Analysis of a Scalar Conservation Law with Space Discontinuous Advection Function in a Bounded Domain

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Abstract

We deal with the scalar conservation law in a one dimensional bounded domain Ω: $\partial_t u + \partial_x(k(x)g(u)) = 0$, associated with a bounded initial value u_0. The function k is supposed to be bounded, discontinuous at $\{x_0 = 0\}$, and with bounded variation. A weak entropy formulation for the Cauchy problem has been introduced by J.D Towers in [11]. In [10] the existence and the uniqueness is proved by N. Seguin and J. Vovelle through a regularization of the function k. We generalize the definition of J.D Towers and we adapt the method developed in [10] to establish an existence and uniqueness property in the case of the homogeneous Dirichlet boundary conditions.

1 Introduction

We are interested in the existence and uniqueness properties for a scalar conservation law made of an hyperbolic first-order quasilinear equation set in a one-dimensional bounded domain Ω, and for any positive finite real T, that can be formally described:

Find a bounded measurable function u on $Q =]0,T[\times\Omega$ such that

$$\left\{\begin{array}{rll} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}(k(x)g(u)) = & 0 & \text{in } Q =]0,T[\times\Omega, \\ u(0,x) = & u_0(x) & \text{on } \Omega, \\ u = & 0 & \text{on (a part of) }]0,T[\times\partial\Omega, \end{array}\right. \tag{1}$$

where k is a discontinuous function at a point x_0 of Ω.
Such an equation arises in the modelling of continuous sedimentation of solid particles in a liquid ([3]) or when one considers a two-phase flow in an heterogeneous porous medium without capillarity effects ([5], [4]).
By normalization, we suppose that $\Omega =]-1,1[$.
The initial condition u_0 belongs to $L^\infty(\Omega)$ and takes values in $[m, M]$ where m and M are two fixed reals, $m < M$.

The flux function g is Lipschitzian on $\mathbb{R}$. We suppose also that:

$$g \text{ changes no more than once its monotony} \tag{2}$$

and satisfies a nondegeneracy condition in the sense of A. Vasseur [12], that is to say:

$$\forall\ \alpha \in \mathbb{R},\ \mathcal{L}\{\lambda \in \mathbb{R},\ g'(\lambda) = \alpha\} = 0. \tag{3}$$

where $\mathcal{L}$ denotes the Lebesgue measure.
The function k is discontinuous at $x_0 = 0$ and $k_{|[-1,0[}$ is an element of $W^{1,+\infty}(]-1,0[)$ while $k_{|]0,1]}$ belongs to $W^{1,+\infty}(]0,1[)$.
Thus, thanks to a Cauchy criterion, we can define:

$$k_L = \lim_{x\to 0^-} k(x) \qquad \text{and} \qquad k_R = \lim_{x\to 0^+} k(x).$$

Eventually, we suppose that:

$$\mathcal{L}\{x \in \Omega,\ k(x) = 0\} = 0. \tag{4}$$

The mathematical formulation for (1) is given in Section 2 through an entropy inequality on the whole Q, using the classical Kruzkov entropy pairs (see [7]) and involving a term that takes into account the jump of k along $\{x_0 = 0\}$. As soon as we are able to transcript in Section 3 the transmission conditions along the interface included in Definition 1, we are able to state, in Section 4, the uniqueness. To do so strong traces for u along the interface $\{x_0 = 0\}$ will be needed. Finally Section 5 is devoted to the existence property for (1) through a suitable regularization of the function k.

2 Definition of an entropy solution

We propose a definition extending that of J.D. Towers ([11]) - also used by N. Seguin and J. Vovelle ([10]) or F. Bachmann ([1]) - to the case where k depends on the space variable and for the homogeneous Dirichlet problem in a bounded interval of $\mathbb{R}$. So we say that:

Definition 1. *A function u of $L^\infty(Q)$ is an entropy solution to problem 1 if:*
(i) $\forall \kappa \in \mathbb{R}$, $\forall \varphi \in \mathcal{C}_c^\infty([0,T[\times\Omega)$, $\varphi \geq 0$,

$$\left\{\begin{aligned} &\int_Q (|u(t,x)-\kappa|\varphi_t(t,x) + k(x)\Phi(u,\kappa)\varphi_x(t,x))dxdt \\ &\qquad\qquad - \int_Q k'(x)sgn(u-\kappa)g(\kappa)\varphi dxdt \\ &+ \int_\Omega |u_0-\kappa|\varphi(0,x)dx + |(k_L-k_R)g(\kappa)|\int_0^T \varphi(t,0)dt \quad \geq 0\ , \end{aligned}\right. \tag{5}$$

where

$$\Phi(u,\kappa) = sgn(u-\kappa)(g(u)-g(\kappa)),$$

(ii) for a.e. t in $]0,T[$, for any real κ,

$$k(1)(sgn(u_1^\tau(t)-\kappa) + sgn(\kappa))(g(u_1^\tau(t)-g(\kappa)) \quad \geq \quad 0, \tag{6}$$

$$k(-1)(sgn(u_{-1}^\tau(t)-\kappa) + sgn(\kappa))(g(u_{-1}^\tau(t)-g(\kappa)) \quad \leq \quad 0. \tag{7}$$

In this definition u_1^τ and u_{-1}^τ denote the traces of u respectively in $(+1)^-$ and $(-1)^+$ in the sense of A. Vasseur [12] (see also Y. Panov [9]). Indeed it follows from [12],

Lemma 1. *Let u be an entropy solution to (1). If for each $(\alpha,\beta) \neq (0,0)$, for a.e. $x \in [-1,1]$, $\mathcal{L}(\{\lambda \mid \alpha + \beta.k(x)g'(\lambda) = 0\}) = 0$, there exists two functions $u_{\pm 1}^\tau$ in $L^\infty(]0,T[)$ such as, for every compact set K of $]0,T[$,*

$$\operatorname*{ess\,lim}_{x\to\pm 1} \int_K |u(t,x) - u_{\pm 1}^\tau(t)|dt = 0. \tag{8}$$

In [9], Y. Panov proved the existence of these strong traces with a continuous flux function, when the boundary is not a characteristic hypersurface. The latter condition is satisfied here under (3) and (4), when we consider the problem (1) separately on $]-1,0[$ and on $]0,1[$.

Remark 1. *Of course, the statement of Lemma 1 also ensures the existence of strong traces for u, γu^+ and γu^- in $L^\infty(]0,T[)$ along $\{x_0 = 0\}$ respectively at right and at left.*

3 Conditions at the interface $\{x_0 = 0\}$

Let us establish that the previous definition ensures the uniqueness. The proof is based on that proposed in [10] and relies essentially on the transmission conditions along $\{x_0 = 0\}$ underlying to entropy inequality (5). Indeed the existence of strong traces for u permits us to state first:

Lemma 2. *Let u in $L^\infty(Q)$ be an entropy solution to (1). So,*
for a.e. t in $]0,T[$, for all real κ,

$$k_L\Phi(\gamma u^-(t),\kappa) - k_R\Phi(\gamma u^+(t),\kappa) + |(k_L - k_R)g(\kappa)| \ \geq\ 0\ . \tag{9}$$

Proof. Let φ be a nonnegative element of $\mathcal{C}_c^\infty(Q)$. We refer to the cut-off function on $\mathbb{R}$, ω_ε, for $\varepsilon > 0$, introduced in [10]:

$$\omega_\varepsilon(x) = \left\{ \begin{array}{lll} 0 & \text{if} \quad 2\varepsilon < |x| & , \\ \dfrac{-|x| + 2\varepsilon}{\varepsilon} & \text{if} \quad \varepsilon \leq |x| \leq 2\varepsilon & , \\ 1 & \text{if} \quad |x| < \varepsilon & . \end{array} \right.$$

such that $\omega_\varepsilon(x) \to 0$ if $x \neq 0$, and $\omega_\varepsilon(0) = 1$ for all ε.

