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# A two-dimensional ruin problem on the positive quadrant: Laplace transform and inversion

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## Abstract

We consider an exit problem of a certain two-dimensional process from a cone, inspired by applications in insurance and queueing theory. One motivation is to study the joint ruin problem for two insurance companies (insurance/reinsurance), or for two branches of the same company, which divide between them both claims and premia in some specified proportions, the goal being to split the risk (in particular when the claims are big). Another motivation is to provide an example of a multi-dimensional ruin model admitting analytic solutions. Indeed, we succeed, in the particular case of exponential claims, to derive the Laplace transform of the perpetual ruin probabilities and subsequently to invert it.

## 1 A two dimensional ruin problem

In this paper we consider a particular two dimensional risk model in which two companies split the amount they pay out of each claim in proportions  $\delta_1$  and  $\delta_2$  where  $\delta_1 + \delta_2 = 1$ , and the premiums according to rates  $c_1$  and  $c_2$ . Let  $U_i$  denote the risk process of the  $i$ 'th company

$$U_i(t) := -\delta_i S(t) + c_i t + u_i, \quad i = 1, 2,$$

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where  $u_i$  denotes the initial reserve and

$$S(t) = \sum_{i=1}^{N(t)} \sigma_i \quad (1)$$

for  $N(t)$  being a Poisson process with intensity  $\lambda$  and the claims  $\sigma_i$  being i.i.d. random variables independent of  $N(t)$  with distribution function  $F(x)$ . We shall denote by  $\mu$  the reciprocals of the means of  $\sigma_i$ , respectively. We shall assume that the second company, to be called reinsurer, gets smaller profits per amount paid, i.e.:

$$p_1 = \frac{c_1}{\delta_1} > \frac{c_2}{\delta_2} = p_2. \quad (2)$$

As usual in risk theory, we assume that  $p_i > \rho := \frac{\lambda}{\mu}$ , which implies that in the absence of ruin,  $U_i(t) \rightarrow \infty$  as  $t \rightarrow \infty$  ( $i = 1, 2$ ). Ruin happens at the time  $\tau = \tau(u_1, u_2)$  when at least one insurance company is ruined:

$$\tau(u_1, u_2) := \inf\{t \geq 0 : U_1(t) < 0 \text{ or } U_2(t) < 0\}, \quad (3)$$

i.e. at the exit time of  $(U_1(t), U_2(t))$  from the positive quadrant. In this paper we will analyse the perpetual or ultimate ruin probability:

$$\psi(u_1, u_2) = P[\tau(u_1, u_2) < \infty]. \quad (4)$$

Although ruin theory under multi-dimensional models admits rarely analytic solutions, we are able to obtain in our problem a closed form solution for (4) if  $\sigma_i$  are exponentially distributed with intensity  $\mu$ .

**Geometrical considerations.** The solution of the two-dimensional ruin problem (4) strongly depends on the relative sizes of the proportions  $\boldsymbol{\delta} = (\delta_1, \delta_2)$  and premium rates  $\boldsymbol{c} = (c_1, c_2)$  – see Figure 1. If, as assumed throughout, the angle of the vector  $\boldsymbol{\delta}$  with the  $u_1$  axis is bigger than that of  $\boldsymbol{c}$ , i.e.  $\delta_2 c_1 > \delta_1 c_2$ , we note that starting with initial capital  $(u_1, u_2) \in \mathcal{C}$  in the cone  $\mathcal{C} = \{(u_1, u_2) : u_2 \leq (\delta_2/\delta_1)u_1\}$  situated below the line  $u_2 = (\delta_2/\delta_1)u_1$ , the process  $(U_1, U_2)$  ends up hitting at time  $\tau$  the  $u_1$  axis. Thus, in the domain  $\mathcal{C}$  ruin occurs iff there is ruin in the one-dimensional problem corresponding to the risk process  $U_2$  with premium  $c_2$  and claims  $\delta_2 \sigma$ .

**One dimensional reduction.** A key observation is that  $\tau$  in (3) is also equal to

$$\tau(u_1, u_2) = \inf\{t \geq 0 : S(t) > b(t)\},$$

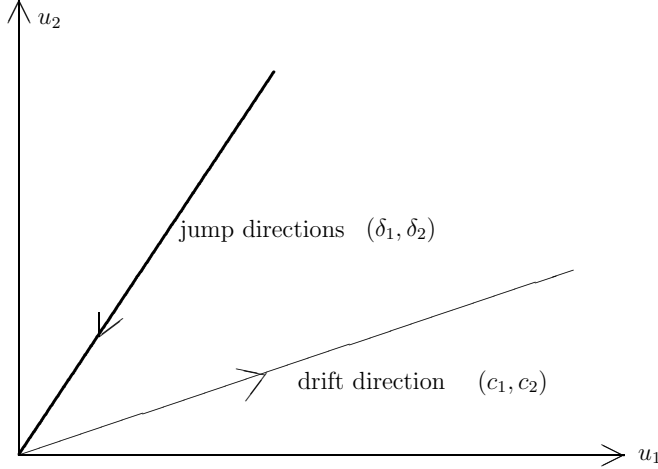


Figure 1: Geometrical considerations

where  $b(t) = \min\{(u_1 + c_1 t)/\delta_1, (u_2 + c_2 t)/\delta_2\}$ . The two dimensional problem (4) may thus be also viewed as a one dimensional crossing problem over a piecewise linear barrier. Note the relation to asset-liability management models, in which regulatory requirements impose prescribed limits of variation for the difference between the assets  $P(t)$  of a company and its liabilities  $S(t)$  (see [5]), and which also translate typically into (several) linear barriers.

