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# Analysis of a Scalar Conservation Law with Space Discontinuous Advection Function in a Bounded Domain

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## Abstract

We deal with the scalar conservation law in a one dimensional bounded domain  $\Omega$ :  $\partial_t u + \partial_x(k(x)g(u)) = 0$ , associated with a bounded initial value  $u_0$ . The function  $k$  is supposed to be bounded, discontinuous at  $\{x_0 = 0\}$ , and with bounded variation. A weak entropy formulation for the Cauchy problem has been introduced by J.D Towers in [11]. In [10] the existence and the uniqueness is proved by N. Seguin and J. Vovelle through a regularization of the function  $k$ . We generalize the definition of J.D Towers and we adapt the method developed in [10] to establish an existence and uniqueness property in the case of the homogeneous Dirichlet boundary conditions.

## 1 Introduction

We are interested in the existence and uniqueness properties for a scalar conservation law made of an hyperbolic first-order quasilinear equation set in a one-dimensional bounded domain  $\Omega$ , and for any positive finite real  $T$ , that can be formally described:

Find a bounded measurable function  $u$  on  $Q = ]0, T[ \times \Omega$  such that

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(k(x)g(u)) = 0 & \text{in } Q = ]0, T[ \times \Omega, \\ u(0, x) = u_0(x) & \text{on } \Omega, \\ u = 0 & \text{on (a part of) } ]0, T[ \times \partial\Omega, \end{cases} \quad (1)$$

where  $k$  is a discontinuous function at a point  $x_0$  of  $\Omega$ .

Such an equation arises in the modelling of continuous sedimentation of solid particles in a liquid ([3]) or when one considers a two-phase flow in an heterogeneous porous medium without capillarity effects ([5], [4]).

By normalization, we suppose that  $\Omega = ]-1, 1[$ .

The initial condition  $u_0$  belongs to  $L^\infty(\Omega)$  and takes values in  $[m, M]$  where  $m$  and  $M$  are two fixed reals,  $m < M$ .

The flux function  $g$  is Lipschitzian on  $\mathbb{R}$ . We suppose also that:

$$g \text{ changes no more than once its monotony} \quad (2)$$

and satisfies a nondegeneracy condition in the sense of A. Vasseur [12], that is to say:

$$\forall \alpha \in \mathbb{R}, \mathcal{L}\{\lambda \in \mathbb{R}, g'(\lambda) = \alpha\} = 0. \quad (3)$$

where  $\mathcal{L}$  denotes the Lebesgue measure.

The function  $k$  is discontinuous at  $x_0 = 0$  and  $k|_{[-1,0]}$  is an element of  $W^{1,+\infty}(-1,0]$  while  $k|_{]0,1]}$  belongs to  $W^{1,+\infty}(]0,1])$ .

Thus, thanks to a Cauchy criterion, we can define:

$$k_L = \lim_{x \rightarrow 0^-} k(x) \quad \text{and} \quad k_R = \lim_{x \rightarrow 0^+} k(x).$$

Eventually, we suppose that:

$$\mathcal{L}\{x \in \Omega, k(x) = 0\} = 0. \quad (4)$$

The mathematical formulation for (1) is given in Section 2 through an entropy inequality on the whole  $Q$ , using the classical Kruzkov entropy pairs (see [7]) and involving a term that takes into account the jump of  $k$  along  $\{x_0 = 0\}$ . As soon as we are able to transcript in Section 3 the transmission conditions along the interface included in Definition 1, we are able to state, in Section 4, the uniqueness. To do so strong traces for  $u$  along the interface  $\{x_0 = 0\}$  will be needed. Finally Section 5 is devoted to the existence property for (1) through a suitable regularization of the function  $k$ .

## 2 Definition of an entropy solution

We propose a definition extending that of J.D. Towers ([11]) - also used by N. Seguin and J. Vovelle ([10]) or F. Bachmann ([1]) - to the case where  $k$  depends on the space variable and for the homogeneous Dirichlet problem in a bounded interval of  $\mathbb{R}$ . So we say that:

**Definition 1.** *A function  $u$  of  $L^\infty(Q)$  is an entropy solution to problem 1 if:*  
*(i)  $\forall \kappa \in \mathbb{R}, \forall \varphi \in \mathcal{C}_c^\infty([0, T] \times \Omega), \varphi \geq 0,$*

$$\left\{ \begin{array}{l} \int_Q (|u(t, x) - \kappa| \varphi_t(t, x) + k(x) \Phi(u, \kappa) \varphi_x(t, x)) dx dt \\ \quad - \int_Q k'(x) \text{sgn}(u - \kappa) g(\kappa) \varphi dx dt \\ + \int_\Omega |u_0 - \kappa| \varphi(0, x) dx + |(k_L - k_R) g(\kappa)| \int_0^T \varphi(t, 0) dt \geq 0, \end{array} \right. \quad (5)$$

where

$$\Phi(u, \kappa) = \text{sgn}(u - \kappa)(g(u) - g(\kappa)),$$

(ii) for a.e.  $t$  in  $]0, T[$ , for any real  $\kappa$ ,

$$k(1)(\text{sgn}(u_1^T(t) - \kappa) + \text{sgn}(\kappa))(g(u_1^T(t)) - g(\kappa)) \geq 0, \quad (6)$$

$$k(-1)(\text{sgn}(u_{-1}^T(t) - \kappa) + \text{sgn}(\kappa))(g(u_{-1}^T(t)) - g(\kappa)) \leq 0. \quad (7)$$

In this definition  $u_1^\tau$  and  $u_{-1}^\tau$  denote the traces of  $u$  respectively in  $(+1)^-$  and  $(-1)^+$  in the sense of A. Vasseur [12] (see also Y. Panov [9]). Indeed it follows from [12],

**Lemma 1.** *Let  $u$  be an entropy solution to (1). If for each  $(\alpha, \beta) \neq (0, 0)$ , for a.e.  $x \in [-1, 1]$ ,  $\mathcal{L}(\{\lambda \mid \alpha + \beta.k(x)g'(\lambda) = 0\}) = 0$ , there exists two functions  $u_{\pm 1}^\tau$  in  $L^\infty(]0, T[)$  such as, for every compact set  $K$  of  $]0, T[$ ,*

$$\operatorname{ess\,lim}_{x \rightarrow \pm 1} \int_K |u(t, x) - u_{\pm 1}^\tau(t)| dt = 0. \quad (8)$$

In [9], Y. Panov proved the existence of these strong traces with a continuous flux function, when the boundary is not a characteristic hypersurface. The latter condition is satisfied here under (3) and (4), when we consider the problem (1) separately on  $] - 1, 0[$  and on  $]0, 1[$ .

**Remark 1.** *Of course, the statement of Lemma 1 also ensures the existence of strong traces for  $u$ ,  $\gamma u^+$  and  $\gamma u^-$  in  $L^\infty(]0, T[)$  along  $\{x_0 = 0\}$  respectively at right and at left.*

### 3 Conditions at the interface $\{x_0 = 0\}$

Let us establish that the previous definition ensures the uniqueness. The proof is based on that proposed in [10] and relies essentially on the transmission conditions along  $\{x_0 = 0\}$  underlying to entropy inequality (5). Indeed the existence of strong traces for  $u$  permits us to state first:

**Lemma 2.** *Let  $u$  in  $L^\infty(Q)$  be an entropy solution to (1). So, for a.e.  $t$  in  $]0, T[$ , for all real  $\kappa$ ,*

$$k_L \Phi(\gamma u^-(t), \kappa) - k_R \Phi(\gamma u^+(t), \kappa) + |(k_L - k_R)g(\kappa)| \geq 0. \quad (9)$$

*Proof.* Let  $\varphi$  be a nonnegative element of  $\mathcal{C}_c^\infty(Q)$ . We refer to the cut-off function on  $\mathbb{R}$ ,  $\omega_\varepsilon$ , for  $\varepsilon > 0$ , introduced in [10]:

$$\omega_\varepsilon(x) = \begin{cases} 0 & \text{if } 2\varepsilon < |x| \\ \frac{-|x| + 2\varepsilon}{\varepsilon} & \text{if } \varepsilon \leq |x| \leq 2\varepsilon \\ 1 & \text{if } |x| < \varepsilon \end{cases} .$$

such that  $\omega_\varepsilon(x) \rightarrow 0$  if  $x \neq 0$ , and  $\omega_\varepsilon(0) = 1$  for all  $\varepsilon$ .

