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On the optimal dividend problem for a spectrally negative Lévy process

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Abstract. In this paper we consider the optimal dividend problem for an insurance company whose risk process evolves as a spectrally negative Lévy process in the absence of dividend payments. The classical dividend problem for an insurance company consists in finding a dividend payment policy that maximizes the total expected discounted dividends. Related is the problem where we impose the restriction that ruin be prevented: the beneficiaries of the dividends must then keep the insurance company solvent by bail-out loans. Drawing on the fluctuation theory of spectrally negative Lévy processes we give an explicit analytical description of the optimal strategy in the set of barrier strategies and the corresponding value function, for either of the problems. Subsequently we investigate when the dividend policy that is optimal amongst all admissible ones takes the form of a barrier strategy.

Keywords: (Doubly) reflected Lévy processes, dividend problem, local time, scale functions, optimal control

MSC 2000: 60J99, 93E20, 60G51

1 Introduction

In classical collective risk theory (e.g. Gerber [11]) the surplus $X = \{X_t, t \geq 0\}$ of an insurance company with initial capital x is described by the Cramér-Lundberg model:

$$X_t = x + \mathtt{d}\,t - \sum_{k=1}^{N_t} C_k, \tag{1.1}$$

where C_k are i.i.d. positive random variables representing the claims made, $N = \{N_t, t \geq 0\}$ is an independent Poisson process modeling the times at which the claims occur, and $\mathtt{d}\,t$ represents the premium income up to time

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t. Under the assumption that the premium income per unit time $\mathtt{d}$ is larger than the average amount claimed $\lambda\mathbb{E}[C_1]$ the surplus in the Cramér-Lundberg model has positive first moment and has therefore the unrealistic property that it converges to infinity with probability one. In answer to this objection De Finetti [10] introduced the dividend barrier model, in which all surpluses above a given level are transferred to a beneficiary. In the mathematical finance and actuarial literature there is a good deal of work on dividend barrier models and the problem of finding an optimal policy for paying out dividends. Gerber & Shiu [12] and Jeanblanc & Shiryaev [15] consider the optimal dividend problem in a Brownian setting. Irbäck [14] and Zhou [26] study constant barriers under the model (1.1). Asmussen, Højgaard and Taksar [3] investigated excess-of-loss reinsurance and dividend distribution policies in a diffusion setting. Azcue and Muler [1] follow a viscosity approach to investigate optimal reinsurance and dividend policies in the Cramér-Lundberg model.

A drawback of the dividend barrier model is that under this model the risk process will down-cross the level zero with probability one. Several ways to combine dividend and ruin considerations are possible; here, we choose one studied in a Brownian motion setting by Harrison and Taylor [13] and Løkka and Zervos [19] involving bail-out loans to prevent ruin, over an infinite horizon.

In this paper we shall approach the dividend problem from the point of view of a general spectrally negative Lévy process. Drawing on the fluctuation theory for spectrally negative Lévy processes, we derive in Sections 3 and 4 expressions for the expectations of the discounted accumulated local time of a reflected and doubly reflected spectrally negative Lévy process, in terms of the scale functions of the Lévy process. Together with known results from the fluctuation theory of spectrally negative Lévy processes and control theory we apply these results in Section 5 to investigate the optimality of barrier dividend strategies for either of the dividend problems. Finally we conclude the paper with some explicit examples in the classical and 'bail-out' setting.

2 Problem setting

Let $X = \{X_t, t \geq 0\}$ be a Lévy process without positive jumps, that is, X is a stationary stochastic process with independent increments that has right-continuous paths with left-limits, only negative jumps and starts at $X_0 = 0$, defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbf{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, where $\mathbf{F} = \{\mathcal{F}_t\}_{t\geq 0}$ is a filtration that satisfies the usual conditions of right-continuity and completeness. Denote by $\{\mathbb{P}_x, x \in \mathbb{R}\}$ the family of probability measures corresponding to a translation of X such that $X_0 = x$, where we write $\mathbb{P} = \mathbb{P}_0$. Let $\mathbb{E}_x$ be expectation with respect to $\mathbb{P}_x$. To avoid trivialities, we exclude the case that X has monotone paths. For background on Lévy processes we refer to Sato [25] and Bertoin [6].

The process X models the risk-process of an insurance company or the cash fund of an investment company before dividends are deducted. Let π be a dividend strategy consisting of a non-decreasing left-continuous $\mathbf{F}$-adapted process

$\pi = \{L_t^\pi, t \geq 0\}$ with $L_0^\pi = 0$, where L_t^π represents the cumulative dividends paid out by the company up till time t. The risk process with initial capital $x > 0$ and controlled by a dividend policy π is then given by $U^\pi = \{U_t^\pi, t \geq 0\}$, where

$$U_t^\pi = X_t - L_t^\pi, \tag{2.1}$$

with $X_0 = x$. Writing $\sigma^\pi = \inf\{t \geq 0 : U_t^\pi < 0\}$ for the time at which ruin occurs, a dividend strategy is called admissible if, at any time before ruin, a lump sum dividend payment is smaller than the size of the available reserves: $L_{t+}^\pi - L_t^\pi < U_t^\pi$ for $t < \sigma^\pi$. Denoting the set of all admissible strategies by Π, the expected value discounted at rate $q > 0$ associated to the dividend policy $\pi \in \Pi$ with initial capital $x > 0$ is given by

$$v_\pi(x) = \mathbb{E}_x \left[\int_0^{\sigma^\pi} \mathrm{e}^{-qt} \mathrm{d}L_t^\pi \right].$$

The objective of the beneficiaries of the insurance company is to maximize $v_\pi(x)$ over all admissible strategies π:

$$v_*(x) = \sup_{\pi \in \Pi} v_\pi(x). \tag{2.2}$$

Consider next the situation where the insurance company is not allowed to go bankrupt and the beneficiary of the dividends is required to inject capital into the insurance company to ensure its risk process stays non-negative. In this setting a dividend policy $\overline{\pi} = \{L^{\overline{\pi}}, R^{\overline{\pi}}\}$ is a pair of non-decreasing $\mathbf{F}$-adapted processes with $R_0^\pi = L_0^\pi = 0$ such that $R^{\overline{\pi}} = \{R_t^{\overline{\pi}}, t \geq 0\}$ is a right-continuous process describing the cumulative amount of injected capital and $L^{\overline{\pi}} = \{L_t^{\overline{\pi}}, t \geq 0\}$ is a left-continuous process representing the cumulative amount of paid dividends. Under policy $\overline{\pi}$ the controlled risk process with initial reserves $x > 0$ satisfies

$$V_t^{\overline{\pi}} = X_t - L_t^{\overline{\pi}} + R_t^{\overline{\pi}},$$

where $X_0 = x$. The set of admissible policies $\overline{\Pi}$ consists of those policies for which $V_t^{\overline{\pi}}$ is non-negative for $t > 0$ and

$$\int_0^\infty \mathrm{e}^{-qt} \mathrm{d}R_t^{\overline{\pi}} < \infty, \qquad \mathbb{P}_x\text{-almost surely.} \tag{2.3}$$

The value associated to the strategy $\overline{\pi} \in \overline{\Pi}$ starting with capital $x > 0$ is then given by

$$\overline{v}_{\overline{\pi}}(x) = \mathbb{E}_x \left[\int_0^\infty \mathrm{e}^{-qt} \mathrm{d}L_t^{\overline{\pi}} - \varphi \int_0^\infty \mathrm{e}^{-qt} \mathrm{d}R_t^{\overline{\pi}} \right],$$

where φ is the cost per unit injected capital, and the associated objective then reads as

$$\overline{v}_*(x) = \sup_{\overline{\pi} \in \overline{\Pi}} \overline{v}_{\overline{\pi}}(x). \tag{2.4}$$

To ensure that the value function is finite and to avoid degeneracies, we assume that $\mathbb{E}_x[X_1] > -\infty$, $q > 0$ and $\varphi > 1$. To illustrate what happens if φ is (close to) one, we consider the case that $\varphi = 1$ and X is given by (1.1). In this setting, it is no more expensive to pay incoming claims from the reserves or by a bail-out loan, and therefore, as a consequence of the positive discount-factor $q > 0$, it is optimal to pay-out all reserves and premiums immediately as dividends and to pay all claims by bail-out loans.

A subclass of possible dividend policies for (2.2), denoted by $\Pi_{\leq C}$, is formed by the set of all strategies $\pi \in \Pi$ under which the controlled risk process U^π stays below the constant level $C \geq 0$, $U^\pi(t) \leq C$ for all $t > 0$. Examples of an element in Π_C is a *constant barrier strategy* π_a at level $a \leq C$ that correspond to reducing the risk process U to the level a if $x > a$, by paying out the amount $(x - a)^+$, and subsequently paying out the minimal amount of dividends to keep the risk process below the level a. Similarly, in problem (2.4), the double barrier strategy $\overline{\pi}_{0,a}$ with a lower barrier at zero and an upper barrier at level a consist in extracting the required amount of capital to bring the risk process down to the level a and subsequently paying out or in the minimal amount of capital required to keep the risk process between 0 and a. In the next section we shall use fluctuation theory of spectrally negative Lévy processes to identify the value functions in problems (2.2) and (2.4) corresponding to the constant barrier strategies π_a and $\overline{\pi}_{0,a}$.

3 Reflected Lévy processes

We first review some fluctuation theory of spectrally negative Lévy processes and refer the reader for more background to Bingham [8], Bertoin [6, 7], Kyprianou [16] and Pistorius [20, 21] and references therein.

3.1 Preliminaries

Since the jumps of a spectrally negative Lévy process X are all non-positive, the moment generating function $\mathbb{E}[\mathrm{e}^{\theta X_t}]$ exists for all $\theta \geq 0$ and is given by $\mathbb{E}[\mathrm{e}^{\theta X_t}] = \mathrm{e}^{t\psi(\theta)}$ for some function $\psi(\theta)$ that is well defined at least on the positive half-axes where it is strictly convex with the property that $\lim_{\theta\to\infty} \psi(\theta) = +\infty$. Moreover, ψ is strictly increasing on $[\Phi(0), \infty)$, where $\Phi(0)$ is the largest root of $\psi(\theta) = 0$. We shall denote the right-inverse function of ψ by $\Phi : [0, \infty) \to [\Phi(0), \infty)$.