In the case that the initial reserves  $u_1$  and  $u_2$  are such that  $(u_1, u_2) \in \mathcal{C}$ , that is,  $u_2/\delta_2 \leq u_1/\delta_1$ , the barrier  $b$  is linear,  $b(t) = (u_2 + c_2 t)/\delta_2$ , the ruin happens always for the second company. Thus, as we already observed, the problem (4) reduces in fact to the classical one-dimensional ultimate ruin problem with premium  $c_2$  and claims  $\delta_2\sigma$ , i.e.

$$\psi(u_1, u_2) = \psi_2(u_2) := P(\tau_2(u_2) < \infty),$$

where  $\tau_2(u_2) = \inf\{t \geq 0 : U_2(t) < 0\}$  and  $\psi_2(u_2)$  is the ruin probability of  $U_2$ , with  $U_2(0) = u_2$ . The Pollaczec-Khinchine formula, well known from the theory of one-dimensional ruin (see e.g. [8] or [1]) yields then an explicit series solution for  $\psi(u_1, u_2) = \psi_2(u_2)$  in the case of a general claims distribution. For phase-type claims  $(\beta, \mathbf{B})$ , i.e. with  $P[\sigma > x] = \beta e^{\mathbf{B}x} \mathbf{1}$ , the ruin probability may be written in a simpler matrix exponential form:

$$\psi_2(u_2) = \boldsymbol{\eta} e^{\delta_2^{-1}(\mathbf{B} + \mathbf{b}\boldsymbol{\eta})u_2} \mathbf{1}$$

with  $\boldsymbol{\eta} = \frac{\lambda}{p_2} \boldsymbol{\beta}(-\mathbf{B})^{-1}$  (see for example (4) in [2]), and in the case of exponential claim sizes with intensity  $\mu$ , it reduces to:

$$\psi_2(u_2) = C_2 e^{-(\gamma_2/\delta_2)u_2}, \quad (5)$$

where  $\gamma_2 = \mu - \lambda\delta_2/c_2 = \mu - \lambda/p_2$  and  $C_2 = \frac{\lambda\delta_2}{\mu c_2} = \frac{\lambda}{\mu p_2}$ .

The rest of the paper is devoted to the opposite case  $u_2/\delta_2 > u_1/\delta_1$  when a barrier composed of two lines is involved. We shall analyze only the case of exponential claim sizes.

## 2 The differential system for the exponential claim sizes case

In this section we provide a differential system for the Laplace transform of the ruin time which will be derived using level crossing arguments. We consider here the Markovian model, that is when  $N(t)$  is a Poisson process with intensity  $\lambda$  and claims sizes are exponentially distributed with parameter  $\mu$ . The memoryless property of interarrival times and claim sizes gives the possibility of embedding our Markovian jump model into a continuous Markov modulated fluid model. This is achieved by a transformation which replaces the jumps by linear movement in the same direction, creating thereby a new *continuous* semi-Markovian model  $\tilde{\mathbf{U}}(t)$  called *fluid embedding*. The smoothed process  $\tilde{\mathbf{U}}(t)$  is a continuous semi-Markovian process, has exactly the same maxima and minima as the original one, and it will cross boundaries continuously. Thus, the generator will be a first order differential operator, and the integro-differential part corresponding to jumps has been removed. The associated fluid model may be more manageable, in so far that the overshoots in first passage problems have been eliminated. In our case, the associated embedding fluid model replaces the jumps in the direction  $(-\delta_1, -\delta_2)$  by linear movement  $(x' = x - \delta_1 t, y' = y - \delta_2 t)$  of duration equal to the size of the jump.

Our new "embedded" process is a semi-Markov process with two states (going out of axes (so-called increasing state) and into the direction of axes (so-called decreasing state)). Denote by  $\phi(u_1, u_2, s)$  and  $\psi(u_1, u_2, s)$  the killed ruin probabilities given that the process starts in a decreasing state and increasing state respectively and write  $\psi_{u_i}$  and  $\phi_{u_i}$  for the partial derivative of  $\psi$  and  $\phi$  with respect to  $u_i$ . We have the following characterisation of  $\phi$  and  $\psi$ .