Thanks to a density argument we may choose  $\varphi \omega_\varepsilon$  as test-function in (5). We pass to the limit when  $\varepsilon$  goes to  $0^+$  by using the Lebesgue dominated convergence Theorem providing that all the terms tend to 0 except  $|k_L - k_R|g(\kappa) \int_0^T \varphi(t, 0) dt$  (which does not depend on  $\varepsilon$ ) and:

$$I_\varepsilon = \int_Q k(x) \Phi(u, \kappa) \varphi \omega'_\varepsilon dx dt .$$

By definition of  $\omega_\varepsilon$ ,

$$I_\varepsilon = \int_0^T \frac{1}{\varepsilon} \int_{-2\varepsilon}^{-\varepsilon} k(x) \Phi(u, \kappa) \varphi dx dt + \int_0^T -\frac{1}{\varepsilon} \int_\varepsilon^{2\varepsilon} k(x) \Phi(u, \kappa) \varphi dx dt,$$

and, by setting  $L_\varepsilon = \int_0^T \frac{1}{\varepsilon} \int_{-2\varepsilon}^{-\varepsilon} |k(x)(\Phi(u, \kappa)\varphi(t, x) - k_L\Phi(\gamma u^-, \kappa)\varphi(t, 0))| dt dx$ ,

we prove that  $\lim_{\varepsilon \rightarrow 0^+} L_\varepsilon = 0$  because  $\Phi(\cdot, \kappa)$  is Lipschitzian on  $[0, 1]$ , and due to the definition of  $k_L$  and  $\gamma u^-$ . As a consequence, we obtain (9).  $\square$

As in [10], a Rankine-Hugoniot condition may be deduced from (9). To do so we need an additional hypothesis on the function  $g$ . So we suppose that:

$$\begin{cases} \exists \kappa_1 \in \mathbb{R}, \kappa_1 \geq \text{ess sup } u, & g(\kappa_1)(k_L - k_R) \leq 0 \\ \exists \kappa_2 \in \mathbb{R}, \kappa_2 \leq \text{ess inf } u, & g(\kappa_2)(k_L - k_R) \geq 0 \end{cases} \quad (10)$$

**Lemma 3.** *Under (10), for a.e.  $t$  in  $]0, T[$ , the following Rankine-Hugoniot condition holds:*

$$k_L g(\gamma u^-(t)) = k_R g(\gamma u^+(t)) . \quad (11)$$

*Proof.* We choose  $\kappa = \kappa_1$  in (9) to obtain:

$$k_R g(\gamma u^+) - k_L g(\gamma u^-) + g(\kappa_1)(k_L - k_R) + |g(\kappa_1)(k_L - k_R)| \geq 0$$

From (10), we deduce that  $k_R g(\gamma u^+) \geq k_L g(\gamma u^-)$ .

By choosing  $\kappa = \kappa_2$  in (9), and using (10), we obtain the reverse inequality.  $\square$

## 4 The uniqueness theorem

First we recall that

**Lemma 4.** *If a bounded mapping  $u$  satisfies (5), then:*

$$\text{esslim}_{t \rightarrow 0^+} \int_{\Omega} |u(t, x) - u_0(x)| dx = 0. \quad (12)$$

We are now able to state an uniqueness property for (1) through a  $T$ -Lipschitzian dependence in  $L^1(Q)$  of a weak entropy solution with respect to corresponding initial data.

**Theorem 1.** *Let  $u$  and  $v$  be two entropy solutions to (1) for initial conditions  $(u_0, v_0)$  in  $(L^\infty(]-1, 1[))^2$ . Then, under (10):*

$$\int_0^T \int_{-1}^1 |u(t, x) - v(t, x)| dx dt \leq T \int_{-1}^1 |u_0(x) - v_0(x)| dx. \quad (13)$$

*Proof.* We use the method of doubling variables due to S. N. Kruzkov (see [7]) by reasoning in two steps: we consider first some test-functions vanishing on a vicinity of  $\{x_0 = 0\}$ . That provides a Kruzkov-type inequality between two entropy solutions from which one the former vanishing hypothesis is released by using (11).

**Lemma 5.** *Let  $u$  and  $v$  be two entropy solutions in  $L^\infty(Q)$  to (1) associated with initial conditions  $u_0$  and  $v_0$  in  $L^\infty(]-1, 1[)$ . For any nonnegative function  $\varphi$  in  $C_c^\infty([0, T[ \times \Omega)$ , vanishing in a neighborhood of  $\{x_0 = 0\}$ ,*

$$\begin{aligned} & \int_Q (|u(t, x) - v(t, x)| \varphi_t(t, x) + k(x)\Phi(u(t, x), v(t, x))\varphi_x(t, x)) dx dt \\ & + \int_{\Omega} |u_0(x) - v_0(x)| \varphi(0, x) dx \geq 0. \end{aligned} \quad (14)$$

*Proof.* Let  $(\rho_j)_{j \in \mathbb{N}^*}$  be a classical sequence of mollifiers in  $\mathbb{R}$ , such that  $\rho_j(x) = \rho_j(-x)$ ,  $\varphi$  an element of  $C_c^\infty([0, T] \times \Omega)$  satisfying the hypotheses of Lemma 5. For  $j \in \mathbb{N}^*$  and  $(t, x, s, y) \in Q \times Q$ , we set:

$$\psi_j(t, x, s, y) = \varphi\left(\frac{t+s}{2}, \frac{x+y}{2}\right) \rho_j(t-s) \rho_j(x-y).$$

To simplify, we denote  $w = \frac{t+s}{2}$ ,  $z = \frac{x+y}{2}$ ,  $u = u(t, x)$ ,  $v = v(t, x)$ ,  $\tilde{v} = v(s, y)$ ,  $q = (t, x)$ ,  $\tilde{q} = (s, y)$ . By choosing  $\kappa = \tilde{v}$  in (5) for  $u$  (respectively  $\kappa = u$  in (5) for  $\tilde{v}$ ) against the test-function  $\psi_j$  and integrating over  $Q$  with respect to  $\tilde{q}$  (respectively  $q$ ), it comes:

$$\begin{aligned} & \int_{Q \times Q} |u - \tilde{v}| \varphi_t(w, z) \rho_j(t-s) \rho_j(x-y) dq d\tilde{q} \\ & - \int_{Q \times Q} \text{sgn}(u - \tilde{v}) (k'(x)g(\tilde{v}) - k'(y)g(u)) \psi_j dq d\tilde{q} \\ & + 2 \int_{\Omega \times \Omega} |u_0(x) - v_0(y)| \varphi\left(\frac{t}{2}, z\right) \rho_j(x-y) \rho_j(t) dq dy \\ & + \int_{Q \times \Omega} (|u - u_0| + |\tilde{v} - \tilde{v}_0|) \varphi\left(\frac{t}{2}, z\right) \rho_j(x-y) \rho_j(t) dq dy \\ & + \int_{Q \times Q} \Phi(u, \tilde{v}) k(x) (\partial_x \varphi)(w, z) \rho_j(t-s) \rho_j(x-y) dq d\tilde{q} \\ & + \int_{Q \times Q} \Phi(u, \tilde{v}) (k(y) - k(x)) \partial_y \psi_j(q, \tilde{q}) dq d\tilde{q} \geq 0. \end{aligned} \tag{15}$$

We will just focus on the second and the sixth line. Indeed there is no difficulty to pass to the limit when  $j$  goes to  $+\infty$  in the other lines by referring to the notion of Lebesgue points for an integrable function on  $Q$  (and by using (12) for the fourth line). Let's study first the sixth line, denoted  $I_j$ . Coming back to the definition of  $\psi_j$  yields

$$I_j = I_{1,j} + I_{2,j},$$

where:

$$\begin{aligned} I_{1,j} &= \int_{Q \times Q} \Phi(u, \tilde{v}) (k(y) - k(x)) \partial_y (\varphi(w, z)) \rho_j(t-s) \rho_j(x-y) dq d\tilde{q}, \\ I_{2,j} &= \int_{Q \times Q} \Phi(u, \tilde{v}) (k(y) - k(x)) \varphi(w, z) \rho_j(t-s) \partial_y (\rho_j(x-y)) dq d\tilde{q}. \end{aligned}$$

By using the notion of Lebesgue points, we state that

$$\lim_{j \rightarrow +\infty} I_{1,j} = 0.$$

Next we write  $I_{2,j} = I_a + I_b$  with:

$$I_a = \int_{Q \times Q} \{\Phi(u, \tilde{v}) - \Phi(u, v)\} (k(y) - k(x)) \varphi(w, z) \rho_j(t-s) \partial_y (\rho_j(x-y)) dq d\tilde{q}$$

and,

$$I_b = \int_{Q \times Q} \Phi(u, v) (k(y) - k(x)) \varphi(w, z) \rho_j(t-s) \partial_y (\rho_j(x-y)) dq d\tilde{q}.$$

