HAL
open science

From Laplacian Transport to Dirichlet-to-Neumann (Gibbs) Semigroups

Valentin A. Zagrebnov

To cite this version:

Valentin A. Zagrebnov. From Laplacian Transport to Dirichlet-to-Neumann (Gibbs) Semigroups. Journal of Mathematical Physics, Analysis, Geometry, 2008, 4 (4), pp.551-568. hal-00219752

HAL Id: hal-00219752

https://hal.science/hal-00219752v1

Submitted on 27 Jan 2008

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Lyapunov Memorial Conference, June 24-30, 2007

FROM LAPLACIAN TRANSPORT TO DIRICHLET-TO-NEUMANN (GIBBS) SEMIGROUPS

Valentin A.Zagrebnov [1]

Université de la Méditerranée
Centre de Physique Théorique - UMR 6207 [2]
Luminy - Case 907, 13288 Marseille, Cedex 09, France

Abstract[3]

The paper gives a short account of some basic properties of *Dirichlet-to-Neumann* operators $\Lambda_{\gamma,\partial\Omega}$ including the corresponding semigroups motivated by the Laplacian transport in anisotropic media ($\gamma \neq I$) and by elliptic systems with dynamical boundary conditions. For illustration of these notions and the properties we use the explicitly constructed *Lax semigroups*. We demonstrate that for a general smooth bounded convex domain $\Omega \subset \mathbb{R}^d$ the corresponding Dirichlet-to-Neumann semigroup $\left\{U(t) := e^{-t\Lambda_{\gamma,\partial\Omega}}\right\}_{t\geq 0}$ in the Hilbert space $L^2(\partial\Omega)$ belongs to the *trace-norm* von Neumann-Schatten ideal for any $t > 0$. This means that it is in fact an *immediate Gibbs* semigroup. Recently Emamirad and Laadnani have constructed a *Trotter-Kato-Chernoff* product-type approximating family $\{(V_{\gamma,\partial\Omega}(t/n))^n\}_{n\geq 1}$ *strongly* converging to the semigroup $U(t)$ for $n \to \infty$. We conclude the paper by discussion of a conjecture about convergence of the *Emamirad-Laadnani approximantes* in the the *trace-norm* topology.

Key words: Laplacian transport, Dirichlet-to-Neumann operators, Lax semigroups, Dirichlet-to-Neumann semigroups, Gibbs semigroups.
PACS: 47A55, 47D03, 81Q10

hal-00219752, version 1 - 27 Jan 2008

[1]E-mail: Valentin.Zagrebnov@cpt.univ-mrs.fr

[2]Université de Provence - Aix-Marseille I, Université de la Méditerranée - Aix-Marseille II, Université du Sud - Toulon - Var, FRUMAM (FR 2291)

[3]An extended version of the author's talk presented on the *Lyapunov Memorial Conference*, June 24-30, 2007 (Kharkov University, Ukraine), which is based on the common project with Professor Hassan Emamirad (Laboratoire de Mathématiques, Université de Poitiers).

Contents

1. Laplacian transport and Dirichlet-to-Neumann operators 2
2. Dirichlet-to-Neumann operators and semigroups 4
3. Product approximations of Dirichlet-to-Neumann semigroups 10
4. Dirichlet-to-Neumann Gibbs semigroups 12
5. Concluding remarks: trace-norm approximations 15
References 17

1. Laplacian transport and Dirichlet-to-Neumann operators

Example 1.1. Is is well-known (see e.g. [LeUl]) that the problem of determining a *conductivity matrix* field $\gamma(x) = [\gamma_{i,j}(x)]_{i,j=1}^d$, for x in a bounded open domain $\Omega \subset \mathbb{R}^d$, is related to "measuring" the elliptic *Dirichlet-to-Neumann* map for associated conductivity equation. Notice that solution of this problem has a lot of practical applications in various domains: geophysics, electrochemistry etc. It is also an important diagnostic tool in medicine, e.g. in the *electrical impedance tomography*; the tissue in the human body is an example of highly anisotropic conductor [BaBr].

Under assumption that there is no sources or sinks of current the potential $v(x),\ x \in \Omega$, for a given voltage $f(\omega),\ \omega \in \partial\Omega$, on the (smooth) boundary $\partial\Omega$ of Ω is a solution of the Dirichlet problem:

$$(\mathbf{P1}) \qquad \begin{cases} \mathrm{div}(\gamma \nabla v) = 0 \quad \text{in } \ \Omega, \\ v|_{\partial\Omega} = f \quad \text{on } \ \partial\Omega. \end{cases}$$

Then the corresponding to (**P1**) Dirichlet-to-Neumann map (operator) $\Lambda_{\gamma,\partial\Omega}$ is defined by

$$\Lambda_{\gamma,\partial\Omega} : f \mapsto \partial v_f / \partial \nu_\gamma := \nu \cdot \gamma \ \nabla v_f \ |_{\partial\Omega} \ . \tag{1.1}$$

Here ν is the unit outer-normal vector to the boundary at $\omega \in \partial\Omega$ and the function $u := u_f$ is solution of the Dirichlet problem (**P1**).

The Dirichlet-to-Neumann operator (1.1) is also called the *voltage-to-current* map, since the function $\Lambda_{\gamma,\partial\Omega} f$ gives the induced current flux trough the boundary $\partial\Omega$. The key (*inverse*) problem is whether on can determine the conductivity matrix γ by knowing electrical boundary measurements, i.e. the corresponding Dirichlet-to-Neumann operator? Unfortunately, this operator does not determine the matrix γ uniquely, see e.g. [GrUl] and references there.

Example 1.2. The problem of electrical current flux in the form (**P1**) is an example of so-called *Laplacian transport*. Besides the voltage-to-current problem the motivation to study this kind of transport comes for instance from the *transfer* across biological membranes, see e.g. [Sap], [GrFiSap].

Let some "species" of concentration $C(x)$, $x \in \mathbb{R}^d$, diffuse in the *isotropic* bulk ($\gamma = I$) from a (distant) source localised on the closed boundary $\partial\Omega_0$ towards

a *semipermeable* compact interface $\partial\Omega$ on which they disappear at a given rate W. Then the *steady* concentration field (Laplacian transport with a diffusion coefficient D) obeys the set of equations:

$$(\mathbf{P2}) \qquad \begin{cases} \Delta C = 0, \ x \in \Omega_0 \setminus \Omega \ , \\ C(\omega_0 \in \partial\Omega_0) = C_0, \ \text{at the source} \ , \\ D \ \partial_\nu C(\omega) = W \ (C(\omega) - 0), \ \text{on the interface} \ \omega \in \partial\Omega \end{cases}$$

Let $C = C_0(1-u)$. Then $\Delta u = 0$, $x \in \Omega$. If we put $\mu := D/W$, then the boundary conditions on $\partial\Omega$ take the form: $(I - \mu\partial_\nu)u \mid_{\partial\Omega} (\omega) = 1 \mid_{\partial\Omega} (\omega)$, where $(1 \mid_{\partial\Omega})(\omega) = \chi_{\partial\Omega}(\omega)$ is characteristic function of the set $\partial\Omega$, and $u(\omega_0) = 0, \omega_0 \in \partial\Omega_0$ on the source boundary.

Consider now the following auxiliary Laplace-Dirichlet problem:

$$\Delta u = 0, \ x \in \Omega_0 \setminus \Omega \ , \quad u \mid_{\partial\Omega} (\omega) = f(\omega \in \partial\Omega) \ \text{ and } \ u \mid_{\partial\Omega_0} (\omega) = 0 \ , \tag{1.2}$$

with solution u_f. Then similar to (1.1) we can associate with the problem (1.2) a Dirihlet-to-Neumann operator

$$\Lambda_{\gamma=I,\partial\Omega} : f \mapsto \partial_\nu u_f \mid_{\partial\Omega} \tag{1.3}$$

with domain $\mathrm{dom}(\Lambda_{I,\partial\Omega})$, which belongs to a certain *Sobolev* space, Section 2.