**Theorem 1** *The vector  $(\psi(u_1, u_2, s), \phi(u_1, u_2, s))^T$  giving the Laplace transforms of two barrier problem in the cone  $\mathcal{C}_2$  between  $x = 0, y = \frac{\delta_2}{\delta_1}x$ , for the embedded continuous semi-Markov process killed with intensity  $s$ , is the solutions of the Feynman-Kac systems:*

$$\begin{pmatrix} c_1 & 0 \\ 0 & -\delta_1 \end{pmatrix} \begin{pmatrix} \psi_{u_1} \\ \phi_{u_1} \end{pmatrix} + \begin{pmatrix} c_2 & 0 \\ 0 & -\delta_2 \end{pmatrix} \begin{pmatrix} \psi_{u_2} \\ \phi_{u_2} \end{pmatrix} + \begin{pmatrix} -\lambda - s & \lambda \\ \mu & -\mu \end{pmatrix} \begin{pmatrix} \psi \\ \phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with the boundary condition:

$$\begin{cases} \psi(u_1, \frac{\delta_2}{\delta_1}u_1) = C_2 e^{-\gamma_2 \frac{\delta_2}{\delta_1}u_1} & \text{for all } u_1 \geq 0, \\ \phi(0, u_2) = 1 & \text{for all } u_2 \geq 0. \end{cases} \quad (6)$$

*Proof.* Conditioning on the first shared claim occurrence epoch we obtain:

$$\begin{aligned} \psi(u_1, u_2, s) &= e^{-\lambda h} e^{-sh} \psi(u_1 + c_1 h, u_2 + c_2 h, s) \\ &+ \int_0^h \lambda e^{-\lambda t} dt \int_0^{\frac{u_1 + c_1 t}{\delta_1} \wedge \frac{u_2 + c_2 t}{\delta_2}} \mu e^{-\mu z} e^{-st} \psi(u_1 + c_1 t - \delta_1 z, u_2 + c_2 t - \delta_2 z, s) dz \\ &+ \int_0^h \lambda e^{-\lambda t} dt \int_{\frac{u_1 + c_1 t}{\delta_1} \wedge \frac{u_2 + c_2 t}{\delta_2}}^{\infty} \mu e^{-\mu z} dz. \end{aligned}$$

Both integrals on the LHS go to 0 as  $h \rightarrow 0$ . This implies that  $\psi$  is a continuous function with respect to  $u_1$  and  $u_2$ . Note that  $\frac{u_1 + c_1 t}{\delta_1} \wedge \frac{u_2 + c_2 t}{\delta_2} = \frac{u_1 + c_1 t}{\delta_1}$  since we live in the upper cone. This gives after simple manipulation:

$$\begin{aligned} &\frac{\psi(u_1 + c_1 h, u_2 + c_2 h, s) - \psi(u_1, u_2, s)}{h} + \frac{e^{-\lambda h} e^{-sh} - 1}{h} \psi(u_1 + c_1 h, u_2 + c_2 h, s) \\ &+ \frac{1}{h} \int_0^h \lambda e^{-\lambda t} dt \int_0^{(u_1 + c_1 t)/\delta_1} \mu e^{-\mu z} e^{-st} \psi(u_1 + c_1 t - \delta_1 z, u_2 + c_2 t - \delta_2 z, s) dz \\ &+ \frac{1}{h} \int_0^h \lambda e^{-\lambda t} dt e^{-\mu(u_1 + c_1 t)/\delta_1} = 0. \end{aligned}$$

Note that last 3 terms on the LHS of above equation have limits as  $h \rightarrow 0$ , since  $\psi$  is a continuous function. Thus  $\psi$  is a differentiable function of  $(u_1, u_2)$ . Taking  $h \rightarrow 0$  we derive:

$$\begin{aligned} &c_1 \psi_{u_1}(u_1, u_2, s) + c_2 \psi_{u_2}(u_1, u_2, s) + (-\lambda - s) \psi(u_1, u_2, s) \\ &+ \lambda \int_0^{u_1/\delta_1} \mu e^{-\mu z} \psi(u_1 - \delta_1 z, u_2 - \delta_2 z, s) dz + \lambda e^{-\mu u_1/\delta_1} = 0 \end{aligned}$$

which is equivalent to

$$c_1\psi_{u_1}(u_1, u_2, s) + c_2\psi_{u_2}(u_1, u_2, s) + (-\lambda - s)\psi(u_1, u_2, s) + \lambda\phi(u_1, u_2, s) = 0,$$

where  $\phi(u_1, u_2, s) = 1 - \mu \int_0^{u_1/\delta_1} \bar{\psi}(u_1 - \delta_1 z, u_2 - \delta_2 z, s) e^{-\mu z} dz$  for non-ruin probability  $\bar{\psi}(u_1, u_2, s) = 1 - \psi(u_1, u_2, s)$ . This gives first equation in Feynman-Kac systems. To derive second one, we use integration-by-parts formula:

$$\begin{aligned} \phi(u_1, u_2, s) &= 1 - \int_0^{u_1/\delta_1} \frac{d}{dz} (-e^{-\mu z}) \bar{\psi}(u_1 - \delta_1 z, u_2 - \delta_2 z, s) dz \\ &= 1 - \bar{\psi}(u_1 - \delta_1 z, u_2 - \delta_2 z, s) (-e^{-\mu z}) \Big|_0^{u_1/\delta_1} \\ &\quad - \int_0^{u_1/\delta_1} \frac{d}{dz} \bar{\psi}(u_1 - \delta_1 z, u_2 - \delta_2 z, s) e^{-\mu z} dz \\ &= 1 - \bar{\psi}(u_1, u_2, s) + \bar{\psi} \left( 0, u_2 - \frac{\delta_2}{\delta_1} u_1, s \right) e^{-\mu u_1/\delta_1} \\ &\quad + \delta_1 \int_0^{u_1/\delta_1} \bar{\psi}_{u_1}(u_1 - \delta_1 z, u_2 - \delta_2 z, s) e^{-\mu z} dz \\ &\quad + \delta_2 \int_0^{u_1/\delta_1} \bar{\psi}_{u_2}(u_1 - \delta_1 z, u_2 - \delta_2 z, s) e^{-\mu z} dz. \end{aligned}$$

Note that,

$$\begin{aligned} -\phi_{u_1}(u_1, u_2, s) &= \mu \int_0^{u_1/\delta_1} \bar{\psi}_{u_1}(u_1 - \delta_1 z, u_2 - \delta_2 z, s) e^{-\mu z} dz \\ &\quad + \frac{\mu}{\delta_1} \bar{\psi} \left( 0, u_2 - \frac{\delta_2}{\delta_1} u_1, s \right) e^{-\mu u_1/\delta_1}. \end{aligned}$$

Moreover,

$$-\phi_{u_2}(u_1, u_2, s) = \mu \int_0^{u_1/\delta_1} \bar{\psi}_{u_2}(u_1 - \delta_1 z, u_2 - \delta_2 z, s) e^{-\mu z} dz$$

and  $1 - \bar{\psi}(u_1, u_2, s) = \psi(u_1, u_2, s)$ . Hence

$$\mu\phi(u_1, u_2, s) = \mu\psi(u_1, u_2, s) - \delta_1\phi_{u_1}(u_1, u_2, s) - \delta_2\phi_{u_2}(u_1, u_2, s)$$

which gives second equation in Feynman-Kac formula. Boundary conditions follows immediately. QED

It will be useful to use the linear transformation:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A} \begin{pmatrix} r \\ w \end{pmatrix} = \begin{pmatrix} \delta_1 & c_1 \\ \delta_2 & c_2 \end{pmatrix} \begin{pmatrix} r \\ w \end{pmatrix}$$

with inverse transformation:

$$\begin{pmatrix} r \\ w \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = d^{-1} \begin{pmatrix} c_2 & -c_1 \\ -\delta_2 & \delta_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

where  $d := \delta_1 c_2 - \delta_2 c_1 < 0$ .

Note that:

- during drift periods,  $r$  is constant and  $w$  increases, at rate  $w' = 1$
- during jump periods,  $w$  is constant and  $r$  decreases, at rate  $r' = -1$ .

Thus, in these new coordinates, the time is split between moving in the direction of the axes, and at any time  $T = T_w + T_r$ , where  $T_w/T_r$  are the total times of winning (upward drifting)/reducing the reserves (jumping). Note that  $w = w_0 + T_w, r = r_0 - T_r$  holds.

In terms of the  $(r, w)$  variables, Feynman-Kac system becomes:

$$\begin{pmatrix} \psi_w \\ -\phi_r \end{pmatrix} + \begin{pmatrix} -\lambda - s & \lambda \\ \mu & -\mu \end{pmatrix} \begin{pmatrix} \psi \\ \phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (7)$$

with

$$\begin{cases} \psi(r, 0) & = C_2 e^{-\gamma_2 \delta_2 r}, \\ \phi(r, -\frac{\delta_1}{c_1} r) & = 1, \end{cases} \quad (8)$$

for all  $r \geq 0$ . Recall that the upper cone is described by inequalities  $r \geq 0$  and  $w \leq 0$ . Following the steps used in the proof of differentiability of  $\psi$  one can prove that  $\psi$  is in class  $\mathcal{C}^2$ .

Eliminating  $\phi$ , we find

$$\psi_{rw} - (\lambda + s)\psi_r + \mu\psi_w - s\mu\psi = 0 \quad (9)$$

with

$$\begin{cases} \psi(r, 0) = C_2 e^{-\gamma_2 \delta_2 r}, \\ \lambda^{-1} \{(\lambda + s)\psi(r, -\frac{\delta_1}{c_1} r) - \psi_w(r, -\frac{\delta_1}{c_1} r)\} = 1. \end{cases} \quad (10)$$



We may remove the linear terms by switching to the function

$$h(r, w) := e^{\mu r} e^{-(\lambda+s)w} \psi(r, w)$$

in terms of which we get:

$$h_{rw} + \mu\lambda h = 0 \tag{11}$$

with the boundary conditions:

$$\begin{cases} h(r, 0) &= C_2 e^{-(\gamma_2 - \mu)r} & \text{for all } r \geq 0, \\ h_w(r, -\frac{\delta_1 r}{c_1}) &= -\lambda e^{\mu r - (\lambda+s)\frac{\delta_1 r}{c_1}} & \text{for all } r \geq 0. \end{cases}$$

As direct solutions of both systems are not obvious, we shall proceed in the next section by obtaining an equation for the double Laplace transform of  $\psi(u_1, u_2)$  in  $u_1, u_2$  and inverting this transform in the complex plane using Bromwich type contours.