For any θ for which $\psi(\theta) = \log \mathbb{E}[\exp \theta X_1]$ is finite we denote by $\mathbb{P}^\theta$ an exponential tilting of the measure $\mathbb{P}$ with Radon-Nikodym derivative with respect to $\mathbb{P}$ given by

$$\left.\frac{\mathrm{d}\mathbb{P}^\theta}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(\theta X_t - \psi(\theta)t\right). \tag{3.1}$$

Under the measure $\mathbb{P}^\theta$ the process X is still a spectrally negative Lévy process

with characteristic function ψ_θ given by

$$\psi_\theta(s) = \psi(s+\theta) - \psi(\theta). \tag{3.2}$$

Denote by σ the Gaussian coefficient and by ν the Lévy measure of X. We recall that if X has bounded variation it takes the form $X_t = \mathtt{d}t - S_t$ for a subordinator S and constant $\mathtt{d} > 0$, also referred to as the infinitesimal drift of X. Throughout the paper we assume that the following (regularity) condition is satisfied:

$$\sigma > 0 \qquad \text{or} \qquad \int_{-1}^{0} x\nu(\mathrm{d}x) = \infty \qquad \text{or} \qquad \nu(\mathrm{d}x) << \mathrm{d}x. \tag{3.3}$$

3.2 Scale functions

For $q \geq 0$, there exists a function $W^{(q)} : [0,\infty) \to [0,\infty)$, called the *q-scale function*, that is continuous and increasing with Laplace transform

$$\int_0^\infty \mathrm{e}^{-\theta x} W^{(q)}(y)\mathrm{d}y = (\psi(\theta) - q)^{-1}, \qquad \theta > \Phi(q). \tag{3.4}$$

The domain of $W^{(q)}$ is extended to the entire real axis by setting $W^{(q)}(y) = 0$ for $y < 0$. For later use we mention some properties of the function $W^{(q)}$ that have been obtained in the literature. On $(0,\infty)$ the function $y \mapsto W^{(q)}(y)$ is right- and left-differentiable and, as shown in [18], under the condition (3.3), it holds that $y \mapsto W^{(q)}(y)$ is continuously differentiable for $y > 0$. The value of the scale function and its derivative in zero can be derived from the Laplace transform (3.4) to be equal to

$$W^{(q)}(0) = 1/\mathtt{d} \text{ and } W^{(q)\prime}(0^+) = (q + \nu(-\infty,0))/\mathtt{d}^2, \tag{3.5}$$

if X has bounded variation, and $W^{(q)}(0) = W(0) = 0$ if X has unbounded variation (see e.g. [16, Exc. 8.5 and Lemma 8.3]). Moreover, if $\sigma > 0$ it holds that $W^{(q)} \in C^\infty(0,\infty)$ with $W^{(q)\prime}(0^+) = 2/\sigma^2$; if X has unbounded variation with $\sigma = 0$, it holds that $W^{(q)\prime}(0^+) = \infty$ (see [21, Lemma 4] and [22, Lemma 1]).

The function $W^{(q)}$ plays a key role in the solution of the two-sided exit problem as shown by the following classical identity. Letting T_a^+, T_a^- be the entrance times of X into (a,∞) and $(-\infty,-a)$ respectively,

$$T_a^+ = \inf\{t \geq 0 : X_t > a\} \qquad T_a^- = \inf\{t \geq 0 : X_t < -a\}$$

and $T_{0,a} = T_0^- \wedge T_a^+$ the first exit time from $[0,a]$ it holds for $y \in [0,a]$ that

$$\mathbb{E}_y\left[\exp(-qT_{0,a})1_{\{T_0^- > T_a^+\}}\right] = W^{(q)}(y)/W^{(q)}(a), \tag{3.6}$$

where 1_A is the indicator of the event A. Closely related to $W^{(q)}$ is the function $Z^{(q)}$ given by

$$Z^{(q)}(y) = 1 + q\overline{W}^{(q)}(y),$$

where $\overline{W}^{(q)}(y) = \int_0^y W^{(q)}(z)\mathrm{d}z$ is the anti-derivative of $W^{(q)}$. The name q-scale function for $W^{(q)}$ and $Z^{(q)}$ is justified as these functions are harmonic for the process X killed upon entering $(-\infty, 0)$, in the sense that

$$\{\mathrm{e}^{-q(t\wedge T_0^-)}Z^{(q)}(X_{t\wedge T_0^-}), t \geq 0\} \text{ and } \{\mathrm{e}^{-q(t\wedge T_0^-)}W^{(q)}(X_{t\wedge T_0^-}), t \geq 0\} \tag{3.7}$$

are martingales, as shown in [21, Prop. 3]. Appealing to this martingale property one can show the following relation between $W^{(q)}$ and its anti-derivative:

Lemma 1 *For $y \in [0, a]$ and $a > 0$ it holds that*

$$\overline{W}^{(q)}(y)/\overline{W}^{(q)}(a) \leq W^{(q)}(y)/W^{(q)}(a).$$

Proof Writing $h(y) = \overline{W}^{(q)}(y)/\overline{W}^{(q)}(a) - W^{(q)}(y)/W^{(q)}(a)$ as

$$h(y) = q^{-1}Z^{(q)}(y)/\overline{W}^{(q)}(a) - W^{(q)}(y)/W^{(q)}(a) - q^{-1}/\overline{W}^{(q)}(a)$$

and using the martingale property of $Z^{(q)}$ and $W^{(q)}$ in conjunction with the optional stopping theorem it follows that $\{\mathrm{e}^{-q(t\wedge T_{0,a})}h(X_{t\wedge T_{0,a}}), t \geq 0\}$ can be written as the sum of a martingale and an increasing process and is thus a sub-martingale. Therefore

$$h(y) \leq \mathbb{E}_y[\mathrm{e}^{-q(t\wedge T_{0,a})}h(X_{t\wedge T_{0,a}})] \leq \mathbb{E}_y[\mathrm{e}^{-qT_{0,a}}h(X_{T_{0,a}})] \leq 0,$$

where the last inequality follows since $h(y) = 0$ for $y \in (-\infty, 0] \cup \{a\}$. □

3.3 Reflection at the supremum

Write I and S for the running infimum and supremum of X respectively, that is,

$$I_t = \inf_{0\leq s\leq t}(X_s \wedge 0) \qquad \text{and} \qquad S_t = \sup_{0\leq s\leq t}(X_s \vee 0),$$

where we used the notations $c \vee 0 = \max\{c, 0\}$ and $c \wedge 0 = \min\{c, 0\}$. By $Y = X - I$ and $\widehat{Y} = S - X$ we denote the Lévy process X reflected at its past infimum I and at its past supremum S, respectively. Denoting by $\eta(q)$ an independent random variable with parameter q, it follows, by duality and the Wiener-Hopf factorisation of X (e.g. Bertoin [6] p. 45 and pp. 188 – 192 respectively), that

$$S_{\eta(q)} \sim Y_{\eta(q)} \sim \exp(\Phi(q)). \tag{3.8}$$

Further, it was shown in [4] and [21] that the Laplace transform of the entrance time τ_a of the reflected process Y into (a, ∞) [resp. the entrance time $\widehat{\tau}_a$ of $\widehat{Y}$ into (a, ∞)] can be expressed in terms of the functions $Z^{(q)}$ and $W^{(q)}$ as follows

$$\mathbb{E}_y[\mathrm{e}^{-q\tau_a}] = \frac{Z^{(q)}(y)}{Z^{(q)}(a)}, \tag{3.9}$$

$$\mathbb{E}_{-y}[\mathrm{e}^{-q\widehat{\tau}_a}] = Z^{(q)}(a-y) - qW^{(q)}(a-y)\frac{W^{(q)}(a)}{W^{(q)\prime}(a)}, \tag{3.10}$$

where $y \in [0,a]$ and where we note that under $\mathbb{P}_y$ $[\mathbb{P}_{-y}]$ it holds that $Y_0 = y$ $[\widehat{Y}_0 = y]$. The identity (3.10) together with the strong Markov property implies the martingale property of

$$\mathrm{e}^{-q(t\wedge\widehat{\tau}_a)}\left\{Z^{(q)}(a-\widehat{Y}_{t\wedge\widehat{\tau}_a}) - qW^{(q)}(a-\widehat{Y}_{t\wedge\widehat{\tau}_a})\frac{W^{(q)}(a)}{W^{(q)\prime}(a)}\right\}. \tag{3.11}$$

Denote by $\pi_a = \{L_t^a, t \leq \sigma_a\}$ the constant barrier strategy at level a and let $U^a = U^{\pi_a}$ be the corresponding risk process. If $U_0^a \in [0,a]$, the strategy π_a corresponds to a reflection of the process $X - a$ at its supremum: for $t \leq \sigma_a$ process L_t^a can be explicitly represented by

$$L_t^a = \sup_{s\leq t}[X_s - a] \vee 0.$$

Note that the process π_a is a Markov local time of U^a at a, that is, π_a is increasing, continuous and adapted such that the support of the Stieltjes measure $\mathrm{d}L_t^a$ is contained in the closure of the set $\{t : U_t^a = a\}$ (See e.g. Bertoin [6, Ch. IV] for background on local times). In the case that $U_0 = x > a$, L_t^a has a jump at $t = 0$ of size $x - a$ to bring U^a to the level a and a similar structure afterwards: $L_t^a = (x-a)1_{\{t>0\}} + \sup_{s\leq t}[X_s - x] \vee 0$.

The following result concerns the value function associated to the dividend policy π_a:

Proposition 1 *Let $a > 0$. For $x \in [0,a]$ it holds that*

$$\mathbb{E}_x\left[\int_0^{\sigma_a} \mathrm{e}^{-qt}\mathrm{d}L_t^a\right] = \mathbb{E}_{x-a}\left[\int_0^{\widehat{\tau}_a} \mathrm{e}^{-qt}\mathrm{d}S_t\right] = \frac{W^{(q)}(x)}{W^{(q)\prime}(a)}, \tag{3.12}$$

where $\sigma_a = \sigma^{\pi_a} = \inf\{t \geq 0 : U_t^a < 0\}$ is the ruin time.