The advantage of this approach is that as soon as the operator (1.3) is defined one can apply it to study the *mixed* boundary value problem (**P2**). This gives in particular the value of the particle flux due to Laplacian transport across the membrane $\partial\Omega$. Indeed, one obtain that $(I + \mu\Lambda_{I,\partial\Omega})u \mid_{\partial\Omega} = 1 \mid_{\partial\Omega}$, and that the local (diffusive) particle *flux* is defined as:

$$\phi \mid_{\partial\Omega} := D \ C_0(-\partial_n u) \mid_{\partial\Omega} = D \ C_0(\Lambda_{I,\partial\Omega}(I + \mu\Lambda_{I,\partial\Omega})^{-1} 1) \mid_{\partial\Omega} \ . \tag{1.4}$$

Then the corresponding total flux across the membrane $\partial\Omega$:

$$\Phi := (\phi, 1)_{L^2(\partial\Omega)} = D \ C_0(\Lambda(I + \mu\Lambda_{I,\partial\Omega})^{-1} 1, 1)_{L^2(\partial\Omega)} \tag{1.5}$$

is experimentally measurable macroscopic response of the system, expressed via transport parameters D, C_0, μ and geometry of $\partial\Omega$. Here $(\cdot,\cdot)_{L^2(\partial\Omega)}$ is scalar product in the Hilbert space $\partial\mathcal{H} := L^2(\partial\Omega)$.

The aim of the present paper is twofold:
(i) to give a short account of some standard results about Dirichlet-to-Neumann operators and related *Dirichlet-to-Neumann semigroups* that solve a certain class of elliptic systems with dynamical boundary conditions;
(ii) to present some recent results concerning the *approximation* theory and the *Gibbs* character of the Dirichlet-to-Neumann semigroups for compact sets Ω with smooth boundaries $\partial\Omega$.

To this end in the next Section 2 we recall some fundamental properties of the Dirichlet-to-Neumann operators and semigroups, we illustrate them by few elementary examples, including the *Lax semigroups* [Lax].

In Section 3 we present the strong *Emamirad-Laadnani approximations* of the Dirichlet-to-Neumann semigroups inspired by the *Chernoff* theory and by its generalizations in [NeZag], [CaZag2].

We show in Section 4 that for compact sets Ω with smooth boundaries $\partial\Omega$ the Dirichlet-to-Neumann semigroups are in fact (immediate) *Gibbs* semigroups [Zag2].

Some recent results and conjectures about approximations of the Dirichlet-to-Neumann (Gibbs) semigroups in operator and *trace-norm* topologies are collected in the last Section 5.

2. Dirichlet-to-Neumann operators and semigroups

2.1 Dirichlet-to-Neumann operators

Let Ω be an open bounded domain in $\mathbb{R}^d$ with a smooth boundary $\partial\Omega$. Let γ be a $C^\infty(\overline{\Omega})$ matrix-valued function on $\overline{\Omega}$, which we call the *Laplacian transport matrix* in domain Ω.

We suppose that the matrix-valued function $\gamma(x) := [\gamma_{i,j}(x)]_{i,j=1}^d$ satisfies the following hypotheses:

(H1) The real coefficients are symmetric and $\gamma_{i,j}(x) = \gamma_{j,i}(x) \in \mathcal{C}^\infty(\overline{\Omega})$.

(H2) There exist two constants $0 < c_1 \leq c_2 < \infty$ such that for all $\xi \in \mathbb{R}^d$, we have

$$c_1\|\xi\|^2 \leq \sum_{i,j=1}^n \xi_i\xi_j\gamma_{i,j}(x) \leq c_2\|\xi\|^2. \tag{2.1}$$

Then the *Dirichlet-to-Neumann* operator $\Lambda_{\gamma,\partial\Omega}$ associated with the Laplacian transport in Ω is defined as follows.

Let $f \in C(\partial\Omega)$ and denote by v_f the *unique* solution (see e.g. [GiTr], Theorem 6.25) of the Dirichlet problem

$$(\mathbf{P1}) \qquad \begin{cases} A_{\gamma,\partial\Omega}\ v := \operatorname{div}(\gamma\ \nabla v) = 0 \quad \text{in } \ \Omega\ , \\ v\ |_{\partial\Omega} = f \quad \text{on } \ \partial\Omega\ , \end{cases}$$

in the Banach space $X := C(\overline{\Omega})$. Here operator $A_{\gamma,\partial\Omega}$ is defined on its maximal domain

$$\operatorname{dom}(A_{\gamma,\partial\Omega}) := \{u \in X : A_{\gamma,\partial\Omega}\ u \in X\}\ . \tag{2.2}$$

Definition 2.1. *The Dirichlet-to-Neumann operator is the map*:

$$\Lambda_{\gamma,\partial\Omega} : f \mapsto \partial v_f/\partial\nu_\gamma = \nu\cdot\gamma\nabla v_f\ |_{\partial\Omega}\ , \tag{2.3}$$

with domain :

$$\begin{aligned} &\operatorname{dom}(\Lambda_{\gamma,\partial\Omega}) = \\ &\{f \in \partial C(\Omega_R) : v_f \in \operatorname{Ker}(A_{\gamma,\partial\Omega}) \ \text{ and } \ |(\nu\cdot\gamma\nabla v_f\ |_{\partial\Omega})| < \infty\}. \end{aligned} \tag{2.4}$$

Here ν denotes the unit outer-normal vector at $\omega \in \partial\Omega$ and v_f is the solution of Dirichlet problem **(P1)**.

The solution $v_f := L_{\partial\Omega}f$ of the problem **(P1)** is called the *γ-harmonic lifting* of f, where $L_{\partial\Omega} : C(\partial\Omega) \mapsto C^2(\Omega) \cap C(\overline{\Omega})$ is called the *lifting* operator with

domain $\mathrm{dom}(L_{\partial\Omega}) = C(\partial\Omega)$. If $T_{\partial\Omega} : C(\overline{\Omega}) \mapsto C(\partial\Omega)$ denotes the *trace* operator on the smooth boundary $\partial\Omega$, i.e. $v\mid_{\partial\Omega} = T_{\partial\Omega}\, v$, then [Eng]:

$$L_{\partial\Omega} = (T_{\partial\Omega}\mid_{\mathrm{Ker}(A_{\gamma,\partial\Omega})})^{-1} \ \text{ and } \ \mathrm{dom}(\Lambda_{\gamma,\partial\Omega}) = T_{\partial\Omega}\{\mathrm{Ker}(A_{\gamma,\partial\Omega})\}\ . \tag{2.5}$$

Remark 2.2. *Let $\partial X := C(\partial\Omega)$. Then (2.5) implies :*

$$T_{\partial\Omega} L_{\partial\Omega}\, u = u\ ,\ u \in \partial X \ \text{ and } \ L_{\partial\Omega} T_{\partial\Omega}\, w = w\ ,\ w \in \mathrm{Ker}(A_{\gamma,\partial\Omega})\ . \tag{2.6}$$

One also gets that the lifting operator is bounded: $L_{\partial\Omega} \in \mathcal{L}(\partial X, X)$, whereas the Dirichlet-to-Neumann operator (2.3) is obviously not.

Now let $\mathcal{H}$ be Hilbert space $L^2(\Omega)$ and $\partial\mathcal{H} := L^2(\partial\Omega)$ denote the *boundary space*. In order that the problem **(P1)** admits a *unique* solution v_f, one has to assume that $f \in W_2^{1/2}(\partial\Omega)$, and then v_f belongs the *Sobolev* space $W_2^1(\Omega)$, see e.g. [Tay, Ch.7]. So, we can define Dirichlet-to-Neumann operator in the Hilbert space $\partial\mathcal{H}$ by (2.3) with domain:

$$\mathrm{dom}(\Lambda_{\gamma,\partial\Omega}) := \{f \in W_2^{1/2}(\partial\Omega) :\ \Lambda_{\gamma,\partial\Omega} f \in \partial\mathcal{H} = L^2(\partial\Omega)\}. \tag{2.7}$$

Proposition 2.3. *The Dirichlet-to-Neumann operator (2.3) with domain (2.7) in the Hilbert space $\partial\mathcal{H}$ is unbounded, non-negative, self-adjoint, first-order elliptic pseudo-differential operator with compact resolvent.*

The complete proof can be found e.g. in [Tay, Ch.7], [Tay1]. Therefore, we give here only some comments on these properties of the Dirichlet-to-Neumann operator (2.3) in $\partial\mathcal{H} = L^2(\partial\Omega)$.