### 3 The Laplace transform

First note that process  $(U_1(t)/\delta_1, U_2(t)/\delta_2)$  has the same ruin probability as the original two-dimensional process  $(U_1(t), U_2(t))$ . Thus it suffices to analyze the case when  $\delta_1 = \delta_2 = 1$  and  $c_1 = p_1, c_2 = p_2$ . Let  $x_i = u_i/\delta_i$ . In this section we will write  $\psi(x_1, x_2)$  for  $\psi(u_1, u_2)$  under above assumptions. By conditioning at the position of the process at time  $T$  (time of crossing two lines composing barrier  $b(t)$ ):

$$T = T(x_1, x_2) = \frac{x_2 - x_1}{p_1 - p_2} \tag{12}$$

it was shown in [3] that if  $x_2 > x_1$ , it holds that

$$\bar{\psi}(x_1, x_2) := 1 - \psi(x_1, x_2) = \int_0^\infty \bar{\psi}_2(z) \tilde{P}_{(x_1, T)}(dz), \tag{13}$$

where

$$\tilde{P}_{(x_1, T)}(dx) = P_{x_1}(\inf_{s \leq T} U_1(s) > 0, U_1(T) \in dx)/dz$$

and

$$\bar{\psi}_2(z) := 1 - \psi_2(z) = 1 - C_2 e^{-\gamma_2 z}. \tag{14}$$

Suprun [7] (see also Bertoin [4], Lem. 1) gives the resolvent of a spectrally negative Lévy process killed as it enters the nonpositive half-line as

$$\begin{aligned} & \frac{1}{q} P_{x_1}(\inf_{s \leq e_q} U_1(s) > 0, U_1(e_q) \in dz) / dz \\ & = \exp\{-q^+(q)z\} W^{(q)}(x_1) - \mathbf{1}_{\{x_1 \geq z\}} W^{(q)}(x_1 - z), \end{aligned} \quad (15)$$

where  $q^+(q)$  largest root of  $\kappa(\alpha) = q$ , where the  $\kappa$  characteristic exponent of  $U_i/\delta_1$ , and  $W^{(q)} : [0, \infty) \rightarrow [0, \infty)$ , called the  $q$ -scale function, is continuous and increasing function with Laplace transform

$$\int_0^\infty e^{-\alpha x} W^{(q)}(y) dy = (\kappa(\alpha) - q)^{-1}, \quad \theta > q^+(q).$$

Note that the characteristic exponent of  $U_1/\delta_1$  is given by

$$\kappa(\alpha) = p_1 \alpha - \frac{\lambda \alpha}{\mu + \alpha}.$$

Now we obtain the Laplace transform of the non-ruin probability with respect to the initial reserves:

$$\tilde{\psi}(p, q) = \int_0^\infty \int_0^\infty e^{-px_1} e^{-qx_2} \bar{\psi}(x_1, x_2) dx_1 dx_2.$$

Note that

$$\tilde{\psi}(p, q) = \int_0^\infty \int_0^{x_1} e^{-px_1} e^{-qx_2} \bar{\psi}(x_1, x_2) dx_2 dx_1 + \int_0^\infty \int_{x_1}^\infty e^{-px_1} e^{-qx_2} \bar{\psi}(x_1, x_2) dx_2 dx_1.$$

The first Laplace transform is given by

$$\int_0^\infty \int_0^{x_1} e^{-px_1} e^{-qx_2} [1 - C_2 e^{-\gamma_2 x_2}] dx_2 dx_1 = \frac{1}{p} \frac{[(1 - C_2)(p + q) + \gamma_2]}{(p + q)(p + q + \gamma_2)} := A.$$

Writing  $s = p + q$  and  $r = (p_1 - p_2)q$  we see from (13) and (15) that the

second Laplace transform is given by

$$\begin{aligned}
\int_0^\infty \int_{x_1}^\infty e^{-px_1} e^{-qx_2} \bar{\psi}(x_1, x_2) dx_2 dx_1 &= (p_1 - p_2) \int_0^\infty e^{-sx_1} dx_1 \int_0^\infty [1 - C_2 e^{-\gamma_2 z}] \\
&\quad \times [e^{-q^+(r)z} W^{(r)}(x_1) - \mathbf{1}_{z \leq x_1} W^{(r)}(x_1 - z)] dz \\
&= \frac{p_1 - p_2}{\kappa(s) - r} \left[ \frac{(1 - C_2)q^+(r) + \gamma_2}{q^+(r)(\gamma_2 + q^+(r))} - \frac{1}{s} + \frac{C_2}{\gamma_2 + s} \right] \\
&= \frac{p_1 - p_2}{\kappa(s) - r} \left[ \frac{\gamma_2(q^+(r)/\mu + 1)}{q^+(r)(\gamma_2 + q^+(r))} - \frac{(\mu + p + q)(1 - C_2)}{(p + q)(\gamma_2 + p + q)} \right] \\
&:= C - B.
\end{aligned}$$