Thanks to a density argument we may choose $\varphi\omega_\varepsilon$ as test-function in (5). We pass to the limit when ε goes to 0^+ by using the Lebesgue dominated convergence Theorem providing that all the terms tend to 0 except $|k_L - k_R|g(\kappa)\int_0^T \varphi(t,0)dt$ (which does not depend on ε) and:

$$I_\varepsilon = \int_Q k(x)\Phi(u,\kappa)\varphi\ {\omega'}_\varepsilon\ dxdt\ .$$

By definition of ω_ε,

$$I_\varepsilon = \int_0^T \frac{1}{\varepsilon}\int_{-2\varepsilon}^{-\varepsilon} k(x)\Phi(u,\kappa)\varphi\ dxdt + \int_0^T -\frac{1}{\varepsilon}\int_\varepsilon^{2\varepsilon} k(x)\Phi(u,\kappa)\varphi\ dxdt,$$

and, by setting $L_\varepsilon = \displaystyle\int_0^T \frac{1}{\varepsilon}\int_{-2\varepsilon}^{-\varepsilon} |k(x)(\Phi(u,\kappa)\varphi(t,x) - k_L\Phi(\gamma u^-,\kappa)\varphi(t,0))|dtdx$, we prove that $\lim_{\varepsilon\to 0^+} L_\varepsilon = 0$ because $\Phi(.,\kappa)$ is Lipschitzian on $[0,1]$, and due to the definition of k_L and γu^-. As a consequence, we obtain (9). $\square$

As in [10], a Rankine-Hugoniot condition may be deduced from (9). To do so we need an additional hypothesis on the function g. So we suppose that:

$$\begin{cases} \exists\ \kappa_1 \in \mathbb{R}, \kappa_1 \geq \operatorname{ess\,sup} u, \quad g(\kappa_1)(k_L - k_R) \leq 0 \\ \exists\ \kappa_2 \in \mathbb{R}, \kappa_2 \leq \operatorname{ess\,inf} u, \quad g(\kappa_2)(k_L - k_R) \geq 0 \end{cases} \tag{10}$$

Lemma 3. *Under (10), for a.e. t in $]0,T[$, the following Rankine-Hugoniot condition holds:*

$$k_L g(\gamma u^-(t)) = k_R g(\gamma u^+(t)) \ . \tag{11}$$

Proof. We choose $\kappa = \kappa_1$ in (9) to obtain:

$$k_R g(\gamma u^+) - k_L g(\gamma u^-) + g(\kappa_1)(k_L - k_R) + |g(\kappa_1)(k_L - k_R)| \geq 0$$

From (10), we deduce that $k_R g(\gamma u^+) \geq k_L g(\gamma u^-)$.
By choosing $\kappa = \kappa_2$ in (9), and using (10), we obtain the reverse inequality. $\square$

4 The uniqueness theorem

First we recall that

Lemma 4. *If a bounded mapping u satisfies (5), then:*

$$\operatorname*{esslim}_{t\to 0^+} \int_\Omega |u(t,x) - u_0(x)|dx = 0. \tag{12}$$

We are now able to state an uniqueness property for (1) through a T-Lipschitzian dependence in $L^1(Q)$ of a weak entropy solution with respect to corresponding initial data.

Theorem 1. *Let u and v be two entropy solutions to (1) for initial conditions (u_0, v_0) in $(L^\infty(]-1,1[)^2$. Then, under (10):*

$$\int_0^T \int_{-1}^1 |u(t,x) - v(t,x)|dxdt \leq T \int_{-1}^1 |u_0(x) - v_0(x)|dx. \tag{13}$$

Proof. We use the method of doubling variables due to S. N. Kruzkov (see [7]) by reasoning in two steps: we consider first some test-functions vanishing on a vicinity of $\{x_0 = 0\}$. That provides a Kruzkov-type inequality between two entropy solutions from which one the former vanishing hypothesis is released by using (11).

Lemma 5. *Let u and v be two entropy solutions in $L^\infty(Q)$ to (1) associated with initial conditions u_0 and v_0 in $L^\infty(]-1,1[)$. For any nonnegative function φ in $\mathcal{C}_c^\infty([0,T[\times\Omega)$, vanishing in a neighborhood of $\{x_0 = 0\}$,*

$$\begin{aligned} &\int_Q (|u(t,x) - v(t,x)|\varphi_t(t,x) + k(x)\Phi(u(t,x),v(t,x))\varphi_x(t,x))dxdt \\ &+ \int_\Omega |u_0(x) - v_0(x)|\varphi(0,x)dx \ \geq 0. \end{aligned} \tag{14}$$

Proof. Let $(\rho_j)_{j\in\mathbb{N}^*}$ be a classical sequence of mollifiers in $\mathbb{R}$, such that $\rho_j(x) = \rho_j(-x)$, φ an element of $\mathcal{C}_c^\infty([0,T[\times\Omega)$ satisfying the hypotheses of Lemma 5. For $j \in \mathbb{N}^*$ and $(t,x,s,y) \in Q\times Q$, we set:

$$\psi_j(t,x,s,y) = \varphi(\tfrac{t+s}{2}, \tfrac{x+y}{2})\rho_j(t-s)\rho_j(x-y).$$

To simplify, we denote $w = \frac{t+s}{2}$, $z = \frac{x+y}{2}$, $u = u(t,x)$, $v = v(t,x)$, $\tilde{v} = v(s,y)$, $q = (t,x)$, $\tilde{q} = (s,y)$. By choosing $\kappa = \tilde{v}$ in (5) for u (respectively $\kappa = u$ in (5) for $\tilde{v}$) against the test-function ψ_j and integrating over Q with respect to $\tilde{q}$ (respectively q), it comes:

$$\begin{aligned}
&\int_{Q\times Q} |u-\tilde{v}|\varphi_t(w,z)\rho_j(t-s)\rho_j(x-y)dqd\tilde{q}\\
&-\int_{Q\times Q} sgn(u-\tilde{v})(k'(x)g(\tilde{v}) - k'(y)g(u))\psi_j dqd\tilde{q}\\
&+2\int_{\Omega\times\Omega} |u_0(x) - v_0(y)|\varphi(\tfrac{t}{2}, z)\rho_j(x-y)\rho_j(t)dqdy\\
&+\int_{Q\times\Omega} (|u-u_0| + |\tilde{v}-\tilde{v}_0|)\varphi(\tfrac{t}{2}, z)\rho_j(x-y)\rho_j(t)dqdy\\
&+\int_{Q\times Q} \Phi(u,\tilde{v})k(x)(\partial_x\varphi)(w,z)\rho_j(t-s)\rho_j(x-y)dqd\tilde{q}\\
&+\int_{Q\times Q} \Phi(u,\tilde{v})(k(y)-k(x))\partial_y\psi_j(q,\tilde{q})\ dqd\tilde{q} \quad \geq 0\ .
\end{aligned} \tag{15}$$

We will just focus on the second and the sixth line. Indeed there is no difficulty to pass to the limit when j goes to $+\infty$ in the other lines by referring to the notion of Lebesgue points for an integrable function on Q (and by using (12) for the forth line). Let's study first the sixth line, denoted I_j. Coming back to the definition of ψ_j yields

$$I_j = I_{1,j} + I_{2,j},$$

where:

$$I_{1,j} = \int_{Q\times Q} \Phi(u,\tilde{v})(k(y)-k(x))\partial_y(\varphi(w,z))\rho_j(t-s)\rho_j(x-y)dqd\tilde{q},$$

$$I_{2,j} = \int_{Q\times Q} \Phi(u,\tilde{v})(k(y)-k(x))\varphi(w,z)\rho_j(t-s)\partial_y(\rho_j(x-y))dqd\tilde{q}.$$