Remark 2.4. (a) *By virtue of definition (2.3) for any $f \in W_2^{1/2}(\partial\Omega)$ one gets:*

$$(f, \Lambda_{\gamma,\partial\Omega} f)_{\partial\mathcal{H}} = \int_{\partial\Omega} d\sigma(\omega)\ \overline{v_f(\omega)}\ \nu\cdot\gamma(\omega)(\nabla v_f)(\omega) = \tag{2.8}$$

$$\int_\Omega dx\ \ \mathrm{div}(\overline{v_f(x)}\ (\gamma\nabla v_f)(x)) = \int_\Omega dx\ (\nabla\overline{v_f(x)}\cdot\gamma\ \nabla v_f)(x)) \geq 0\ ,$$

since the matrix γ verifies **(H2)**. *Thus, operator $\Lambda_{\gamma,\partial\Omega}$ is non-negative.*
(b) *In fact to ensure the existence of the trace $T_{\partial\Omega}(\nu\cdot\gamma\nabla(L_{\partial\Omega} f))$ one has initially to define operator $\Lambda_{\gamma,\partial\Omega}$ for $f \in W_2^{3/2}(\partial\Omega)$. Then Dirichlet-to-Neumann operator is a self-adjoint extension with domain (2.7) and moreover it is a bounded map $\Lambda_{\gamma,\partial\Omega} : W_2^{1/2}(\partial\Omega) \mapsto W_2^{-1/2}(\partial\Omega)$.*
(c) *By (2.8) and since derivatives of the first-order are involved in (2.3) one can conclude that this operator should be elliptic and pseudo-differential. If $\gamma(x) = I$, then $\Lambda_{I,\partial\Omega}$ is, roughly, the operator $(-\Delta_{\partial\Omega})^{1/2}$, where $\Delta_{\partial\Omega}$ is the Laplace-Beltrami operator on $\partial\Omega$, with corresponding induced metric* [Tay, Ch.7], [Tay1].
(d) *Compactness of the imbedding $W_2^{1/2}(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$ implies the compactness of the resolvent of $\Lambda_{\gamma,\partial\Omega}$.*

By (a) and (d) the spectrum $\sigma(\Lambda_{\gamma,\partial\Omega})$ of the Dirichlet-to-Neumann operator is a set of non-negative increasing eigenvalues $\{\lambda_k\}_{k=1}^\infty$. The rate of increasing is given by the Weyl asymptotic formula, see e.g. [Hor], [Tay]:

Proposition 2.5. *Let $\Lambda_{\gamma,\partial\Omega}(x,\xi)$, for $(x,\xi)\in T^*\partial\Omega$, be the symbol of the first-order, elliptic pseudo-differential Dirichlet-to-Neumann operator $\Lambda_{\gamma,\partial\Omega}$. Then the asymptotic behaviour of the corresponding eigenvalues as $k\to\infty$ has the form:*

$$\lambda_k \sim \left\{\frac{k}{C(\partial\Omega,\Lambda_\gamma)}\right\}^{1/(d-1)} ,$$

where

$$C(\partial\Omega,\Lambda_\gamma) := \frac{1}{(2\pi)^{d-1}}\int_{\Lambda_{\gamma,\partial\Omega}(x,\xi)\le 1} dx\, d\xi \ .$$

Another important result is due to Hislop and Lutzer [HiLu]. It concerns a localization (*rapid decay*) of the γ-harmonic lifting of the corresponding eigenfunctions.

Proposition 2.6. *Let $\{\phi_k\}_{k=1}^\infty$ be eigenfunctions of the Dirichlet-to-Neumann operator: $\Lambda_{\gamma,\partial\Omega}=\lambda_k\phi_k$ with $\|\phi_k\|_{L^2(\partial\Omega)}=1$. Let $v_{\phi_k}:=L_{\partial\Omega}\phi_k$ be γ-harmonic lifting of ϕ_k to Ω corresponding to the problem* **(P1)**. *Then for any compact $\mathcal{C}\subset\phi_k$ and $x\in\mathcal{C}$ one gets the representation:*

$$|v_{\phi_k}(x)| = \psi(x,p,\mathcal{C})/{\lambda_k}^p \tag{2.9}$$

with arbitrary large $p>0$. Here $\psi(x,p,\mathcal{C})$ is a decreasing function of the distance $\mathrm{dist}(x,\,\partial\Omega)$.

Since by the Weyl asymptotic formula we have $\lambda_k = O(k^{1/(d-1)})$, the decay implied by the estimate (2.9) is algebraic.

Conjecture 2.7. [HiLu] *In fact the order of decay instead of $\psi(x,p,\mathcal{C})/{\lambda_k}^p$ is exponential: $O(\exp[-\,k\,\mathrm{dist}(\mathcal{C},\,\partial\Omega)])$.*

2.2 Example of a Dirichlet-to-Neumann operator

To illustrate the results mentioned above we consider a simple example which will be useful below for contraction of the *Lax semigroups*.

Consider a homogeneous isotropic case: $\gamma(x)=I$, and let $\Omega=\Omega_R:=\{x\in\mathbb{R}^{d=3}:\ \|x\|<R\}$. Then $A_{\gamma,\partial\Omega_R}=\Delta_{\partial\Omega_R}$ and for the *harmonic lifting* of

$$f(\omega)=\sum_{l,m} f^{(R)}_{l,m}\ Y_{l,m}(\theta,\varphi)\in W_2^{1/2}(\partial\Omega_R)\ ,$$

we obtain:

$$v_f(r,\theta,\varphi)=\sum_{l,m}\left(\frac{r}{R}\right)^l\ f^{(R)}_{l,m}\ Y_{l,m}(\theta,\varphi), \tag{2.10}$$

since the *spherical* functions $\{Y_{l,m}\}_{l=0,|m|\le l}^\infty$ form a complete orthonormal basis in the Hilbert space $\partial\mathcal{H}=L^2(\partial\Omega_R,\ d\theta\,\sin\theta\,d\varphi)$.

Definition (2.3) and (2.10) imply that non-negative, self-adjoint, first-order elliptic pseudo-differential Dirichlet-to-Neumann operator

$$(\Lambda_{I,\partial\Omega_R} f)(\omega = (R,\theta,\varphi)) = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \left(\frac{l}{R}\right) \; f_{l,m}^{(R)} \; Y_{l,m}(\theta,\varphi), \tag{2.11}$$

has discrete spectrum $\sigma(\Lambda_{I,\partial\Omega_R}) := \{\lambda_{l,m} = \; l/R\}_{l=0,|m|\leq l}^{\infty}$ with spherical eigenfunctions:

$$(\Lambda_{I,\partial\Omega_R} Y_{l,m})(R,\theta,\varphi) = \left(\frac{l}{R}\right) \; Y_{l,m}(\theta,\varphi), \tag{2.12}$$

and multiplicity m. The operator (2.11) is obviously unbounded and it has a compact resolvent.

Remark 2.8. *Since by virtue of (2.10) the γ-harmonic lifting of the eigenfunction $Y_{l,m}$ to the ball Ω_R is*

$$v_{Y_{l,m}}(r,\theta,\varphi) = \left(\frac{r}{R}\right)^l \; Y_{l,m}(\theta,\varphi) \; ,$$

one can check the localization (Proposition 2.6) and Conjecture about the exponential decay explicitly. For distances: $0 < \mathrm{dist}(x,\partial\Omega_R) = R - r \ll R$, one obtains $|v_{Y_{l,m}}(r,\theta,\varphi)| = O(e^{-l(R-r)/R})$.

