gap of the linear Boltzmann operator for hard-spheres

$\mathcal{L}_{\mathsf{hs}}$ through a suitable comparison of the entropy production functionals $\mathcal{D}_{\mathsf{hs}}$ and $\mathcal{D}_{\mathsf{max}}$. This is the subject of the following section.

3.3. **Entropy estimate for hard-spheres.** The goal of this subsection is to show that the entropy production functional $\mathcal{D}_{\mathsf{hs}}$ for hard-spheres relates to the one for Maxwell molecules $\mathcal{D}_{\mathsf{max}}$. More precisely we shall show that

Proposition 3.3. *The entropy production functionals $\mathcal{D}_{\mathsf{hs}}$ and $\mathcal{D}_{\mathsf{max}}$ are related by:*

$$\mathcal{D}_{\mathsf{hs}}(f) \geqslant C^\star \, \mathcal{D}_{\mathsf{max}}(f), \qquad \forall f \in \mathscr{D}(\mathcal{L}_{\mathsf{hs}})$$

for some explicit constant $C^\star$ depending only on α and β.

Remark 3.4. *The idea of searching for such an inequality was already present in* [2]*, but here the method of proof is different and simpler: one does not need any triangular inequality between collisions, and the proof reduces to a careful study of a convolution integral.*

Remark 3.5. *Note that in the hard-spheres case, the operator $\mathcal{L}_{\mathsf{hs}}$ is unbounded. For a careful study of its properties (compactness of the non-local part, definition of the associated C^0-semigroup of contraction in the Hilbert space $L^2(\mathcal{M}^{-1})$) we refer to* [1].

Proof. Let $f \in \mathscr{D}(\mathcal{L}_{\mathsf{hs}})$. We set $u_1 = 0$ in this proof without restriction since this only amounts to a space translation.

We introduce the following parametrization, for fixed $n \in \mathbb{S}^2$, $v = rn + \bar{v}$, $v^\star = r^\star n + \bar{v}$, $w = r_w n + \bar{w}$, $w^\star = r_{w^\star} n + \bar{w}$, where $r, r^\star, r_w$ and $r_{w^\star}$ are real numbers and $\bar{v}, \bar{w}$ are orthogonal to n. Simple computations show that

$$r_w = \frac{r^\star}{2\alpha(1-\beta)} + \left(1 - \frac{1}{2\alpha(1-\beta)}\right) r,$$

while

$$r_{w^\star} = \left(\frac{1}{2\alpha(1-\beta)} - \frac{1-\alpha}{\alpha}\right) r^\star + \left(\frac{1}{\alpha} - \frac{1}{2\alpha(1-\beta)}\right) r.$$

Therefore, r_w and $r_{w^\star}$ only depend on r and $r^\star$. Then if we denote θ the angle between $\widetilde{q}$ and n, we get from Prop. 2.3, where we set $g = \frac{f}{\mathcal{M}}$,

$$\mathcal{D}_{\mathsf{hs}}(f) = \frac{1}{2\pi} \int_{\mathbb{S}^2} \int_{r,r^\star \in \mathbb{R}} \int_{\bar{v},\bar{w} \in n^\perp} |q| \, \cos\theta \Big[g(r^\star n + \bar{v}) - g(rn + \bar{v}) \Big]^2 \\ \mathcal{M}_1(r_w n + \bar{w}) \, \mathcal{M}(rn + \bar{v}) \, \mathrm{d}\bar{v} \, \mathrm{d}\bar{w} \, \mathrm{d}r \, \mathrm{d}r^\star \, \mathrm{d}n$$

with

$$|q| = \left(|\bar{v} - \bar{w}|^2 + (2\kappa)^{-2} |r^\star - r|^2 \right)^{1/2}$$

and

$$\cos\theta = \frac{(2\kappa)^{-1} |r^\star - r|}{\left(|\bar{v} - \bar{w}|^2 + (2\kappa)^{-2} |r^\star - r|^2 \right)^{1/2}},$$

where we recall that $\kappa = \alpha(1-\beta)$. We split the integral into two parts according to $|r^\star - r| \geqslant \varrho_0 > 0$ or $|r^\star - r| \leqslant \varrho_0$ where ϱ_0 is a positive parameter to be determine latter. Using the

fact that $|q| \geqslant |r^\star - r|/2\kappa$, one has the following estimate for the first part of the integral

$$\frac{1}{2\pi}\int_{\mathbb{S}^2} \mathrm{d}n \int_{\{|r^\star - r|\geqslant \varrho_0\}} \mathrm{d}r\ \mathrm{d}r^\star \int_{\bar v,\bar w\in n^\perp} |q|\cos\theta\, \mathcal{M}(rn+\bar v)\,\mathcal{M}_1(r_w n+\bar w) \Big[g(r^\star n+\bar v)-g(rn+\bar v)\Big]^2\,\mathrm{d}\bar v\,\mathrm{d}\bar w$$
$$\geqslant \frac{(2\kappa)^{-1}\,\varrho_0}{2\pi}\int_{\mathbb{S}^2}\mathrm{d}n\int_{\{|r^\star-r|\geqslant\varrho_0\}}\mathrm{d}r\,\mathrm{d}r^\star \int_{\bar v,\bar w\in n^\perp}\cos\theta\,\mathcal{M}(rn+\bar v)\,\mathcal{M}_1(r_w n+\bar w)\Big[g(r^\star n+\bar v)-g(rn+\bar v)\Big]^2\,\mathrm{d}\bar v\,\mathrm{d}\bar w$$

which corresponds (up to the multiplicative factor $(2\kappa)^{-1}\,\varrho_0$) to the integral for $|r-r^\star|\geqslant\varrho_0$ corresponding to Maxwell molecules, *i.e.*,

$$-\int_{\mathbb{R}^3}\chi_{\{|r-r^\star|\geqslant\varrho_0\}}\mathcal{L}_{\mathsf{hs}}(f)f\mathcal{M}^{-1}\,\mathrm{d}v \geqslant -\frac{\varrho_0}{2\kappa}\int_{\mathbb{R}^3}\chi_{\{|r-r^\star|\geqslant\varrho_0\}}\mathcal{L}_{\mathsf{max}}(f)f\mathcal{M}^{-1}\,\mathrm{d}v. \tag{3.8}$$

Concerning now the second part of the integral (corresponding to $|r^\star - r|\leqslant\varrho_0$), we use that $|q|\geqslant|\bar v-\bar w|$ and we isolate the integration over $\bar w$:

$$-\int_{\mathbb{R}^3}\chi_{\{|r-r^\star|\leqslant\varrho_0\}}\mathcal{L}_{\mathsf{hs}}(f)f\mathcal{M}^{-1}\,\mathrm{d}v \geqslant \frac{1}{2\pi}\int_{\mathbb{S}^2}\mathrm{d}n\int_{\{|r^\star-r|\leqslant\varrho_0\}}\mathrm{d}r\,\mathrm{d}r^\star\,|r^\star-r| \int_{\bar v\in n^\perp}(2\kappa)^{-1}\Big(\frac{m_1}{2\pi\Theta_1}\Big)^{-3/2}\Big(\int_{\bar w\in n^\perp}\frac{|\bar v-\bar w|\,\mathcal{M}_1(\bar w)}{(|\bar v-\bar w|^2+(2\kappa)^{-2}|r^\star-r|^2)^{1/2}}\,\mathrm{d}\bar w\Big) \mathcal{M}(rn+\bar v)\,\mathcal{M}_1(r_w n)\Big[g(r^\star n+\bar v)-g(rn+\bar v)\Big]^2\,\mathrm{d}\bar v$$

where we used the fact that, since $\bar w$ is orthogonal to n,

$$\mathcal{M}_1(r_w n+\bar w)=\Big(\frac{m_1}{2\pi\Theta_1}\Big)^{-3/2}\mathcal{M}_1(\bar w)\mathcal{M}_1(r_w n).$$

Setting $\xi=|r^\star-r|/2\kappa$, if one were able to prove that there is a constant C such that

$$\int_{\mathbb{R}^2}\frac{|\bar v-\bar w|\,\mathcal{M}_1(\bar w)}{(|\bar v-\bar w|^2+\xi^2)^{1/2}}\,\mathrm{d}\bar w \geqslant C\int_{\mathbb{R}^2}\frac{\mathcal{M}_1(\bar w)}{(|\bar v-\bar w|^2+\xi^2)^{1/2}}\,\mathrm{d}\bar w \tag{3.9}$$

uniformly for $\bar v\in\mathbb{R}^2$ and $\xi\in[0,\varrho_0/2\kappa]$, then one would obtain the desired estimate (by doing all the previous transformations backward):

$$-\int_{\mathbb{R}^3}\chi_{\{|r-r^\star|\leqslant\varrho_0\}}\mathcal{L}_{\mathsf{hs}}(f)f\mathcal{M}^{-1}\,\mathrm{d}v \geqslant -C\int_{\mathbb{R}^3}\chi_{\{|r-r^\star|\leqslant\varrho_0\}}\mathcal{L}_{\mathsf{max}}(f)f\mathcal{M}^{-1}\,\mathrm{d}v.$$