Proof By spatial homogeneity of the Lévy process X, it follows that the ensemble $\{U_t^a, L_t^a, t \leq \sigma_a; U_0 = x\}$ has the same law as $\{a - \widehat{Y}_t, S_t, t \leq \widehat{\tau}_a; \widehat{Y}_0 = a - x\}$. Noting that $\widehat{Y}_0 = a - x$ precisely if $X_0 = x - a$ (since then $S_0 = 0$), the first equality of (3.12) is seen to hold true. Using excursion theory it was shown in the proof of [4, Thm. 1] that

$$\mathbb{E}_0\left[\int_0^{\widehat{\tau}_a} \mathrm{e}^{-qt}\mathrm{d}S_t\right] = \frac{W^{(q)}(a)}{W^{(q)\prime}(a)}. \tag{3.13}$$

Applying the strong Markov property of $\widehat{Y}$ at $\widehat{\tau}_0 = \inf\{t \geq 0 : \widehat{Y}_t = 0\}$ and using that $\{\widehat{Y}_t, t \leq \widehat{\tau}_0\}$ is in law equal to $\{-X_t, t \leq T_0^+, X_0 = -\widehat{Y}_0 = x - a\}$ we find that

$$\begin{aligned}\mathbb{E}_{x-a}\left[\int_0^{\widehat{\tau}_a} \mathrm{e}^{-qt}\mathrm{d}S_t\right] &= \mathbb{E}_{x-a}[\mathrm{e}^{-q\widehat{\tau}_0}1_{\{\widehat{\tau}_0<\widehat{\tau}_a\}}]\mathbb{E}_0\left[\int_0^{\widehat{\tau}_a} \mathrm{e}^{-qt}\mathrm{d}S_t\right] \\ &= \mathbb{E}_{x-a}[\mathrm{e}^{-qT_0^+}1_{\{T_0^+<T_a^-\}}]\mathbb{E}_0\left[\int_0^{\widehat{\tau}_a} \mathrm{e}^{-qt}\mathrm{d}S_t\right].\end{aligned}$$

Inserting the identities (3.13) and (3.6) into this equation finishes the proof. □

Let us complement the previous result by considering what happens in the case that the barrier is taken to be 0. If X has unbounded variation, 0 is regular for $(-\infty, 0)$ so that U^0 immediately enters the negative half-axis and $\mathbb{P}_0(\sigma_0 = 0) = 1$, and the rhs of (3.12) is zero (if $x = a = 0$). If $\nu(-\infty, 0)$ is finite, U_0 enters $(-\infty, 0)$ when the first jump occurs so that σ_0 is exponential with mean $\nu(-\infty, 0)^{-1}$ and

$$\mathbb{E}_0\left[\int_0^{\sigma_0} \mathrm{e}^{-qt}\mathrm{d}L_t^0\right] = \mathtt{d}\mathbb{E}_0\left[\int_0^{\sigma_0} \mathrm{e}^{-qt}\mathrm{d}t\right] = \frac{\mathtt{d}}{q + \nu(-\infty, 0)}. \tag{3.14}$$

If ν is infinite but X has bounded variation, the validity of (3.14) follows by approximation. Combining these observations with (3.5), we note that (3.12) remains valid for $x = a = 0$ if $W^{(q)\prime}(a)$ for $a = 0$ is understood to be $W^{(q)\prime}(0^+)$.

In view of (3.8), (3.13) and since $a \mapsto \widehat{\tau}_a$ is non-decreasing with $\lim_{a\to\infty} \widehat{\tau}_a = +\infty$ a.s., we note for later reference that $W^{(q)}/W^{(q)\prime}$ is an increasing function on $(0, \infty)$ with limit

$$\lim_{a\to\infty} \frac{W^{(q)}(a)}{W^{(q)\prime}(a)} = \mathbb{E}_0[S_{\eta(q)}] = \frac{1}{\Phi(q)}. \tag{3.15}$$

3.4 Martingales and overshoot

In the sequel we shall need the following identities of expected discounted overshoots and related martingales in terms of the anti-derivative $\overline{Z}^{(q)}(y)$ of $Z^{(q)}(y)$ which is for $y \in \mathbb{R}$ defined by

$$\overline{Z}^{(q)}(y) = \int_0^y Z^{(q)}(z)\mathrm{d}z = y + q\int_0^y \int_0^z W^{(q)}(w)\mathrm{d}w\mathrm{d}z.$$

Note that $\overline{Z}^{(q)}(y) = y$ for $y < 0$, since we set $W^{(q)}(y) = 0$ for $y < 0$.

Proposition 2 *If $\psi'(0^+) > -\infty$, then the processes*

$$\mathrm{e}^{-q(t\wedge T_0^-)}\left\{\overline{Z}^{(q)}(X_{t\wedge T_0^-}) + \psi'(0^+)/q\right\}$$

and

$$\mathrm{e}^{-q(t\wedge\widehat{\tau}_a)}\left\{\overline{Z}^{(q)}(a - \widehat{Y}_{t\wedge\widehat{\tau}_a}) + \psi'(0^+)/q - W^{(q)}(a - \widehat{Y}_{t\wedge\widehat{\tau}_a})Z^{(q)}(a)/W^{(q)\prime}(a)\right\},$$

are martingales. In particular, it holds that for $y \in [0, a]$ and $x \geq 0$,

$$\mathbb{E}_{y-a}[\mathrm{e}^{-q\widehat{\tau}_a}(a - \widehat{Y}_{\widehat{\tau}_a})] = \overline{Z}^{(q)}(y) - \psi'(0^+)\overline{W}^{(q)}(y) - CW^{(q)}(y), \tag{3.16}$$

$$\mathbb{E}_x[\mathrm{e}^{-qT_0^-}X_{T_0^-}] = \overline{Z}^{(q)}(x) - \psi'(0^+)\overline{W}^{(q)}(x) + DW^{(q)}(x), \tag{3.17}$$

where $D = [\psi'(0^+)\Phi(q) - q]/\Phi(q)^2$ and $C = [Z^{(q)}(a) - \psi'(0^+)W^{(q)}(a)]/W^{(q)\prime}(a)$.

Proof We first show the validity of the identities (3.17) and (3.16). Writing $W_v^{(q)}$ and $Z_v^{(q)}$ for the ('tilted') q-scale functions of X under $\mathbb{P}^v$ we read off from [4, Thm.1] and [17, Thm. 4] that for $\kappa := q - \psi(v) \geq 0$, $x \geq 0$ and $y \leq a$ it holds that

$$\mathbb{E}_{y-a}[\mathrm{e}^{-q\widehat{\tau}_a - v(\widehat{Y}_{\widehat{\tau}_a} - a)}] = \mathrm{e}^{vy}\left[Z_v^{(\kappa)}(y) - C_v W_v^{(\kappa)}(y)\right], \tag{3.18}$$

$$\mathbb{E}_x[\mathrm{e}^{-qT_0^- + vX_{T_0^-}}] = \mathrm{e}^{vx}\left[Z_v^{(\kappa)}(x) - D_v W_v^{(\kappa)}(x)\right], \tag{3.19}$$

where $D_v = \kappa/[\Phi(q) - v]$ and $C_v = [\kappa W_v^{(\kappa)}(a) + vZ_v^{(\kappa)}(a)]/[W_v^{(\kappa)\prime}(a) + vW_v^{(\kappa)}(a)]$. The 'tilted' scale functions can be linked to non-tilted scale functions via the relation $\mathrm{e}^{vy}W_v^{(q-\psi(v))}(y) = W^{(q)}(y)$ from [4, Remark 4]. This relation implies that $\mathrm{e}^{vy}[W_v^{(\kappa)\prime}(y) + vW_v^{(\kappa)}(y)] = W^{(q)\prime}(y)$ and

$$Z_v^{(\kappa)}(y) = 1 + \kappa \int_0^y \mathrm{e}^{-vz} W^{(q)}(z)\mathrm{d}z.$$

In view of these relations it is a matter of algebra to verify that the right-derivatives with respect to v in $v = 0$ of D_v, C_v and $\ell(v) := \mathrm{e}^{vy}Z_v^{(q-\psi(v))}(y)$ are respectively equal to the constants D and C given in the statement of the Proposition and

$$\ell'(0^+) = \overline{Z}^{(q)}(y) - \psi'(0^+)\overline{W}^{(q)}(y).$$

Differentiating (3.18) and (3.19) and inserting the derived results we arrive at the equations (3.17) and (3.16).

Write now h_1, h_2 for the right-hand sides of (3.17) and (3.16) respectively. From the overshoot identities (3.17) and (3.16) and the definition for $y < 0$ of $W^{(q)}(y)$, $\overline{Z}^{(q)}(y)$ and $\overline{W}^{(q)}(y)$, it is straightforward to verify that

$$\mathbb{E}_x[\mathrm{e}^{-qT_0^-} h_1(X_{T_0^-})] = h_1(x), \qquad \mathbb{E}_{y-a}[\mathrm{e}^{-q\widehat{\tau}_a} h_2(a - \widehat{Y}_{\widehat{\tau}_a})] = h_2(y).$$

The strong Markov property then implies that for $t \geq 0$,

$$\mathbb{E}_x[\mathrm{e}^{-qT_0^-} h_1(X_{T_0^-})|\mathcal{F}_t] = \mathrm{e}^{-q(t\wedge T_0^-)} h_1(X_{t\wedge T_0^-}),$$
$$\mathbb{E}_{y-a}[\mathrm{e}^{-q\widehat{\tau}_a} h_2(a - \widehat{Y}_{\widehat{\tau}_a})|\mathcal{F}_t] = \mathrm{e}^{-q(t\wedge\widehat{\tau}_a)} h_2(a - \widehat{Y}_{t\wedge\widehat{\tau}_a})$$

and, in view of (3.7) and (3.11), the stated martingale properties follow. □

4 Doubly reflected Lévy processes

Now we turn to the computation of the value function corresponding to the constant barrier strategy $\overline{\pi}_{0,a} = \{L_t^a, R_t^0, t \geq 0\}$ that consists of imposing 'reflecting' barriers L^a and R^0 at a and 0 respectively. When the initial capital $X_0 = x \in [0, a]$ the risk process $V_t^a := V_t^{\overline{\pi}_{0,a}}$ is a doubly reflected spectrally negative Lévy process. Informally, this process moves as a Lévy process whilst

it is inside $[0,a]$ but each time it attempts to down-cross 0 or up-cross a it is 'regulated' to keep it inside the interval $[0,a]$. In [20] a path-wise construction of a doubly reflected Lévy process was given, showing that V^a is a strong Markov process. See also Asmussen [2, XIV.3] for a discussion of processes with two reflecting barriers in the context of queueing models. It was shown ([20, Thm. 1]) that a version of the potential measure $\tilde{U}^q(x,\mathrm{d}y) = \int_0^\infty \mathbb{P}_x(V_t^a \in \mathrm{d}y)$ of V^a is given by $\tilde{U}^q(x,\mathrm{d}y) = \tilde{u}^q(x,0)\delta_0(\mathrm{d}y) + \tilde{u}^q(x,y)\mathrm{d}y$ where δ_0 is the pointmass in zero, $\tilde{u}^q(x,0) = Z^{(q)}(a-x)W^{(q)}(0)/(qW^{(q)}(a))$ and

$$\tilde{u}^q(x,y) = \frac{Z^{(q)}(a-x)W^{(q)\prime}(y)}{qW^{(q)}(a)} - W^{(q)}(y-x), \qquad x,y \in [0,a], y \neq 0. \tag{4.1}$$