By using the notion of Lebesgue points, we state that

$$\lim_{j\to+\infty} I_{1,j} = 0.$$

Next we write $I_{2,j} = I_a + I_b$ with:

$$I_a = \int_{Q\times Q} \{\Phi(u,\tilde{v}) - \Phi(u,v)\}\,(k(y)-k(x))\varphi(w,z)\rho_j(t-s)\partial_y(\rho_j(x-y))dqd\tilde{q}$$

and,

$$I_b = \int_{Q\times Q} \Phi(u,v)(k(y)-k(x))\varphi(w,z)\rho_j(t-s)\partial_y(\rho_j(x-y))dqd\tilde{q}.$$

Let us first consider I_b. We denote

$$T(q,\tilde{q}) = \Phi(u,v)(k(y)-k(x))\varphi(w,z)\rho_j(t-s)\partial_y(\rho_j(x-y)),$$

$$Q_- =]0,T[\times]-1,0[\text{ and } Q_+ =]0,T[\times]0,1[,$$

$$I_{b,1} = \int_{Q_-\times Q_-} T(q,\tilde{q})dqd\tilde{q}\ ,\ I_{b,2} = \int_{Q_-\times Q_+} T(q,\tilde{q})dqd\tilde{q}$$

$$I_{b,3} = \int_{Q_+\times Q_-} T(q,\tilde{q})dqdq \text{ and } I_{b,4} = \int_{Q_+\times Q_+} T(q,\tilde{q})dqd\tilde{q}.$$

Then, $I_b = I_{b,1} + I_{b,2} + I_{b,3} + I_{b,4}$, so we just need to study $I_{b,1}$ and $I_{b,2}$, the arguments for $I_{b,3}$ and $I_{b,4}$ being similar. We integrate by parts $I_{b,1}$ with respect to y to obtain:

$$\begin{aligned} I_{b,1} = \quad & -\int_{Q_-\times Q_-} \Phi(u,v)k'(y)\varphi(w,z)\rho_j(t-s)\rho_j(s-y)dqd\tilde{q} \\ & -\tfrac{1}{2}\int_{Q_-\times Q_-} \Phi(u,v)(k(y)-k(x))\varphi_y(w,z)\rho_j(t-s)\rho_j(s-y)dqd\tilde{q} \\ & +\int_{Q_-}\int_0^T \Phi(u,v)(k_L-k(x))\varphi(w,\frac{x}{2})\rho_j(t-s)\rho_j(x)dqds \end{aligned}$$

When j goes to $+\infty$, the two last terms tend to 0, owing to the continuity of k on $]-1,0[$ and to the definition of k_L. Moreover, since $k_{|[-1,0[}$ belongs to $W^{1,+\infty}([-1,0[)$ and φ is continuous, the first term tends to:

$$-\int_{Q_-} \Phi(u(t,x),v(t,x))k'(x)\varphi(t,x)dq.$$

Similarly, $\lim_{j\to+\infty} I_{b,4} = -\int_{Q_+} \Phi(u(t,x),v(t,x))k'(x)\varphi(t,x)dq.$

By definition of ρ_j, $I_{b,2}$ is equal to:

$$\int_0^T\int_{-\frac{1}{j}}^0\int_0^T\int_0^{\frac{1}{j}} \Phi(u,v)(k(y)-k(x))\varphi(w,z)\rho_j(t-s)\partial_y(\rho_j(x-y))dqd\tilde{q}$$

As φ vanishes on a neighborhood of $\{x_0=0\}$, from a certain j_0, $I_{b,2}$ vanishes and it is the same for $I_{b,3}$. Eventually:

$$\lim_{j\to+\infty} I_b = -\int_Q \Phi(u,v)k'(x)\varphi(t,x)dq$$

We study now I_a. By using the same decomposition as for I_b, it appears four integrals whose two vanish (because φ vanishes on a vicinity of $\{x_0=0\}$) and it only leads to consider the term, denoting by $I_{a,1}$:

$$\int_{Q_-\times Q_-} \{\Phi(u,\tilde{v})-\Phi(u,v)\}(k(y)-k(x))\varphi(w,z)\rho_j(t-s)\partial_y(\rho_j(x-y))dqd\tilde{q}$$

By using the Lipschitz condition for ϕ and k, we highlight a nonnegative constant C_1 independent from j, such that:

$$|I_{a,1}| \leq C_1 \int_{Q_- \times Q_-} |v(s,y) - v(t,x)||x-y|\rho_j(t-s)|\partial_y(\rho_j(x-y))|dqd\tilde{q}$$

This way, due to the definition of ρ_j, there exists a nonnegative constant C_2 such that:

$$|I_{a,1}| \leq C_2 j^2 \int_{\{|t-s|\leq \frac{1}{j}, |x-y|\leq \frac{1}{j}\}} |v(t,x) - v(s,y)|dqd\tilde{q},$$

so that, $\lim_{j\to+\infty} I_{a,1} = 0$, and as a consequence

$$\lim_{j\to+\infty} I_a = 0.$$

To sum up:

$$\lim_{j\to+\infty} I_j = \lim_{j\to+\infty} (I_{1,j} + I_a + I_b) = -\int_Q \Phi(u(t,x), v(t,x))k'(x)\varphi(t,x)dq.$$

We study now the j-limit of the second line in (15) that is:

$$L_j = -\int_{Q\times Q} sgn(u-\tilde{v})(k'(x)g(\tilde{v}) - k'(y)g(u))\psi_j dqd\tilde{q}.$$

We write

$$L_j = L_{1,j} - L_{2,j}$$

with

$$L_{1,j} = \int_{Q\times Q} k'(x)\Phi(u,\tilde{v})\psi_j dqd\tilde{q} \text{ and}$$

and

$$L_{2,j} = \int_{Q\times Q} g(u)sgn(u-\tilde{v})(k'(x) - k'(y))\psi_j dqd\tilde{q}.$$

On the one hand, it is clear that:

$$\lim_{j\to+\infty} L_{1,j} = \int_Q k'(x)\Phi(u(t,x), v(t,x))\varphi(t,x)dq.$$

On the other hand, as for the study of I_a and I_b we share $L_{2,j}$ into four terms whose two vanishes (φ vanishing on a neighborhood of $\{x_0 = 0\}$) so that we only consider:

$$L_{2,a} = \int_{Q_-\times Q_-} g(u(t,x))sgn(u(t,x) - v(s,y))(k'(x) - k'(y))\psi_j dqd\tilde{q}$$

and,

$$L_{2,b} = \int_{Q_+\times Q_+} g(u(t,x))sgn(u(t,x) - v(s,y))(k'(x) - k'(y))\psi_j dqd\tilde{q}.$$

We observe that:

$$|L_{2,a}| \leq C\|g\|_\infty \|\varphi\|_\infty \int_{\Omega^-\times\Omega^-} |k'(x) - k'(y)|\rho_j(x-y)dxdy,$$

where $\Omega^- =]-1,0[$.
So that, since k' belongs to $L^\infty([-1,0[)$, $\lim_{j\to+\infty} L_{2,a} = 0$ and it is the same for $L_{2,b}$.
To summarize, $\lim_{j\to+\infty} L_j = \int_Q k'(x)\Phi(u(t,x), v(t,x))\varphi(t,x)dq$, and (14) follows that completes the proof of Lemma 5. $\square$

Now we state that:

Lemma 6. *Under (10), the Kruzkov inequality (14) still holds for φ in $\mathcal{C}_c^\infty([0,T[\times\Omega)$, $\varphi \geq 0$.*

Proof. Thanks to a density argument we can choose in (14) the test function $\varphi(1-\omega_\varepsilon)$ where ω_ε is defined in the proof of Lemma 2. By taking the ε-limit, it comes:

$$\int_Q (|u-v|\varphi_t + k(x)\Phi(u,v)\varphi_x)dxdt + \int_\Omega |u_0 - v_0|\varphi(0,x)dx \ \geq J,$$

with:

$$J = \int_0^T (k_L\Phi(\gamma u^-, \gamma v^-) - k_R\Phi(\gamma u^+, \gamma v^+))\varphi(t,0)dt.$$

Inequality (9) shows that J is nonnegative. Indeed let us study, for *a.e.* t of $]0,T[$, the sign of:

$$I = k_L\Phi(\gamma u^-, \gamma v^-) - k_R\Phi(\gamma u^+, \gamma v^+) \ .$$

We just focus on the case when $\gamma u^+ - \gamma v^+$ and $\gamma u^- - \gamma v^-$ have an opposite sign. Otherwise due to (11), that is satisfied because of (10), $I = 0$. When $sgn(\gamma u^+ - \gamma v^+) = -sgn(\gamma u^- - \gamma v^-) \neq 0$, by using (11), we have:

$$I = 2k_L\Phi(\gamma u^-, \gamma v^-) = -2k_R\Phi(\gamma u^+, \gamma v^+)$$

We suppose that $k_L - k_R > 0$, $\gamma v^+ < \gamma u^+$ and $\gamma u^- < \gamma v^-$, the study of the other cases being similar. So,

$$I = -2k_L(g(\gamma u^-) - g(\gamma v^-)) = -2k_R(g(\gamma u^+) - g(\gamma v^+))$$

Here we must consider some different situations.