Note that  $A - B$  is equal to

$$\frac{(1 - C_2)p_2}{p(\kappa(s) - r)} = \frac{(\mu + p + q)(p_2 - \rho)}{pp_1(z_1(q) - p)(z_2(q) - p)},$$

where

$$\begin{aligned}
z_1(q) &= \frac{-(p_2q + p_1(q + \gamma_1)) - \sqrt{(p_2q + p_1(q + \gamma_1))^2 - 4p_1qp_2(q + \gamma_2)}}{2p_1}, \\
z_2(q) &= \frac{-(p_2q + p_1(q + \gamma_1)) + \sqrt{(p_2q + p_1(q + \gamma_1))^2 - 4p_1qp_2(q + \gamma_2)}}{2p_1}.
\end{aligned}$$

Similarly,  $C$  can be written as

$$\begin{aligned}
&\frac{p_1 - p_2}{\kappa(s) - r} \frac{\gamma_2(q^+(r)/\mu + 1)}{q^+(r)(\gamma_2 + q^+(r))} \\
&= \frac{(\mu + p + q)(p_2 - \rho)}{pp_1(z_1(q) - p)(z_2(q) - p)} \left[ p \frac{(p_1 - p_2)(q^+(q(p_1 - p_2)) + \mu)}{p_2(q^+(q(p_1 - p_2))(\gamma_2 + q^+(q(p_1 - p_2))))} \right].
\end{aligned}$$

Thus, the Laplace transform is given by

$$\tilde{\psi}(p, q) = \frac{(\mu + p + q)(p_2 - \rho)(1 + ph(q))}{pp_1(p - z_1(q))(p - z_2(q))}, \quad (16)$$

where

$$h(q) = \frac{(p_1 - p_2)(q^+(q(p_1 - p_2)) + \mu)}{p_2(q^+(q(p_1 - p_2))(\gamma_2 + q^+(q(p_1 - p_2))))}.$$

Noting that  $q^+(q(p_1 - p_2))$  is the largest root of  $\kappa(\alpha) = q(p_1 - p_2)$  and  $z_2(q)$  is the largest root of  $\kappa(v + q) = q(p_1 - p_2)$  we identify

$$q^+(q(p_1 - p_2)) = z_2(q) + q.$$

Hence

$$h(q) = \frac{(p_1 - p_2)(z_2(q) + q + \mu)}{p_2(q + z_2(q))(\gamma_2 + q + z_2(q))}. \quad (17)$$

## 4 Spectral representation

In this subsection we invert the Laplace transform found in the previous section. To perform the inversion we shall employ the method of residues. For an overview of the theory of Laplace transforms see e.g. [9]; a recent application to one-dimensional ruin is given in [6]. The residues method leads to a spectral representation of the non-ruin probability  $\bar{\psi}(x_1, x_2)$  given in Theorem 2.

First we extend  $\tilde{\psi}(p, q)$  to  $\text{Re } \underline{p} < 0$ ,  $\text{Re } q < 0$ . Then Mellin's formula gives the following expression for  $\bar{\psi}(x_1, x_2)$ :

$$\bar{\psi}(x_1, x_2) = \left( \frac{1}{2\pi i} \right)^2 \int_{\alpha - i\infty}^{\alpha + i\infty} \int_{\beta(q) - i\infty}^{\beta(q) + i\infty} \tilde{\psi}(p, q) e^{x_2 q} e^{x_1 p} dp dq \quad (18)$$

for  $\alpha > 0$  and  $\beta(q) > z_2(q)$ .

The Laplace transform  $\tilde{\psi}(p, q)$  is regular in

$$P = \{(p, q) : q \notin [q_+, q_-], q \neq 0, q \neq -\gamma_2, p \neq z_1(q), p \neq z_2(q)\},$$

where

$$q_+ = -\frac{1}{p_1 - p_2}(\sqrt{\lambda} + \sqrt{p_1 \mu})^2,$$

$$q_- = -\frac{1}{p_1 - p_2}(\sqrt{\lambda} - \sqrt{p_1 \mu})^2.$$

Consider the two-dimensional contour that is the product of contour given in Figure 2 (for  $q$ ) and the contour consisting of line  $[\alpha - iR, \alpha + iR]$  and semicircle (for  $p$ ). Note that  $|z_2(q)| < |q|$  for large  $|q|$ , thus  $|h(q)|$  is dominated by  $A_0/|q|$  for certain constant  $A_0$  and it tends to 0 as  $|q| \rightarrow \infty$ . Thus