Let us first consider  $I_b$ . We denote

$$\begin{aligned} T(q, \tilde{q}) &= \Phi(u, v)(k(y) - k(x))\varphi(w, z)\rho_j(t-s)\partial_y(\rho_j(x-y)), \\ Q_- &= ]0, T[\times]-1, 0[ \text{ and } Q_+ = ]0, T[\times]0, 1[, \\ I_{b,1} &= \int_{Q_- \times Q_-} T(q, \tilde{q})dq d\tilde{q}, \quad I_{b,2} = \int_{Q_- \times Q_+} T(q, \tilde{q})dq d\tilde{q} \\ I_{b,3} &= \int_{Q_+ \times Q_-} T(q, \tilde{q})dq d\tilde{q} \text{ and } I_{b,4} = \int_{Q_+ \times Q_+} T(q, \tilde{q})dq d\tilde{q}. \end{aligned}$$

Then,  $I_b = I_{b,1} + I_{b,2} + I_{b,3} + I_{b,4}$ , so we just need to study  $I_{b,1}$  and  $I_{b,2}$ , the arguments for  $I_{b,3}$  and  $I_{b,4}$  being similar. We integrate by parts  $I_{b,1}$  with respect to  $y$  to obtain:

$$\begin{aligned} I_{b,1} &= - \int_{Q_- \times Q_-} \Phi(u, v)k'(y)\varphi(w, z)\rho_j(t-s)\rho_j(s-y)dq d\tilde{q} \\ &\quad - \frac{1}{2} \int_{Q_- \times Q_-} \Phi(u, v)(k(y) - k(x))\varphi_y(w, z)\rho_j(t-s)\rho_j(s-y)dq d\tilde{q} \\ &\quad + \int_{Q_-} \int_0^T \Phi(u, v)(k_L - k(x))\varphi(w, \frac{x}{2})\rho_j(t-s)\rho_j(x)dq ds \end{aligned}$$

When  $j$  goes to  $+\infty$ , the two last terms tend to 0, owing to the continuity of  $k$  on  $] -1, 0[$  and to the definition of  $k_L$ . Moreover, since  $k|_{[-1, 0[}$  belongs to  $W^{1,+\infty}([-1, 0[)$  and  $\varphi$  is continuous, the first term tends to:

$$- \int_{Q_-} \Phi(u(t, x), v(t, x))k'(x)\varphi(t, x)dq.$$

Similarly,  $\lim_{j \rightarrow +\infty} I_{b,4} = - \int_{Q_+} \Phi(u(t, x), v(t, x))k'(x)\varphi(t, x)dq.$

By definition of  $\rho_j$ ,  $I_{b,2}$  is equal to:

$$\int_0^T \int_{-\frac{1}{j}}^0 \int_0^T \int_0^{\frac{1}{j}} \Phi(u, v)(k(y) - k(x))\varphi(w, z)\rho_j(t-s)\partial_y(\rho_j(x-y))dq d\tilde{q}$$

As  $\varphi$  vanishes on a neighborhood of  $\{x_0 = 0\}$ , from a certain  $j_0$ ,  $I_{b,2}$  vanishes and it is the same for  $I_{b,3}$ . Eventually:

$$\lim_{j \rightarrow +\infty} I_b = - \int_Q \Phi(u, v)k'(x)\varphi(t, x)dq$$

We study now  $I_a$ . By using the same decomposition as for  $I_b$ , it appears four integrals whose two vanish (because  $\varphi$  vanishes on a vicinity of  $\{x_0 = 0\}$ ) and it only leads to consider the term, denoting by  $I_{a,1}$ :

$$\int_{Q_- \times Q_-} \{\Phi(u, \tilde{v}) - \Phi(u, v)\} (k(y) - k(x))\varphi(w, z)\rho_j(t-s)\partial_y(\rho_j(x-y))dq d\tilde{q}$$

By using the Lipschitz condition for  $\phi$  and  $k$ , we highlight a nonnegative constant  $C_1$  independent from  $j$ , such that:

$$|I_{a,1}| \leq C_1 \int_{Q_- \times Q_-} |v(s, y) - v(t, x)| |x - y| \rho_j(t - s) |\partial_y(\rho_j(x - y))| dq d\tilde{q}$$

This way, due to the definition of  $\rho_j$ , there exists a nonnegative constant  $C_2$  such that:

$$|I_{a,1}| \leq C_2 j^2 \int_{\{|t-s| \leq \frac{1}{j}, |x-y| \leq \frac{1}{j}\}} |v(t, x) - v(s, y)| dq d\tilde{q},$$

so that,  $\lim_{j \rightarrow +\infty} I_{a,1} = 0$ , and as a consequence

$$\lim_{j \rightarrow +\infty} I_a = 0.$$

To sum up:

$$\lim_{j \rightarrow +\infty} I_j = \lim_{j \rightarrow +\infty} (I_{1,j} + I_a + I_b) = - \int_Q \Phi(u(t, x), v(t, x)) k'(x) \varphi(t, x) dq.$$

We study now the  $j$ -limit of the second line in (15) that is:

$$L_j = - \int_{Q \times Q} \text{sgn}(u - \tilde{v})(k'(x)g(\tilde{v}) - k'(y)g(u)) \psi_j dq d\tilde{q}.$$

We write

$$L_j = L_{1,j} - L_{2,j}$$

with

$$L_{1,j} = \int_{Q \times Q} k'(x) \Phi(u, \tilde{v}) \psi_j dq d\tilde{q} \text{ and}$$

and

$$L_{2,j} = \int_{Q \times Q} g(u) \text{sgn}(u - \tilde{v})(k'(x) - k'(y)) \psi_j dq d\tilde{q}.$$

On the one hand, it is clear that:

$$\lim_{j \rightarrow +\infty} L_{1,j} = \int_Q k'(x) \Phi(u(t, x), v(t, x)) \varphi(t, x) dq.$$

On the other hand, as for the study of  $I_a$  and  $I_b$  we share  $L_{2,j}$  into four terms whose two vanishes ( $\varphi$  vanishing on a neighborhood of  $\{x_0 = 0\}$ ) so that we only consider:

$$L_{2,a} = \int_{Q_- \times Q_-} g(u(t, x)) \text{sgn}(u(t, x) - v(s, y)) (k'(x) - k'(y)) \psi_j dq d\tilde{q}$$

and,

$$L_{2,b} = \int_{Q_+ \times Q_+} g(u(t, x)) \text{sgn}(u(t, x) - v(s, y)) (k'(x) - k'(y)) \psi_j dq d\tilde{q}.$$

We observe that:

$$|L_{2,a}| \leq C \|g\|_\infty \|\varphi\|_\infty \int_{\Omega^- \times \Omega^-} |k'(x) - k'(y)| \rho_j(x - y) dx dy,$$

where  $\Omega^- = ] - 1, 0[$ .

So that, since  $k'$  belongs to  $L^\infty([-1, 0])$ ,  $\lim_{j \rightarrow +\infty} L_{2,a} = 0$  and it is the same for  $L_{2,b}$ .

To summarize,  $\lim_{j \rightarrow +\infty} L_j = \int_Q k'(x) \Phi(u(t, x), v(t, x)) \varphi(t, x) dq$ , and (14) follows that completes the proof of Lemma 5.  $\square$



Now we state that:

**Lemma 6.** *Under (10), the Kruzkov inequality (14) still holds for  $\varphi$  in  $\mathcal{C}_c^\infty([0, T[ \times \Omega)$ ,  $\varphi \geq 0$ .*

*Proof.* Thanks to a density argument we can choose in (14) the test function  $\varphi(1 - \omega_\varepsilon)$  where  $\omega_\varepsilon$  is defined in the proof of Lemma 2. By taking the  $\varepsilon$ -limit, it comes:

$$\int_Q (|u - v| \varphi_t + k(x) \Phi(u, v) \varphi_x) dx dt + \int_\Omega |u_0 - v_0| \varphi(0, x) dx \geq J,$$

with:

$$J = \int_0^T (k_L \Phi(\gamma u^-, \gamma v^-) - k_R \Phi(\gamma u^+, \gamma v^+)) \varphi(t, 0) dt.$$

Inequality (9) shows that  $J$  is nonnegative. Indeed let us study, for *a.e.*  $t$  of  $]0, T[$ , the sign of:

$$I = k_L \Phi(\gamma u^-, \gamma v^-) - k_R \Phi(\gamma u^+, \gamma v^+).$$

We just focus on the case when  $\gamma u^+ - \gamma v^+$  and  $\gamma u^- - \gamma v^-$  have an opposite sign. Otherwise due to (11), that is satisfied because of (10),  $I = 0$ . When  $\text{sgn}(\gamma u^+ - \gamma v^+) = -\text{sgn}(\gamma u^- - \gamma v^-) \neq 0$ , by using (11), we have:

$$I = 2k_L \Phi(\gamma u^-, \gamma v^-) = -2k_R \Phi(\gamma u^+, \gamma v^+)$$

We suppose that  $k_L - k_R > 0$ ,  $\gamma v^+ < \gamma u^+$  and  $\gamma u^- < \gamma v^-$ , the study of the other cases being similar. So,

$$I = -2k_L(g(\gamma u^-) - g(\gamma v^-)) = -2k_R(g(\gamma u^+) - g(\gamma v^+))$$

Here we must consider some different situations.