For $t \geq 0$, V_t^a can be expressed in terms of X, L^a and R^0 as

$$V_t^a = X_t - L_t^a + R_t^0 \tag{4.2}$$

for some increasing adapted processes L^a and R^0 such that the supports of the Stieltjes measures $\mathrm{d}L_t^a$ and $\mathrm{d}R_t^0$ are included in the closures of the sets $\{t : V_t^a = a\}$ and $\{t : V_t^a = 0\}$ respectively. For completeness we extend the construction in [20] to a simultaneous construction of the processes L^a, R^0 and V^a, when $X_0 \in [0,a]$:

0. Set $\sigma = T_{0,a}$. For $t < \sigma$, set $L_t^a = R_t^0 = 0$ and $V_t^a = X_t$.

   If $X_\sigma \leq 0$ set $\xi := X_\sigma$ and go to step 2; else set $L_\sigma^a = 0$ and $V_\sigma^a = a$ and go to step 1.

1. Set $Z_t = X_t - X_\sigma$. For $\sigma < t < \sigma' := \inf\{u \geq \sigma : Z_u \leq -a\}$, set

   $$L_t^a = L_\sigma^a + \sup_{\sigma \leq s \leq t} [Z_s \vee 0], \qquad V_t^a = a + Z_t - (L_t^a - L_\sigma^a)$$

   and let $R_t^0 = R_\sigma^0$. Set $\sigma := \sigma'$ and $\xi = X_{\sigma'} - X_\sigma + a$ and go to step 2.

2. Set $Z_t = X_t - X_\sigma$. For $\sigma \leq t \leq \sigma'' := \inf\{u > \sigma : Z_u = a\}$, set

   $$R_t^0 = R_{\sigma^-}^0 - \xi - \inf_{\sigma \leq s \leq t} Z_s \wedge 0 \qquad V_t^a = Z_t + R_t^0 - R_{\sigma^-}^0$$

   and let $L_t^a = L_\sigma^a$. Set $\sigma := \sigma''$ and go to step 1.

It can be verified by induction that the process V constructed in this way satisfies $V_t \in [0,a]$ and L^a and R^0 are processes with the required properties such that (4.2) holds.

Remark. If the initial capital $x > a$, then above construction can be easily adapted: in step 0 set $L_0^a = 0, V_0^a = x$ and $L_{0+}^a = x - a, V_{0+}^a = a$ and in step 1 set $\sigma = 0$ and replace L_0^a by L_{0+}^a and repeat the rest of the construction.

In the next result, the expectations of the Laplace-Stieltjes transforms of L^a and R^0 are identified:

Theorem 1 *Let $a > 0$. For $x \in [0, a]$ and $q > 0$ it holds that*

$$\mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-qt}\mathrm{d}L_t^a\right] = Z^{(q)}(x)/[qW^{(q)}(a)], \tag{4.3}$$

$$\mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-qt}\mathrm{d}R_t^0\right] = -\overline{Z}^{(q)}(x) - \frac{\psi'(0^+)}{q} + \frac{Z^{(q)}(a)}{qW^{(q)}(a)}Z^{(q)}(x), \tag{4.4}$$

where the expression in (4.4) is understood to be $+\infty$ if $\psi'(0^+) = -\infty$.

Remark. If X has bounded variation we can also consider the strategy of immediately paying out all dividends and paying all incoming claims with bail-out loans – this corresponds to keeping the risk process constant equal to zero. Denoting the the 'reflecting barriers' corresponding to this case by L^0 and R^0 respectively, one can directly verify that

$$\mathbb{E}_0[L^0_{\eta(q)}] = \mathtt{d}/q, \qquad \mathbb{E}_0[R^0_{\eta(q)}] = (\mathtt{d} - \psi'(0^+))/q. \tag{4.5}$$

Proof We first prove equation (4.3). Denote by $f(u)$ its left-hand side and write $\tau'_b = \inf\{t \geq 0 : V_t^a = b\}$ for the first hitting time of $\{b\}$. We shall derive a recursion for $f(x)$ by considering one cycle of the process V^a. More specifically, applying the strong Markov property of V^a at τ'_0 we find that

$$f(x) = \mathbb{E}_x\left[\int_0^{\tau'_0} \mathrm{e}^{-qt}\mathrm{d}L_t^a\right] + \mathbb{E}_x[\mathrm{e}^{-q\tau'_0}]f(0). \tag{4.6}$$

Since $\{V_t^a, t < \tau'_0, V_0^a = x\}$ has the same law as $\{a - \widehat{Y}_t, t < \widehat{\tau}_a, \widehat{Y}_0 = a - x\}$ the first term and first factor in the second term in (4.6) are equal to (3.12) and (3.10) (with $y = a - x$) respectively. By the fact that no local time is collected until V^a reaches the level a, we find by the strong Markov property that $f(0) = \mathbb{E}_0[\mathrm{e}^{-q\tau'_a}]f(a)$, where $\mathbb{E}_0[\mathrm{e}^{-q\tau'_a}] = Z^{(q)}(a)^{-1}$ in view of (3.9) and the fact that $\{V_t^a, t \leq \tau'_a, V_0^a = x\}$ has the same law as $\{Y_t, t \leq \tau_a, Y_0 = x\}$. Inserting all the three formulas into (4.6) results in the equation

$$f(x) = \frac{W^{(q)}(x)}{W^{(q)\prime}(a)} + f(a)\left(\frac{Z^{(q)}(x)}{Z^{(q)}(a)} - q\frac{W^{(q)}(x)W^{(q)}(a)}{Z^{(q)}(a)W^{(q)\prime}(a)}\right). \tag{4.7}$$

As this relation remains valid for $x = a$, we are led to a recursion for $f(a)$ the solution of which reads as $f(a) = Z^{(q)}(a)/[qW^{(q)}(a)]$. Inserting $f(a)$ back in (4.7) finishes the proof of (4.3).

Now we turn to the expected discounted local time of the process V^a collected at the lower reflection boundary R^0. Writing $g(x)$ for the left-hand side of (4.4) and applying the strong Markov property of V^a at τ'_a shows that

$$g(x) = \mathbb{E}_x\left[\int_0^{\tau'_a} \mathrm{e}^{-qt}\mathrm{d}R_t^0\right] + \mathbb{E}_x[\mathrm{e}^{-q\tau'_a}]g(a) \tag{4.8}$$

with $g(a) = \mathbb{E}_a[\mathrm{e}^{-q\tau'_0}\Delta R^0_{\tau'_0}] + \mathbb{E}_a[\mathrm{e}^{-q\tau'_0}]g(0)$, where $\Delta R^0_{\tau'_0} = R^0_{\tau'_0} - R^0_{\tau'_0-}$ denotes the jump of R at τ'_0. Appealing to the fact that $\{V_t^a, t < \tau'_0, V_0^a = x\}$ and

$\{V_t^a, t \leq \tau_a', V_0^a = x\}$ have the same distribution as $\{a - \widehat{Y}_t, t < \widehat{\tau}_a, \widehat{Y}_0 = a - x\}$ and $\{Y_t, t \leq \tau_a, Y_0 = x\}$ respectively, the Laplace transforms of τ_a', τ_0' and the expectation involving $\Delta R^0_{\tau_0'}$ can be identified by (3.9) (with $y = x$), (3.10) (with $y = 0$) and (3.16) respectively. The rest of the proof is devoted to the computation of the first term on the right-hand side of (4.8). Invoking the strong Markov property shows that

$$\begin{aligned}
-\mathbb{E}_x\left[\int_0^{\tau_a'} \mathrm{e}^{-qt}\mathrm{d}R_t^0\right] &= \mathbb{E}_x\left[\int_0^{\tau_a} \mathrm{e}^{-qt}\mathrm{d}I_t\right] \\
&= \mathbb{E}_x\left[\int_0^{\infty} \mathrm{e}^{-qt}\mathrm{d}I_t\right] - \mathbb{E}_x[\mathrm{e}^{-q\tau_a}]\mathbb{E}_a\left[\int_0^{\infty} \mathrm{e}^{-qt}\mathrm{d}I_t\right] \\
&= k(x) - \frac{Z^{(q)}(x)}{Z^{(q)}(a)}k(a),
\end{aligned}$$

where $k(x) = \mathbb{E}_x[I_{\eta(q)}]$ satisfies

$$\begin{aligned}
\mathbb{E}_x[I_{\eta(q)}] &= \mathbb{E}_x[\mathrm{e}^{-qT_0^-} X_{T_0^-}] + \mathbb{E}_x[\mathrm{e}^{-qT_0^-}]\mathbb{E}_0[I_{\eta(q)}] \\
&= \overline{Z}^{(q)}(x) - \Phi(q)^{-1}Z^{(q)}(x) + \psi'(0^+)/q.
\end{aligned}$$

In the last line we inserted the identity (3.17) and (3.19) (with $v = 0$). Further we used that $\mathbb{E}_0[I_{\eta(q)}] = \mathbb{E}_0[X_{\eta(q)}] - \mathbb{E}_0[(X - I)_{\eta(q)}]$ where $\mathbb{E}_0[X_{\eta(q)}] = \psi'(0^+)/q$ (from the definition of ψ) and $\mathbb{E}_0[(X - I)_{\eta(q)}] = 1/\Phi(q)$ from (3.8). Inserting the found identities into (4.8) and taking x to be zero in (4.8) yields a recursion for $g(0)$, which can be solved explicitly in terms of the scale functions. After some algebra one arrives at

$$g(0) = -\frac{\psi'(0^+)}{q} + \frac{Z^{(q)}(a)}{qW^{(q)}(a)}.$$

Substituting this expression back into (4.8) results in (4.4). □

5 Optimal dividend strategies

When solving the dividend problems our method draws on classical optimal control theory: we mention e.g. Jeanblanc and Shiryaev [15] and Harrison and Taylor [13] who deal with the classical dividend problem and a storage system in a Brownian motion setting, respectively. In these papers it was shown that if the state process follows a Brownian motion with drift the optimal strategy takes the form of a barrier strategy. In view of the fact that our state process is a Markov process we consider below barrier strategies and investigate their optimality amongst all admissible strategies in the classical dividend problem (2.2) and the bail-out problem (2.4).