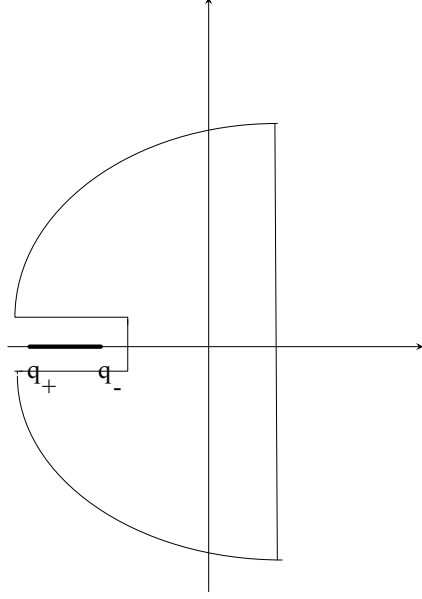


Figure 2: Bromwich contour

$|\tilde{\psi}(p, q)| \sim (A_1|p|^2 + A_2|q|)/(A_3|p|^3 + A_4|p|^2|q| + A_5|p||q|^2)$  for certain constants  $A_i$  ( $i = 1, \dots, 5$ ) that goes to 0 as  $|p| \rightarrow \infty$  or  $|q| \rightarrow \infty$ . Hence Mellin's integral (18) equals to:

$$\begin{aligned} \left(\frac{1}{2\pi i}\right)^2 \oint \oint \tilde{\psi}(p, q) e^{x_2 q} e^{x_1 p} dp dq &= \left(\frac{1}{2\pi i}\right)^2 \int_{q_+ \rightarrow q_-} \oint \tilde{\psi}(p, q) e^{x_2 q} e^{x_1 p} dp dq \\ &= \left(\frac{1}{2\pi i}\right)^2 \int_{q_- \rightarrow q_+} \oint \tilde{\psi}(p, q) e^{x_2 q} e^{x_1 p} dp dq, \end{aligned}$$

where integrals  $\int_{q_+ \rightarrow q_-}$  and  $\int_{q_- \rightarrow q_+}$  are integrals along lines of the contour from  $q_+$  to  $q_-$  and from  $q_-$  to  $q_+$  respectively.

In the sequel we will analyze two separate cases:

$$\rho < \frac{p_2^2}{p_1}, \quad (19)$$

$$\rho > \frac{p_2^2}{p_1} \quad (20)$$

In the first case

$$z_2(-\gamma_2) = \frac{\mu}{p_2} \left( \frac{p_2^2}{p_1} - \rho \right) \geq 0$$

and  $z_1(-\gamma_2) = 0$  and in the second case  $z_2(-\gamma_2) = 0$  and

$$z_1(-\gamma_2) = \frac{\mu}{p_2} \left( \frac{p_2^2}{p_1} - \rho \right) \leq 0.$$

In the first case from Cauchy theorem we have

$$\begin{aligned} & \left( \frac{1}{2\pi i} \right)^2 \oint \oint \tilde{\psi}(p, q) e^{x_2 q} e^{x_1 p} dq dp \\ &= \text{Res}_{p=0, q=0} \tilde{\psi}(p, q) e^{p x_1} e^{q x_2} + \text{Res}_{p=0, q=-\gamma_2} \tilde{\psi}(p, q) e^{p x_1} e^{q x_2} \\ & \quad + \text{Res}_{p=z_2(-\gamma_2), q=-\gamma_2} \tilde{\psi}(p, q) e^{p x_1} e^{q x_2} + \text{Res}_{p=z_1(0), q=0} \tilde{\psi}(p, q) e^{p x_1} e^{q x_2} \end{aligned}$$

and in the second:

$$\begin{aligned} & \left( \frac{1}{2\pi i} \right)^2 \oint \oint \tilde{\psi}(p, q) e^{x_2 q} e^{x_1 p} dq dp \\ &= \text{Res}_{p=0, q=0} \tilde{\psi}(p, q) e^{p x_1} e^{q x_2} + \text{Res}_{p=0, q=-\gamma_2} \tilde{\psi}(p, q) e^{p x_1} e^{q x_2} \\ & \quad + \text{Res}_{p=z_1(-\gamma_2), q=-\gamma_2} \tilde{\psi}(p, q) e^{p x_1} e^{q x_2} + \text{Res}_{p=z_1(0), q=0} \tilde{\psi}(p, q) e^{p x_1} e^{q x_2}. \end{aligned}$$

Formally we take contour from Figure 2 with  $q_{\pm} + \epsilon$  and  $q \pm \epsilon i$  for  $q \in [q_+, q_-]$ . Then take a limit with  $\epsilon \downarrow 0$ . Note that  $\text{Arg}[(z - q_+)(z - q_-)] = \text{Arg}(z - q_+) + \text{Arg}(z - q_-)$  implies that if

$$z = q + i\epsilon, \quad z \in [q_+, q_-], \quad \epsilon > 0,$$

then

$$\lim_{\epsilon \rightarrow 0} \text{Arg}(z - q_+) = \pi, \quad \lim_{\epsilon \rightarrow 0} \text{Arg}(z - q_-) = 0;$$

if

$$z = q - i\epsilon, \quad z \in [q_+, q_-], \quad \epsilon > 0,$$

then

$$\lim_{\epsilon \rightarrow 0} \text{Arg}(z - q_+) = -\pi, \quad \lim_{\epsilon \rightarrow 0} \text{Arg}(z - q_-) = 0.$$