To study the convolution integral of (3.9), we make a second splitting between $|\bar w-\bar v|\geqslant\varrho_1>0$ and $|\bar w-\bar v|\leqslant\varrho_1$ (for some $\varrho_1>0$). It gives

$$\int_{\bar w\in\mathbb{R}^2}\frac{|\bar v-\bar w|\,\mathcal{M}_1(\bar w)}{(|\bar v-\bar w|^2+\xi^2)^{1/2}}\,\mathrm{d}\bar w \geqslant \int_{\{|\bar w-\bar v|\geqslant\varrho_1\}}\frac{|\bar v-\bar w|\,\mathcal{M}_1(\bar w)}{(|\bar v-\bar w|^2+\xi^2)^{1/2}}\,\mathrm{d}\bar w \geqslant \varrho_1\int_{\{|\bar w-\bar v|\geqslant\varrho_1\}}\frac{\mathcal{M}_1(\bar w)}{(|\bar v-\bar w|^2+\xi^2)^{1/2}}\,\mathrm{d}\bar w$$
$$\geqslant \varrho_1\Big(\int_{\mathbb{R}^2}\frac{\mathcal{M}_1(\bar w)}{(|\bar v-\bar w|^2+\xi^2)^{1/2}}\,\mathrm{d}\bar w-\int_{\{|\bar w-\bar v|\leqslant\varrho_1\}}\frac{\mathcal{M}_1(\bar w)}{(|\bar v-\bar w|^2+\xi^2)^{1/2}}\,\mathrm{d}\bar w\Big).$$

Then we use the obvious estimates

$$\forall\, \bar v \in \mathbb{R}^2,\ \forall\, \xi \in [0, \varrho_0/2\kappa],\ \int_{\mathbb{R}^2} \frac{\mathcal{M}_1(\bar w)}{(|\bar v - \bar w|^2 + \xi^2)^{1/2}}\, \mathrm{d}\bar w \geqslant \frac{C(\kappa, \varrho_0)}{1 + |\bar v|}$$

for an explicit constant $C(\kappa, \varrho_0) > 0$ depending only on κ, ϱ_0, and

$$\forall\, \bar v \in \mathbb{R}^2,\ \forall\, \xi \in [0, \varrho_0/2\kappa],\ \int_{\{|\bar w - \bar v| \leqslant \varrho_1\}} \frac{\mathcal{M}_1(\bar w)}{(|\bar v - \bar w|^2 + \xi^2)^{1/2}}\, \mathrm{d}\bar w \leqslant C(\kappa, \varrho_1)\, e^{-|\bar v|^2}.$$

for an explicit constant $C(\kappa, \varrho_1) > 0$ going to 0 as ϱ_1 goes to 0. It yields for ϱ_1 small enough (depending on ϱ_0)

$$\int_{\{|\bar w - \bar v| \leqslant \varrho_1\}} \frac{\mathcal{M}_1(\bar w)}{(|\bar v - \bar w|^2 + \xi^2)^{1/2}}\, \mathrm{d}\bar w \leqslant \frac{1}{2} \int_{\mathbb{R}^2} \frac{\mathcal{M}_1(\bar w)}{(|\bar v - \bar w|^2 + \xi^2)^{1/2}}\, \mathrm{d}\bar w. \tag{3.10}$$

for any $\bar v \in \mathbb{R}^2, \xi \in [0, \varrho_0/2\kappa]$ (we also refer to the Appendix A of this paper for a construction of the parameter ϱ_1). Consequently, for this choice of ϱ_1 we obtain (3.9) with $C = \varrho_1/2$, *i.e.*,

$$-\int_{\mathbb{R}^3} \chi_{\{|r - r^\star| \leqslant \varrho_0\}} \mathcal{L}_{\mathsf{hs}}(f) f \mathcal{M}^{-1}\, \mathrm{d}v \geqslant -\frac{\varrho_1}{2} \int_{\mathbb{R}^3} \chi_{\{|r - r^\star| \leqslant \varrho_0\}} \mathcal{L}_{\mathsf{max}}(f) f \mathcal{M}^{-1}\, \mathrm{d}v.$$

This, together with estimate (3.8), yield

$$\mathcal{D}_{\mathsf{hs}}(f) \geqslant \min\left\{\frac{\varrho_0}{2\kappa}, \frac{\varrho_1}{2}\right\} \mathcal{D}_{\mathsf{max}}(f)$$

which concludes the proof. □

Remark 3.6. *The constant $C^\star$ from the proof can be optimized according to the parameter ϱ_0, by expliciting ϱ_1 as a function of ϱ_0. Precisely, making use of Lemma A.1 given in the Appendix,*

$$C^\star = \min\left\{\frac{\varrho_0}{2\kappa}, \frac{\varrho_1}{2}\right\} \geqslant \frac{\eta}{\sqrt{5}}$$

with $\eta = \sqrt{\frac{2\Theta_1}{m_1}}\, \mathrm{erf}^{-1}\left(\frac{1}{2}\right)$ where erf^{-1} denotes the inverse error function, $\mathrm{erf}^{-1}(\frac{1}{2}) \simeq 0.4769$. Notice that this lower bound for $C^\star$ does not depend on the parameters α, β.

Remark 3.7. *The above Proposition provides an estimate of the spectral gap of $\mathcal{L}_{\mathsf{hs}}$ in $L^2(\mathcal{M}^{-1})$. Precisely, we recall from* [1] *that the spectrum of $\mathcal{L}_{\mathsf{hs}}$ is made of continuous (essential) spectrum $\{\lambda \in \mathbb{R}\,;\, \lambda \leqslant -\nu_0\}$ where $\nu_0 = \inf_{v \in \mathbb{R}^3} \sigma_{\mathsf{hs}}(v) > 0$ and a decreasing sequence of real eigenvalues with finite algebraic multiplicities which unique possible cluster point is $-\nu_0$. Then, since 0 is an eigenvalue of $\mathcal{L}_{\mathsf{hs}}$ associated to $\mathcal{M}$, one sees from the above Proposition that the spectral gap μ_{hs} of $\mathcal{L}_{\mathsf{hs}}$ satisfies*

$$\mu_{\mathsf{hs}} := \min\left\{\lambda : -\lambda \in (-\nu_0, 0), -\lambda \in \mathfrak{S}(\mathcal{L}_{\mathsf{hs}}) \setminus \{0\}\right\} \geqslant C^\star \mu_{\mathsf{max}} \geqslant \frac{\eta \min\{\kappa, 2\kappa(1-\kappa)\}}{\sqrt{5}}.$$

To summarize, one gets the following coercivity estimate for the Dirichlet form:

Theorem 3.8. *For $Q = \mathcal{L}_{\mathsf{hs}}$ or $\mathcal{L}_{\mathsf{max}}$, one has the following:*

$$-\int_{\mathbb{R}^3} Q(f)(v) f(v) \mathcal{M}^{-1}(v)\, \mathrm{d}v \geqslant \mu \|f - \varrho_f \mathcal{M}\|^2_{L^2(\mathcal{M}^{-1})}, \qquad \forall f \in \mathscr{D}(Q)$$

where, $\varrho_f = \displaystyle\int_{\mathbb{R}^3} f(v)\, \mathrm{d}v$, and $\mu = \mu_{\mathsf{max}}$ whenever $Q = \mathcal{L}_{\mathsf{max}}$ while, for hard-spheres interactions, i.e., *$Q = \mathcal{L}_{\mathsf{hs}}$, one has $\mu \geqslant C^\star \mu_{\mathsf{max}}$.*

Proof. If $\varrho_f = 0$, the proof follows directly from Proposition 3.3 and (3.7). Now, if f is a given function with non-zero mean ϱ_f, set $h = f - \varrho_f \mathcal{M}$. Then, $\varrho_h = 0$ so that

$$-\int_{\mathbb{R}^3} Q(h)(v)h(v)\mathcal{M}^{-1}(v)\,\mathrm{d}v \geqslant \mu\|h\|^2_{L^2(\mathcal{M}^{-1})}.$$

This leads to the result since $Q(h) = Q(f)$ and $\displaystyle\int_{\mathbb{R}^3} Q(f)\,\mathrm{d}v = 0$. □