5.1 Classical dividend problem

From Proposition 1 we read off that the value functions corresponding to barrier strategies π_a at the levels $a > 0$ are given by

$$v_a(x) \quad = \quad v_{\pi_a}(x) = \begin{cases} \frac{W^{(q)}(x)}{W^{(q)\prime}(a)}, & 0 \leq x \leq a, \\ \\ x - a + \frac{W^{(q)}(a)}{W^{(q)\prime}(a)}, & x > a, \end{cases} \tag{5.1}$$

and the strategy of taking out all dividends immediately has value $v_0(x) = x + W^{(q)}(0)/W^{(q)\prime}(0^+)$. To complete the description of the candidate optimal barrier solution we specify the level c^* of the barrier as

$$c^* = \inf\{a > 0 : W^{(q)\prime}(a) \leq W^{(q)\prime}(x) \ \text{ for all } x\}, \tag{5.2}$$

where $\inf \emptyset = \infty$. Note that, if $W^{(q)}$ is twice continuously differentiable on $(0, \infty)$ (which is in general not the case) and $c^* > 0$, then c^* satisfies

$$W^{(q)\prime\prime}(c^*) = 0, \tag{5.3}$$

so that in that case the optimal level c^* is such that the value function is C^2 on $(0, \infty)$. Recalling that $W^{(q)\prime}(0^+)$ is infinite if X has no Brownian component and the mass of its Lévy ν is infinite, we infer from the definition (5.2) of c^* that in this case $c^* > 0$ irrespective of the sign of the drift $\mathbb{E}_x[X_1]$. In comparison, if X is a Brownian motion with drift μ, c^* is positive or zero according to whether the drift μ is positive or non-positive. (See also Section 6 for other specific examples).

Denote by Γ the extended generator of the process X, which acts on C^2 functions f with compact support as

$$\Gamma f(x) = \frac{\sigma^2}{2} f''(x) + c f'(x) + \int_{-\infty}^{0} \left[f(x+y) - f(x) - f'(x) y 1_{\{|y|<1\}} \right] \nu(\mathrm{d}y),$$

where ν is the Lévy measure of X and σ^2 denotes the Gaussian coefficient and $c = \mathtt{d} + \int_{-1}^{0} y\nu(\mathrm{d}y)$ if the jump-part has bounded variation; see Çinlar et al. [9, Thm. 7.14] and Sato [25, Ch. 6, Thm. 31.5]. Note that by the properties of $W^{(q)}$ given in Section 3.2, it follows that v_{c^*} is C^2 on $(0, \infty)$ if $\sigma > 0$ [and $C^1(0, \infty)$ if X has bounded variation, respectively]. The following result concerns optimality of the barrier strategy π_{c^*} for the classical dividend problem.

Theorem 2 *Assume that $\sigma > 0$ or that X has bounded variation or, otherwise, suppose that $v_{c^*} \in C^2(0, \infty)$. If $q > 0$, then $c^* < \infty$ and the following hold true:*

(i) π_c^* is the optimal strategy in the set $\Pi_{\leq c^*}$ and $v_{c^*} = \sup_{\pi \in \Pi_{\leq c^*}} v_\pi$.

(ii) If $(\Gamma v_{c^*} - q v_{c^*})(x) \leq 0$ for $x > c^*$, the value function and optimal strategy of (2.2) are given by $v_* = v_{c^*}$ and $\pi_* = \pi_{c^*}$ respectively.

Remark. If the condition $(\Gamma v_{c^*} - q v_{c^*})(x) \leq 0$ is not satisfied for all $x \geq c^*$, but if $c^* > 0$ and one can construct a function v on $[0, \infty)$ that satisfies the HJB (5.8) (see for a precise statement Proposition 5 below), the strategy π_{c^*} is optimal for 'small' initial reserves, i.e. it is optimal to apply the barrier strategy $\pi_{c^*} \circ \theta_t$ whenever $U_t \in [0, c^*]$ (where θ denotes the shift operator) and it holds that $v(x) = v_*(x) = v_{c^*}(x)$ for $x \in [0, c^*]$. This observation agrees with the description of the optimal value function in the setting of the Cramér-Lundberg model, obtained in Azcue and Muler [1, section 9] using viscosity methods.

5.2 Dividends and bail-out

In the 'bail-out' setting and under the assumption that $\psi'(0^+) > -\infty$, we read off from Theorem 1 that the value function corresponding to the strategy $\overline{\pi}_{0,a}$ of putting reflecting barriers at the levels 0 and $a > 0$ is given by $\overline{v}_{\overline{\pi}_{0,a}} = \overline{v}_a$ where

$$\overline{v}_a(x) = \begin{cases} \varphi(\overline{Z}^{(q)}(x) + \psi'(0^+)/q) + Z^{(q)}(x) \left[\frac{1 - \varphi Z^{(q)}(a)}{q W^{(q)}(a)} \right], & 0 \leq x \leq a, \\ x - a + \overline{v}_a(a), & x > a. \end{cases} \tag{5.4}$$

In particular, if X is a Lévy process of bounded variation with drift $\mathtt{d}$,

$$\overline{v}_0(x) = x + [\varphi \psi'(0^+) + (1 - \varphi)\mathtt{d}]/q \tag{5.5}$$

is the value function corresponding to keeping the risk process identically equal to zero. The barrier level is specified as

$$d^* = \inf\{a > 0 : G(a) := [\varphi Z^{(q)}(a) - 1] W^{(q)\prime}(a) - \varphi q W^{(q)}(a)^2 \leq 0\} \tag{5.6}$$

Below, in Lemma 2, we shall show that if $\nu(-\infty, 0) \leq \frac{q}{\varphi - 1}$ and there is no Brownian component, then $d^* = 0$, else $d^* > 0$.

The constructed solution v_{d^*} can be identified as the value function of the optimal dividend problem (2.4). As a consequence, in the bail-out setting the optimal strategy takes the form of a barrier strategy for any initial capital:

Theorem 3 *Let $q > 0$ and suppose that $\psi'(0^+) < \infty$. Then $d^* < \infty$ and the value function and optimal strategy of (2.4) are given by $\overline{v}_*(x) = \overline{v}_{d^*}(x)$ and $\overline{\pi}_* = \overline{\pi}_{0,d^*}$, respectively.*

5.3 Optimal barrier strategies

As a first step in proving Theorems 2 and 3 we show optimality of π_{c^*} and $\overline{\pi}_{0,d^*}$ across the respective set of barrier strategies:

Proposition 3 *Let $q > 0$.*

(i) It holds that $c^* < \infty$ and π_{c^*} is an optimal barrier strategy, that is,

$$v_a(x) \leq v_{c^*}(x), \qquad x, a \geq 0.$$

(ii) Suppose that $\psi'(0^+) < \infty$. It holds that $d^* < \infty$ and $\overline{\pi}_{0,d^*}$ is the optimal barrier strategy, that is,

$$\overline{v}_a(x) \leq \overline{v}_{d^*}(x), \qquad x, a \geq 0.$$

To prove Proposition 3 we use the following facts regarding c^* and d^*:

Lemma 2 *Suppose that $q > 0$. (i) It holds that $c^* < \infty$.*
(ii) If $\nu(-\infty, 0) \leq q/(\varphi - 1)$ and $\sigma = 0$, then $d^* = 0$, else $d^* > 0$.

Proof of Lemma 2 (i) Recall that $W^{(q)\prime}(y)$ is non-negative and continuous for $y > 0$ and increases to ∞ as $y \to \infty$. Therefore either $W^{(q)\prime}(y)$ attains its finite minimum at some $y \in (0, \infty)$ or $W^{(q)\prime}(0^+) \leq W^{(q)\prime}(y)$ for all $y \in (0, \infty)$.

(ii) Write $H(a) = \mathbb{E}_0[\mathrm{e}^{-q\widehat{\tau}_a}]$ and recall that $H(a)$ is given by (3.10) [with $y = 0$]. It is a matter of algebra to verify that $G(a) = 0$ in (5.6) can be rewritten as $F(a) = 0$ where

$$F(a) := [\varphi H(a) - 1]W^{(q)\prime}(a)/[qW^{(q)}(a)^2]. \tag{5.7}$$

Since $a \mapsto \widehat{\tau}_a$ is monotonically increasing with $\lim_{a\to\infty} \widehat{\tau}_a = \infty$ almost surely, it follows that $H(a)$ monotonically decreases to zero as $a \to \infty$. Therefore $F(a) \leq 0$ for all $a > 0$ if $F(0^+) \leq 0$. Further, as F is continuous, it follows as a consequence of the intermediate value theorem that $F(a) = 0$ has a root in $(0, \infty)$ if $F(0^+) \in (0, \infty]$. In view of the fact that both $W^{(q)}(0^+) > 0$ and $W^{(q)\prime}(0^+) < \infty$ hold true precisely if X is a compound Poisson process, we see that $F(0^+) \leq 0$ if and only if both $\sigma = 0$ and $\nu(-\infty, 0) \leq q/(\varphi - 1)$ are satisfied. The statement (ii) follows. □

Proof of Proposition 3 (i) Since, by Lemma 2(i), $c^* < \infty$, the second statement follows since it is easily verified that the functions $a \mapsto [W^{(q)\prime}(a)]^{-1}$ and $a \mapsto W^{(q)}(a)[W^{(q)\prime}(a)]^{-1} - a$ both attain their maximum over $a \in (0, \infty)$ in $a = c^*$.