1.  $\gamma u^- < \gamma v^- < \gamma v^+ < \gamma u^+$ .

Then (9) can be written, for any  $\kappa$  of  $[\gamma u^-, \gamma u^+]$ ,

$$-k_L(g(\gamma u^-) - g(\kappa)) - k_R(g(\gamma u^+) - g(\kappa)) + (k_L - k_R)|g(\kappa)| \geq 0. \quad (16)$$

- if  $g(\gamma v^-) \geq 0$ , by choosing  $\kappa = \gamma v^-$  in (16), we have:

$$-2k_L(g(\gamma u^-) - g(\gamma v^-)) \geq 0, \text{ so } I \geq 0.$$

- if  $g(\gamma v^+) \leq 0$ , by choosing  $\kappa = \gamma v^+$  in (16), we obtain:

$$-2k_R(g(\gamma u^+) - g(\gamma v^+)) \geq 0, \text{ so } I \geq 0.$$

- if  $g(\gamma v^-) < 0$  and  $g(\gamma v^+) > 0$ , we deduce from (11), as  $k_L - k_R > 0$ , that  $k_R < 0$  and  $k_L > 0$ . If we suppose that  $I = -2k_L(g(\gamma u^-) - g(\gamma v^-)) < 0$ , then  $g(\gamma u^-) > g(\gamma v^-)$ . But we also have  $I = -2k_R(g(\gamma u^+) - g(\gamma v^+))$ , that implies that  $g(\gamma u^+) < g(\gamma v^+)$ . Consequently  $g$  changes at least twice its monotony that contradicts the assumption (2).

2.  $\gamma u^- < \gamma v^+ < \gamma v^- < \gamma u^+$ .

As in the previous case, if  $g(\gamma v^-) \geq 0$  or  $g(\gamma v^+) \leq 0$ , we have  $I \geq 0$ . If  $g(\gamma v^-) < 0$  and  $g(\gamma v^+) > 0$ , there exists  $\alpha$  in  $] \gamma v^+, \gamma v^- [$  such that  $g(\alpha) = 0$ . Choosing  $\kappa = \alpha$  in (9) written successively for  $u$  and  $v$  yields to  $-2k_R g(\gamma u^+) \geq 0$  and  $2k_R g(\gamma v^+) \geq 0$ . Then  $I \geq 0$ .

3.  $\gamma u^- < \gamma v^+ < \gamma u^+ < \gamma v^-$ .

- if  $g(\gamma v^+) \geq 0$  or  $g(\gamma u^+) \leq 0$ , from (9),  $I \geq 0$ .
- if  $g(\gamma v^+) < 0$  and  $g(\gamma u^+) > 0$ , there exists  $\beta$  in  $] \gamma v^+, \gamma u^+ [$ , such that  $g(\beta) = 0$ . By choosing  $\kappa = \beta$  in (9), written for  $u$  and  $v$ , we show that  $I \geq 0$ .

All the others cases may be reduced to one of the previous situations.  $\square$

Now, in order to prove (13), thanks to Lemma 6 we choose in (14) the test-function  $\varphi$  such that, for any  $(t, x)$  in  $[0, T[ \times \Omega$ ,

$$\varphi(t, x) = \theta(t) \alpha_\varepsilon(x), \quad \varepsilon > 0$$

where  $\theta \in \mathcal{C}_c^\infty([0, T[)$  and  $\alpha_\varepsilon$  is an element of  $\mathcal{C}_c^\infty(\Omega)$  such that  $\alpha_\varepsilon = 1$  on  $] -1 + \varepsilon, 1 - \varepsilon [$  and  $|\alpha'_\varepsilon| \leq \frac{2}{\varepsilon}$ . We obtain:

$$\int_Q \{ |u - v| \theta'(t) \alpha_\varepsilon(x) + k(x) \Phi(u, v) \theta(t) \alpha'_\varepsilon(x) \} dx dt + \int_\Omega |u_0 - v_0| \theta(0) \alpha_\varepsilon(x) dx \geq 0.$$

There is no difficulty to pass to the limit when  $\varepsilon$  goes to  $0^+$ . We just point out that due to the properties of  $(\alpha_\varepsilon)_\varepsilon$  and to the definition of  $v_{\pm 1}^\tau$  and  $u_{\pm 1}^\tau$ ,

$$\lim_{\varepsilon \rightarrow 0^+} \int_Q k(x) \Phi(u, v) \theta(t) \alpha'_\varepsilon(x) dx = \int_0^T \{ k(-1) \Phi(u_{-1}^\tau, v_{-1}^\tau) - k(1) \Phi(u_1^\tau, v_1^\tau) \} \theta(t) dt.$$

It comes

$$\int_Q |u - v| \theta'(t) dx dt + \int_\Omega |u_0 - v_0| \theta(0) dx \geq \int_0^T k(1) \Phi(u_1^\tau, v_1^\tau) \theta(t) dt - \int_0^T k(-1) \Phi(u_{-1}^\tau, v_{-1}^\tau) \theta(t) dt.$$

Let's prove now that:

$$\int_0^T k(1) \Phi(u_1^\tau, v_1^\tau) \theta(t) dt - \int_0^T k(-1) \Phi(u_{-1}^\tau, v_{-1}^\tau) \theta(t) dt \geq 0.$$

By coming back to Definition 1, we know that, for a.e  $t$  in  $]0, T[$ , for all  $\kappa$  in  $\mathbb{R}$ :

$$\begin{aligned} k(1) (\text{sgn}(u_1^\tau(t) - \kappa) + \text{sgn}(\kappa)) (g(u_1^\tau(t)) - g(\kappa)) &\geq 0, \\ k(1) (\text{sgn}(v_1^\tau(t) - \kappa) + \text{sgn}(\kappa)) (g(v_1^\tau(t)) - g(\kappa)) &\geq 0. \end{aligned}$$

Thus, a.e. on  $]0, T[$ ,

• if  $u_1^\tau(t)$  and  $v_1^\tau(t)$  have the same sign, we choose  $\kappa = v_1^\tau$  in the first inequality (or  $\kappa = u_1^\tau(t)$  in the second one) to obtain:

$$k(1) \Phi(u_1^\tau(t), v_1^\tau(t)) \geq 0,$$

• if  $u_1^\tau(t)$  and  $v_1^\tau(t)$  have an opposite sign, choosing  $\kappa = 0$  in the two inequalities gives:

$$k(1)\Phi(u_1^\tau(t), v_1^\tau(t)) \geq 0.$$

Hence, for *a.e.*  $t$  in  $]0, T[$ ,

$$k(1)\Phi(u_1^\tau(t), v_1^\tau(t))\theta(t) \geq 0.$$

Similarly, for *a.e.*  $t$  in  $]0, T[$ ,

$$k(-1)\Phi(u_{-1}^\tau, v_{-1}^\tau)\theta(t) \leq 0.$$

This way,

$$\int_Q |u - v|\theta_t(t)dxdt + \int_\Omega |u_0 - v_0|\theta(0)dx \geq 0.$$

The conclusion follows from classical arguments which completes the proof of Theorem 1.  $\square$