1. $\gamma u^- < \gamma v^- < \gamma v^+ < \gamma u^+$.
   Then (9) can be written, for any κ of $[\gamma u^-, \gamma u^+]$,

$$-k_L(g(\gamma u^-) - g(\kappa)) - k_R(g(\gamma u^+) - g(\kappa)) + (k_L - k_R)|g(\kappa)| \geq 0. \quad (16)$$

. if $g(\gamma v^-) \geq 0$, by choosing $\kappa = \gamma v^-$ in (16), we have:

$$-2k_L(g(\gamma u^-) - g(\gamma v^-)) \geq 0, \text{ so } I \geq 0.$$

. if $g(\gamma v^+) \leq 0$, by choosing $\kappa = \gamma v^+$ in (16), we obtain:

$$-2k_R(g(\gamma u^+) - g(\gamma v^+)) \geq 0, \text{ so } I \geq 0.$$

. if $g(\gamma v^-) < 0$ and $g(\gamma v^+) > 0$, we deduce from (11), as $k_L - k_R > 0$, that $k_R < 0$ and $k_L > 0$. If we suppose that $I = -2k_L(g(\gamma u^-) - g(\gamma v^-)) < 0$, then $g(\gamma u^-) > g(\gamma v^-)$. But we also have $I = -2k_R(g(\gamma u^+) - g(\gamma v^+))$, that implies that $g(\gamma u^+) < g(\gamma v^+)$. Consequently g changes at least twice its monotony that contradicts the assumption (2).

2. $\gamma u^- < \gamma v^+ < \gamma v^- < \gamma u^+$.

   As in the previous case, if $g(\gamma v^-) \geq 0$ or $g(\gamma v^+) \leq 0$, we have $I \geq 0$. If $g(\gamma v^-) < 0$ and $g(\gamma v^+) > 0$, there exists α in $]\gamma v^+, \gamma v^-[$ such that $g(\alpha) = 0$. Choosing $\kappa = \alpha$ in (9) written successively for u and v yields to $-2k_R g(\gamma u^+) \geq 0$ and $2k_R g(\gamma v^+) \geq 0$. Then $I \geq 0$.

3. $\gamma u^- < \gamma v^+ < \gamma u^+ < \gamma v^-$.

   . if $g(\gamma v^+) \geq 0$ or $g(\gamma u^+) \leq 0$, from (9), $I \geq 0$.

   . if $g(\gamma v^+) < 0$ and $g(\gamma u^+) > 0$, there exists β in $]\gamma v^+, \gamma u^+[$, such that $g(\beta) = 0$. By choosing $\kappa = \beta$ in (9), written for u and v, we show that $I \geq 0$.

All the others cases may be reduced to one of the previous situations. □

Now, in order to prove (13), thanks to Lemma 6 we choose in (14) the test-function φ such that, for any (t, x) in $[0, T[\times\Omega$,

$$\varphi(t, x) = \theta(t)\alpha_\varepsilon(x), \ \varepsilon > 0$$

where $\theta \in \mathcal{C}_c^\infty([0, T[)$ and α_ε is an element of $\mathcal{C}_c^\infty(\Omega)$ such that $\alpha_\varepsilon = 1$ on $]-1+\varepsilon, 1-\varepsilon[$ and $|\alpha'_\varepsilon| \leq \frac{2}{\varepsilon}$. We obtain:

$$\int_Q \{|u-v|\theta'(t)\alpha_\varepsilon(x) + k(x)\Phi(u, v)\theta(t)\alpha'_\varepsilon(x)\}dxdt + \int_\Omega |u_0 - v_0|\theta(0)\alpha_\varepsilon(x)dx \geq 0.$$

There is no difficulty to pass to the limit when ε goes to 0^+. We just point out that due to the properties of $(\alpha_\varepsilon)_\varepsilon$ and to the definition of $v_{\pm 1}^\tau$ and $u_{\pm 1}^\tau$,

$$\lim_{\varepsilon \to 0^+} \int_Q k(x)\Phi(u, v)\theta(t)\alpha'_\varepsilon(x)dq = \int_0^T \{k(-1)\Phi(u_{-1}^\tau, v_{-1}^\tau) - k(1)\Phi(u_1^\tau, v_1^\tau)\}\theta(t)dt.$$

It comes

$$\begin{gathered}\int_Q |u-v|\theta'(t)dxdt + \int_\Omega |u_0 - v_0|\theta(0)dx \ \geq \\ \int_0^T k(1)\Phi(u_1^\tau, v_1^\tau)\theta(t)dt - \int_0^T k(-1)\Phi(u_{-1}^\tau, v_{-1}^\tau)\theta(t)dt.\end{gathered}$$

Let's prove now that:

$$\int_0^T k(1)\Phi(u_1^\tau, v_1^\tau)\theta(t)dt - \int_0^T k(-1)\Phi(u_{-1}^\tau, v_{-1}^\tau)\theta(t)dt \ \geq \ 0 \ .$$

By coming back to Definition 1, we know that, for $a.e$ t in $]0, T[$, for all κ in $\mathbb{R}$:

$$\begin{aligned} k(1)(sgn(u_1^\tau(t) - \kappa) + sgn(\kappa))(g(u_1^\tau(t)) - g(\kappa)) &\geq 0, \\ k(1)(sgn(v_1^\tau(t) - \kappa) + sgn(\kappa))(g(v_1^\tau(t)) - g(\kappa)) &\geq 0. \end{aligned}$$

Thus, $a.e.$ on $]0, T[$,

. if $u_1^\tau(t)$ and $v_1^\tau(t)$ have the same sign, we choose $\kappa = v_1^\tau$ in the first inequality (or $\kappa = u_1^\tau(t)$ in the second one) to obtain:

$$k(1)\Phi(u_1^\tau(t), v_1^\tau(t)) \geq 0,$$

. if $u_1^\tau(t)$ and $v_1^\tau(t)$ have an opposite sign, choosing $\kappa = 0$ in the two inequalities gives:

$$k(1)\Phi(u_1^\tau(t), v_1^\tau(t)) \geq 0.$$

Hence, for *a.e.* t in $]0, T[$,

$$k(1)\Phi(u_1^\tau(t), v_1^\tau(t))\theta(t) \geq 0.$$

Similarly, for *a.e.* t in $]0, T[$,

$$k(-1)\Phi(u_{-1}^\tau, v_{-1}^\tau)\theta(t) \leq 0.$$

This way,

$$\int_Q |u - v|\theta_t(t)dxdt + \int_\Omega |u_0 - v_0|\theta(0)dx \geq 0.$$

The conclusion follows from classical arguments which completes the proof of Theorem 1. □