Adopting now the entropy language, one obtains the following relaxation rate, which is also new in the context of linear Boltzmann equation:

Corollary 3.9. *Let $f_0(v) \in L^2(\mathbb{R}^3_v, \mathcal{M}^{-1}(v)\,\mathrm{d}v)$ be given and let $f(t)$ be the unique solution of* (1.1) *with initial condition $f(t=0) = f_0$. Then, for any $t \geqslant 0$, one has the following*

$$\left\|f(t) - \mathcal{M}\right\|_{L^2(\mathcal{M}^{-1})} \leqslant \exp(-\mu t)\left\|f_0 - \mathcal{M}\right\|_{L^2(\mathcal{M}^{-1})}, \qquad \forall t \geqslant 0,$$

where $\mu = \mu_{\mathsf{max}}$ when $Q = Q_{\mathsf{max}}$ while, for hard-spheres interactions, i.e., *$Q = Q_{\mathsf{hs}}$, one has $\mu \geqslant C^\star \mu_{\mathsf{max}}$.*

We state another corollary of the above Theorem 3.8 in which we strengthen the coercivity estimate:

Corollary 3.10. *For $Q = \mathcal{L}_{\mathsf{hs}}$ or $\mathcal{L}_{\mathsf{max}}$, there exists $c_\sigma > 0$ such that*

$$-\int_{\mathbb{R}^3} Q(f)(v)f(v)\mathcal{M}^{-1}(v)\,\mathrm{d}v \geqslant c_\sigma\|(f - \varrho_f\mathcal{M})\sqrt{\sigma}\|^2_{L^2(\mathcal{M}^{-1})} \qquad \forall f \in \mathscr{D}(Q)$$

where, $\varrho_f = \displaystyle\int_{\mathbb{R}^3} f(v)\,\mathrm{d}v$ and $\sigma(v)$ is the collision frequency associated to Q.

Proof. If $Q = \mathcal{L}_{\mathsf{max}}$, since $\sigma_{\mathsf{max}}(v) = 1$ the estimate is nothing but Theorem 3.8. Let us consider now the hard-spheres case, $Q = \mathcal{L}_{\mathsf{hs}}$. Arguing as in the proof of Theorem 3.8, it suffices to prove the result for $f \perp \mathcal{M}$, *i.e.*, whenever $\varrho_f = 0$. We recall from [1] that $\mathcal{L}_{\mathsf{hs}}$ splits as

$$\mathcal{L}_{\mathsf{hs}} f = \mathscr{K} f - \sigma_{\mathsf{hs}} f, \quad f \in \mathscr{D}(\mathcal{L}_{\mathsf{hs}})$$

where $\mathscr{K}$ is a bounded (and compact) operator in $L^2(\mathcal{M}^{-1})$. We then have

$$\begin{aligned}\|f\sqrt{\sigma_{\mathsf{hs}}}\|^2_{L^2(\mathcal{M}^{-1})} &= \int_{\mathbb{R}^3_v} \mathscr{K}(f)\, f\, \mathcal{M}^{-1}\,\mathrm{d}v - \int_{\mathbb{R}^3_v} \mathcal{L}_{\mathsf{hs}}(f)\, f\, \mathcal{M}^{-1}\,\mathrm{d}v \\ &\leqslant \|\mathscr{K}\|\|f\|^2_{L^2(\mathcal{M}^{-1})} + \mathscr{D}_{\mathsf{hs}}(f) \leqslant \Big(\frac{\|\mathscr{K}\|}{\mu_{\mathsf{max}} C^\star} + 1\Big)\mathscr{D}_{\mathsf{hs}}(f)\end{aligned}$$

where $\|\mathscr{K}\|$ stands for the norm of $\mathscr{K}$ as a bounded operator on $L^2(\mathcal{M}^{-1})$ and we used Theorem 3.8. The corollary follows with

$$c_\sigma = \frac{C^\star \mu_{\mathsf{max}}}{\|\mathscr{K}\| + C^\star \mu_{\mathsf{max}}}.$$

□

Remark 3.11. *Here again, as in Prop. 3.3, the constant $c_\sigma > 0$ can be quantitatively estimated using for instance the estimate $\|\mathscr{K}\| \leqslant \frac{2\pi}{(1+\tau)^2}\sqrt{\frac{\pi\Theta_1}{m_1}}$ that can be deduced without major difficulty from the explicit expression of $\mathscr{K}$ provided in* [1] *with $\tau = (1-2\kappa)/\kappa > 0$.*

Remark 3.12. *Recalling that σ_{hs} behaves like $(1+|v|)$, the above corollary allows to control from below the entropy production functional by the weighted $L^2((1+|v|)\mathcal{M}^{-1}, \mathrm{d}v)$ norm. Such a weighted estimate shall be very useful for the diffusion approximation.*

4. **Diffusion Approximation.** We shall assume again in this whole section that $u_1 = 0$. From the results of the previous section, it is possible to derive some exact convergence results for the solution of the re-scaled linear kinetic Boltzmann equation

$$\varepsilon \partial_t f_\varepsilon(t,x,v) + v \cdot \nabla_x f_\varepsilon(t,x,v) = \frac{1}{\varepsilon} Q(f_\varepsilon)(t,x,v), \tag{4.1}$$

with initial condition $f_\varepsilon(x,v,0) = f_0(x,v) \geqslant 0$, with $(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3$. Note that all the analysis we perform here is also valid if the spatial domain denotes the three-dimensional torus $\mathbb{T}^3_x$. One shall prove that f_ε converges, as $\varepsilon \to 0$, to $\mathcal{M}(v)\varrho$ where $\varrho = \varrho(t,x)$ is the solution to the (parabolic) diffusion equation:

$$\begin{cases} \partial_t \varrho(t,x) = \nabla_x \cdot \left(\mathsf{D} \nabla_x \varrho(t,x) + u_1 \, \rho \right), & t \geqslant 0, \ x \in \mathbb{R}^3, \\ \varrho(0,x) = \varrho_0(x) = \displaystyle\int_{\mathbb{R}^3_v} f_0(x,v) \, \mathrm{d}v \end{cases} \tag{4.2}$$

where the diffusion coefficient D depends on the model we investigate (hard-sphere interactions or Maxwell molecules). One shall adopt here the strategy of [12, 14]. Namely, to prove the convergence of the solution to (4.1) towards the solution ϱ of (4.2), the idea is to use the *a priori* estimate given by the production of entropy, as in [14] where this idea was applied to discrete velocity models of the Boltzmann equation. Let us define the number density and the current vector

$$\varrho_\varepsilon(t,x) = \int_{\mathbb{R}^3_v} f_\varepsilon(t,x,v) \, \mathrm{d}v, \qquad j_\varepsilon(t,x) = \frac{1}{\varepsilon} \int_{\mathbb{R}^3_v} f_\varepsilon(t,x,v) \, v \ \mathrm{d}v.$$

We also define h_ε as

$$h_\varepsilon(t,x,v) = \frac{1}{\varepsilon} \Big(f_\varepsilon(t,x,v) - \varrho_\varepsilon(t,x) \mathcal{M}(v) \Big).$$

Integrating (4.1) with respect to x and v and using the fact that the mean of $Q(f_\varepsilon)$ is zero, one gets the mass conservation identity

$$\int_{\mathbb{R}^3_x \times \mathbb{R}^3_v} f_\varepsilon(x,v,t) \, \mathrm{d}x \, \mathrm{d}v = \int_{\mathbb{R}^3_x \times \mathbb{R}^3_v} f_0(x,v) \, \mathrm{d}x \, \mathrm{d}v, \tag{4.3}$$

which means (using the fact that the equation preserves non-negativity) that, for any $T > 0$, the sequence $\varrho_\varepsilon(x,t)$ is bounded in $L^\infty(0,T;L^1(\mathbb{R}^3_x))$. Now, multiplying (4.1) by $f_\varepsilon \mathcal{M}^{-1}$ and integrating over $\mathbb{R}^3_x \times \mathbb{R}^3_v$, we get

$$\begin{aligned} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3_x \times \mathbb{R}^3_v} f_\varepsilon^2(t,x,v) \mathcal{M}^{-1}(v) \, \mathrm{d}x \, \mathrm{d}v + \frac{1}{2\varepsilon} \int_{\mathbb{R}^3_x \times \mathbb{R}^3_v} \mathrm{div}_x \Big(v f_\varepsilon^2(t,x,v) \Big) \mathcal{M}^{-1}(v) \, \mathrm{d}x \, \mathrm{d}v \\ - \frac{1}{\varepsilon^2} \int_{\mathbb{R}^3_x \times \mathbb{R}^3_v} f_\varepsilon \, Q(f_\varepsilon) \mathcal{M}^{-1} \, \mathrm{d}x \, \mathrm{d}v = 0. \end{aligned} \tag{4.4}$$