## 5 Existence of an entropy solution

The proof relies on a suitable regularization  $k_\varepsilon$ ,  $\varepsilon > 0$ , of the function  $k$  and uses a compactness argument for the sequence  $(k_\varepsilon\Phi(u_\varepsilon, \kappa))_{\varepsilon>0}$ , where  $u_\varepsilon$  is the weak entropy solution to the corresponding mollified problem. We will consider two different situations:

$$\begin{cases} \exists \alpha \in \mathbb{R}^-, \forall x \leq \alpha, (k_L - k_R)g(x) \geq 0, \\ \exists \beta \in \mathbb{R}^+, \forall x \geq \beta, (k_L - k_R)g(x) \leq 0 \end{cases} \quad (17)$$

$$g(m) = g(M) = 0. \quad (18)$$

**Remark 2.** *As well (17) as (18) implies (10).*

**Remark 3.** *When we take into account (2) and (3), we observe that (18) includes the class of strictly convex (or strictly concave) functions that vanish at  $m$  and  $M$ . In addition (17) is fulfilled as soon as  $g$  is strictly monotone and vanishes at a point.*

In this framework we establish that:

**Theorem 2.** *The following assertions hold:*

- (i) *Under (17) Problem (1) admits at least one entropy solution  $u$ .*
- (ii) *Under (18) the problem (1) admits at least one entropy solution  $u$  such that, for *a.e.*  $(t, x)$  in  $Q$ ,  $m \leq u(t, x) \leq M$ .*

**Remark 4.** *Under (2) and (3), the assumption (17) or (18) on  $g$  implies that*

$$\exists \kappa_0 \in \mathbb{R} \text{ such that } \Phi(\cdot, \kappa_0) \text{ is strictly monotone on } \mathbb{R}.$$

*That is a key point that provides the strong convergence of the approximating sequence  $(u_\varepsilon)_{\varepsilon>0}$ .*

We suppose first that the initial condition  $u_0$  is smooth. Then through a Cauchy criterion in  $L^1(Q)$  we come back to the situation of  $u_0$  in  $L^\infty(\Omega)$ .

### 5.1 First step: $u_0 \in \mathcal{C}_c^\infty(\Omega)$

We apply the ideas introduced in [10] (also used in [1]) that is to consider a regular approximation of the function  $k$ . Let  $(k_\varepsilon)_\varepsilon$  be a sequence of smooth functions such as, for every positive  $\varepsilon$ ,  $k_\varepsilon = k$  out of  $]-\varepsilon, \varepsilon[$  and  $k_\varepsilon$  is monotone on  $[-\varepsilon, \varepsilon]$  (depending on the sign of  $k_L - k_R$ ). That implies:

$$\forall x \in \mathbb{R}^*, k_\varepsilon(x) \rightarrow k(x) \text{ and } |k_\varepsilon|_{BV(\mathbb{R})} \leq |k|_{BV(\mathbb{R})}.$$

Then we denote  $u_\varepsilon$  the unique entropy solution (see [2]) to the regularized problem:

Find a measurable and bounded function  $u$  in  $BV(Q) \cap \mathcal{C}([0, T]; L^1(\Omega))$  such that formally

$$\begin{cases} \frac{\partial u_\varepsilon}{\partial t} + \frac{\partial}{\partial x}(k_\varepsilon(x)g(u_\varepsilon)) = 0 & \text{on } Q, \\ u_\varepsilon(0, x) = u_0(x) & \text{on } \Omega, \\ u = 0 & \text{on a part of } ]0, T[ \times \partial\Omega. \end{cases} \quad (19)$$

**Lemma 7. (i)** *Under (17), for any  $t$  in  $]0, T[$ , we set:*

$$R(t) = (\|u_0\|_\infty + \max(|\alpha|, \beta))e^{M_k M_g t} + \frac{e^{M_k M_g t} - 1}{M_g} |g(0)|$$

where  $M_k = \max(\|k'\|_{L^\infty(]-1, 0])}, \|k'\|_{L^\infty(]0, 1])})$  and  $M_g = \text{Lip}(g)$ .

Then

$$|u_\varepsilon(t, x)| \leq R(t), \text{ a.e. on } \Omega.$$

**(ii)** *Under (18),  $m \leq u_\varepsilon(t, x) \leq M$ , for a.e.  $(t, x) \in Q$ .*

**(iii)** *Under (17) or (18), there exists a constant  $C > 0$ , such that:*

$$|k_\varepsilon(\Phi(u_\varepsilon, \kappa))|_{BV(Q)} \leq C(|u_0|_{BV(\Omega)} + |k|_{BV(Q)})$$

*Proof. (i)* Let  $\mu > 0$ . We introduce the viscous problem associated with (19):

Find  $u_{\varepsilon, \mu}$  in  $L^2(0, T; H^2(\Omega)) \cap \mathcal{C}([0, T]; H^1(\Omega))$ ,  $\partial_t u_{\varepsilon, \mu} \in L^2(Q)$  such that,

$$\begin{cases} \partial_t u_{\varepsilon, \mu} + \partial_x(k_\varepsilon(x)g(u_{\varepsilon, \mu})) = \mu \partial_{xx} u_{\varepsilon, \mu} & \text{a.e. in } Q, \\ u_{\varepsilon, \mu} = u_0 & \text{on } \Omega, \\ u(t, 1) = u(t, -1) = 0 & \text{for any } t \in ]0, T[. \end{cases} \quad (20)$$

It is known that (20) admits a unique solution that converges toward  $u_\varepsilon$  in  $\mathcal{C}([0, T]; L^1(\Omega))$  when  $\mu$  tends to 0. We multiply (20) with  $(u_{\varepsilon, \mu} - R(t))^+$ , we integrate over  $Q_s = ]0, s[ \times \Omega$ ,  $s \in ]0, T[$ , and perform the following transformations:

$$\begin{aligned} \mu \int_{Q_s} \partial_{xx} u_{\varepsilon, \mu} (u_{\varepsilon, \mu} - R(t))^+ dx dt &= -\mu \int_{Q_s} [\partial_x (u_{\varepsilon, \mu} - R(t))^+]^2, \\ \int_{Q_s} \partial_t u_{\varepsilon, \mu} (u_{\varepsilon, \mu} - R(t))^+ dt dx &= \frac{1}{2} \|(u_{\varepsilon, \mu}(s, \cdot) - R(s))^+\|_{L^2(\Omega)}^2 \\ &\quad + \int_{Q_s} R'(t) (u_{\varepsilon, \mu} - R(t))^+ dt dx, \end{aligned}$$

since  $u_{\varepsilon, \mu}(0, \cdot) = u_0 \leq R(0)$ . Moreover

$$\begin{aligned} \int_{Q_s} \partial_x(k_\varepsilon(x)g(u_{\varepsilon,\mu}))(u_{\varepsilon,\mu} - R(t))^+ dt dx &= \int_{Q_s} k'_\varepsilon(x)g(R(t))(u_{\varepsilon,\mu} - R(t))^+ dt dx \\ &+ \int_{Q_s} \partial_x(k_\varepsilon(x)(g(u_{\varepsilon,\mu}) - g(R(t)))(u_{\varepsilon,\mu} - R(t))^+ dt dx \end{aligned}$$

where we use an integration by parts in the last term on the right-hand side. Then the Young inequality gives:

$$\begin{aligned} \left| \int_{Q_s} \partial_x(k_\varepsilon(x)(g(u_{\varepsilon,\mu}) - g(R(t)))(u_{\varepsilon,\mu} - R(t))^+ dt dx \right| &\leq \mu \int_{Q_s} [\partial_x(u_{\varepsilon,\mu} - R(t))^+]^2 \\ &+ \frac{1}{4\mu} \int_{Q_s} (k_\varepsilon M_g)^2 ((u_{\varepsilon,\mu} - R(t))^+)^2. \end{aligned}$$

Gathering all terms yields to:

$$\begin{aligned} &\frac{1}{2} \| (u_{\varepsilon,\mu}(s, \cdot) - R(s))^+ \|_{L^2(\Omega)}^2 + \int_{Q_s} [R'(t) + k'_\varepsilon g(R(t))](u_{\varepsilon,\mu} - R(t))^+ dt dx \\ &\leq \frac{(\|k\|_\infty M_g)^2}{4\mu} \int_{Q_s} ((u_{\varepsilon,\mu} - R(t))^+)^2 dx dt. \end{aligned}$$

Now let's show that the term

$$\Psi(t, x) = R'(t) + k'_\varepsilon g(R(t))$$

is nonnegative. Therefore, thanks to the Gronwall's Lemma the conclusion will follow.

On  $[-\varepsilon, \varepsilon]$ , by definition of  $R(t)$ ,  $R(t) \geq \beta$  and  $R'(t) \geq 0$ . So:

if  $k_L \leq k_R$ ,  $k'_\varepsilon \geq 0$ , and by (17),  $g(R(t)) \geq 0$ . Then  $\Psi(t, x) \geq 0$ .

if  $k_R \leq k_L$ ,  $k'_\varepsilon \leq 0$ , and by (17),  $g(R(t)) \leq 0$ . Then  $\Psi(t, x) \geq 0$ .