5 Existence of an entropy solution

The proof relies on a suitable regularization k_ε, $\varepsilon > 0$, of the function k and uses a compactness argument for the sequence $(k_\varepsilon \Phi(u_\varepsilon, \kappa))_{\varepsilon>0}$, where u_ε is the weak entropy solution to the corresponding mollified problem. We will consider two different situations:

$$\begin{cases} \exists \alpha \in \mathbb{R}^-, \ \forall x \leq \alpha, \ (k_L - k_R)g(x) \geq 0, \\ \exists \beta \in \mathbb{R}^+, \ \forall x \geq \beta, \ (k_L - k_R)g(x) \leq 0 \end{cases} \tag{17}$$

$$g(m) = g(M) = 0. \tag{18}$$

Remark 2. *As well (17) as (18) implies (10).*

Remark 3. *When we take into account (2) and (3), we observe that (18) includes the class of srictly convex (or strictly concave) functions that vanish at m and M. In addition (17) is fulfilled as soon as g is strictly monotone and vanishes at a point.*

In this framework we establish that:

Theorem 2. *The following assertions hold:*

(i) *Under (17) Problem (1) admits at least one entropy solution u.*

(ii) *Under (18) the problem (1) admits at least one entropy solution u such that, for a.e. (t, x) in Q, $m \leq u(t, x) \leq M$.*

Remark 4. *Under (2) and (3), the assumption (17) or (18) on g implies that*

$$\exists\ \kappa_0 \in \mathbb{R} \text{ such that } \Phi(., \kappa_0) \text{ is strictly monotone on } \mathbb{R}.$$

That is a key point that provides the strong convergence of the approximating sequence $(u_\varepsilon)_{\varepsilon>0}$.

We suppose first that the initial condition u_0 is smooth. Then through a Cauchy criterion in $L^1(Q)$ we come back to the situation of u_0 in $L^\infty(\Omega)$.

5.1 First step: $u_0 \in \mathcal{C}_c^\infty(\Omega)$

We apply the ideas introduced in [10] (also used in [1]) that is to consider a regular approximation of the function k. Let $(k_\varepsilon)_\varepsilon$ be a sequence of smooth functions such as, for every positive ε, $k_\varepsilon = k$ out of $]-\varepsilon, \varepsilon[$ and k_ε is monotone on $[-\varepsilon, \varepsilon]$ (depending on the sign of $k_L - k_R$). That implies:

$$\forall\, x \in \mathbb{R}^*, k_\varepsilon(x) \;\to\; k(x) \;\text{ and } |k_\varepsilon|_{BV(\mathbb{R})} \leq |k|_{BV(\mathbb{R})}.$$

Then we denote u_ε the unique entropy solution (see [2]) to the regularized problem:

Find a measurable and bounded function u in $BV(Q) \cap \mathcal{C}([0,T]; L^1(\Omega))$ such that formally

$$\left\{\begin{array}{rll} \dfrac{\partial u_\varepsilon}{\partial t} + \dfrac{\partial}{\partial x}(k_\varepsilon(x) g(u_\varepsilon)) = & 0 & \text{on } Q, \\ u_\varepsilon(0,x) = & u_0(x) & \text{on } \Omega, \\ u = & 0 & \text{on a part of }]0,T[\times\partial\Omega. \end{array}\right. \tag{19}$$

Lemma 7. **(i)** *Under (17), for any t in $]0,T[$, we set:*

$$R(t) = (\|u_0\|_\infty + \max(|\alpha|, \beta)) e^{M_k M_g t} + \frac{e^{M_k M_g t} - 1}{M_g} |g(0)|$$

where $M_k = \max(\|k'\|_{L^\infty(]-1,0[)}, \|k'\|_{L^\infty(]0,1[)})$ and $M_g = Lip(g)$.

Then

$$|u_\varepsilon(t,x)| \leq R(t), \;\; a.e. \; on \; \Omega.$$

(ii) *Under (18), $m \leq u_\varepsilon(t,x) \leq M$, for a.e.$(t,x) \in Q$.*

(iii) *Under (17) or (18), there exists a constant $C > 0$, such that:*

$$|k_\varepsilon(\Phi(u_\varepsilon, \kappa))|_{BV(Q)} \;\leq\; C(|u_0|_{BV(\Omega)} + |k|_{BV(Q)})$$

Proof. **(i)** Let $\mu > 0$. We introduce the viscous problem associated with (19):

Find $u_{\varepsilon,\mu}$ in $L^2(0,T; H^2(\Omega)) \cap \mathcal{C}([0,T]; H^1(\Omega))$, $\partial_t u_{\varepsilon,\mu} \in L^2(Q)$ such that,

$$\left\{\begin{array}{rcll} \partial_t u_{\varepsilon,\mu} + \partial_x(k_\varepsilon(x) g(u_{\varepsilon,\mu})) & = & \mu \partial_{xx} u_{\varepsilon,\mu} & a.e. \text{ in } Q, \\ u_{\varepsilon,\mu} & = & u_0 & \text{on } \Omega, \\ u(t,1) = u(t,-1) & = & 0 & \text{for any } t \in]0,T[. \end{array}\right. \tag{20}$$

It is known that (20) admits an unique solution that converges toward u_ε in $\mathcal{C}([0,T]; L^1(\Omega))$ when μ tends to 0. We multiply (20) with $(u_{\varepsilon,\mu} - R(t))^+$, we integrate over $Q_s =]0,s[\times\Omega$, $s \in]0,T[$, and perform the following transformations:

$$\mu \int_{Q_s} \partial_{xx} u_{\varepsilon,\mu} (u_{\varepsilon,\mu} - R(t))^+ dxdt = -\mu \int_{Q_s} [\partial_x (u_{\varepsilon,\mu} - R(t))^+]^2,$$

$$\int_{Q_s} \partial_t u_{\varepsilon,\mu} (u_{\varepsilon,\mu} - R(t))^+ dtdx = \frac{1}{2} \|(u_{\varepsilon,\mu}(s,.) - R(s))^+\|^2_{L^2(\Omega)} + \int_{Q_s} R'(t)(u_{\varepsilon,\mu} - R(t))^+ dtdx,$$

since $u_{\varepsilon,\mu}(0,.) = u_0 \leq R(0)$. Moreover

$$\int_{Q_s} \partial_x(k_\varepsilon(x)g(u_{\varepsilon,\mu}))(u_{\varepsilon,\mu}-R(t))^+dtdx = \int_{Q_s} k'_\varepsilon(x)g(R(t))(u_{\varepsilon,\mu}-R(t))^+dtdx$$
$$+\int_{Q_s} \partial_x(k_\varepsilon(x)(g(u_{\varepsilon,\mu})-g(R(t)))(u_{\varepsilon,\mu}-R(t))^+dtdx$$

where we use an integration by parts in the last term on the right-hand side. Then the Young inequality gives:

$$\left|\int_{Q_s} \partial_x(k_\varepsilon(x)(g(u_{\varepsilon,\mu})-g(R(t)))(u_{\varepsilon,\mu}-R(t))^+dtdx\right| \leq \mu \int_{Q_s} [\partial_x(u_{\varepsilon,\mu}-R(t))^+]^2$$
$$+\frac{1}{4\mu}\int_{Q_s}(k_\varepsilon M_g)^2((u_{\varepsilon,\mu}-R(t))^+)^2.$$

Gathering all terms yields to:

$$\frac{1}{2}\|(u_{\varepsilon,\mu}(s,.)-R(s))^+\|^2_{L^2(\Omega)}+\int_{Q_s}[R'(t)+k'_\varepsilon g(R(t))](u_{\varepsilon,\mu}-R(t))^+dtdx$$
$$\leq \frac{(\|k\|_\infty M_g)^2}{4\mu}\int_{Q_s}((u_{\varepsilon,\mu}-R(t))^+)^2dxdt.$$

Now let's show that the term

$$\Psi(t,x) = R'(t) + k'_\varepsilon g(R(t))$$

is nonnegative. Therefore, thanks to the Gronwall's Lemma the conclusion will follow.
On $[-\varepsilon,\varepsilon]$, by definition of $R(t)$, $R(t) \geq \beta$ and $R'(t) \geq 0$. So:

if $k_L \leq k_R$, $k'_\varepsilon \geq 0$, and by (17), $g(R(t)) \geq 0$. Then $\Psi(t,x) \geq 0$.

if $k_R \leq k_L$, $k'_\varepsilon \leq 0$, and by (17), $g(R(t)) \leq 0$. Then $\Psi(t,x) \geq 0$.