Now, because of the divergence form of the integrand, one sees that the second term in (4.4) is zero while, because of Corollary 3.10,

$$\begin{aligned} & -\frac{1}{\varepsilon^2} \int_{\mathbb{R}^3_x \times \mathbb{R}^3_v} f_\varepsilon \, Q(f_\varepsilon) \mathcal{M}^{-1} \, \mathrm{d}x \, \mathrm{d}v \\ & \geqslant \frac{c_\sigma}{\varepsilon^2} \int_{\mathbb{R}^3_x} \| f_\varepsilon(t,x,\cdot) - \varrho_\varepsilon(t,x) \mathcal{M} \|^2_{L^2(\mathbb{R}^3_v, \sigma(v)\mathcal{M}^{-1}(v) \, \mathrm{d}v)} \, \mathrm{d}x \\ & = c_\sigma \int_{\mathbb{R}^3_x \times \mathbb{R}^3_v} h_\varepsilon^2(t,x,v) \mathcal{M}^{-1}(v) \sigma(v) \, \mathrm{d}x \, \mathrm{d}v. \end{aligned} \tag{4.5}$$

Consequently, Eq. (4.4), together with (4.5), leads to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3_x\times\mathbb{R}^3_v} f_\varepsilon^2(x,v,t)\mathcal{M}^{-1}(v)\,\mathrm{d}x\,\mathrm{d}v \leqslant -c_\sigma\int_{\mathbb{R}^3_x\times\mathbb{R}^3_v} h_\varepsilon^2(x,v,t)\sigma(v)\mathcal{M}^{-1}(v)\,\mathrm{d}x\,\mathrm{d}v.$$

Defining therefore the following Hilbert space:

$$\mathcal{H} = L^2(\mathbb{R}^3_x\times\mathbb{R}^3_v, \mathcal{M}^{-1}(v)\,\mathrm{d}x\,\mathrm{d}v)$$

endowed with its natural norm $\|\cdot\|_{\mathcal{H}}$, one has

$$\left\|f_\varepsilon(t)\right\|^2_{\mathcal{H}} + 2c_\sigma\int_0^t \left\|h_\varepsilon(s)\sqrt{\sigma}\right\|^2_{\mathcal{H}}\,\mathrm{d}s \leqslant \left\|f_0\right\|^2_{\mathcal{H}}, \qquad \forall t\geqslant 0. \tag{4.6}$$

We obtain the following *a priori* bounds:

Proposition 4.1. *For any $\varepsilon>0$, let $f_\varepsilon(t)$ denotes the unique solution to* (4.1) *with $f_0\in\mathcal{H}$, $f_0\geqslant 0$. Then, for any $0\leqslant T<\infty$*

1. *The sequence $(f_\varepsilon)_\varepsilon$ is bounded in $L^\infty(0,T;\mathcal{H})$,*
2. *the sequence $(\sqrt{\sigma}h_\varepsilon)_\varepsilon$ is bounded in $L^2(0,T;\mathcal{H})$,*
3. *the density sequence $(\varrho_\varepsilon)_\varepsilon$ is bounded in $L^\infty(0,T;L^1(\mathbb{R}^3_x)\cap L^2(\mathbb{R}^3_x))$,*
4. *the current sequence $(j_\varepsilon)_\varepsilon$ is bounded in $\left[L^2((0,T)\times\mathbb{R}^3_x)\right]^3$.*

Proof. The first two points are direct consequences of (4.6) with

$$\sup_{\varepsilon>0}\left\|f_\varepsilon\right\|_{L^\infty(0,T;\mathcal{H})} \leqslant \left\|f_0\right\|_{x,v}, \qquad \sup_{\varepsilon>0}\left\|\sqrt{\sigma}h_\varepsilon\right\|_{L^2(0,T;\mathcal{H})} \leqslant (2c_\sigma)^{-1/2}\left\|f_0\right\|_{x,v}.$$

Now, Eq. (4.3) proves that the number density sequence $(\varrho_\varepsilon)_\varepsilon$ is bounded in $L^\infty(0,T;L^1(\mathbb{R}^3_x))$ and, according to Cauchy-Schwarz inequality,

$$0\leqslant\varrho_\varepsilon(t,x)\leqslant\left(\int_{\mathbb{R}^3_v} f_\varepsilon^2(t,x,v)\mathcal{M}^{-1}(v)\,\mathrm{d}v\right)^{1/2}$$

we see from point *(1)* that $(\varrho_\varepsilon)_\varepsilon$ is also bounded in $L^\infty(0,T;L^2(\mathbb{R}^3_x))$. Finally, since $f_\varepsilon=\varrho_\varepsilon\mathcal{M}+\varepsilon h_\varepsilon$ and $\int_{\mathbb{R}^3_v} v\mathcal{M}(v)\,\mathrm{d}v=0$, one has

$$\int_0^T\mathrm{d}t\int_{\mathbb{R}^3_x}\left|j_\varepsilon(t,x)\right|^2\,\mathrm{d}x = \int_0^T\mathrm{d}t\int_{\mathbb{R}^3_x}\mathrm{d}x\left|\int_{\mathbb{R}^3_v} vh_\varepsilon(t,x,v)\,\mathrm{d}v\right|^2$$

while, from Cauchy-Schwarz inequality and the fact that σ is bounded from below

$$\left|\int_{\mathbb{R}^3_v} vh_\varepsilon(t,x,v)\,\mathrm{d}v\right|^2 \leqslant \left(\int_{\mathbb{R}^3_v}|v|^2\mathcal{M}(v)\,\mathrm{d}v\right)\left(\int_{\mathbb{R}^3_v} h_\varepsilon^2\mathcal{M}^{-1}\,\mathrm{d}v\right)$$

so that

$$\int_0^T\mathrm{d}t\int_{\mathbb{R}^3_x}\left|j_\varepsilon(t,x)\right|^2\,\mathrm{d}x \leqslant \frac{3\Theta^\#}{m}\int_0^T\|h_\varepsilon(t)\|^2_{x,v}\,\mathrm{d}t$$

and the conclusion follows from point *(2)*. □

Remark 4.2. *Since $f_\varepsilon=\varepsilon h_\varepsilon+\varrho_\varepsilon\mathcal{M}$, noticing that $\int_{\mathbb{R}^3_v}\sigma(v)\mathcal{M}(v)\,\mathrm{d}v<\infty$, one deduces from the above points (2) and (3) and that the sequence $(\sqrt{\sigma}f_\varepsilon)_\varepsilon$ is bounded in $L^2(0,T;\mathcal{H})$.*

For any $T>0$, we define

$$\mathbf{\Omega}_T=(0,T)\times\mathbb{R}^3_v\times\mathbb{R}^3_x \quad\text{and}\quad \mathrm{d}\mu_T = \mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t.$$

The bounds provided by Prop. 4.1 allows to assume that, up to a subsequence,

$$f_\varepsilon\rightharpoonup f \quad\text{in}\quad L^2(\mathbf{\Omega}_T;\sigma\mathcal{M}^{-1}\,\mathrm{d}\mu_T), \qquad\qquad h_\varepsilon\rightharpoonup h \quad\text{in}\quad L^2(\mathbf{\Omega}_T;\sigma\mathcal{M}^{-1}\,\mathrm{d}\mu_T);$$

$$\varrho_\varepsilon \rightharpoonup \varrho \quad \text{in} \quad L^2((0,T)\times\mathbb{R}^3_x), \qquad\qquad j_\varepsilon \rightharpoonup j \quad \text{in} \quad \left[L^2((0,T)\times\mathbb{R}^3_x)\right]^3.$$

Let $\Psi \in L^2(\mathbf{\Omega}_T, \sigma^{-1}\mathcal{M}\,\mathrm{d}\mu_T) = \left[L^2(\mathbf{\Omega}_T, \sigma\mathcal{M}^{-1}\,\mathrm{d}\mu_T)\right]^\star$ be given. Since $\sigma = \sigma_{\mathsf{max}}$ is constant while $\sigma = \sigma_{\mathsf{hs}}$ behaves asymptotically like $(1+|v|)$, one easily has from Cauchy-Schwarz

$$\varphi(t,x) = \int_{\mathbb{R}^3_v} \mathcal{M}(v)\Psi(t,x,v)\,\mathrm{d}v \in L^2((0,T)\times\mathbb{R}^3_x),$$

and therefore

$$\lim_{\varepsilon\to 0}\int_0^T \mathrm{d}t \int_{\mathbb{R}^3_x} \varrho_\varepsilon(t,x)\varphi(t,x)\,\mathrm{d}x = \int_0^T \mathrm{d}t \int_{\mathbb{R}^3_x} \varrho(t,x)\varphi(t,x)\,\mathrm{d}x.$$