Besides, on  $] -1, \varepsilon[ \cup ] \varepsilon, 1[$ ,  $k_\varepsilon = k$ . Moreover:

$$R'(t) = M_g M_k R(t) + M_k |g(0)|, \text{ and } k'g(R(t)) \geq -M_k M_g R(t) + k'g(0).$$

So

$$\Psi(t, x) \geq k'g(0) + M_k |g(0)| \geq 0.$$

To show that  $u_\varepsilon(t, x) \geq -R(t)$ , we multiply (20) with  $(u_{\varepsilon,\mu} + R(t))^-$  and we use the same techniques as before, especially the first line in (17).

(ii) The proof refers to (i) and basically lies on the fact that  $g(m) = g(M) = 0$ .

(iii) The BV-estimate is based on the maximum principle and the following estimates that are classical (see [10]):

$$\mu \int_Q |\partial_{xx} u_{\varepsilon,\mu}|^2 dx dt \leq C_1, \quad \int_\Omega |\partial_t u_{\varepsilon,\mu}| dx \leq C_2 (|k_\varepsilon|_{BV(\Omega)} + \mu |u_0|_{L^1(\Omega)})$$

where  $C_1$  and  $C_2$  are constants independent from  $\mu$ .

As, in  $\mathcal{D}'(Q)$ ,

$$\partial_x(k_\varepsilon \Phi(u_{\varepsilon,\mu}, \kappa)) \leq -\partial_t |u_{\varepsilon,\mu} - \kappa| - k'_\varepsilon \text{sgn}(u_{\varepsilon,\mu} - \kappa) g(\kappa) + \mu \partial_{xx} |u_{\varepsilon,\mu} - \kappa|,$$

when  $\mu$  tends to 0, the conclusion follows.  $\square$

As a consequence of Lemma 7, and of the compactness embedding of  $BV(Q)$  into  $L^1(Q)$  there exists  $\chi$  in  $L^\infty(Q) \cap BV(Q)$  and a subsequence of  $(k_\varepsilon \Phi(u_\varepsilon, \kappa))_\varepsilon$  such that  $(k_\varepsilon \Phi(u_\varepsilon, \kappa))_\varepsilon \rightarrow \chi$  a.e. on  $Q$ . Let  $\kappa \in \mathbb{R}$ . By remarking that  $k_\varepsilon \Phi(u_\varepsilon, \kappa) = (k_\varepsilon - k) \Phi(u_\varepsilon, \kappa) + k \Phi(u_\varepsilon, \kappa)$ , we deduce:

$$k \Phi(u_\varepsilon, \kappa) \longrightarrow \chi \quad \text{a.e. in } Q.$$

As  $\mathcal{L}\{x \in \Omega, k(x) = 0\} = 0$ ,

$$\Phi(u_\varepsilon, \kappa) \longrightarrow \frac{\chi}{k} \quad a.e. \text{ in } Q.$$

Thanks to Remark 4, there exists a subsequence of  $(u_\varepsilon)_\varepsilon$  that tends in  $L^1(Q)$  and *a.e.* on  $Q$  to a limit denoted  $u$ . Consequently, up to a subsequence,  $(k_\varepsilon \Phi(u_\varepsilon, \kappa))_\varepsilon$  converges toward  $k\Phi(u, \kappa)$  in  $L^1(Q)$  and *a.e.* on  $Q$ .

Now we have to establish that  $u$  is an entropy solution to (1). First we prove that  $u$  fulfills (5). To this purpose, we introduce the regularized entropy pairs, for any  $\kappa \in \mathbb{R}$ , and any real  $\tau$ :

$$\Phi_\eta(\tau) = \int_\kappa^\tau \text{sgn}_\eta(r - \kappa)g'(r)dr \quad \text{and} \quad I_\eta(\tau) = \int_\kappa^\tau \text{sgn}_\eta(r - \kappa)dr,$$

where  $\text{sgn}_\eta$  denotes the Lipschitzian approximation of the function  $\text{sgn}$  given for any positive  $\eta$  and any nonnegative real  $x$  by  $\text{sgn}_\eta(x) = \min(\frac{x}{\eta}, 1)$  and  $\text{sgn}_\eta(-x) = \text{sgn}(x)$ .

By coming back to (20) and considering the test-function  $v = \text{sgn}_\eta(u_\varepsilon, \mu - \kappa)\varphi$ ,  $\varphi \in \mathcal{C}_c^\infty([0, T] \times \Omega)$ ,  $\varphi \geq 0$ , we can take the limit on  $\mu$  with classical arguments. So we establish that  $u_\varepsilon$  fulfills the regularized entropy inequality for all  $\varphi$  in  $\mathcal{C}_c^\infty([0, T] \times \Omega)$ ,

$$\begin{aligned} & \int_Q I_\eta(u_\varepsilon)\varphi_t dxdt + \int_Q k_\varepsilon(x)\Phi_\eta(u_\varepsilon)\varphi_x dxdt \\ & + \int_Q k'_\varepsilon(x)(\Phi_\eta(u_\varepsilon) - I'_\eta(u_\varepsilon)g(u_\varepsilon))\varphi dxdt + \int_\Omega I_\eta(u_0)\varphi(0, x)dx \geq 0. \end{aligned} \quad (21)$$

We want to pass to the limit in (21), first with respect to  $\varepsilon$  and then with respect to  $\eta$ . The difficulty is only concentrated in the first term of the second line. That is why we write (with  $dq = dxdt$ ):

$$\begin{aligned} & \int_Q k'_\varepsilon(x)(\Phi_\eta(u_\varepsilon) - I'_\eta(u_\varepsilon)g(u_\varepsilon))\varphi dq = \int_0^T \int_{-1}^{-\varepsilon} k'_\varepsilon(x)(\Phi_\eta(u_\varepsilon) - I'_\eta(u_\varepsilon)g(u_\varepsilon))\varphi dq \\ & + \int_0^T \int_{-\varepsilon}^\varepsilon k'_\varepsilon(x)(\Phi_\eta(u_\varepsilon) - I'_\eta(u_\varepsilon)g(u_\varepsilon))\varphi dq + \int_0^T \int_\varepsilon^1 k'_\varepsilon(x)(\Phi_\eta(u_\varepsilon) - I'_\eta(u_\varepsilon)g(u_\varepsilon))\varphi dq. \end{aligned} \quad (22)$$

However, owing to the definition of  $k_\varepsilon$ ,

$$\int_0^T \int_{-1}^{-\varepsilon} k'_\varepsilon(x)(\Phi_\eta(u_\varepsilon) - I'_\eta(u_\varepsilon)g(u_\varepsilon))\varphi dq = \int_0^T \int_{-1}^{-\varepsilon} k'(x)(\Phi_\eta(u_\varepsilon) - I'_\eta(u_\varepsilon)g(u_\varepsilon))\varphi dq, \quad (23)$$

and

$$\int_0^T \int_\varepsilon^1 k'_\varepsilon(x)(\Phi_\eta(u_\varepsilon) - I'_\eta(u_\varepsilon)g(u_\varepsilon))\varphi dq = \int_0^T \int_\varepsilon^1 k'(x)(\Phi_\eta(u_\varepsilon) - I'_\eta(u_\varepsilon)g(u_\varepsilon))\varphi dq. \quad (24)$$

In addition, by referring to the definition of  $\Phi_\eta$  and  $I_\eta$ ,

$$\int_0^T \int_{-\varepsilon}^\varepsilon k'_\varepsilon(x)(\Phi_\eta(u_\varepsilon) - I'_\eta(u_\varepsilon)g(u_\varepsilon))\varphi dq = - \int_0^T \int_{-\varepsilon}^\varepsilon k'_\varepsilon g(\kappa) \text{sgn}_\eta(u_\varepsilon - \kappa)\varphi dq$$

$$+ \int_0^T \int_{-\varepsilon}^{\varepsilon} k'_\varepsilon \varphi \left( \int_\kappa^{u_\varepsilon} \operatorname{sgn}_\eta(\tau - \kappa) g'(\tau) d\tau - (g(u_\varepsilon) - g(\kappa)) \operatorname{sgn}_\eta(u_\varepsilon - \kappa) \right) dq.$$