Besides, on $]-1,\varepsilon[\cup]\varepsilon,1[$, $k_\varepsilon = k$. Moreover:

$R'(t) = M_gM_kR(t) + M_k|g(0)|$, and $k'g(R(t)) \geq -M_kM_gR(t) + k'g(0)$.

So

$\Psi(t,x) \geq k'g(0) + M_k|g(0)| \geq 0.$

To show that $u_\varepsilon(t,x) \geq -R(t)$, we multiply (20) with $(u_{\varepsilon,\mu}+R(t))^-$ and we use the same techniques as before, especially the first line in (17).

(ii) The proof refers to **(i)** and basically lies on the fact that $g(m) = g(M) = 0$.

(iii) The BV-estimate is based on the maximum principle and the following estimates that are classical (see [10]):

$\mu \int_Q |\partial_{xx}u_{\varepsilon,\mu}|^2dxdt \leq C_1 \quad , \quad \int_\Omega |\partial_t u_{\varepsilon,\mu}|dx \leq C_2(|k_\varepsilon|_{BV(\Omega)} + \mu|u_0|_{L^1(\Omega)})$

where C_1 and C_2 are constants independent from μ.
As, in $\mathcal{D}'(Q)$,

$\partial_x(k_\varepsilon\Phi(u_{\varepsilon,\mu},\kappa)) \leq -\partial_t|u_{\varepsilon,\mu}-\kappa| - k'_\varepsilon sgn(u_{\varepsilon,\mu}-\kappa)g(\kappa) + \mu\partial_{xx}|u_{\varepsilon,\mu}-\kappa|,$

when μ tends to 0, the conclusion follows. □

As a consequence of Lemma 7, and of the compactness embedding of $BV(Q)$ into $L^1(Q)$ there exists χ in $L^\infty(Q)\cap BV(Q)$ and a subsequence of $(k_\varepsilon\Phi(u_\varepsilon,\kappa))_\varepsilon$ such that $(k_\varepsilon\Phi(u_\varepsilon,\kappa))_\varepsilon \to \chi$ *a.e.* on Q. Let $\kappa \in \mathbb{R}$. By remarking that $k_\varepsilon\Phi(u_\varepsilon,\kappa) = (k_\varepsilon - k)\Phi(u_\varepsilon,\kappa) + k\Phi(u_\varepsilon,\kappa)$, we deduce:

$$k\Phi(u_\varepsilon,\kappa) \longrightarrow \chi \quad a.e. \text{ in } Q\ .$$

As $\mathcal{L}\{x \in \Omega,\ k(x) = 0\} = 0$,

$$\Phi(u_\varepsilon, \kappa) \longrightarrow \frac{\chi}{k} \quad a.e. \text{ in } Q.$$

Thanks to Remark 4, there exists a subsequence of $(u_\varepsilon)_\varepsilon$ that tends in $L^1(Q)$ and $a.e.$ on Q to a limit denoted u. Consequently, up to a subsequence, $(k_\varepsilon \Phi(u_\varepsilon, \kappa))_\varepsilon$ converges toward $k\Phi(u, \kappa)$ in $L^1(Q)$ and $a.e.$ on Q.

Now we have to establish that u is an entropy solution to (1). First we prove that u fulfills (5). To this purpose, we introduce the regularized entropy pairs, for any $\kappa \in \mathbb{R}$, and any real τ:

$$\Phi_\eta(\tau) = \int_\kappa^\tau sgn_\eta(r-\kappa) g'(r) dr \quad \text{and} \quad I_\eta(\tau) = \int_\kappa^\tau sgn_\eta(r-\kappa) dr,$$

where sgn_η denotes the Lipschitzian approximation of the function sgn given for any positive η and any nonnegative real x by $sgn_\eta(x) = \min(\frac{x}{\eta}, 1)$ and $sgn(-x) = sgn(x)$.

By coming back to (20) and considering the test-function $v = sgn_\eta(u_{\varepsilon,\mu} - \kappa)\varphi$, $\varphi \in \mathcal{C}_c^\infty([0,T[\times\Omega)$, $\varphi \geq 0$, we can take the limit on μ with classical arguments. So we establish that u_ε fulfills the regularized entropy inequality for all φ in $\mathcal{C}_c^\infty([0,T[\times\Omega)$,

$$\begin{aligned} &\int_Q I_\eta(u_\varepsilon)\varphi_t dxdt + \int_Q k_\varepsilon(x)\Phi_\eta(u_\varepsilon)\varphi_x dxdt \\ &+ \int_Q k'_\varepsilon(x)(\Phi_\eta(u_\varepsilon) - I'_\eta(u_\varepsilon)g(u_\varepsilon))\varphi dxdt + \int_\Omega I_\eta(u_0)\varphi(0,x)dx \quad \geq \quad 0. \end{aligned} \tag{21}$$

We want to pass to the limit in (21), first with respect to ε and then with respect to η. The difficulty is only concentrated in the first term of the second line. That is why we write (with $dq = dxdt$):

$$\begin{aligned} \int_Q k'_\varepsilon(x)(\Phi_\eta(u_\varepsilon) - I'_\eta(u_\varepsilon)g(u_\varepsilon))\varphi dq &= \int_0^T\int_{-1}^{-\varepsilon} k'_\varepsilon(x)(\Phi_\eta(u_\varepsilon) - I'_\eta(u_\varepsilon)g(u_\varepsilon))\varphi dq \\ + \int_0^T\int_{-\varepsilon}^{\varepsilon} k'_\varepsilon(x)(\Phi_\eta(u_\varepsilon) - I'_\eta(u_\varepsilon)g(u_\varepsilon))\varphi dq &+ \int_0^T\int_{\varepsilon}^{1} k'_\varepsilon(x)(\Phi_\eta(u_\varepsilon) - I'_\eta(u_\varepsilon)g(u_\varepsilon))\varphi dq. \end{aligned} \tag{22}$$

However, owing to the definition of k_ε,

$$\int_0^T\int_{-1}^{-\varepsilon} k'_\varepsilon(x)(\Phi_\eta(u_\varepsilon) - I'_\eta(u_\varepsilon)g(u_\varepsilon))\varphi dq = \int_0^T\int_{-1}^{-\varepsilon} k'(x)(\Phi_\eta(u_\varepsilon) - I'_\eta(u_\varepsilon)g(u_\varepsilon))\varphi dq, \tag{23}$$

and

$$\int_0^T\int_{\varepsilon}^{1} k'_\varepsilon(x)(\Phi_\eta(u_\varepsilon) - I'_\eta(u_\varepsilon)g(u_\varepsilon))\varphi dq = \int_0^T\int_{\varepsilon}^{1} k'(x)(\Phi_\eta(u_\varepsilon) - I'_\eta(u_\varepsilon)g(u_\varepsilon))\varphi dq. \tag{24}$$

In addition, by referring to the definition of Φ_η and I_η,

$$\int_0^T\int_{-\varepsilon}^{\varepsilon} k'_\varepsilon(x)(\Phi_\eta(u_\varepsilon) - I'_\eta(u_\varepsilon)g(u_\varepsilon))\varphi dq = -\int_0^T\int_{-\varepsilon}^{\varepsilon} k'_\varepsilon g(\kappa) sgn_\eta(u_\varepsilon - \kappa)\varphi dq$$

$$+\int_0^T\int_{-\varepsilon}^{\varepsilon} k_\varepsilon'\varphi(\int_\kappa^{u_\varepsilon} sgn_\eta(\tau-\kappa)g'(\tau)d\tau-(g(u_\varepsilon)-g(\kappa))sgn_\eta(u_\varepsilon-\kappa))dq.$$