2.3 Dirichlet-to-Neumann semigroups on ∂X

To define the Dirichlet-to-Neumann semigroups on the *boundary* Banach space $\partial X = C(\partial\Omega)$ we can follow the line of reasoning of [Esc] or [Eng]. To this end consider in $X = C(\Omega)$ the following elliptic system with *dynamical boundary conditions*

$$(\mathbf{P2}) \qquad \begin{cases} \mathrm{div}(\gamma\nabla u(t,\cdot)) = 0 & \text{in } (0,\infty)\times\Omega, \\ \partial u(t,\cdot)/\partial t + \partial u(t,\cdot)/\partial\nu_\gamma = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ u(0,\cdot) = f & \text{on } \partial\Omega. \end{cases}$$

Proposition 2.9. *The problem* **(P2)** *has a unique solution $u_f(t,x)$ for any $f \in C(\partial\Omega)$. Its trace on the boundary $\partial\Omega$ has the form:*

$$u_f(t,\omega) := (T_{\partial\Omega} u_f(t,\cdot))(\omega) = (U(t)f)(\omega) \; , \tag{2.13}$$

where the family of operators $\{U(t) = e^{-t\Lambda_{\gamma,\partial\Omega}}\}_{t\geq 0}$ is a C_0-semigroup generated by the Dirichlet-to-Neumann operator of the problem **(P1)**.

The following key result about the properties of the Dirichlet-to-Neumann semigroups on the boundary Banach space $\partial X = C(\partial\Omega)$ is due to Escher-Engel [Esc],[Eng] and Emamirad-Laadnani [EmLa]:

Proposition 2.10. *The semigroup $\{U(t) = e^{-t\Lambda_{\gamma,\partial\Omega}}\}_{t\geq 0}$ is analytic, compact, positive, irreducible and Markov C_0-semigroup of contractions on $C(\partial\Omega)$.*

Remark 2.11. *The complete proof can be found in the papers quoted above. So, here we make only some comments and hints concerning the Proposition 2.10.*

2.4 Dirichlet-to-Neumann semigroups on $\partial\mathcal{H}$

The Dirichlet-to-Neumann semigroup $\{U(t) = e^{-t\Lambda_{\gamma,\partial\Omega}}\}_{t\geq 0}$ on $\partial\mathcal{H}$ is defined by self-adjoint and non-negative Dirichlet-to-Neumann generator $\Lambda_{\gamma,\partial\Omega}$ of Proposition 2.3.

Proposition 2.12. *The Dirichlet-to-Neumann semigroup $\{U(t) = e^{-t\Lambda_{\gamma,\partial\Omega}}\}_t$ on the Hilbert space $\partial\mathcal{H}$ is a holomorphic quasi-sectorial contraction with values in the trace-class $\mathfrak{C}_1(\partial\mathcal{H})$ for $\mathrm{Re}\,(t) > 0$.*

Remark 2.13. *The first part of the statement follows from Proposition 2.3. Since the generator $\Lambda_{\gamma,\partial\Omega}$ is self-adjoint and non-negative, the semigroup $\{U(t)\}_t$ is holomorphic and quasi-sectorial contraction for $\mathrm{Re}\,(t) > 0$, see e.g.* [CaZag1], [Zag1]. *Compactness of the resolvent of $\Lambda_{\gamma,\partial\Omega}$ implies the compactness of $\{U(t)\}_{t>0}$, but to prove the last part of the statement we need a supplementary argument about asymptotic behaviour of its eigenvalues given by the Weyl asymptotic formula, Proposition 2.5.*

This behaviour of eigenvalues implies the second part of the Proposition 2.12:

Lemma 2.14. *The Dirichlet-to-Neumann semigroup $U(t)$ has values in the trace-class $\mathfrak{C}_1(\partial\mathcal{H})$ for any $t > 0$.*

Proof. Since the Dirichlet-to-Neumann operator $\Lambda_{\gamma,\partial\Omega}$ is self-adjoint, we have to prove that

$$\|U(t)\|_1 = \sum_{k\geq 1} \mathrm{e}^{-t\lambda_k} < \infty \ , \tag{2.14}$$

for $t > 0$. Here $\|\cdot\|_1$ denotes the norm in the trace-class $\mathfrak{C}_1(\partial\mathcal{H})$. Then the Weyl asymptotic formula implies that there exists bounded M and function $r(k)$ such that

$$\begin{aligned}\sum_{k\geq 1} \mathrm{e}^{-t\lambda_k} &\leq \sum_{k\geq 1} \exp\{-t[(k/c)^{\frac{1}{d-1}} + r(k)]\} \\ &\leq \mathrm{e}^{tM} \sum_{k\geq 1} \exp\{-t(k/c)^{\frac{1}{d-1}}\}.\end{aligned}$$

Here $c := C(\partial\Omega, \Lambda_\gamma)$ and the last sum converges for any $t > 0$, which proves the equation (2.14). □

2.5 Example: Lax semigroups

A beautiful example of explicit representation of the Dirichlet-to-Neumann semigroup (2.13) is due to Lax [Lax], Ch.36.

Let $\gamma(x) = I$, and $\Omega = \Omega_R$, see Section 2.2. Following [Lax] we define the mapping:

$$K(t) : v(x) \mapsto v(e^{-t/R}\, x) \ \text{ for any } \ u \in C(\Omega_R) \ , \tag{2.15}$$

which is a semigroup for the parameter $t \geq 0$ in the Banach space $X = C(\Omega_R)$:

$$(K(\tau)K(t)v)(x) = v(e^{-\tau/R}\, e^{-t/R}\, x) = v(e^{-(\tau+t)/R}\, x) \ , \ \tau, t \geq 0 \ , \ x \in \Omega_R \ . \tag{2.16}$$

Remark 2.15. *It is clear that if $v(x)$ is $(\gamma = I)$-harmonic in $C(\Omega_R)$, then the function: $x \mapsto v(e^{-t/R}\ x)$ is also harmonic. Therefore,*

$$u_f(t,x) := v_f(e^{-t/R}\ x) = (K(t)L_{\partial\Omega_R}f)(x) = (L_{\partial\Omega_R}f_t)(x)\ ,\ x \in \Omega_R\ , \tag{2.17}$$

is the harmonic lifting of the function $f_t(\omega) := v_f(e^{-t/R}\ \omega)$, $\omega \in \partial\Omega_R$, where v_f solves the problem **(P1)** *for $\gamma = I$. Since in the spherical coordinates $x = (r,\theta,\varphi)$ one has:*

$$\partial v_f(t,x)/\partial t = -\partial_r v_f(e^{-t/R}r,\theta,\varphi)e^{-t/R}\ (r/R)$$

and

$$\partial v_f(t,R,\theta,\varphi)/\partial\nu_I = \partial_r v_f(e^{-t/R}r,\theta,\varphi)e^{-t/R}\ ,$$

we get that $\partial u_f(t,\omega)/\partial t + \partial u_f(t,\omega)/\partial\nu_I = 0$, i.e. the function (2.17) is a solution of the problem **(P2)**.