We look for a majoration of the right-hand side of this equality. First,

$$\left| \int_0^T \int_{-\varepsilon}^{\varepsilon} k'_\varepsilon g(\kappa) \operatorname{sgn}_\eta(u_\varepsilon - \kappa) \varphi dq \right| \leq |g(\kappa)| \int_0^T \int_{-\varepsilon}^{\varepsilon} |k'_\varepsilon| \varphi dq.$$

Now we turn on to the estimate of the term  $|D(u_\varepsilon)|$  where:

$$D(u_\varepsilon) = \int_\kappa^{u_\varepsilon} \operatorname{sgn}_\eta(\tau - \kappa) g'(\tau) d\tau - (g(u_\varepsilon) - g(\kappa)) \operatorname{sgn}_\eta(u_\varepsilon - \kappa).$$

More precisely,  $|D(u_\varepsilon)| \leq C_g \eta$  with  $C_g = 2M_g$ . Indeed, let us fix  $\eta$  and  $\kappa$ . Then *a.e.* on  $Q$ :

• if  $u_\varepsilon \geq \kappa + \eta$ ,

$$D(u_\varepsilon) = \int_\kappa^{\eta+\kappa} \frac{\tau - \kappa}{\eta} g'(\tau) d\tau + \int_{\kappa+\eta}^{u_\varepsilon} g'(\tau) d\tau - (g(u_\varepsilon) - g(\kappa))$$

and

$$|D(u_\varepsilon)| \leq \eta \|M_g + |g(\kappa + \eta) - g(\kappa)|,$$

because  $0 \leq \frac{\tau - \kappa}{\eta} \leq 1$ . Then we use the Lipschitz condition for  $g$ .

• if  $\kappa - \eta \leq u_\varepsilon \leq \kappa + \eta$ ,

$$D(u_\varepsilon) = \int_\kappa^{u_\varepsilon} \frac{\tau - \kappa}{\eta} g'(\tau) d\tau - \frac{u_\varepsilon - \kappa}{\eta} (g(u_\varepsilon) - g(\kappa))$$

and

$$|D(u_\varepsilon)| \leq M_g |u_\varepsilon - \kappa| + M_g \frac{|u_\varepsilon - \kappa|^2}{\eta} \leq 2M_g \eta.$$

• if  $u_\varepsilon \leq \kappa - \eta$ ,

$$D(u_\varepsilon) = \int_{u_\varepsilon}^{\kappa-\eta} g'(\tau) d\tau + \int_{\kappa-\eta}^{\kappa} \frac{\tau - \kappa}{\eta} g'(\tau) d\tau + g(u_\varepsilon) - g(\kappa),$$

and  $|D(u_\varepsilon)| \leq C_g \eta$ , as in the first case.

Eventually we deduce that the term:

$$\int_0^T \int_{-\varepsilon}^{\varepsilon} k'_\varepsilon \varphi \left( \int_\kappa^{u_\varepsilon} \operatorname{sgn}_\eta(\tau - \kappa) g'(\tau) d\tau - (g(u_\varepsilon) - g(\kappa)) \operatorname{sgn}_\eta(u_\varepsilon - \kappa) \right) dq$$

is bounded by:

$$C_g \eta \int_0^T \int_{-\varepsilon}^{\varepsilon} |k'_\varepsilon| \varphi dx dt,$$

and

$$\int_0^T \int_{-\varepsilon}^{\varepsilon} k'_\varepsilon(x) (\Phi_\eta(u_\varepsilon) - I'_\eta(u_\varepsilon) g(u_\varepsilon)) \varphi dq \leq (C_g \eta + |g(\kappa)|) \int_0^T \int_{-\varepsilon}^{\varepsilon} |k'_\varepsilon| \varphi dx dt. \quad (25)$$

As  $k_\varepsilon$  is monotone on  $[-\varepsilon, \varepsilon]$ , for  $\varepsilon$  small enough,  $|k'_\varepsilon| = \operatorname{sgn}(k_R - k_L) k'_\varepsilon$ . So,

$$\int_0^T \int_{-\varepsilon}^{\varepsilon} |k'_\varepsilon| \varphi dq = \operatorname{sgn}(k_R - k_L) \int_0^T \int_{-\varepsilon}^{\varepsilon} k'_\varepsilon \varphi dq$$

Then we integrate by parts to obtain:

$$\begin{aligned} \int_0^T \int_{-\varepsilon}^{\varepsilon} |k'_\varepsilon| \varphi dq &= -\operatorname{sgn}(k_R - k_L) \int_0^T \int_{-\varepsilon}^{\varepsilon} k_\varepsilon \varphi_x dq \\ &+ \operatorname{sgn}(k_R - k_L) \int_0^T (k(\varepsilon)\varphi(t, \varepsilon) - k(-\varepsilon)\varphi(t, -\varepsilon)) dt. \end{aligned} \quad (26)$$

Finally, from (21), (22), (23), (24), (25) and (26), for any positive  $\eta$  and  $\varepsilon$ , we have:

$$\begin{aligned} &-\operatorname{sgn}(k_R - k_L)(C_g\eta + |g(\kappa)|) \int_0^T \int_{-\varepsilon}^{\varepsilon} k_\varepsilon \varphi_x dx dt + \int_Q k_\varepsilon(x) \Phi_\eta(u_\varepsilon) \varphi_x dx dt \\ &\int_0^T \int_{-1}^{-\varepsilon} k'(x) (\Phi_\eta(u_\varepsilon) - I'_\eta(u_\varepsilon)g(u_\varepsilon)) \varphi dq + \int_\Omega I_\eta(u_0) \varphi(0, x) dx \\ &+ \int_0^T \int_\varepsilon^1 k'(x) (\Phi_\eta(u_\varepsilon) - I'_\eta(u_\varepsilon)g(u_\varepsilon)) \varphi dq + \int_Q I_\eta(u_\varepsilon) \varphi_t dx dt \\ &+ \operatorname{sgn}(k_R - k_L)(C_g\eta + |g(\kappa)|) \int_0^T (k(\varepsilon)\varphi(t, \varepsilon) - k(-\varepsilon)\varphi(t, -\varepsilon)) dt \geq 0. \end{aligned}$$

We take now the  $\varepsilon$ -limit. Clearly, because  $(u_\varepsilon)_\varepsilon$  goes to  $u$  in  $L^1(Q)$  and since  $I_\eta$  and  $\Phi_\eta$  are Lipschitzian,

$$\lim_{\varepsilon \rightarrow 0^+} \int_Q (I_\eta(u_\varepsilon) \varphi_t + k_\varepsilon(x) \Phi_\eta(u_\varepsilon) \varphi_x) dq = \int_Q (I_\eta(u) \varphi_t + k(x) \Phi_\eta(u) \varphi_x) dq.$$

Thanks to the definition of  $k_L$  and  $k_R$  and to the continuity of  $\varphi$ ,

$$\lim_{\varepsilon \rightarrow 0^+} \int_0^T (k(\varepsilon)\varphi(t, \varepsilon) - k(-\varepsilon)\varphi(t, -\varepsilon)) dt = (k_R - k_L) \int_0^T \varphi(t, 0) dt.$$

Moreover,  $k_\varepsilon \varphi_x$  being bounded independently with respect to  $\varepsilon$ ,

$$\lim_{\varepsilon \rightarrow 0^+} \int_0^T \int_{-\varepsilon}^{\varepsilon} k_\varepsilon \varphi_x dx dt = 0.$$

So, for any positive  $\eta$ , the following inequality holds:

$$\begin{aligned} &\int_Q (I_\eta(u) \varphi_t + k(x) \Phi_\eta(u) \varphi_x) dx dt + \int_Q k'(x) (\Phi_\eta(u) - I'_\eta(u)g(u)) \varphi dx dt \\ &+ \int_\Omega I_\eta(u_0) \varphi(0, x) dx + (|g(\kappa)| + C_g\eta) |k_R - k_L| \int_0^T \varphi(t, 0) dt \geq 0. \end{aligned} \quad (27)$$

We take the limit with respect to  $\eta$  through the Lebesgue dominated convergence Theorem, providing that  $u$  fulfills (5).