Thus, writing $f_\varepsilon = \varrho_\varepsilon\mathcal{M} + \varepsilon h_\varepsilon$, one checks that

$$\lim_{\varepsilon\to 0}\int_{\mathbf{\Omega}_T} f_\varepsilon\Psi\,\mathrm{d}\mu_T = \int_0^T \mathrm{d}t \int_{\mathbb{R}^3_x} \varrho(t,x)\varphi(t,x)\,\mathrm{d}x = \int_{\mathbf{\Omega}_T} \varrho\mathcal{M}\Psi\,\mathrm{d}\mu_T,$$

i.e., $f_\varepsilon \rightharpoonup \varrho\mathcal{M}$ in $L^2(\mathbf{\Omega}_T, \sigma\mathcal{M}^{-1}\,\mathrm{d}\mu_T)$. In particular, $f(t,x,v) = \varrho(t,x)\mathcal{M}(v)$. Moreover,

$$\lim_{\varepsilon\to 0}\int_{\mathbf{\Omega}_T} h_\varepsilon\Psi\,\mathrm{d}\mu_T = \int_{\mathbf{\Omega}_T} h\Psi\,\mathrm{d}\mu_T. \tag{4.7}$$

for any $\Psi = \Psi(t,x,v) \in L^2(\mathbf{\Omega}_T, \sigma^{-1}\mathcal{M}\,\mathrm{d}\mu_T)$. Now, choosing Ψ independent of v, one sees that

$$\int_{\mathbb{R}^3_v} h(t,x,v)\,\mathrm{d}v = 0, \qquad \forall t>0, \quad x\in\mathbb{R}^3_x.$$

Finally, using in (4.7) a test function $\Psi(t,x,v) = v\varphi(t,x)$ with $\varphi \in L^2((0,T)\times\mathbb{R}^3_x)$, we deduces from the weak convergence of j_ε to j that

$$j(t,x) = \int_{\mathbb{R}^3_v} vh(t,x,v)\,\mathrm{d}v.$$

Finally, integrating equation (4.1) over $\mathbb{R}^3_v$ leads to the continuity equation

$$\partial_t\varrho_\varepsilon(t,x) + \mathrm{div}_x j_\varepsilon(t,x) = 0, \qquad\qquad \forall\varepsilon>0. \tag{4.8}$$

We deduce at the limit that

$$\partial_t\varrho(t,x) + \mathrm{div}_x j(t,x) = 0, \qquad\qquad t>0, \quad x\in\mathbb{T}^3_x \tag{4.9}$$

in the *distributional sense*. We summarize these first results in the following:

Proposition 4.3. *Under the assumptions of Proposition 4.1, for any $T>0$, up to a subsequence,*

i) (ϱ_ε) *converges weakly in* $L^2((0,T)\times\mathbb{R}^3_x)$ *to some* ϱ*;*

ii) (h_ε) *converges weakly in* $L^2(\mathbf{\Omega}_T, \sigma\mathcal{M}^{-1}\,\mathrm{d}\mu_T)$ *to some function* h *with* $\displaystyle\int_{\mathbb{R}^3_v} h(t,x,v)\,\mathrm{d}v = 0$*;*

iii) (f_ε) *converges weakly to* $\varrho\mathcal{M}$ *in* $L^2(\mathbf{\Omega}_T, \sigma\mathcal{M}^{-1}\,\mathrm{d}\mu_T)$*;*

iv) (j_ε) *converges weakly to* $\displaystyle j(t,x) = \int_{\mathbb{R}^3_v} vh(t,x,v)\,\mathrm{d}v$ *in* $\left[L^2((0,T)\times\mathbb{R}^3_x)\right]^3$.

where ϱ and j are related by (4.9).

The problem of the diffusion approximation is then reduced to the one of finding a suitable relation, similar to the *classical Fick's law*, linking the current $j(t,x)$ to the gradient of the density $\varrho(t,x)$. Such a Fick's law (and the corresponding coefficient) shall depend heavily on the collision kernel.

4.1. **Maxwell molecules.** When dealing with Maxwell molecules, *i.e.*, whenever $Q = \mathcal{L}_{\mathsf{max}}$, it is possible to obtain an explicit expression for the diffusion coefficient. Precisely, multiplying equation (4.1) by v and integrating over $\mathbb{R}^3_v$ gives

$$\varepsilon^2\,\partial_t j_\varepsilon(t,x) + \left(\int_{\mathbb{R}^3_v} (v\otimes v) : \nabla_x f_\varepsilon(t,x,v)\,\mathrm{d}v\right) = \frac{1}{\varepsilon}\int_{\mathbb{R}^3_v} \mathcal{L}_{\mathsf{max}}(f_\varepsilon)\,v\,\mathrm{d}v \tag{4.10}$$

Now, as we already saw it (see (3.2)):

$$\int_{\mathbb{R}^3_v} \mathcal{L}_{\mathsf{max}}(f_\varepsilon)\,v\,\mathrm{d}v = -\alpha(1-\beta)\varepsilon j_\varepsilon = -\lambda_{0,1}\varepsilon j_\varepsilon.$$

Then, recalling that $f_\varepsilon(t,x,v) = \varrho_\varepsilon(t,x)\mathcal{M}(v) + \varepsilon h_\varepsilon(t,x,v)$, Eq. (4.10) becomes

$$\varepsilon^2\,\partial_t j_\varepsilon(t,x) + \mathbf{A} : \nabla_x \varrho_\varepsilon(t,x) + \varepsilon\left(\int_{\mathbb{R}^3_v} (v\otimes v) : \nabla_x h_\varepsilon(t,x,v)\,\mathrm{d}v\right) = -\lambda_{0,1}\,j_\varepsilon \tag{4.11}$$

where $\mathbf{A}$ is the matrix of directional temperatures associated to the distribution $\mathcal{M}$:

$$\mathbf{A} = \int_{\mathbb{R}^3_v} (v\otimes v)\,\mathcal{M}(v)\,\mathrm{d}v = \frac{\Theta^\#}{m}\mathbf{Id} = \mathrm{diag}\left(\frac{\Theta^\#}{m}\,;\frac{\Theta^\#}{m}\,;\frac{\Theta^\#}{m}\right).$$

One may rewrite (4.11) as

$$\frac{\Theta^\#}{m}\nabla_x\varrho_\varepsilon(t,x) + \lambda_{0,1} j_\varepsilon(t,x) = -\varepsilon^2\,\partial_t j_\varepsilon(t,x) - \varepsilon\left(\int_{\mathbb{R}^3_v} (v\otimes v) : \nabla_x h_\varepsilon(t,x,v)\,\mathrm{d}v\right).$$

Choosing a test-function $\psi \in \mathscr{C}^\infty_c((0,T)\times\mathbb{R}^3_x)$, the above equation reads in its distributional form:

$$\begin{aligned}\frac{\Theta^\#}{m}\int_0^T \mathrm{d}t \int_{\mathbb{R}^3_x} \nabla_x\psi(t,x)\varrho_\varepsilon(t,x)\,\mathrm{d}x - \lambda_{0,1}\int_0^T \mathrm{d}t\int_{\mathbb{R}^3_x}\psi(t,x) j_\varepsilon(t,x)\,\mathrm{d}t = \\ -\varepsilon^2\int_0^T \mathrm{d}t\int_{\mathbb{R}^3_x}\partial_t\psi(t,x) j_\varepsilon(t,x)\,\mathrm{d}x - \varepsilon\int_{\Omega_T} h_\varepsilon(t,x,v)(v\otimes v) : \nabla_x\psi(t,x)\,\mathrm{d}\mu_T\end{aligned}$$

and, by virtue of the bounds in Prop. 4.1, the right-hand side converges to zero as $\varepsilon\to 0$ and one gets at the limit:

$$j(t,x) = -\frac{\Theta^\#}{m\lambda_{0,1}}\nabla_x\varrho(t,x) \tag{4.12}$$

in the distributional sense. *The above formula provides the so-called Fick's law for Maxwell's molecules*. One deduces the following Theorem:

Theorem 4.4. *Let $f_0 \in \mathcal{H}$ and, for any $\varepsilon > 0$, let $f_\varepsilon(t,x,v)$ denotes the solution to* (4.1). *Then, for any $T > 0$, up to a sequence, f_ε converges strongly in $L^2_{\mathrm{loc}}(\mathbf{\Omega}_T\,;\,\mathcal{M}^{-1}\,\mathrm{d}\mu_T)$ towards $\varrho(t,x)\mathcal{M}(v)$, where $\varrho(t,x)$ is the solution to the diffusion equation*

$$\partial_t\varrho = \nabla_x\cdot\left(\frac{\Theta^\#}{m\lambda_{0,1}}\nabla_x\varrho(t,x)\right), \qquad \varrho(t=0,x) = \int_{\mathbb{R}^3_v} f_0(x,v)\,\mathrm{d}v. \tag{4.13}$$

Proof. We already proved that f_ε converges weakly to $\varrho\mathcal{M}$ in $L^2((0,T)\,;\,\mathcal{H})$. To prove the strong convergence, since

$$\int_0^T \|f_\varepsilon(t) - \varrho_\varepsilon(t,x)\mathcal{M}\|^2_{\mathcal{H}} = \varepsilon^2\int_0^T \|h_\varepsilon(t)\|^2_{\mathcal{H}} \to 0$$

it suffices to prove that $\varrho_\varepsilon\mathcal{M}$ converges strongly to $\varrho\mathcal{M}$ in $\mathcal{H}$. This is equivalent to prove that ϱ_ε converges strongly to ϱ in $L^2(0,T\,;\,L^2_{\mathrm{loc}}(\mathbb{R}^3_x))$. This is done in the spirit of [14] and [12] by using a compensated-compactness argument. Precisely, let us define the following

vectors of $\mathbb{R}^3_x \times \mathbb{R}^+_t$: $\mathbf{U}_\varepsilon = (j_\varepsilon, \varrho_\varepsilon)$ and $\mathbf{V}_\varepsilon = (0, \varrho_\varepsilon)$. From (4.8), one sees that $\mathrm{div}_{x,t}\mathbf{U}_\varepsilon = 0$, in particular $(\mathrm{div}_{x,t}\mathbf{U}_\varepsilon)_\varepsilon$ is bounded in $L^2(\mathbb{R}^3_x \times \mathbb{R}^+_t)$. Now, from (4.11), one sees that $\mathbf{A} : \nabla_x \varrho_\varepsilon$ is a bounded family in $L^2((0,T)\times\mathbb{R}^3_x)$. Since $\mathbf{A} = \frac{\Theta^\#}{m}\mathbf{Id}$, it is clear that

$$\mathrm{curl}\,\mathbf{V}_\varepsilon = \begin{pmatrix} 0 & -{}^T\nabla_x\varrho_\varepsilon \\ \nabla_x\varrho_\varepsilon & 0 \end{pmatrix}$$

is bounded in $[L^2_{\mathrm{loc}}((0,T)\times\mathbb{R}^3_x)]^{4\times 4}$. Now, from the div-curl Lemma [17, 22], $\mathbf{U}_\varepsilon \cdot \mathbf{V}_\varepsilon = \varrho_\varepsilon^2$ converges to ϱ^2 in $\mathcal{D}'((0,T)\times\mathbb{R}^3_x)$.

Moreover, we already saw that ϱ_ε is bounded in $L^\infty(0,T\,;\,L^2(\mathbb{R}^3_x))$ from which we deduce the strong convergence of ϱ_ε to ϱ in $L^2((0,T)\,;\,L^2_{\mathrm{loc}}(\mathbb{R}^3_x))$. □

Remark 4.5. *As already pointed out in* [21], *the dependence of the diffusivity* $\mathsf{D}_{\max} := \Theta^\#/m\lambda_{0,1}$ *on the inelasticity parameter* β *shows that inelasticity tends to slow down the diffusive process.*

4.2. **Hard spheres.** When dealing with hard-spheres interactions, it appears difficult to obtain an explicit expression of the diffusion coefficient. Nevertheless, its existence can be deduced from Theorem 3.8. Indeed, a direct consequence of the Fredholm Alternative is the following:

Proposition 4.6. *For any* $i = 1,2,3$, *the equation*

$$\mathcal{L}_{\mathsf{hs}}(\chi_i) = v_i\mathcal{M}(v), \qquad\qquad v \in \mathbb{R}^3$$

has a unique solution $\chi_i \in L^2(\sigma(v)\mathcal{M}^{-1}(v)\,\mathrm{d}v)$, *such that* $\langle \chi_i, \mathcal{M}\rangle = \displaystyle\int_{\mathbb{R}^3_v} \chi_i(v)dv = 0$ *for any* $i = 1,2,3$.

Remark 4.7. *Note that the above Proposition holds true only because we assumed the bulk velocity* u_1 *to be zero*, i.e., $\int_{\mathbb{R}^3_v} v\mathcal{M}\,\mathrm{d}v = 0$. *If one deals with a non-zero bulk velocity* u_1, *then if one denotes* $a(v) = v - u_1$, χ_i *then solves* $\mathcal{L}_{\mathsf{hs}}(\chi_i) = a_i(v)\mathcal{M}(v)$ *(see also* (4.15)*), and moreover the limit diffusion equation includes in this case an additional drift term* $u_1 \cdot \nabla_x\rho$, *see Eq.* (4.2).

Then, setting $\chi = (\chi_1, \chi_2, \chi_2)$ one defines the diffusion matrix:

$$\mathbf{D} := -\int_{\mathbb{R}^3_v} v \otimes \chi(v)\,\mathrm{d}v \in \mathbb{R}^{3\times 3}.$$

Adapting the result of [20], the diffusion matrix is given by $\mathbf{D} = \mathrm{diag}(\mathsf{D}_{\mathsf{hs}}, \mathsf{D}_{\mathsf{hs}}, \mathsf{D}_{\mathsf{hs}})$ for some *positive constant* $\mathsf{D}_{\mathsf{hs}} > 0$, namely,

$$\mathsf{D}_{\mathsf{hs}} = -\int_{\mathbb{R}^3_v} v_1\chi_1(v)\,\mathrm{d}v = -\int_{\mathbb{R}^3_v} \mathcal{L}_{\mathsf{hs}}(\chi_1)\chi_1\mathcal{M}^{-1}\,\mathrm{d}v \geqslant \mu\|\chi_1\|^2_{L^2(\mathcal{M}^{-1})}.$$

Remark 4.8. *Note that, when dealing with Maxwell molecules, for any* $i = 1,2,3$, *the function* χ_i *appearing in Proposition 4.6 is given by* $\chi_i = -\frac{1}{\lambda_{0,1}}v_i\mathcal{M}$ *and we find again the expression of the diffusion matrix* $\mathbf{D} = \frac{\Theta^\#}{m\lambda_{0,1}}\mathbf{Id}$.

Recall that, for any $T > 0$, we defined $\mathbf{\Omega}_T = (0,T)\times\mathbb{R}^3_v\times\mathbb{R}^3_x$ and $\mathrm{d}\mu_T = \mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t$. Then, for any $\phi \in L^\infty(\mathbb{R}^3_v)$ and any $\psi \in \mathscr{C}^\infty_c((0,T)\times\mathbb{R}^3_x)$, multiplying Eq. (4.1) by $\phi(v)\psi(t,x)$

and integrating over $\mathbf{\Omega}_T$ one has

$$\int_{\mathbf{\Omega}_T} \varrho_\varepsilon(t,x)\mathcal{M}(v)\,(v\cdot\nabla_x\psi(t,x))\,\phi(v)\,\mathrm{d}\mu_T + \int_{\mathbf{\Omega}_T} \mathcal{L}_{\mathsf{hs}}(h_\varepsilon)\phi(v)\psi(t,x)\,\mathrm{d}\mu_T = \varepsilon\Big(\int_{\mathbf{\Omega}_T} \phi\, f_\varepsilon \partial_t\psi\ \mathrm{d}\mu_T + \int_{\mathbf{\Omega}_T} h_\varepsilon(v\cdot\nabla_x\psi)\phi\,\mathrm{d}\mu_T\Big). \quad (4.14)$$

In particular, by virtue of Propositions 4.1 and 4.3, one sees that

$$\lim_{\varepsilon\to 0}\Big(\int_{\mathbf{\Omega}_T} \mathcal{M}(v)\varrho_\varepsilon(t,x)\,(v\cdot\nabla_x\psi(t,x))\,\phi(v)\,\mathrm{d}\mu_T + \int_{\mathbf{\Omega}_T} \mathcal{L}_{\mathsf{hs}}(h_\varepsilon)\phi(v)\psi(t,x)\,\mathrm{d}\mu_T\Big) = 0.$$