Hence, according to (2.13) and (2.17) the operator family:

$$S(t) := T_{\partial\Omega_R}K(t)L_{\partial\Omega_R}\ ,\ t \geq 0\ , \tag{2.18}$$

defines the Dirichlet-to-Neumann semigroup corresponding to the problem **(P2)** for $\gamma(x) = I$, and $\Omega = \Omega_R$, which is known as the *Lax semigroup*. By virtue of (2.17) and (2.18) the action of this semigroup is known explicitly:

$$(S(t)f)(\omega) = v_f(e^{-t/R}\omega)\ ,\ \omega \in \partial\Omega_R\ . \tag{2.19}$$

Notice that the semigroup relation:

$$S(\tau)S(t) = T_{\partial\Omega_R}K(\tau)L_{\partial\Omega_R}T_{\partial\Omega_R}K(t)L_{\partial\Omega_R} = S(\tau+t)\ , \tag{2.20}$$

follows from the properties of lifting and trace operators (see Remark 2.2), from identity (2.16) and definition (2.18). One finds generator $\Lambda_{\gamma=I,\partial\Omega_R}$ of this semigroup from the limit:

$$0 = \lim_{t\to 0}\sup_{\omega\in\partial\Omega_R}|\frac{1}{t}(f - S(t)f)(\omega) - (\Lambda_{\gamma=I,\partial\Omega_R}f)(\omega)| = \tag{2.21}$$

$$\lim_{t\to 0}\sup_{\omega\in\partial\Omega_R}|\frac{1}{t}(v_f(R,\theta,\varphi) - v_f(e^{-t/R}\ R,\theta,\varphi)) - (\Lambda_{\gamma=I,\partial\Omega_R}f)(R,\theta,\varphi)|.$$

Then operator

$$(\Lambda_{\gamma=I,\partial\Omega_R}f)(R,\theta,\varphi) = \partial_r v_f(r = R,\theta,\varphi) \tag{2.22}$$

for any function f from domain:

$$\mathrm{dom}(\Lambda_{I,\partial\Omega_R}) = \{f \in \partial C(\Omega_R) : v_f \in \mathrm{Ker}(A_{I,\partial\Omega_R})\,\mathrm{and}\,|(\partial_r v_f)\ |_{\partial\Omega_R}\ | < \infty\} \tag{2.23}$$

is identical to (2.4) for the case: $\gamma = I$ and $\partial\Omega = \partial\Omega_R$. Therefore, generator (2.22) of the Lax semigroup is the the Dirichlet-to-Neumann operator in this particular case of the Banach space $\partial X = C(\partial\Omega_R)$.

Similarly we can consider the Lax semigroup (2.18) in the Hilbert space $\partial\mathcal{H} = L^2(\partial\Omega_R,\ d\theta\,\sin\theta\,d\varphi)$. Since generator of this semigroup is a particular case of

the Dirichlet-to-Neumann operator (2.11), by (2.12) and (2.10) we again obtain the corresponding action in the explicit form:

$$(S(t)f)(\omega) = \tag{2.24}$$
$$(e^{-t\Lambda_{I,\partial\Omega_R}}f)(\omega)) = \sum_{l=0}^{\infty}\sum_{m=-l}^{m=l}\sum_{s=0}^{\infty}\frac{(-t)^s}{s!}\left(\frac{l}{R}\right)^s f_{l,m}^{(R)}\, Y_{l,m}(\theta,\varphi) =$$
$$\sum_{l=0}^{\infty}\sum_{m=-l}^{m=l}(e^{-\ t/R})^l\ f_{l,m}^{(R)}\ Y_{l,m}(\theta,\varphi) = v_f(e^{-t/R}\omega)\ ,\ \omega\in\partial\Omega_R\ ,$$

which coincides with (2.19).

Notice that for $t>0$ the Lax semigroups have their values in the trace-class $\mathfrak{C}_1(\partial\mathcal{H})$. This explicitly follows from (2.12), i.e. from the fact that the spectrum of the semigroup generator $\sigma(\Lambda_{I,\partial\Omega_R}) := \{\lambda_{l,m} = \ l/R\}_{l=0,|m|\le l}^{\infty}$ is discrete and

$$\mathrm{Tr}\, S(t) = \sum_{l=0}^{\infty}(2l+1)\ e^{-l/R} < \infty\ . \tag{2.25}$$

The last is proven in the whole generality in Theorem 2.14.

3. Product approximations of Dirichlet-to-Neumann semigroups

3.1 Approximating family

Since in contrast to the Lax semigroup ($\gamma = I$) the action of the general Dirichlet-to-Neumann semigroup for $\gamma\neq I$ is known only implicitly (2.13), it is useful to construct converging approximations, which are simpler for calculations and analysis.

One of them is the Emamirad-Laadnani *approximation* [EmLa], which is motivated by the explicit action (2.19), (2.24) of the Lax semigroup

$$(S(t)f)(\omega) = (T_{\partial\Omega_R}K(t)L_{\partial\Omega_R}f)(\omega) = v_f(e^{-t/R}\omega),\ \omega\in\partial\Omega_R\ , \tag{3.1}$$
$$K_R(t) : v(x)\mapsto v(e^{-t/R}\ x)\ \text{ for any }\ v\in C(\Omega_R)\,(\text{or}\,\mathcal{H}(\Omega_R))\ .$$

The suggestion of [EmLa] consists in substitution of the family $\{K(t)\}_{t\ge0}$ by the $\gamma-$*deformed* operator family:

$$K_{\gamma,R}(t) : v(x)\mapsto v(e^{-(t/R)\ \gamma(x)}\ x)\ \text{ for any }\ v\in C(\Omega_R)\,(\text{or}\,\mathcal{H}(\Omega_R))\ . \tag{3.2}$$

Definition 3.1. *For the ball Ω_R the Emamirad-Laadnani approximating family $\{V_{\gamma,R}(t) := V_{\gamma,\partial\Omega_R}(t)\}_{t\ge0}$ is defined by*

$$(V_{\gamma,R}(t)f)(\omega) := (T_{\partial\Omega_R}K_\gamma(t)L_{\partial\Omega_R}f)(\omega) = v_f(e^{-(t/R)\ \gamma(\omega)}\ \omega),\ \omega\in\partial\Omega_R\ . \tag{3.3}$$

Remark 3.2. (a) *Notice that the approximating family (3.3) is not a semigroup:*

$$(V_{\gamma,R}(t)V_{\gamma,R}(s)f)(\omega) = (T_{\partial\Omega_R}K_\gamma(t)L_{\partial\Omega_R}\widetilde{f}(s))(\omega) = \tag{3.4}$$
$$v_{\widetilde{f}(s)}(e^{-(t/R)\ \gamma(\omega)}\ \omega) \neq v_f(e^{-((t+s)/R)\ \gamma(\omega)}\ \omega) = (V_{\gamma,R}(t+s))f)(\omega)\ .$$

(b) *This family is strongly continuous at* $t = 0$*:*

$$\lim_{t\searrow 0} V_{\gamma,R}(t)f = f \quad \text{for any} \quad f \in \partial X \,(\text{or}\, \partial\mathcal{H}) \ . \tag{3.5}$$

(c) *By definition (3.3) this family has derivative at* $t = +0$*:*

$$(\partial_t V_{\gamma,R}(t)f)(\omega)\,|_{t=0} = -\nu(\omega)\cdot\gamma(\omega)(\nabla v_f)(\omega) = -(\Lambda_{\gamma,\partial\Omega_R} f)(\omega) \ , \tag{3.6}$$

which for any $f \in \mathrm{dom}(\Lambda_{\gamma,\partial\Omega_R})$ *coincides with the (minus) Dirichlet-to-Neumann operator (2.3).*

3.2 Strong approximation of the Dirichlet-to-Neumann semigroups

By virtue of Remark 3.2 the Emamirad-Laadnani approximation family verifies the conditions of the Chernoff *approximation theorem* (Theorem 1.1, [Che]):

Proposition 3.3. *Let* $\{\Phi(s)\}_{s\geq 0}$ *be a family of linear contractions on a Banach space* $\mathfrak{B}$ *and let* X_0 *be the generator of a* C_0*-contraction semigroup. Define* $X(s) := s^{-1}(I - \Phi(s))$*,* $s > 0$*. Then for* $s \to +0$ *the family* $\{X(s)\}_{s>0}$ *converges strongly in the resolvent sense to the operator* X_0 *if and only if the sequence* $\{\Phi(t/n)^n\}_{n\geq 1}$*,* $t > 0$*, converges strongly to* e^{-tX_0} *as* $n \to \infty$*, uniformly on any compact* t*-intervals in* $\mathbb{R}^1_+$*.*

Notice that $\{V_{\gamma,R}(t)\}_{t\geq 0}$ in the Banach space ∂X is the family of contractions because of the *maximum principle* for the γ-harmonic functions v_f. Since the Dirichlet-to-Neumann operator (2.3) is densely defined and closed, Remark 3.2 (c) implies that the family $X(s) := s^{-1}(I - V_{\gamma,R}(s))$ converges for $s \to +0$ to $X_0 = \Lambda_{\gamma,\partial\Omega_R}$ in the strong resolvent sense.