(ii) It is straightforward to verify that the derivatives of $a \mapsto \overline{v}_a(a) - a$ and $a \mapsto [1 - \varphi Z^{(q)}(a)]/[qW^{(q)}(a)]$ in $a > 0$ are equal to $F(a)$ and $F(a) \times [Z^{(q)}(a)]$ respectively, where $F(a)$ is given in (5.7). From the proof of Lemma 2 and the definition of d^* we see that $F(a) \leq 0$ for $a > d^*$, and if $d^* > 0$, $F(d^*) = 0$ and $F(a) > 0$ for $0 < a < d^*$. Thus it follows that $[1 - \varphi Z^{(q)}(a)]/W^{(q)}(a)$ and $\overline{v}_a(a) - a$ (and therefore $\overline{v}_a(x)$) attain their maximum over $a \in (0, \infty)$ in d^*. □

For later use we also collect the following properties of v_{c^*} and $\overline{v}_{d^*}$:

Lemma 3 *Let $x, a > 0$. The following are true: (i) $v'_{c^*}(x) \geq 1$.*

(ii) $1 \leq \overline{v}'_{d^*}(x) \leq \varphi$. Further, if $d^* > 0$, $\overline{v}'_{d^*}(d^{*-}) = 1$ and $\overline{v}'_{d^*}(0^+) = \varphi$ [resp. $\overline{v}'_{d^*}(0^+) < \varphi$] if X has unbounded [resp. bounded] variation.

(iii) $a \mapsto \overline{v}_a(x)$ is monotone decreasing for $a > d^*$.

Proof of Lemma 3 (i) Since, by Lemma 2, $c^* < \infty$, the statement follows from the definition of c^*.

(ii) In view of Lemma 2 and the argument in Proposition 3 it follows that if $d^* > 0$ and $0 < x < d^*$,

$$\begin{aligned} 1 &= \varphi Z^{(q)}(x) + W^{(q)}(x)[1 - \varphi Z^{(q)}(x)]/W^{(q)}(x) \\ &\leq \varphi Z^{(q)}(x) + W^{(q)}(x)[1 - \varphi Z^{(q)}(d^*)]/W^{(q)}(d^*) = \overline{v}'_{d^*}(x). \end{aligned}$$

Also, if $d^* > 0$ and $0 < x < d^*$, it holds that

$$\begin{aligned} (\overline{v}'_{d^*}(x) - \varphi)W^{(q)}(d^*) &= \varphi(Z^{(q)}(x) - 1)W^{(q)}(d^*) + W^{(q)}(x)[1 - \varphi Z^{(q)}(d^*)] \\ &= \varphi q[\overline{W}^{(q)}(x)W^{(q)}(d^*) - W^{(q)}(x)\overline{W}^{(q)}(d^*)] + W^{(q)}(x)(1 - \varphi) \leq 0, \end{aligned}$$

where in the second line we used Lemma 1, so that $v'_{d^*}(x) \leq \varphi$. The other statements of (ii) follow from the definitions of $\overline{v}_{d^*}$ and $Z^{(q)}$ and the form of $W^{(q)}(0)$ (see (3.5)).

(iii) The assertion follows since, from the proof of Proposition 3, $(\mathrm{d}\overline{v}_a/\mathrm{d}a)(x)$ has the same sign as $F(a)$ and $F(a) \leq 0$ for $a > d^*$. □

5.4 Verification theorems

To investigate the optimality of the barrier strategy π_{c^*} across all admissible strategies Π for the classical dividend problem (2.2) we are led, by standard Markovian arguments, to consider the following variational inequality:

$$\begin{aligned} \max\{\Gamma w(x) - qw(x), 1 - w'(x)\} = 0 \qquad & x > 0, \\ w(x) = 0 \quad & x < 0, \end{aligned} \tag{5.8}$$

where Γ is the extended generator of X.

Similarly, for the 'bail-out' problem (2.4) we are led to the variational inequality equation

$$\begin{aligned} \max\{\Gamma w(x) - qw(x), 1 - w'(x)\} = 0 \qquad & x > 0, \\ w'(x) \leq \varphi \qquad x > 0, \qquad w'(x) = \varphi \qquad & x \leq 0. \end{aligned} \tag{5.9}$$

The next step to establish the optimality of the barrier strategies amongst all admissible strategies is to prove the following verification results. In the case of (5.8) we shall only prove a local verification theorem.

Proposition 4 *Let $w : [0, \infty) \to \mathbb{R}$ be continuous.*

(i) Let $C \in (0, \infty]$, suppose $w(0) = w(0^+) \geq 0$ and extend w to the negative half-line by setting $w(x) = 0$ for $x < 0$. Suppose w is C^2 on $(0, C)$ [if X has unbounded variation] or is C^1 on $(0, C)$ [if X has bounded variation]. If w satisfies (5.8) for $x \in (0, C)$, then $w \geq \sup_{\pi \in \Pi_{\leq C}} v_\pi$. In particular, if $C = \infty$, $w \geq v_*$.

(ii) Suppose $w \in C^2[0, \infty)$ and set $w(x) = w(0) + \varphi x$ for $x \leq 0$. If w satisfies (5.9), then $w \geq \overline{v}_*$.

The proof follows below. Inspired by properties of v_{c^*} and with the smoothness required to apply the appropriate version of Itô's in mind, we weaken now the assumptions of above Proposition on the solution w. Let $\mathcal{P} = (p_1, p_2, \ldots, p_N)$ with $0 < p_1 < \ldots < p_N$ be a finite subset of $(0, \infty)$ and let $w : [0, \infty) \to [0, \infty)$ be continuous. If X has bounded variation, suppose that $w \in C^1(0, \infty) \backslash \mathcal{P}$ with finite left- and right-derivatives for $x \in \mathcal{P}$ and that w satisfies the HJB (5.8) where w' is understood to be w'_-. If X has unbounded variation suppose that $w \in C^2(0, \infty) \backslash \mathcal{P}$ with finite left- and right-second derivatives for $x \in \mathcal{P}$ and that w satisfies the HJB (5.8) where w'' is understood to be the weak derivative of w'. The following result complements Theorem 9.4 in Azcue and Muler [1]:

Proposition 5 *Suppose w is as described. If $w'(0^+) > 1$, then $c^* > 0$ and $w(x) = v_*(x) = v_{c^*}(x)$ for $x \in [0, c^*]$.*

Proof of Proposition 4 (ii) Let $\overline{\pi} \in \overline{\Pi}$ be any admissible policy and denote by $L = L^{\overline{\pi}}, R = R^{\overline{\pi}}$ the corresponding pair of cumulative dividend and cumulative loss processes respectively and by $V = V^{\overline{\pi}}$ the corresponding risk process. By an application of Itô's lemma to $\mathrm{e}^{-qt} w(V_t)$ it can be verified that

$$\begin{aligned} \mathrm{e}^{-qt} w(V_t) - w(V_0) &= J_t + \int_0^t \mathrm{e}^{-qs} w'(V_{s^-}) \mathrm{d}R_s^c - \int_0^t \mathrm{e}^{-qs} w'(V_{s^-}) \mathrm{d}L_s^c \\ &+ \int_0^t \mathrm{e}^{-qs} (\Gamma w - qw)(V_{s^-}) \mathrm{d}s + M_t, \end{aligned} \tag{5.10}$$

where M_t is a local martingale with $M_0 = 0$, R^c and L^c are the path-wise continuous parts of R and L, respectively, and J_t is given by

$$J_t = \sum_{s \leq t} \mathrm{e}^{-qs} \left[w(A_s + B_s) - w(A_s) \right] \mathbf{1}_{\{B_s \neq 0\}}, \tag{5.11}$$

where $A_s = V_{s^-} + \Delta X_s$ and $B_s = \Delta(R - L)_s$ denotes the jump of $R - L$ at time s. Note that $1 \leq w'(x) \leq \varphi$ holds for all $x \in \mathbb{R}$. In particular, we see that $w(A_s + B_s) - w(A_s) \leq \varphi \Delta R_s - \Delta L_s$, so that the first three terms on the rhs of (5.10) are bounded above by $\varphi \int_0^t \mathrm{e}^{-qs} \mathrm{d}R_s - \int_0^t \mathrm{e}^{-qs} \mathrm{d}L_s$. Let T_n the first time absolute value of any of the five terms on the rhs of (5.10) exceeds the value n, so that, in particular, T_n is a localizing sequence for M. Applying (5.10) at T_n, taking expectations and using that, on $[0, \infty)$, w is bounded below by some constant ,$-M$ say, $1 \leq w'(x) \leq \varphi$ and $(\Gamma w - qw)(x) \leq 0$ for $x > 0$, it follows after rearranging that

$$\begin{aligned} w(x) &\geq \mathbb{E}_x \left[\int_0^{T_n} \mathrm{e}^{-qs} \mathrm{d}L_s - \varphi \int_0^{T_n} \mathrm{e}^{-qs} \mathrm{d}R_s \right] + \mathbb{E}_x[\mathrm{e}^{-qT_n} w(V_{T_n})] \\ &\geq \mathbb{E}_x \left[\int_0^{T_n} \mathrm{e}^{-qs} \mathrm{d}L_s - \varphi \int_0^{\infty} \mathrm{e}^{-qs} \mathrm{d}R_s \right] - M \mathbb{E}_x[\mathrm{e}^{-qT_n}]. \end{aligned}$$

Letting $n \to \infty$, the condition (2.3) in conjunction with the monotone convergence theorem then implies that $\overline{v}_{\overline{\pi}}(x) \leq w(x)$. Since $\overline{\pi}$ was arbitrary it follows w dominates the value function $\overline{v}_*$.