Lastly, let us establish that  $u$  satisfies (6)-(7). To this purpose, we use the functions  $H_\eta$  and  $Q_\eta$  defined in [8] for any  $\tau, \kappa \in \mathbb{R}$ , by:

$$H_\eta(\tau, \kappa) = ((\operatorname{dist}(\tau, I[0, \kappa]))^2 + \eta^2)^{\frac{1}{2}} - \eta$$

and

$$Q_\eta(\tau, \kappa) = \int_\kappa^\tau \partial_1 H_\eta(\lambda, \kappa) g'(\lambda) d\lambda$$



where  $I[0, \kappa]$  denotes the closed interval bounded by 0 and  $\kappa$ . The sequence  $(H_\eta, Q_\eta)_\eta$  converges uniformly to  $(\text{dist}(\tau, I[0, \kappa]), \mathcal{G}(\tau, 0, \kappa))$  where:

$$\mathcal{G}(\tau, 0, \kappa) = \frac{1}{2}(\Phi(\tau, 0) + \Phi(\kappa, 0) + \Phi(\tau, \kappa)).$$

By taking in (20) the test-function  $\partial_1 H_\eta(u_\varepsilon, \kappa)\varphi$ , for any function  $\varphi \in \mathcal{C}_c^\infty(]0, T[ \times \bar{\Omega})$ , we obtain for any positive  $\eta$  and  $\varepsilon$  the following inequality:

$$\begin{aligned} & \int_Q H_\eta(u_\varepsilon, \kappa)\varphi_t dxdt + \int_Q k_\varepsilon Q_\eta(u_\varepsilon, \kappa)\varphi_x dxdt \\ & \int_Q k'_\varepsilon(x)(Q_\eta(u_\varepsilon, \kappa) - \partial_1 H_\eta(u_\varepsilon, \kappa)g(u_\varepsilon))\varphi dxdt \geq 0. \end{aligned}$$

If we only consider functions  $\varphi$  vanishing in a neighborhood of  $\{x_0 = 0\}$  containing  $[-\varepsilon, \varepsilon]$  (that will not be restrictive in the sequel), we can take the  $\varepsilon$ -limit without difficulty to obtain:

$$\begin{aligned} & \int_Q H_\eta(u, \kappa)\varphi_t dxdt + \int_Q k Q_\eta(u, \kappa)\varphi_x dxdt \\ & \int_Q k'(x)(Q_\eta(u, \kappa) - \partial_1 H_\eta(u, \kappa)g(u))\varphi dxdt \geq 0. \end{aligned} \quad (28)$$

Then, for  $(t, x) \in ]0, T[ \times \bar{\Omega}$ , we choose in (28) a sequence of test-functions defined by  $\varphi_n(t, x) = \beta(t)\alpha_n(x)$  with  $\beta \in \mathcal{C}_c^\infty(]0, T[)$ ,  $\beta \geq 0$ , and  $\alpha_n \in \mathcal{C}_c^\infty(\bar{\Omega})$  such as  $\alpha_n \geq 0$ ,  $\alpha_n(x) = 0$  on  $] -1, 1 - \frac{1}{n}[$ ,  $\alpha_n(1) = 1$  and  $\|\alpha'_n\|_\infty \leq n$ . On the one hand, by reasoning as in [8] we make sure that

$$\lim_{n \rightarrow \infty} \int_{1-\frac{1}{n}}^1 \int_0^T \alpha'_n(x)k(x)Q_\eta(u, \kappa)\beta(t)dt dx \text{ exists and is nonnegative.}$$

On the other hand by using the definition of  $u_1^\tau$ ,

$$\lim_{n \rightarrow \infty} \int_{1-\frac{1}{n}}^1 \int_0^T \alpha'_n(x)k(x)Q_\eta(u, \kappa)\beta(t)dt dx = \int_0^T k(1)Q_\eta(u_1^\tau(t), \kappa)\beta(t)dt.$$

Finally, when  $\eta$  goes to  $0^+$ ,

$$\int_0^T k(1)\mathcal{G}(u_1^\tau, 0, \kappa)\beta(t)dt \geq 0.$$

To conclude we just emphasize that the previous inequality is equivalent for all  $\kappa$  in  $I[0, u_1^\tau]$ , to:

$$\text{sgn}(u_1^\tau)k(1)(g(u_1^\tau) - g(\kappa)) \geq 0,$$

that is namely (6) when  $\kappa$  is reduced to belong to  $I(0, u_1^\tau)$ .

In the same way, by choosing  $\varphi(t, x) = \beta(t)\delta_n(x)$  in (28), with  $\beta \in \mathcal{C}_c^\infty(]0, T[)$ ,  $\beta \geq 0$ , and  $\delta_n \in \mathcal{C}_c^\infty(\bar{\Omega})$  such as  $\delta_n \geq 0$ ,  $\delta_n(x) = 0$  on  $] -1 + \frac{1}{n}, 1[$ ,  $\delta_n(-1) = 1$  and  $\|\delta'_n\|_\infty \leq n$ , by using the definition of  $u_{-1}^\tau$ , we establish (7).

## 5.2 Second step: $u_0 \in L^\infty(\Omega)$

We use a mollification process to come back to the first step. Indeed, for  $j \in \mathbb{N}^*$ , we consider the sequence  $(u_0^j)_j$  such that  $u_0^j$  belongs to  $C_c^\infty(\Omega)$  and  $(u_0^j)$  tends to  $u_0$  in  $L^1(\Omega)$ . We denote  $u^j$  the entropy solution to (1) associated with the initial condition  $u_0^j$  so that, for any  $j$ ,  $u^j$  fulfills (27) and (28). The comparison result (13) ensures that the sequence  $(u_j)_j$  is a Cauchy sequence in  $L^1(Q)$  and so tends to a limit, denoted  $u$ . Then the  $j$ -limit in (27) and (28) warrants that  $u$  is an entropy solution to (1).

To conclude, we point out that (17) or (18) implies (10), so that:

**Corollary 1.** *Assume that (17) or (18) holds. Then (1) has an unique entropy solution.*

## 6 Generalisation

In this section we keep the same assumptions on  $g$  but we consider that  $k$  has a finite number of discontinuities. Let  $D = \{1, \dots, n-1\}$ ,  $n \neq 0$ ,  $x_0 = -1$ ,  $x_n = 1$ . We suppose that:

$$k \text{ is discontinuous at } x_i, i \in D, \quad (29)$$

while  $k|_{]x_i, x_{i+1}[} \in W^{1,+\infty}(]x_i, x_{i+1}[)$ . Of course we need a new definition of an entropy solution which has to be equivalent to Definition 1 when  $D$  is reduced to one point. So we say that:

**Definition 2.** *Under (29), a function  $u$  of  $L^\infty(Q)$  is an entropy solution to Problem (1) if  $u$  satisfies (6)-(7) and if,  $\forall \kappa \in \mathbb{R}$ ,  $\forall \varphi \in C_c^\infty([0, T[ \times \Omega)$ ,  $\varphi \geq 0$ ,*

$$\left\{ \begin{array}{l} \int_Q (|u(t, x) - \kappa| \varphi_t(t, x) + k(x) \Phi(u, \kappa) \varphi_x(t, x)) dx dt \\ \quad - \int_Q k'(x) \operatorname{sgn}(u - \kappa) g(\kappa) \varphi dx dt \\ + \int_\Omega |u_0 - \kappa| \varphi(0, x) dx + \sum_{i \in D} |(k_i^+ - k_i^-) g(\kappa)| \int_0^T \varphi(t, x_i) dt \geq 0, \end{array} \right. \quad (30)$$

where

$$k_i^+ = \lim_{x \rightarrow x_i^+} k(x) \quad \text{and} \quad k_i^- = \lim_{x \rightarrow x_i^-} k(x)$$

We denote  $\gamma u_i^+$  and  $\gamma u_i^-$  the strong traces in  $L^\infty(]0, T[)$  at  $\{x = x_i\}$ . By using the same techniques as before we can state the following theorem:

**Theorem 3.** *Under (18), when  $k$  satisfies (29), there exists a unique entropy solution to (1). Moreover, at every point  $x_i$ ,  $i \in D$ ,  $u$  satisfies the Rankine-Hugoniot condition:*

$$k_i^+ g(\gamma u_i^+) = k_i^- g(\gamma u_i^-)$$

In addition, we adapt (17) under the form:

$$\left\{ \begin{array}{l} \text{for } i, j \in D, i \neq j, \operatorname{sgn}(k_i^+ - k_i^-) = \operatorname{sgn}(k_j^+ - k_j^-), \\ \exists \alpha \in \mathbb{R}^-, \forall x \leq \alpha, (k_1^- - k_1^+) g(x) \geq 0, \\ \exists \beta \in \mathbb{R}^+, \forall x \geq \beta, (k_1^- - k_1^+) g(x) \leq 0. \end{array} \right. \quad (31)$$

This condition is satisfied when  $g$  is strictly monotone and the next corollary holds:

**Corollary 2.** *Assume that (31) is satisfied. Then problem (1) has a unique entropy solution, in the sense of Definition 2.*

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