We look for a majoration of the right-hand side of this equality. First,

$$\left|\int_0^T\int_{-\varepsilon}^{\varepsilon} k_\varepsilon' g(\kappa)sgn_\eta(u_\varepsilon-\kappa)\varphi dq\right| \leq |g(\kappa)|\int_0^T\int_{-\varepsilon}^{\varepsilon}|k_\varepsilon'|\varphi dq.$$

Now we turn on to the estimate of the term $|D(u_\varepsilon)|$ where:

$$D(u_\varepsilon)=\int_\kappa^{u_\varepsilon} sgn_\eta(\tau-\kappa)g'(\tau)d\tau-(g(u_\varepsilon)-g(\kappa))sgn_\eta(u_\varepsilon-\kappa)).$$

More precisely, $|D(u_\varepsilon)|\leq C_g\eta$ with $C_g=2M_g$. Indeed, let us fix η and κ. Then $a.e.$ on Q:

- if $u_\varepsilon \geq \kappa+\eta$,

$$D(u_\varepsilon)=\int_\kappa^{\eta+\kappa}\frac{\tau-\kappa}{\eta}g'(\tau)d\tau+\int_{\kappa+\eta}^{u_\varepsilon} g'(\tau)d\tau-(g(u_\varepsilon)-g(\kappa))$$

and

$$|D(u_\varepsilon)|\leq \eta\|M_g+|g(\kappa+\eta)-g(\kappa)|,$$

because $0\leq \frac{\tau-\kappa}{\eta}\leq 1$. Then we use the Lipschitz condition for g.

- if $\kappa-\eta\leq u_\varepsilon\leq\kappa+\eta$,

$$D(u_\varepsilon)=\int_\kappa^{u_\varepsilon}\frac{\tau-\kappa}{\eta}g'(\tau)d\tau-\frac{u_\varepsilon-\kappa}{\eta}(g(u_\varepsilon)-g(\kappa))$$

and

$$|D(u_\varepsilon)|\leq M_g|u_\varepsilon-\kappa|+M_g\frac{|u_\varepsilon-\kappa|^2}{\eta}\leq 2M_g\eta.$$

- if $u_\varepsilon\leq\kappa-\eta$,

$$D(u_\varepsilon)=\int_{u_\varepsilon}^{\kappa-\eta} g'(\tau)d\tau+\int_{\kappa-\eta}^{\kappa}\frac{\tau-\kappa}{\eta}g'(\tau)d\tau+g(u_\varepsilon)-g(\kappa),$$

and $|D(u_\varepsilon)|\leq C_g\eta$, as in the first case.

Eventually we deduce that the term:

$$\int_0^T\int_{-\varepsilon}^{\varepsilon} k_\varepsilon'\varphi(\int_\kappa^{u_\varepsilon} sgn_\eta(\tau-\kappa)g'(\tau)d\tau-(g(u_\varepsilon)-g(\kappa))sgn_\eta(u_\varepsilon-\kappa))dq$$

is bounded by:

$$C_g\eta\int_0^T\int_{-\varepsilon}^{\varepsilon}|k_\varepsilon'|\varphi dxdt,$$

and

$$\int_0^T\int_{-\varepsilon}^{\varepsilon} k_\varepsilon'(x)(\Phi_\eta(u_\varepsilon)-I_\eta'(u_\varepsilon)g(u_\varepsilon))\varphi dq\leq (C_g\eta+|g(\kappa)|)\int_0^T\int_{-\varepsilon}^{\varepsilon}|k_\varepsilon'|\varphi dxdt. \tag{25}$$

As k_ε is monotone on $[-\varepsilon,\varepsilon]$, for ε small enough, $|k_\varepsilon'|=sgn(k_R-k_L)k_\varepsilon'$. So,

$$\int_0^T\int_{-\varepsilon}^{\varepsilon}|k_\varepsilon'|\varphi dq=sgn(k_R-k_L)\int_0^T\int_{-\varepsilon}^{\varepsilon}k_\varepsilon'\varphi dq$$

Then we integrate by parts to obtain:

$$\int_0^T\int_{-\varepsilon}^{\varepsilon} |k'_\varepsilon|\varphi dq = -\, sgn(k_R-k_L)\int_0^T\int_{-\varepsilon}^{\varepsilon} k_\varepsilon\varphi_x dq \\ +\, sgn(k_R-k_L)\int_0^T (k(\varepsilon)\varphi(t,\varepsilon)-k(-\varepsilon)\varphi(t,-\varepsilon))dt\ . \tag{26}$$

Finally, from (21), (22), (23), (24), (25) and (26), for any positive η and ε, we have:

$$\begin{aligned}
&-sgn(k_R-k_L)(C_g\eta+|g(\kappa)|)\int_0^T\int_{-\varepsilon}^{\varepsilon} k_\varepsilon\varphi_x dxdt + \int_Q k_\varepsilon(x)\Phi_\eta(u_\varepsilon)\varphi_x dxdt\\
&\int_0^T\int_{-1}^{-\varepsilon} k'(x)(\Phi_\eta(u_\varepsilon)-I'_\eta(u_\varepsilon)g(u_\varepsilon))\varphi dq + \int_\Omega I_\eta(u_0)\varphi(0,x)dx\\
&+\int_0^T\int_{\varepsilon}^{1} k'(x)(\Phi_\eta(u_\varepsilon)-I'_\eta(u_\varepsilon)g(u_\varepsilon))\varphi dq + \int_Q I_\eta(u_\varepsilon)\varphi_t dxdt\\
&+sgn(k_R-k_L)(C_g\eta+|g(\kappa)|)\int_0^T (k(\varepsilon)\varphi(t,\varepsilon)-k(-\varepsilon)\varphi(t,-\varepsilon))dt\ \geq 0.
\end{aligned}$$

We take now the ε-limit. Clearly, because $(u_\varepsilon)_\varepsilon$ goes to u in $L^1(Q)$ and since I_η and Φ_η are Lipschitzian,

$$\lim_{\varepsilon\to 0^+}\int_Q (I_\eta(u_\varepsilon)\varphi_t + k_\varepsilon(x)\Phi_\eta(u_\varepsilon)\varphi_x)dq = \int_Q (I_\eta(u)\varphi_t + k(x)\Phi_\eta(u)\varphi_x)dq.$$

Thanks to the definition of k_L and k_R and to the continuity of φ,

$$\lim_{\varepsilon\to 0^+}\int_0^T (k(\varepsilon)\varphi(t,\varepsilon)-k(-\varepsilon)\varphi(t,-\varepsilon))dt = (k_R-k_L)\int_0^T \varphi(t,0)dt.$$

Moreover, $k_\varepsilon\varphi_x$ being bounded independently with respect to ε,

$$\lim_{\epsilon\to 0^+}\int_0^T\int_{-\varepsilon}^{\varepsilon} k_\varepsilon\varphi_x dxdt = 0.$$

So, for any positive η, the following inequality holds:

$$\int_Q (I_\eta(u)\varphi_t + k(x)\Phi_\eta(u)\varphi_x)dxdt + \int_Q k'(x)(\Phi_\eta(u)-I'_\eta(u)g(u))\varphi dxdt \\ +\int_\Omega I_\eta(u_0)\varphi(0,x)dx + (|g(\kappa)|\ +\ C_g\eta)|k_R-k_L|\int_0^T \varphi(t,0)dt\ \geq 0. \tag{27}$$

We take the limit with respect to η through the Lebesgue dominated convergence Theorem, providing that u fulfills (5).