(i) Let $\pi \in \Pi_{\leq C}$ be any admissible policy and denote by $L = L^\pi$ and $U = U^\pi$ the corresponding cumulative dividend process and risk process, respectively. If X has unbounded variation, w is C^2 and we are allowed to apply Itô's lemma (e.g. [23, Thm. 32]) to $\mathrm{e}^{-q(t\wedge\sigma^\pi)}w(U_{t\wedge\sigma^\pi})$, using that $U_t \leq C$. If X has bounded variation, w is C^1 and we apply the change of variable formula (e.g. [23, Thm. 31]). Following then an analogous line of reasoning as in (ii) we find that

$$w(x) \geq \mathbb{E}_x\left[\int_0^{T_n'\wedge\sigma^\pi} \mathrm{e}^{-qs}\mathrm{d}L_s\right] \tag{5.12}$$

for some increasing sequence of stopping times T_n' with $T_n' \to \infty$ a.s. Taking $n \to \infty$ in (5.12) yields, in view of the monotone convergence theorem and the fact that $w \geq 0$, that,

$$w(x) \geq \mathbb{E}_x\left[\int_0^{\sigma^\pi} \mathrm{e}^{-qs}\mathrm{d}L_s\right].$$

Since the previous display holds for arbitrary $\pi \in \Pi_{\leq C}$, it follows that $w(x) \geq \sup_{\pi\in\Pi_{\leq C}} v_\pi(x)$ and the proof is finished. □

Proof of Proposition 5 Noting that w is smooth enough for an application of the appropriate version of Itô's lemma (as follows from the (proof of) the Itô-Tanaka-Meyer formula, see e.g. Protter [23]), it can be verified, as in Proposition 4, that $w \geq v_*$.

Putting $m = \inf\{x > 0 : w'(x^-) = 1\}$, it follows from the assumptions that $m \in (0, \infty)$ or $m = \infty$. The latter case can be ruled out as follows. If $m = \infty$, it follows by applying Itô's lemma to $\mathrm{e}^{-q(t\wedge T_0^-)}w(X_{t\wedge T_0^-})$ that

$$w(x) = \mathbb{E}_x[\mathrm{e}^{-q(T_0^-\wedge T_n')}w(X_{T_0^-\wedge T_n'})]$$

for some increasing sequence of stopping time T_n' with $T_n' \to \infty$. Letting $n \to \infty$, the right-hand-side converges to zero, which leads to a contradiction in view of the fact that $w \geq v_*$. Therefore we see that $m \in (0, \infty)$. Applying Itô's lemma to $\mathrm{e}^{-q(t\wedge\sigma^\pi)}w(U_{t\wedge\sigma^\pi})$ with $\pi = \pi_m$ and using that w satisfies the HJB equation (5.8), we find that

$$w(x) = \mathbb{E}_x\left[\int_0^{T_n'\wedge\sigma^{\pi_m}} \mathrm{e}^{-qs}\mathrm{d}L_s\right] + \mathbb{E}_x[\mathrm{e}^{-q(\sigma^{\pi_m}\wedge T_n'')}w(U_{\sigma^{\pi_m}\wedge T_n''})]$$

for some increasing sequence of stopping time T_n'' with $T_n'' \to \infty$. Letting $n \to \infty$ and using that $w(U_{\sigma^{\pi_m}\wedge T_n''})$ is bounded (since $U^{\pi_m} \leq m$) and $w(U_{\sigma^{\pi_m}}) = 0$, it follows that $w(x) = v_m(x)$ for $x \in [0, m]$. Since, on the one hand, Proposition 3 implies that $v_m \leq v_{c^*}$, while, on the other hand, $w \geq v_*$, we deduce that $m = c^*$ and $v_*(x) = v_{c^*}(x)$ for $x \in [0, c^*]$ where $c^* > 0$. □

5.5 Proofs of Theorems 2 and 3

We set $v_{c^*}(x) = 0$ for $x < 0$ and extend $\overline{v}_{d^*}$ to the negative half-axis by setting $\overline{v}_{d^*}(x) = \overline{v}_{d^*}(x) + \varphi x$ for $x < 0$. Recalling that $W^{(q)}(x) = 0$, $Z^{(q)}(x) = 1$ and $\overline{Z}^{(q)}(x) = x$ for $x < 0$, we see that these extensions are natural extensions of the formulas (5.1) and (5.4) and satisfy the HJB equations (5.8) and (5.9) for $x < 0$. The proofs of Theorems 2 and 3 are based on the following lemmas:

Lemma 4 *If $c^* > 0$, $(\Gamma v_{c^*} - q v_{c^*})(x) = 0$ for $x \in (0, c^*)$*

Lemma 5 *It holds that $(\Gamma \overline{v}_{d^*} - q\overline{v}_{d^*})(x) \leq 0$ [resp. $= 0$] if $x > 0$ [resp. if $d^* > 0$ and $x \in (0, d^*)$].*

Proof of Theorem 2 (i) In view of Lemmas 4 and 5 it follows that the function v_{c^*} satisfies the respective variational inequality (5.8) for $x \in (0, c^*)$. Therefore, Proposition 4 implies the optimality of the strategies π_c^* in the set $\Pi_{\leq c^*}$.

(ii) If the condition of Theorem 2 (ii) holds, then, in view of the observations of part (i), it follows that v_{c^*} satisfies the variational inequalities (5.8) for $x \in (0, \infty)$. By Proposition 4 it then follows that $v_c^* = v_*$. □

Proof of Theorem 3 In view of Lemma 5 it follows that the function $\overline{v}_{d^*}$ satisfies the variational inequality (5.9). Therefore, Proposition 4(ii) implies that $\overline{v}_* = \overline{v}_{d^*}$ and the strategy $\overline{\pi}_{0,d^*}$ is optimal. □

Proof of Lemma 4 Suppose that $c^* > 0$. Since $\mathrm{e}^{-q(t\wedge T_{0,c^*})} W^{(q)}(X_{t\wedge T_{0,c^*}})$ is a martingale, $\mathrm{e}^{-q(t\wedge T_{0,c^*})} v_{c^*}(X_{t\wedge T_{0,c^*}})$ inherits this martingale property by definition of v_{c^*}. Since v_{c^*} is smooth enough to apply the appropriate version of Itô's lemma ([23, Thm. 31] is applicable if X has bounded variation since then $v_{c^*} \in C^1(0, c^*)$ and [23, Thm. 32] if X has unbounded variation as then $v_{c^*} \in C^2(0, c^*)$) it then follows that $\Gamma v_{c^*}(y) - q v_{c^*}(y) = 0$ for $0 < y < c^*$. □

Proof of Lemma 5 First let $d^* > 0$. In view of Proposition 2 and the martingale property (3.7), it follows that the process $\mathrm{e}^{-q(t\wedge T_{0,d^*})} \overline{v}_{d^*}(X_{t\wedge T_{0,d^*}})$ is a martingale. An application of Itô's lemma, which we are allowed to apply as $Z^{(q)} \in C^2(0, \infty)$, then yields that $\Gamma \overline{v}_{d^*}(y) - q\overline{v}_{d^*}(y) = 0$ for $0 < y < d^*$.

Let now $d^* \geq 0$ and fix $a > d^*$ and $V_0 = x \in (0, a)$. Note that $s \mapsto L_s^a$ can be taken to be continuous in this case and that the support of Stieltjes measure $\mathrm{d}L_s^a$ is contained in the set $\{s : V_{s^-}^a = a\}$. Further, in this case R^0 jumps at time s if and only if X jumps at time s and ΔX_s is larger than $V_{s^-}^a$. Thus $\Delta R_s^0 = -\min\{0, V_{s^-}^a + \Delta X_s\}$ and the measure $\mathrm{d}(R^0)_s^c$ has support inside $\{s : V_{s^-}^a = 0\}$. In view of these observations, an application of Itô's lemma to $\mathrm{e}^{-qt}\overline{v}_{d^*}(V_t^a)$ as in (5.10) shows that

$$\begin{aligned} \mathrm{e}^{-qt}\overline{v}_{d^*}(V_t^a) - \overline{v}_{d^*}(x) &= \int_0^t \mathrm{e}^{-qs}\overline{v}'_{d^*}(0^+)\mathrm{d}(R^0)_s^c + \varphi \sum_{s\leq t} \mathrm{e}^{-qs}\Delta R_s^0 \mathbf{1}_{\{\Delta R_s^0 > 0\}} \\ -\int_0^t \mathrm{e}^{-qs}\overline{v}'_{d^*}(a^-)\mathrm{d}L_s^a &+ \int_0^t \mathrm{e}^{-qs}(\Gamma\overline{v}_{d^*} - q\overline{v}_{d^*})(V_{s^-}^a)\mathrm{d}s + M_t, \end{aligned} \tag{5.13}$$

where we used that in (5.10) $J_t = \varphi \sum_{s\leq t} \mathrm{e}^{-qs}\Delta R_s^0 \mathbf{1}_{\{\Delta R_s^0 > 0\}}$ since, by definition of the extended function $\overline{v}_{d^*}$ on $(-\infty, 0]$, it follows that $\overline{v}_{d^*}(x+y) - \overline{v}_{d^*}(x) = \varphi y$

if $x=-y$, $x<0$. Since $\overline{v}_{d^*}\in C^2(0,\infty)$ and V^a takes values in $[0,a]$, it follows that in this case M is a martingale. Further, $\overline{v}'_{d^*}(a^-)=1$ and either $(R^0)^c=0$ (if X has bounded variation) or $\overline{v}'_{d^*}(0^+)=\varphi$ (if X has unbounded variation). Taking then expectations and letting $t\to\infty$ in (5.13) shows, in view of the dominated convergence theorem and the first part of the proof, that

$$\begin{aligned}\overline{v}_a(x)-\overline{v}_{d^*}(x) &= \mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-qs}(\Gamma\overline{v}_{d^*}-q\overline{v}_{d^*})(V^a_{s^-})\mathrm{d}s\right] \\ &= \int_{d^*}^a(\Gamma\overline{v}_{d^*}-q\overline{v}_{d^*})(y)\tilde{U}^q_a(x,\mathrm{d}y),\end{aligned} \tag{5.14}$$

where $\tilde{U}^q_a(x,\mathrm{d}y)=\tilde{U}^q(x,\mathrm{d}y)$ is the resolvent measure of V^a given in (4.1). In view of Lemma 3(iii) the left-hand side of (5.14) is non-positive. Since $y\mapsto(\Gamma\overline{v}_{d^*}-q\overline{v}_{d^*})(y)$ is continuous for $y>0$ (and equal to 0 on $(0,d^*)$) and $\tilde{U}^q_a(x,\mathrm{d}y)$ is absolutely continuous on $(0,a)$ with positive density (see (4.1)), it follows that there exists an $\epsilon>0$ such that $(\Gamma\overline{v}_{d^*}-q\overline{v}_{d^*})(y)\le 0$ for $y\in(d^*,d^*+\epsilon)$.