Now, one deduces easily as in [12] that

$$\int_{\mathbf{\Omega}_T} \varrho(t,x)\,(v\cdot\nabla_x\psi(t,x))\,\mathcal{M}(v)\phi(v)\,\mathrm{d}\mu_T = -\int_{\mathbf{\Omega}_T} \mathcal{L}_{\mathsf{hs}}(h)\phi(v)\psi(t,x)\,\mathrm{d}\mu_T, \quad (4.15)$$

which means that, in the distributional sense,

$$\mathrm{div}_x(v\mathcal{M}(v)\varrho(t,x)) = \mathcal{L}_{\mathsf{hs}}(h), \qquad t>0, x\in\mathbb{R}^3_x.$$

Since h is of zero $\mathbb{R}^3_v$–average, Proposition 4.6 asserts that

$$h(t,x,v) = -\chi(v)\cdot\nabla_x\varrho(t,x)$$

and Proposition 4.3 *(iv)* leads to

$$j(t,x) = \int_{\mathbb{R}^3_v} vh(t,x,v)\,\mathrm{d}v = \mathbf{D}:\nabla_x\varrho(t,x).$$

We then obtain the following:

Theorem 4.9. *Let $0\leqslant f_0(x,v)\in L^2(\mathbb{R}^3_x\times\mathbb{R}^3_v, \mathcal{M}^{-1}\,\mathrm{d}v)$ be given and let f_ε be the associated sequence of solution to* (4.1) *where $Q = Q_{\mathsf{hs}}$. Then, up to a subsequence, f_ε converges strongly in $L^2_{\mathrm{loc}}(\mathbf{\Omega}_T, \mathcal{M}^{-1}\,\mathrm{d}\mu_T)$ to $\varrho(t,x)\mathcal{M}$ where $\varrho\geqslant 0$ is the solution to the parabolic diffusion equation* (4.2) *where the diffusion coefficient D_{hs} is given by*

$$\mathsf{D}_{\mathsf{hs}} := -\int_{\mathbb{R}^3_v} v_1\chi_1(v)\,\mathrm{d}v \in \mathbb{R}^{3\times 3}$$

with χ_1 defined in Prop. 4.6.

Proof. We already proved that f_ε converges weakly to $\varrho\mathcal{M}$ in $L^2((0,T),\mathcal{H})$ and the strategy to prove the strong convergence is that used in Theorem 4.4. Precisely, we define again $\mathbf{U}_\varepsilon = (j_\varepsilon,\varrho_\varepsilon)$ and $\mathbf{V}_\varepsilon = (0,\varrho_\varepsilon)$ and observes that again $(\mathrm{div}_{x,t}\mathbf{U}_\varepsilon)_\varepsilon$ is bounded in $L^2(\mathbb{R}^3_x\times\mathbb{R}^+_t)$. Now, from (4.14), with $\phi(v) = \dfrac{v}{|v|}$ and setting $\Gamma = \displaystyle\int_{\mathbb{R}^3_v}\frac{v\otimes v}{|v|}\mathcal{M}(v)\,\mathrm{d}v$, one sees that ϱ_ε satisfies:

$$\Gamma:\nabla_x\varrho_\varepsilon = \int_{\mathbb{R}^3_v}\mathcal{L}_{\mathsf{hs}}(h_\varepsilon)\frac{v}{|v|}\,\mathrm{d}v - \varepsilon\Big(\partial_t\int_{\mathbb{R}^3_v}\frac{v}{|v|}f_\varepsilon\,\mathrm{d}v + \mathrm{div}_x\Big[\int_{\mathbb{R}^3_v}\frac{v\otimes v}{|v|}h_\varepsilon\,\mathrm{d}v\Big]\Big)$$

so that $\Gamma:\nabla_x\varrho_\varepsilon$ lies in a bounded subset of $L^2_{\mathrm{loc}}((0,T)\times\mathbb{R}^3_x)$. Since Γ is invertible, one proceeds as in the proof of Theorem 4.4 that ϱ_ε converges strongly to ϱ in $L^2_{\mathrm{loc}}((0,T)\times\mathbb{R}^3_x)$. □

As we saw it, the diffusivity D_{hs} associated to hard-spheres interactions is not explicitly computable, the solution χ not being explicit. It is however possible to obtain a quantitative estimate of D_{hs} in terms of known quantities (*i.e.*, that do not involve χ_1):

Proposition 4.10. *One has the following estimate:*

$$\frac{\Theta^{\#}}{c_{\mathsf{hs}} m} \leqslant \mathsf{D}_{\mathsf{hs}} \leqslant \frac{\Theta^{\#}}{\lambda_{0,1} C^{\star} m}$$

where $C^{\star}$ is the constant provided by Prop. 3.3 and $c_{\mathsf{hs}} = -\dfrac{\langle \mathcal{L}_{\mathsf{hs}}(v_1\mathcal{M}), v_1\mathcal{M}\rangle}{\|v_1\mathcal{M}\|^2_{L^2(\mathcal{M}^{-1})}} > 0.$

Proof. We begin with the lower bound of D_{hs}. For any $s \in \mathbb{R}$, let

$$P(s) = \langle \mathcal{L}_{\mathsf{hs}}(\chi_1 + s v_1\mathcal{M}), \chi_1 + s v_1\mathcal{M}\rangle.$$

Since $\mathcal{L}_{\mathsf{hs}}$ is negative, one has $P(s) \leqslant 0$ for any $s \in \mathbb{R}$. Moreover,

$$\begin{aligned} P(s) &= \langle \mathcal{L}_{\mathsf{hs}}(\chi_1), \chi_1\rangle + 2s\langle \mathcal{L}_{\mathsf{hs}}(\chi_1), v_1\mathcal{M}\rangle + s^2\langle \mathcal{L}_{\mathsf{hs}}(v_1\mathcal{M}), v_1\mathcal{M}\rangle \\ &= -\mathsf{D}_{\mathsf{hs}} + 2s\|v_1\mathcal{M}\|^2_{L^2(\mathcal{M}^{-1})} + s^2\langle \mathcal{L}_{\mathsf{hs}}(v_1\mathcal{M}), v_1\mathcal{M}\rangle. \end{aligned}$$

We get therefore that

$$\mathsf{D}_{\mathsf{hs}} \geqslant 2s\|v_1\mathcal{M}\|^2_{L^2(\mathcal{M}^{-1})} + s^2\langle \mathcal{L}_{\mathsf{hs}}(v_1\mathcal{M}), v_1\mathcal{M}\rangle, \qquad \forall s \in \mathbb{R}.$$

With the definition of c_{hs} (note that $c_{\mathsf{hs}} > 0$ since $\mathcal{L}_{\mathsf{hs}}$ is negative and $v_1\mathcal{M} \perp \mathcal{M}$), we get

$$\mathsf{D}_{\mathsf{hs}} \geqslant \left(2s - c_{\mathsf{hs}} s^2\right)\|v_1\mathcal{M}\|^2_{L^2(\mathcal{M}^{-1})}, \qquad \forall s \in \mathbb{R}.$$

Optimizing with respect to s, one sees that

$$\mathsf{D}_{\mathsf{hs}} \geqslant \frac{1}{c_{\mathsf{hs}}}\|v_1\mathcal{M}\|^2_{L^2(\mathcal{M}^{-1})} = \frac{\Theta^{\#}}{c_{\mathsf{hs}} m}.$$

To get an upper bound for D_{hs}, we use the fact that, thanks to (3.2),

$$\mathsf{D}_{\mathsf{hs}} = -\langle \chi_1, v_1\mathcal{M}\rangle = \lambda_{0,1}^{-1}\langle \mathcal{L}_{\mathsf{max}}(\chi_1), v_1\mathcal{M}\rangle.$$

Now, as above, for any $s \in \mathbb{R}$, define $Q(s) = \langle \mathcal{L}_{\mathsf{max}}(s\chi_1 + v_1\mathcal{M}), s\chi_1 + v_1\mathcal{M}\rangle$. Here again, $Q(s) \leqslant 0$ for any $s \in \mathbb{R}$ and

$$\begin{aligned} Q(s) &= s^2\langle \mathcal{L}_{\mathsf{max}}(\chi_1), \chi_1\rangle + 2s\langle \mathcal{L}_{\mathsf{max}}(\chi_1), v_1\mathcal{M}\rangle + \langle \mathcal{L}_{\mathsf{max}}(v_1\mathcal{M}), v_1\mathcal{M}\rangle \\ &= s^2\langle \mathcal{L}_{\mathsf{max}}(\chi_1), \chi_1\rangle + 2\lambda_{0,1}\mathsf{D}_{\mathsf{hs}} s - \lambda_{0,1}\|v_1\mathcal{M}\|^2_{L^2(\mathcal{M}^{-1})}. \end{aligned}$$