Hence considering continuous spectrum  $p = z_1(q)$  and  $p = z_2(q)$  for  $q \in [q_+, q_-]$  we have

$$\begin{aligned}
& \left(\frac{1}{2\pi i}\right)^2 \int_{q_+ \rightarrow q_-} \oint \tilde{\psi}(p, q) e^{x_2 q} e^{x_1 p} dp dq \\
& + \left(\frac{1}{2\pi i}\right)^2 \int_{q_- \rightarrow q_+} \oint \tilde{\psi}(p, q) e^{x_2 q} e^{x_1 p} dp dq \\
& = \frac{1}{2\pi i} \int_{q_+}^{q_-} \left( \tilde{\psi}_+(z^+(q), q) e^{x_2 q} e^{x_1 z^+(q)} - \tilde{\psi}_-(z^-(q), q) e^{x_2 q} e^{x_1 z^-(q)} \right) dq \\
& - \frac{1}{2\pi i} \int_{q_+}^{q_-} \left( \tilde{\psi}_+(z^-(q), q) e^{x_2 q} e^{x_1 z^-(q)} - \tilde{\psi}_-(z^+(q), q) e^{x_2 q} e^{x_1 z^+(q)} \right) dq,
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\psi}_+(p, q) &= \frac{(\mu + p + q)(p_2 - \rho)(1 + ph^+(q))}{pp_1(z^+(q) - z^-(q))}, \\
\tilde{\psi}_-(p, q) &= \frac{(\mu + p + q)(p_2 - \rho)(1 + ph^-(q))}{pp_1(z^-(q) - z^+(q))}
\end{aligned}$$

and  $h_{\pm}(q)$  is a  $h(q)$  when we put  $z^{\pm}$  instead of  $z_2(q)$  and

$$z^-(q) = a(q) - ib(q), \quad z^+(q) = a(q) + ib(q) \quad (21)$$

for

$$\begin{aligned}
a(q) &= \frac{-(p_1\mu - \lambda + p_2q + p_1q)}{2p_1}, \\
b(q) &= \frac{\sqrt{4p_1(p_2q\mu + p_2q^2 - \lambda q) - (p_1\mu - \lambda + p_2q + p_1q)^2}}{2p_1}.
\end{aligned}$$

Straightforward calculations gives:

$$\text{Res}_{p=0, q=0} \tilde{\psi}(p, q) e^{px_1} e^{qx_2} = 1, \quad \text{Res}_{p=z_1(0), q=0} \tilde{\psi}(p, q) e^{px_1} e^{qx_2} = -C_1 e^{-\gamma_1 x_1}.$$

Note that  $\lim_{q \rightarrow -\gamma_2} (q + \gamma_2) h(q) = 0$  in the first case and  $\lim_{q \rightarrow -\gamma_2} (q + \gamma_2) h(q) = p_1 z_1(-\gamma_2) / (p_2 \gamma_2)$  in the second case. In the first case (19) we have then

$$\text{Res}_{p=0, q=-\gamma_2} \tilde{\psi}(p, q) e^{px_1} e^{qx_2} = 0, \quad \text{Res}_{p=z_2(-\gamma_2), q=-\gamma_2} \tilde{\psi}(p, q) e^{px_1} e^{qx_2} = 0$$

and in the second case (20)

$$\text{Res}_{p=0, q=-\gamma_2} \tilde{\psi}(p, q) e^{px_1} e^{qx_2} = -C_2 e^{-\gamma_2 x_2}$$

and

$$\text{Res}_{p=z_1(-\gamma_2), q=-\gamma_2} \tilde{\psi}(p, q) e^{px_1} e^{qx_2} = \frac{p_2}{p_1} e^{-\gamma_3 x_1} e^{-\gamma_2 x_2},$$

where  $\gamma_3 = \frac{\mu}{p_2} \left( \rho - \frac{p_2^2}{p_1} \right)$ . From above we derive the main Theorem 2.

**Theorem 2** *Let  $x_2 > x_1$ . If  $\rho < \frac{p_2^2}{p_1}$  holds, then*

$$\bar{\psi}(x_1, x_2) = 1 - C_1 e^{-\gamma_1 x_1} + \omega(x_1, x_2),$$

where

$$\omega(x_1, x_2) = \frac{1}{4\pi} \int_{q_+}^{q_-} (h^- - h^+)(q) (f(z^+(q), q) - f(z^-(q), q)) dq$$

with  $z^+(q)$  and  $z^-(q)$  given in (21) and

$$f(p, q) = \frac{(\mu + p + q)(p_2 - \rho)}{p_1 b(q)} e^{px_1} e^{qx_2}$$

and  $h^+(q) = g(z^+(q), q)$ ,  $h^-(q) = g(z^-(q), q)$ , where

$$g(p, q) = \frac{(p_1 - p_2)(p + q + \mu)}{p_2(q + p)(\gamma_2 + q + p)}.$$

If  $\rho > \frac{p_2^2}{p_1}$  holds, then

$$\bar{\psi}(x_1, x_2) = 1 - C_1 e^{-\gamma_1 x_1} - C_2 e^{-\gamma_2 x_1} + \frac{p_2}{p_1} e^{-\gamma_3 x_1} e^{-\gamma_2 x_2} + \omega(x_1, x_2).$$

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