Lastly, let us establish that u satisfies (6)-(7). To this purpose, we use the functions H_η and Q_η defined in [8] for any τ, $\kappa\in\mathbb{R}$, by:

$$H_\eta(\tau,\kappa) = \left((\mathrm{dist}(\tau, I[0,\kappa]))^2+\eta^2\right)^{\frac{1}{2}} - \eta$$

and

$$Q_\eta(\tau,\kappa) = \int_\kappa^\tau \partial_1 H_\eta(\lambda,\kappa)g'(\lambda)d\lambda$$

where $I[0,\kappa]$ denotes the closed interval bounded by 0 and κ. The sequence $(H_\eta, Q_\eta)_\eta$ converges uniformly to $(\mathrm{dist}(\tau, I[0,\kappa]), \mathcal{G}(\tau,0,\kappa))$ where:

$$\mathcal{G}(\tau,0,\kappa) = \frac{1}{2}(\Phi(\tau,0) + \Phi(\kappa,0) + \Phi(\tau,\kappa)).$$

By taking in (20) the test-function $\partial_1 H_\eta(u_\varepsilon,\kappa)\varphi$, for any function $\varphi \in \mathcal{C}_c^\infty(]0,T[\times\overline{\Omega})$, we obtain for any positive η and ε the following inequality:

$$\begin{aligned}
&\int_Q H_\eta(u_\varepsilon,\kappa)\varphi_t dxdt + \int_Q k_\varepsilon Q_\eta(u_\varepsilon,\kappa)\varphi_x dxdt \\
&\int_Q k'_\varepsilon(x)(Q_\eta(u_\varepsilon,\kappa) - \partial_1 H_\eta(u_\varepsilon,\kappa) g(u_\varepsilon))\varphi dxdt \quad \geq \quad 0.
\end{aligned}$$

If we only consider functions φ vanishing in a neighborhood of $\{x_0 = 0\}$ containing $[-\varepsilon,\varepsilon]$ (that will not be restictive in the sequel), we can take the ε-limit without difficulty to obtain:

$$\begin{aligned}
&\int_Q H_\eta(u,\kappa)\varphi_t dxdt + \int_Q k Q_\eta(u,\kappa)\varphi_x dxdt \\
&\int_Q k'(x)(Q_\eta(u,\kappa) - \partial_1 H_\eta(u,\kappa) g(u))\varphi dxdt \quad \geq \quad 0.
\end{aligned} \tag{28}$$

Then, for $(t,x) \in]0,T[\times\overline{\Omega}$, we choose in (28) a sequence of test-functions defined by $\varphi_n(t,x) = \beta(t)\alpha_n(x)$ with $\beta \in \mathcal{C}_c^\infty(]0,T[)$, $\beta \geq 0$, and $\alpha_n \in \mathcal{C}_c^\infty(\overline{\Omega})$ such as $\alpha_n \geq 0$, $\alpha_n(x) = 0$ on $]-1, 1-\frac{1}{n}[$, $\alpha_n(1) = 1$ and $\|\alpha'_n\|_\infty \leq n$. On the one hand, by reasoning as in [8] we make sure that

$$\lim_{n\to\infty} \int_{1-\frac{1}{n}}^1 \int_0^T \alpha'_n(x) k(x) Q_\eta(u,\kappa)\beta(t) dtdx \text{ exists and is nonnegative.}$$

On the other hand by using the definition of u_1^τ,

$$\lim_{n\to\infty} \int_{1-\frac{1}{n}}^1 \int_0^T \alpha'_n(x) k(x) Q_\eta(u,\kappa)\beta(t) dtdx = \int_0^T k(1) Q_\eta(u_1^\tau(t),\kappa)\beta(t)dt.$$

Finally, when η goes to 0^+,

$$\int_0^T k(1)\mathcal{G}(u_1^\tau, 0, \kappa)\beta(t)dt \geq 0.$$

To conclude we just emphasize that the previous inequality is equivalent for all κ in $I[0,u_1^\tau]$, to:

$$sgn(u_1^\tau)k(1)(g(u_1^\tau) - g(\kappa)) \geq 0,$$

that is namely (6) when κ is reduced to belong to $I(0,u_1^\tau)$.
In the same way, by choosing $\varphi(t,x) = \beta(t)\delta_n(x)$ in (28), with $\beta \in \mathcal{C}_c^\infty(]0,T[)$, $\beta \geq 0$, and $\delta_n \in \mathcal{C}_c^\infty(\overline{\Omega})$ such as $\delta_n \geq 0$, $\delta_n(x) = 0$ on $]-1+\frac{1}{n}, 1[$, $\delta_n(-1) = 1$ and $\|\delta'_n\|_\infty \leq n$, by using the definition of u_{-1}^τ, we establish (7).

5.2 Second step: $u_0 \in L^\infty(\Omega)$

We use a mollification process to come back to the first step. Indeed, for $j \in \mathbb{N}^*$, we consider the sequence $(u_0^j)_j$ such that u_0^j belongs to $\mathcal{C}_c^\infty(\Omega)$ and (u_0^j) tends to u_0 in $L^1(\Omega)$. We denote u^j the entropy solution to (1) associated with the initial condition u_0^j so that, for any j, u^j fulfills (27) and (28). The comparison result (13) ensures that the sequence $(u_j)_j$ is a Cauchy sequence in $L^1(Q)$ and so tends to a limit, denoted u. Then the j-limit in (27) and (28) warrants that u is an entropy solution to (1).

To conclude, we point out that (17) or (18) implies (10), so that:

Corollary 1. *Assume that (17) or (18) holds. Then (1) has an unique entropy solution.*

6 Generalisation

In this section we keep the same assumptions on g but we consider that k has a finite number of discontinuities. Let $D = \{1, ..., n-1\}$, $n \neq 0$, $x_0 = -1$, $x_n = 1$. We suppose that:

$$k \text{ is discontinuous at } x_i, i \in D, \tag{29}$$

while $k_{|]x_i,x_{i+1}[} \in W^{1,+\infty}(]x_i, x_{i+1}[)$. Of course we need a new definition of an entropy solution which has to be equivalent to Definition 1 when D is reduced to one point. So we say that:

Definition 2. *Under (29), a function u of $L^\infty(Q)$ is an entropy solution to Problem (1) if u satisfies (6)-(7) and if, $\forall \kappa \in \mathbb{R}$, $\forall \varphi \in \mathcal{C}_c^\infty([0,T[\times\Omega)$, $\varphi \geq 0$,*

$$\left\{ \begin{array}{r} \displaystyle\int_Q (|u(t,x)-\kappa|\varphi_t(t,x) + k(x)\Phi(u,\kappa)\varphi_x(t,x))dxdt \\ \displaystyle - \int_Q k'(x) sgn(u-\kappa) g(\kappa)\varphi dxdt \\ \displaystyle + \int_\Omega |u_0-\kappa|\varphi(0,x)dx + \sum_{i\in D} |(k_i^+ - k_i^-)g(\kappa)| \int_0^T \varphi(t,x_i)dt \quad \geq 0 \end{array} \right. , \tag{30}$$

where

$$k_i^+ = \lim_{x\to x_i^+} k(x) \quad \text{and} \quad k_i^- = \lim_{x \to x_i^-} k(x)$$

We denote γu_i^+ and γu_i^- the strong traces in $L^\infty(]0,T[)$ at $\{x = x_i\}$. By using the same techniques as before we can state the following theorem:

Theorem 3. *Under (18), when k satisfies (29), there exists a unique entropy solution to (1). Moreover, at every point x_i, $i \in D$, u satisfies the Rankine-Hugoniot condition:*

$$k_i^+ g(\gamma u_i^+) = k_i^- g(\gamma u_i^-)$$

In addition, we adapt (17) under the form:

$$\left\{ \begin{array}{l} \text{for } i,j \in D, i \neq j, sgn(k_i^+ - k_i^-) = sgn(k_j^+ - k_j^-), \\ \exists \alpha \in \mathbb{R}^-,\ \forall x \leq \alpha,\ (k_1^- - k_1^+)g(x) \geq 0, \\ \exists \beta \in \mathbb{R}^+,\ \forall x \geq \beta,\ (k_1^- - k_1^+)g(x) \leq 0. \end{array} \right. \tag{31}$$

This condition is satisfied when g is strictly monotone and the next corollary holds:

Corollary 2. *Assume that (31) is satisfied. Then problem (1) has a unique entropy solution, in the sense of Definition 2.*

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