The similar arguments are valid for the case of the Hilbert space $\partial\mathcal{H}$. By virtue of Remark 2.4 the Dirichlet-to-Neumann operator $\Lambda_{\gamma,\partial\Omega}$ is non-negative and self-adjoint. This implies again that (3.3) is the family of contractions in $\partial\mathcal{H}$ and that by Remark 3.2 (c) the family $X(s) := s^{-1}(I - V_{\gamma,R}(s))$ converges for $s \to +0$ to $X_0 = \Lambda_{\gamma,\partial\Omega_R}$ in the *strong* resolvent sense.

Resuming the above observations we obtain the *strong* approximation of the Dirichlet-to-Neumann semigroup $U(t)$:

Corollary 3.4. [EmLa]

$$\lim_{n\to\infty} (V_{\gamma,R}(t/n))^n f = U(t)f \ , \quad \textit{for every } f \in \partial X \ \textit{ or } \partial\mathcal{H} \ , \tag{3.7}$$

uniformly on any compact t*-intervals in* $(0, \infty)$*.*

The Emamirad-Laadnani approximation theorem (Corollary 3.4) has the following important extension to more *general* geometry than ball [EmLa].

Definition 3.5. *We say that a bounded smooth domain* Ω *in* $\mathbb{R}^d$ *has the* property of the interior ball*, if for any* $\omega \in \partial\Omega$ *there exists a tangent to* $\partial\Omega$ *at* ω *plane* $\mathcal{T}_\omega$*, and such that one can construct a ball tangent to* $\mathcal{T}_\omega$ *at* ω*, which is totally included in* Ω*.*

If Ω has this property, then with any point $\omega \in \partial\Omega$, one can associate a *unique* point x_ω, which is the center of the *biggest* ball $B(x_\omega, r_\omega)$ of radius r_ω included in Ω. For any $0 < r \leq r_\omega$, we can construct the approximating family $V_r(t)$ related to the ball $B(x_{r,\omega}, r) := \{x \in \Omega : |x - x_{r,\omega}| \leq r\}$ of radius r, which is centered on the line perpendicular to $\mathcal{T}_\omega$ at the point $\omega \in \partial\Omega$, i.e. $x_{r,\omega} = (r/r_\omega)x_\omega + (1 - r/r_\omega)\omega$. Then we define

$$(V_{\gamma,r}(t)f)(\omega) := T_{\partial\Omega}\ v_f\left(x_{r,\omega} + \mathrm{e}^{-(t/r)\gamma(\omega)}(r\ \nu_\omega)\right)\ . \tag{3.8}$$

Here ν_ω is the outer-normal vector at ω, the function $v_f = L_{\partial\Omega}f$ is the γ-harmonic lifting of the boundary condition f on $\partial\Omega$, and $T_{\partial\Omega}$ is the trace operator:

$$T_{\partial\Omega} : H^1(\Omega) \ni v \longmapsto v\ |_{\partial\Omega} \in H^{1/2}(\partial\Omega). \tag{3.9}$$

Remark 3.6. *Notice that:*
(a) *since* $\nu_\omega = (\omega - x_{r,\omega})/r$, *one gets* $(V_{\gamma,r}(t = 0)f)(\omega) := (T_{\partial\Omega}\ v_f)(\omega) = f(\omega)$ *;*
(b) *by virtue of (3.8) the strong derivative at* $t = 0$ *has the form:*

$$(\partial_t V_{\gamma,r}(t = 0)f)(\omega) = -\gamma(\omega)\nu_\omega \cdot (\nabla v_f)(\omega) = -(\Lambda_{\gamma,\partial\Omega}f)(\omega),$$

see (3.6).

Proposition 3.7. [EmLa] *Let* Ω *has the property of interior ball, and let*

$$\inf_{\omega\in\partial\Omega}\{r > 0 : B(x_\omega, r_\omega) \subset \Omega\} > 0,$$
$$\sup_{\omega\in\partial\Omega}\{r > 0 : B(x_\omega, r_\omega) \subset \Omega\} < \infty\ .$$

For any $0 < s \leq 1$ *we define* V_{γ,sr_ω}, *i.e.*

$$V_{\gamma,sr_\omega}f(\omega) = v_f\left(x_{s,\omega} + \mathrm{e}^{-(t/(sr_\omega))\gamma(\omega)}(sr_\omega\ \nu_\omega)\right)\ , \tag{3.10}$$

where $x_{s,\omega} = sx_\omega + (1 - s)\omega$. *Then for any* $0 < s \leq 1$

$$\lim_{n\to\infty}(V_{\gamma,sr_\omega}(t/n))^n f = U(t)f\ , \quad \textit{for every } f \in \partial X \textit{ or } \partial\mathcal{H}\ , \tag{3.11}$$

uniformly on any compact t*-intervals in* $(0, \infty)$.

Remark 3.8. *By Definition 3.1 for the ball* Ω_R *and constant matrix-valued function* $\gamma(x) = I$ *one obviously have* $V_{\gamma=I,R}(t) = S(t) = U(t)$. *On the other hand, for a general smooth domain* Ω *with geometry verifying the conditions of Proposition 3.7, one is obliged to consider the family of approximations* V_{γ,sr_ω} *even for the homogeneous case* $\gamma = I$.

4. Dirichlet-to-Neumann Gibbs semigroups

4.1 Gibbs semigroups

Since by Lemma 2.14 for any Dirichlet-to-Neumann semigroup we obtain: $U(t > 0) \in \mathfrak{C}_1(\partial\mathcal{H})$, then one can check that it is in fact a *Gibbs* semigroup. To this end we recall main definitions and some results that we need for the proof, see e.g. [Zag2].

Let $\mathfrak{H}$ be a separable, infinite-dimensional complex Hilbert space. We denote by $\mathcal{L}(\mathfrak{H})$ the algebra of all bounded operators on $\mathfrak{H}$ and by $\mathfrak{C}_\infty(\mathfrak{H}) \subset \mathcal{L}(\mathfrak{H})$ the subspace of all *compact* operators. The $\mathfrak{C}_\infty(\mathfrak{H})$ is a $*$-*ideal* in $\mathcal{L}(\mathfrak{H})$, that is: if $A \in \mathfrak{C}_\infty(\mathfrak{H})$, then $A^* \in \mathfrak{C}_\infty(\mathfrak{H})$ and, if $A \in \mathfrak{C}_\infty(\mathfrak{H})$ and $B \in \mathcal{L}(\mathfrak{H})$, then $AB \in \mathfrak{C}_\infty(\mathfrak{H})$ and $BA \in \mathfrak{C}_\infty(\mathfrak{H})$. We say that a compact operator $A \in \mathfrak{C}_\infty(\mathfrak{H})$ belongs to the *von Neumann-Schatten* $*$-ideal $\mathfrak{C}_p(\mathfrak{H})$ for a certain $1 \le p < \infty$, if the norm

$$\|A\|_p := \left(\sum_{n\ge 1} s_n(A)^p\right)^{1/p} < \infty, \tag{4.1}$$

where $s_n(A) := \sqrt{\lambda_n(A^*A)}$ are the *singular* values of A, defined by the eigenvalues $\{\lambda_n(\cdot)\}_{n\ge 1}$ of non-negative self-adjoint operator A^*A. Since the norm $\|A\|_p$ is a non-increasing function of $p > 0$, one gets:

$$\|A\|_1 \ge \|A\|_p \ge \|A\|_q > \|A\|_\infty (= \|A\|), \tag{4.2}$$

for $1 \le p \le q < \infty$. Then for the von Neumann-Schatten ideals this implies inclusions:

$$\mathfrak{C}_1(\mathfrak{H}) \subseteq \mathfrak{C}_p(\mathfrak{H}) \subseteq \mathfrak{C}_q(\mathfrak{H}) \subset \mathfrak{C}_\infty(\mathfrak{H}). \tag{4.3}$$