Assume now that there exists a $y>\epsilon$ such that $\Gamma\overline{v}_{d^*}(y)-q\overline{v}_{d^*}(y)>0$. We show that this assumption leads to a contradiction. By continuity of $y\mapsto\Gamma\overline{v}_{d^*}(y)-q\overline{v}_{d^*}(y)$, the assumption implies that there exist $a>b\ge d^*+\epsilon$ such that $(\Gamma\overline{v}_{d^*}-q\overline{v}_{d^*})(y)\le 0$ for $y\in(d^*,b)$ and $(\Gamma\overline{v}_{d^*}-q\overline{v}_{d^*})(y)>0$ for $y\in(b,a)$. From the previous display, applied with a and b, it follows that the difference $\overline{v}_a(x)-\overline{v}_b(x)$ is equal to

$$\int_b^a(\Gamma\overline{v}_{d^*}-q\overline{v}_{d^*})(y)\tilde{U}^q_a(x,\mathrm{d}y)+\int_{d^*}^b(\Gamma\overline{v}_{d^*}-q\overline{v}_{d^*})(y)[\tilde{U}^q_a(x,\mathrm{d}y)-\tilde{U}^q_b(x,\mathrm{d}y)]. \tag{5.15}$$

As $\tilde{U}^q_a(x,\mathrm{d}y)=\tilde{U}^q(x,\mathrm{d}y)$ has strictly positive density for $y>0$ (see (4.1)), we see that the first term is strictly positive. Also, we see from (4.1) that $\frac{\mathrm{d}}{\mathrm{d}a}\tilde{u}^q_a(x,y)$ has the same sign as $\frac{\mathrm{d}}{\mathrm{d}a}Z^{(q)}(a-x)/qW^{(q)}(a)$ which is given by

$$\frac{\mathrm{d}}{\mathrm{d}a}\frac{Z^{(q)}(a-x)}{qW^{(q)}(a)} = \frac{W^{(q)\prime}(a)}{qW^{(q)}(a)^2}[-\mathbb{E}_x[\mathrm{e}^{-q\widehat{\tau}_a}]]<0,$$

where we used (3.10). Thus, $\tilde{u}^q_a(x,y)\le\tilde{u}^q_b(x,y)$ for $a>b>d^*$ and also the second term of (5.15) is positive. On the other hand, since $a\mapsto\overline{v}_a(x)$ is monotonically decreasing, $\overline{v}_a(x)-\overline{v}_b(x)\le 0$ and we arrive at a contradiction. Thus, it holds that $\Gamma\overline{v}_{d^*}-q\overline{v}_{d^*}(y)\le 0$ for all $y>0$. □

6 Examples

6.1 Small claims: Brownian motion

If $X_t=\sigma B_t+\mu t$ is a Brownian motion with drift μ (a standard model for small claims) then

$$W^{(q)}(x)=\frac{1}{\sigma^2\delta}[\mathrm{e}^{(-\omega+\delta)x}-\mathrm{e}^{-(\omega+\delta)x}],$$

where $\delta = \sigma^{-2}\sqrt{\mu^2 + 2q\sigma^2}$ and $\omega = \mu/\sigma^2$. It is a matter of calculus to verify that

$$W^{(q)\prime\prime}(x) = 2\sigma^{-2}[qW^{(q)}(x) - \mu W^{(q)\prime}(x)]$$

from which it follows that if $\mu \leq 0$, $W^{(q)\prime}(x)$ attains its minimum over $[0, \infty)$ in $x = 0$. Thus in the classical setting it is optimal to take out all dividends immediately if $\mu \leq 0$; if $\mu > 0$ it follows that $c^* > 0$ and it holds that $W^{(q)\prime\prime}(c^*) = 0$, so that $W^{(q)}(c^*)/W^{(q)\prime}(c^*) = \mu/q$, as Gerber and Shiu [12] have found before, and the optimal level c^* is explicitly given by

$$c^* = \log \left| \frac{\delta + \omega}{\delta - \omega} \right|^{1/\delta}.$$

Since $\frac{\sigma^2}{2} v''_{c^*}(x) + \mu v'_{c^*}(x) - q v_{c^*}(x) < 0$ for $x > c^*$, it follows by Theorem 2 that π_{c^*} is the optimal strategy as shown found before by Jeanblanc and Shiryaev [15]. In the 'bail-out' setting $d^* \in (0, \infty)$ solves $G(a) = 0$ where G is given in (5.6) with

$$Z^{(q)}(y) = y + \frac{2q}{\sigma^2} + \frac{q}{\sigma^2 \delta} \left[\frac{1}{\omega + \delta} \mathrm{e}^{-(\omega+\delta)y} - \frac{1}{\delta - \omega} \mathrm{e}^{(-\omega+\delta)y} \right]$$

and

$$W^{(q)\prime}(y) = \frac{1}{\sigma^2 \delta} \left[(\omega + \delta)\mathrm{e}^{-(\omega+\delta)y} + (\delta - \omega)\mathrm{e}^{(-\omega+\delta)y} \right].$$

The relation between the classical and bail-out strategies in this Brownian setting is studied in Løkka and Zervos [19].

6.2 Stable claims

We model X as

$$X_t = \sigma Z_t,$$

where Z is a standard stable process of index $\alpha \in (1, 2]$ and $\sigma > 0$. Its cumulant is given by $\psi(\theta) = (\sigma\theta)^\alpha$. By inverting the Laplace transform $(\psi(\theta) - q)^{-1}$, Bertoin [5] found that the q-scale function is given by

$$W^{(q)}(y) = \alpha \frac{y^{\alpha-1}}{\sigma^\alpha} E'_\alpha \left(q \frac{y^\alpha}{\sigma^\alpha} \right), \qquad y > 0,$$

and hence $Z^{(q)}(y) = E_\alpha(q(y/\sigma)^\alpha)$ for $y > 0$, where E_α is the Mittag-Leffler function of index α

$$E_\alpha(y) = \sum_{n=0}^{\infty} \frac{y^n}{\Gamma(1 + \alpha n)}, \qquad y \in \mathbb{R}.$$

The form of the value functions v_{c^*} and $\overline{v}_{d^*}$ follows by inserting the expressions for the scale functions in equations (5.1)–(5.4). The optimal levels c^*, d^* are given by

$$c^* = \sigma q^{-1/\alpha} u(\alpha)^{1/\alpha}, \quad d^* = \sigma q^{-1/\alpha} v(\alpha, q)^{1/\alpha},$$

where $u(\alpha) > 0$ and $v(\alpha, q) > 0$ are positive roots of the respective equations

$$\begin{aligned}(\alpha-1)(\alpha-2)E'_\alpha(u) + 3\alpha(\alpha-1)uE''_\alpha(u) + \alpha^2u^2E'''_\alpha(u) &= 0,\\ \varphi qv(E'_\alpha(v))^2 + [(\alpha-1)E'_\alpha(v) + \alpha vE''_\alpha(v)][1-\varphi E_\alpha(v)] &= 0.\end{aligned}$$

6.3 Cramér-Lundberg model with exponential jumps

Suppose X is given by the Cramér-Lundberg model (1.1) with exponential jump sizes, that is, X is a deterministic drift p (the premium income) minus a compound Poisson process (with jump intensity λ and jump sizes C_k that are exponentailly distributed with mean $1/\mu$) such that X has positive drift i.e. $p > \lambda/\mu$. Then $\psi(\theta) = p\theta - \lambda\theta/(\mu+\theta)$ and the scale function $W^{(q)}$ is given by

$$W^{(q)}(x) = p^{-1}\left(A_+ \mathrm{e}^{q^+(q)x} - A_- \mathrm{e}^{q^-(q)x}\right),$$

where $A_\pm = \frac{\mu+q^\pm(q)}{q^+(q)-q^-(q)}$ with $q^+(q) = \Phi(q)$ and $q^-(q)$ the smallest root of $\kappa(\theta) = q$:

$$q^\pm(q) = \frac{q+\lambda-\mu p \pm \sqrt{(q+\lambda-\mu p)^2 + 4pq\mu}}{2p}.$$

Then from (5.3) we have that $c^* = 0$ if $W^{(q)\prime\prime}(0) \le 0 \Leftrightarrow p\lambda\mu \le (q+\lambda)^2$. If $p\lambda\mu > (q+\lambda)^2$,

$$c^* = \frac{1}{q^+(q)-q^-(q)} \log \frac{q^-(q)^2(\mu+q^-(q))}{q^+(q)^2(\mu+q^+(q))}.$$

Since it is readily verified that $\Gamma v_{c^*}(x) - qv_{c^*}(x) < 0$ for $x > c^*$, Theorem 2(ii) implies that π_{c^*} is the optimal strategy.

Further, if $\lambda(\varphi-1) \le q$ then $d^* = 0$. Otherwise $d^* > 0$ satisfies $G(d^*) = 0$ where G is given in (5.6).

6.4 Jump-diffusion with hyper-exponential jumps

Let $X = \{X_t, t \ge 0\}$ be a jump-diffusion given by

$$X_t = \mu t + \sigma W_t - \sum_{i=1}^{N_t} Y_i,$$

where $\sigma > 0$, N is a Poisson process with intensity $\lambda > 0$ and $\{Y_i\}$ is a sequence of i.i.d. random variables with hyper-exponential distribution

$$F(y) = 1 - \sum_{i=1}^{n} A_i \mathrm{e}^{-\alpha_i y}, \qquad y \ge 0,$$

where $A_i > 0$; $\sum_{i=1}^n A_i = 1$; and $0 < \alpha_1 < \ldots < \alpha_n$. In [4] it was shown that the function $Z^{(q)}$ of X is given by

$$Z^{(q)}(x) = \sum_{i=0}^{n+1} D_i(q)\mathrm{e}^{\theta_i(q)x},$$

where $\theta_i = \theta_i(q)$ are the roots of $\psi(\theta) = q$, where $\theta_{n+1} > 0$ and the rest of the roots are negative, and where

$$D_i(q) = \prod_{k=1}^{n} (\theta_i(q)/\alpha_k + 1) \Bigg/ \prod_{k=0, k\neq i}^{n+1} (\theta_i(q)/\theta_k(q) - 1).$$

If $c^* > 0$, it is a non-negative root x of

$$\sum_{i=0}^{n+1} \theta_i(q)^3 D_i(q) \mathrm{e}^{\theta_i(q)x} = 0.$$

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