Now, according to Prop. 3.3, $\langle \mathcal{L}_{\mathsf{max}}(\chi_1), \chi_1\rangle \geqslant \frac{1}{C^{\star}}\langle \mathcal{L}_{\mathsf{hs}}(\chi_1), \chi_1\rangle = -\mathsf{D}_{\mathsf{hs}}/C^{\star}$ so that

$$0 \geqslant Q(s) \geqslant -\frac{\mathsf{D}_{\mathsf{hs}}}{C^{\star}} s^2 + 2\lambda_{0,1}\mathsf{D}_{\mathsf{hs}} s - \lambda_{0,1}\|v_1\mathcal{M}\|^2_{L^2(\mathcal{M}^{-1})}, \qquad \forall s \in \mathbb{R}.$$

Optimizing the right-hand side with respect to $s \in \mathbb{R}$, we get

$$\lambda_{0,1}\|v_1\mathcal{M}\|^2_{L^2(\mathcal{M}^{-1})} \geqslant \lambda_{0,1}^2 C^{\star}\mathsf{D}_{\mathsf{hs}}$$

which gives the desired upper bound. □

Remark 4.11. *It is possible to provide some upper bound for c_{hs}. Namely, using the fact that there exists $\nu_1 > 0$ such that $\sigma(v) \leqslant \nu_1(1 + |v|)$, it is easy to see that*

$$c_{\mathsf{hs}} \leqslant \frac{1}{\|v_1\mathcal{M}\|^2_{L^2(\mathcal{M}^{-1})}} \int_{\mathbb{R}^3} \sigma(v)^2 v_1^2 \mathcal{M}(v)\,\mathrm{d}v \leqslant \frac{m\nu_1}{\Theta^{\#}} \int_{\mathbb{R}^3} (1 + |v|)^4 \mathcal{M}(v)\,\mathrm{d}v.$$

This very rough estimate could certainly be strengthened. Note also that the upper bound for D_{hs} reads as

$$\mathsf{D}_{\mathsf{hs}} \leqslant \frac{\tau(1-\alpha)(1-\beta)}{\alpha(1-\beta)(1-\alpha(1-\beta))} \frac{\sqrt{m_1\Theta_1}}{m}$$

where $\tau = \dfrac{\sqrt{5}}{\mathrm{erf}^{-1}(1/2)\,\sqrt{2}} \simeq 3.3154$ *is a numerical constant and we used the lower bound of* $C^\star$ *provided by Remark 3.6.*

Appendix A. We provide here a constructive proof of the coefficient ϱ_1 appearing in Proposition 3.3 with the aim of finding quantitative estimates for the coefficient $C^\star$ in Prop. 3.3. Namely, recalling that $\kappa = \alpha(1-\beta)$, one has

Lemma A.1. *Given* $\varrho_0 > 0$, *there exists* $\varrho_1 > 0$ *such that*

$$\int_{\{|\bar w-\bar v|\leqslant \varrho_1\}} \frac{\mathcal{M}_1(\bar w)}{(|\bar v-\bar w|^2+\xi^2)^{1/2}}\,\mathrm{d}\bar w \leqslant \frac12 \int_{\mathbb{R}^2} \frac{\mathcal{M}_1(\bar w)}{(|\bar v-\bar w|^2+\xi^2)^{1/2}}\,\mathrm{d}\bar w. \tag{A.1}$$

for any $\bar v \in \mathbb{R}^2, \xi \in [0, \varrho_0/2\kappa]$. *Moreover, setting* $\eta = \sqrt{\frac{2\Theta_1}{m_1}}\mathrm{erf}^{-1}(\frac12)$, *where* erf^{-1} *is the inverse error function, one has*

$$\varrho_1 \geqslant \left(\eta^2 - \frac{\varrho_0^2}{4\kappa^2}\right)^{1/2}, \qquad \forall 0 < \varrho_0 < 2\kappa\eta.$$

Proof. We assume without loss of generality that $u_1 = 0$. Let $\varrho_0 > 0$ be given. Let us fix $\bar v = (\bar v_1, \bar v_2) \in \mathbb{R}^2$ and $\xi \in [0, \varrho_0/2\kappa]$. Using polar coordinates, it is clear that

$$\left(\frac{m_1}{2\pi\Theta_1}\right)^{-3/2} \int_{\{|\bar w-\bar v|\leqslant\varrho_1\}} \frac{\mathcal{M}_1(\bar w)}{(|\bar v-\bar w|^2+\xi^2)^{1/2}}\,\mathrm{d}\bar w =$$
$$\exp(-a|\bar v|^2)\int_0^{\varrho_1} \frac{r\exp(-ar^2)}{\sqrt{r^2+\xi^2}}\,\mathrm{d}r \int_0^{2\pi} \exp\Big(-2ar(v_1\cos\theta + v_2\sin\theta)\Big)\,\mathrm{d}\theta$$

where $a = m_1/(2\Theta_1)$. Therefore, a sufficient condition (independent of $\bar v$) for (A.1) to hold is that

$$\int_0^{\varrho_1} \frac{r\exp(-ar^2)}{\sqrt{r^2+\xi^2}}\,\mathrm{d}r \leqslant \frac12 \int_0^{\infty} \frac{r\exp(-ar^2)}{\sqrt{r^2+\xi^2}}\,\mathrm{d}r, \qquad \forall \xi \in [0, \varrho_0/2\kappa].$$

It is not difficult to see that this is equivalent to

$$\mathrm{erf}\Big(\sqrt{a(\varrho_1^2+\xi^2)}\Big) - \mathrm{erf}(\sqrt{a}\xi) \leqslant \frac12 - \frac12\mathrm{erf}(\sqrt{a}\xi), \qquad \forall \xi \in [0, \varrho_0/2\kappa]$$

where erf is the error function $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\displaystyle\int_0^x \exp(-t^2)\,\mathrm{d}t$, $x \geqslant 0$. This allows to define a function:

$$z:\ \xi \in \mathbb{R}_+ \mapsto z(\xi)$$

where $z(\xi)$ is the nonnegative solution to the identity

$$\sqrt{a(z^2+\xi^2)} = \mathrm{erf}^{-1}\left(\frac12 + \frac12\mathrm{erf}(\sqrt{a}\xi)\right) \tag{A.2}$$

where erf^{-1} is the inverse error function. Clearly the Lemma is proven provided

$$\varrho_1 := \min\{z(\xi),\ \xi \in [0, \varrho_0/2\kappa]\} > 0.$$

Note that, according to (A.2), the function $z(\cdot)$ is continuously differentiable and there is some $\zeta \in [0, \varrho_0/2\kappa]$ such that $\min\{z(\xi),\ \xi \in [0, \varrho_0/2\kappa]\} = z(\zeta)$. In particular $z'(\zeta) = 0$ and one checks, thanks to (A.2), that

$$z'(\xi) = \frac12\sqrt{z^2+\xi^2}\exp(az^2) - \xi, \qquad \forall \xi \geqslant 0.$$

In particular, $z'(\zeta) = 0$ is equivalent to

$$4\zeta^2 \exp(-az^2(\zeta)) = z^2(\zeta) + \zeta^2, \tag{A.3}$$

and one sees that $z^2(\zeta) = 0$ should imply $\zeta = 0$ whereas, according to (A.2), $z(0) \neq 0$. Consequently, all the local extrema of z are positive. Therefore,

$$\varrho_1 = z(\zeta) = \min\{z(\xi),\ 0 \leqslant \xi \leqslant \varrho_0/2\kappa\} > 0$$

which achieves to prove that (A.1) holds true for some $\varrho_1 > 0$. It remains now to provide some estimate for ϱ_1. Precisely, defining

$$\eta = \frac{1}{\sqrt{a}} \mathrm{erf}^{-1}\left(\frac{1}{2}\right),$$

we see from (A.2) that $z^2(\xi) + \xi^2 \geqslant \eta^2$, for any $\xi \geqslant 0$, so that $\varrho_1^2 \geqslant \eta^2 - \frac{\varrho_0^2}{4\kappa^2}$ for any $\varrho_0 \in (0, 2\kappa\eta)$, which achieves to prove the lemma. □

Remark A.1. *According to the above Lemma, with the choice of* $\varrho_0 = \frac{2\kappa\eta}{\sqrt{5}}$, *one obtains that* $\min\left(\frac{\varrho_0}{2\kappa}, \frac{\varrho_1}{2}\right) \geqslant \frac{\eta}{\sqrt{5}}$.

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