Let $p^{-1} = q^{-1} + r^{-1}$. Then by virtue of the *Hölder inequality* applied to (4.1) one gets: $\|AB\|_p \le \|A\|_q \|B\|_r$, if $A \in \mathfrak{C}_q(\mathfrak{H})$ and $B \in \mathfrak{C}_r(\mathfrak{H})$. Consequently we obtain:

Lemma 4.1. *The operator A belongs to the trace-class $\mathfrak{C}_1(\mathfrak{H})$ if and only if there exists two* (Hilbert-Schmidt) *operators $K_1,\ K_2 \in \mathfrak{C}_2(\mathfrak{H})$, such that $A = K_1\ K_2$. Similarly, if $K \in \mathfrak{C}_p(\mathfrak{H})$, then $K^p \in \mathfrak{C}_1(\mathfrak{H})$.*

Let K be integral operator in the Hilbert space $L^2(D,\mu)$. It is a Hilbert-Schmidt operator if and only if its kernel $k(x,y) \in L^2(D\times D, \mu\times\mu)$ and then one gets the estimate: $\|K\|_2 \le \|k\|_{L^2(D\times D,\mu\times\mu)}$.

The proof is quite straightforward and can be found in, e.g., [Kat], [Sim].

Definition 4.2. [Zag2] *Let $\{G(t)\}_{t\ge 0}$ be a C_0-semigroup on $\mathfrak{H}$ with $\{G(t)\}_{t>0} \subset \mathfrak{C}_\infty(\mathfrak{H})$. It is called immediate Gibbs semigroup, if $G(t) \in \mathfrak{C}_1(\mathfrak{H})$ for any $t > 0$ and it is called eventually Gibbs semigroup, if there is $t_0 > 0$, such that $G(t) \in \mathfrak{C}_1(\mathfrak{H})$ for any $t \ge t_0$.*

Remark 4.3. (a) *Notice that by Lemma 4.1 any C_0-semigroup such that one has $\{G(t)\}_{t>0} \subset \mathfrak{C}_p(\mathfrak{H})$ for some $p < \infty$, is an immediate Gibbs semigroup.*
(b) *Since compact C_0-semigroups are norm-continuous for any $t > 0$, the immediate Gibbs semigroups are $\|\cdot\|_1$-norm continuous for $t > 0$.*

For more details of the Gibbs semigroups properties we refer to the book [Zag2].

Corollary 4.4. *By virtue of Proposition 2.12, Definition 4.2 and Remark 4.3 the Dirichlet-to-Neumann semigroup $\{U(t) = e^{-t\Lambda_{\gamma,\partial\Omega}}\}_t$ on the Hilbert space $\partial\mathcal{H}$ is a $\|\cdot\|_1$-holomorphic quasi-sectorial immediate Gibbs for* $\mathrm{Re}\,(t) > 0$.

4.2 Compact and Tr-norm approximating family

Proposition 4.5. [EmLa] *For the ball Ω_R the Emamirad-Laadnani approximating family $\{V_{\gamma,R}(t)\}_{t\geq 0}$ consists of compact operators on the Banach space $\partial X = C(\partial\Omega_R)$ for any $t > 0$.*

The proof follows from Definition 3.1 by *Arzela-Ascoli* criterium of compactness, since representation (3.3) and conditions on γ imply the uniform bound and equicontinuity of the sets $\{V_{\gamma,R}(t)(\partial X)\}_t$ for any $t > 0$.

For the case of the Hilbert space we recall the following useful condition for characterization of the Tr-class operators [Zag2].

Proposition 4.6. *If $A \in \mathcal{L}(\mathfrak{H})$ and $\sum_{j=1}^{\infty} \|Ae_j\| < \infty$ for an orthonormal basis $\{e_j\}_{j=1}^{\infty}$ of $\mathfrak{H}$, then $A \in \mathfrak{C}_1(\mathfrak{H})$.*

Theorem 4.7. *On the Hilbert space $\partial\mathcal{H} = L^2(\partial\Omega_R)$ the approximating family $\{V_{\gamma,R}(t)\}_{t>0} \subset \mathfrak{C}_1(\partial\mathcal{H})$.*

Proof: Since the eigenfunctions $\{\phi_k\}_{k=1}^{\infty}$ of the self-adjoint Dirichlet-to-Neumann operator $\Lambda_{\gamma,\partial\Omega_R}$ form an orthonormal basis in $L^2(\partial\Omega_R)$, we apply Proposition 4.6 for this basis.

Let $\partial\Omega_{t,\gamma,R} := \{x_\omega := e^{-(t/R)\ \gamma(\omega)}\ \omega\}_{\omega\in\partial\Omega_R}$. By representation (3.3) and by estimate (2.9) one obtains

$$\begin{aligned} \|V_{\gamma,R}(t)\phi_k\|^2 &= \int_{\partial\Omega_R} d\sigma(\omega)|v_{\phi_k}(x_\omega)|^2 \\ &\leq |\partial\Omega_R| \sup_{\omega\in\partial\Omega_R} \psi(x_\omega, p, \partial\Omega_{t,\gamma,R})^2/k^{2p/(d-1)} \ . \end{aligned} \tag{4.4}$$

Then by hypothesis **(H2)** on the matrix γ one gets for the norm of the vector x_ω in $\mathbb{R}^d$ the estimate:

$$\|x_\omega\| \leq \|e^{-(t/R)\ \gamma}\|\ R \leq e^{-c_1(t/R)}\ R\ .$$

Hence, for any $t > 0$ the $\mathrm{dist}(x_\omega, \partial\Omega_R) \geq (1 - e^{-c_1(t/R)})R > 0$, which implies for the estimates in (2.9) and in (4.4) that

$$0 < \inf_{\omega\in\partial\Omega_R} \psi(x_\omega, p, \partial\Omega_{t>0,\gamma,R}) \leq \sup_{\omega\in\partial\Omega_R} \psi(x_\omega, p, \partial\Omega_{t>0,\gamma,R}).$$

Then for $2p/(d-1) > 1$ the estimate (4.4) ensures the convergence of the series in the inequality:

$$\|V_{\gamma,R}(t)\|_1 \leq \sum_{k=1}^{\infty} \|V_{\gamma,R}(t)\phi_k\|,$$

which finishes the proof. □

5. Concluding remarks: trace-norm approximations

The strong Emamirad-Laadnani approximation theorem (Corollary 3.4) and the results of Section 4.2 proving that Dirichlet-to-Neumann semigroup $U(t)$ and approximants $V_{\gamma,\partial\Omega}(t/n)^n$ belong to $\mathfrak{C}_1(\partial\mathcal{H})$, for all $n \geq 1$ and $t > 0$, motivate the following conjecture:

Conjecture 5.1. [EmZa] *The Emamirad-Laadnani approximation theorem is valid in the* Tr*-norm topology of* $\mathfrak{C}_1(\partial\mathcal{H})$.

Remark 5.2. *Notice that the strong approximation of the Dirichlet-to-Neumann Gibbs semigroup* $U(t)$ *by the* Tr*-class family* $(V_{\gamma,\partial\Omega}(t/n))^n$ *does not lift automatically the topology of convergence to, e.g., operator-norm approximation* [Zag2].

Therefore, to prove the Conjecture 5.1 one needs additional arguments similar to those of [CaZag2]. To this end we put the difference in question $\Delta_n(t) := (V_{\gamma,\partial\Omega}(t/n))^n - U(t)$ in the following form:

$$\begin{aligned} \Delta_n(t) &= \{(V_{\gamma,\partial\Omega}(t/n))^{k_n} - (U(t/n))^{k_n}\}(V_{\gamma,R}(t/n))^{m_n} \\ &+ (U(t/n))^{k_n}\{(V_{\gamma,\partial\Omega}(t/n))^{m_n} - (U(t/n))^{m_n}\}. \end{aligned} \tag{5.1}$$

Here for any $n > 1$, we define two variables $k_n = [n/2]$ and $m_n = [(n+1)/2]$, where $[x]$ denotes the *integer part* of $x \geq 0$, i.e., $n = k_n + m_n$. Then for the estimate of $\Delta_n(t)$ in the $\mathfrak{C}_1(\partial\mathcal{H})-$topology one gets:

$$\begin{aligned} \|\Delta_n(t)\|_1 &\leq \|(V_{\gamma,\partial\Omega}(t/n))^{k_n} - (U(t/n))^{k_n}\| \, \|(V_{\gamma,\partial\Omega}(t/n))^{m_n}\|_1 \\ &+ \|(U(t/n))^{k_n}\|_1 \, \|(V_{\gamma,\partial\Omega}(t/n))^{m_n} - (U(t/n))^{m_n}\|. \end{aligned} \tag{5.2}$$

In spite of Remark 5.2, the *explicit* representation of approximants $\{(V_{\gamma,\partial\Omega}(t/n))^n\}_{n\geq 1}$ allows to prove the corresponding operator-norm estimate.

Theorem 5.3. [EmZa] *Let* $V_{\gamma,\partial\Omega_R}(t)$ *be defined by (3.3). Then one gets the estimate:*

$$\|(V_{\gamma,\partial\Omega_R}(t/n))^n - U(t)\| \leq \varepsilon(n) \ , \quad \lim_{n\to\infty} \varepsilon(n) = 0, \tag{5.3}$$

uniformly for any t*-compact in* $\mathbb{R}^1_+$.

To establish (5.3) we use the "telescopic" representation:

$$\begin{aligned} &(V_{\gamma,\partial\Omega_R}(t/n))^n - U(t) = \\ &\sum_{s=0}^{n-1} (V_{\gamma,\partial\Omega_R}(t/n))^{(n-s-1)}\{V_{\gamma,\partial\Omega_R}(t/n) - U(t/n)\}(U(t/n))^s \ , \end{aligned} \tag{5.4}$$

and the operator-norm estimate of $\{V_{\gamma,\partial\Omega_R}(t/n) - U(t/n)\}$ for large n.

The next auxiliary result establishes a relation between family of operators $V_{\gamma,\partial\Omega_R}(t)$ and the Dirichlet-to-Neumann semigroup $U(t)$.

Lemma 5.4. [EmZa] *There exists a bounded operator* $W_{\gamma,\partial\Omega_R}(t)$ *on* $\partial\mathcal{H}$ *such that*

$$V_{\gamma,\partial\Omega_R}(t) = W_{\gamma,\partial\Omega_R}(t)U(t) \ , \tag{5.5}$$

for any $t \geq 0$.

Now we return to the main inequality (5.2). To estimate the *first* term in the right-hand side of (5.2) we need Theorem 5.3 and the *Ginibre-Gruber* inequality [CaZag2]:

$$\|(V_{\gamma,\partial\Omega}(t/n))^{m_n}\|_1 \leq C\ U(m_n t/n)\ .$$

To establish the latter we use representation (5.5) given by Lemma 5.4.

To estimate the *second* term one needs only the result of Theorem 5.3. All together this gives a proof of Conjecture 5.1 at least for the ball Ω_R.

Acknowledgements

This paper is based on the lecture given by the author on the *Lyapunov Memorial Conference*, June 24-30, 2007 (Kharkov University, Ukraine). I would like to express my gratitude to organizers and in particular to Prof. Leonid A.Pastur for invitation and for support.

The conference talk, as well as the present account, are a part of the common project with Prof. Hassan Emamirad, I would like to thank him for a fruitful and pleasant collaboration.

References

[BaBr] D.C. Barber and B.H. Brown, Applied potential tomography. *J.Phys. E* **17** (1984), 723-733.

[CaZag1] V. Cachia, V.A. Zagrebnov, Operator-norm approximation of semigroups by quasi-sectorial contractions. *J. Funct. Anal.* **180** (2001), 176-194.

[CaZag2] V. Cachia and V.A. Zagrebnov, Trotter product formula for nonself-adjoint Gibbs semigroups. *J. London Math.Soc.* **64** (2001), 436-444.

[Che] P.R. Chernoff, Product formulas, nonlinear semigroups and addition of unbounded operators. *Mem. Am. Math. Soc.* **140** (1974), 1-121.

[EmLa] H. Emamirad and I. Laadnani, An approximating family for the Dirichelet-to-Neumann semigroup. *Adv. Diff. Equ.* **11** (2006), 241-257.

[EmZa] H. Emamirad and V.A. Zagrebnov, On the Gibbs character of Dirichelet-to-Neumann semigroups. (*In preparation*).

[Eng] K.-J. Engel, The Laplacian on $C(\overline{\Omega})$ with generalized Wenzell boundary conditions. *Arch.Math.* **81** (2003), 548-558.

[Esc] J. Escher, The the Dirichelet-to-Neumann operator on continuous functions. *Annali della Scuola Norm. Sup. Pisa* **21** (1994), 235-266.

[GiTr] D. Gilbarg and N.S. Trudinger, *Partial Differential Equations of Second Order* Springer-Verlag, Heidelberg-Berlin. 1983.

[GrFiSap] D.S. Grebenkov, M.Filoche and B. Sapoval, Mathematical basis for a general theory of Laplacian transport towards irregular interfaces. *Phys.Rev. E* **73** (2006), 021103-9.

[GrUl] A. Geeenleaf and G. Uhlmann, Local uniqueness for the Dirichelet-to-Neumann map via the two plane transform. *Duke Math.J.* **108** (2001), 559-617.

[HiLu] P. Hislop and C. Lutzer, Spectral asymptotics of the Dirichelet-to-Neumann map on multiply connected domains in $\mathbb{R}^n$. *Inverse Problems* **17** (2001), 1717-1741.

[Hor] L. Hörmander, *The Analysis of Linear Partial Differential Equations IV.* Springer-Verlag, New-York. 1985.

[Kat] T. Kato, *Perturbation Theory for Linear Operators* Springer-Verlag, Heidelberg-Berlin. 1966.

[Lax] P. Lax, *Functional Analysis* Wiley Inter-Science, N.Y. 2002.

[LeUl] J. Lee and G. Uhlmann, Determining anisotropic real-analytic conductivities by boundary measurement. *Comm.Pure Appl.Math.* **42** (1989), 1097-1112.

[NeZag] H.Neidhardt and V.A. Zagrebnov Trotter-Kato Product Formula and Operator-Norm Convergence. *Commun.Math.Phys.* **205** (1999), 129-159.

[Sap] B. Sapoval, General formulation of Laplacian transfer across irregular sufaces. *Phys.Rev.Lett.* **73** (1994), 3314-3316.

[Sim] B. Simon, *Trace Ideals and Their Applications* . Amer. Math. Soc. Providence. 2005.

[Tay] M.E. Taylor, *Partial Differential Equations II: Qualitative Studies of Linear Equations.* Springer-Verlag, New-York. 1996.

[Tay1] M.E. Taylor, *Preudodifferential Operators.* Princeton University Press, New Jersey. 1996.

[Zag1] V.A. Zagrebnov, Quasi-Sectorial Contractions. *J.Func.Anal.* (2008). (*In press*).

[Zag2] V.A. Zagrebnov, *Gibbs Semigroups.* Leuven Notes in Mathematical and Theoretical Physics. Vol.10, Leuven University Press, Leuven 2002.