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Teodor Banica, Julien Bichon

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# QUANTUM AUTOMORPHISM GROUPS OF VERTEX-TRANSITIVE GRAPHS OF ORDER $\leq 11$

TEODOR BANICA AND JULIEN BICHON

ABSTRACT. We study quantum automorphism groups of vertex-transitive graphs having less than 11 vertices. With one possible exception, these can be obtained from cyclic groups  $\mathbb{Z}_n$ , symmetric groups  $S_n$  and quantum symmetric groups  $\mathcal{Q}_n$ , by using various product operations. The exceptional case is that of the Petersen graph, and we present some questions about it.

## INTRODUCTION

A remarkable fact, discovered by Wang in [15], is that the symmetric group  $S_n$  has a quantum analogue  $\mathcal{Q}_n$ . For  $n \geq 4$  this quantum group is bigger than  $S_n$ , and fits into Woronowicz's formalism in [16].

The quantum group  $\mathcal{Q}_n$  is best understood via its representation theory: with suitable definitions, it appears as Tannakian realisation of the Temperley-Lieb algebra ([3]). This elucidates a number of questions regarding the Cayley graph, fusion rules, amenability, etc. More generally, this puts  $\mathcal{Q}_n$  into the framework of free quantum groups of Van Daele and Wang ([13]), where a whole machinery, inspired from work of Gromov, Jones, Voiculescu, Wassermann, Weingarten is now available.

The study of  $\mathcal{Q}_n$ , and of free quantum groups in general, focuses now on more technical aspects: matrix models ([5], [6]), ergodic actions ([9], [11]), harmonic analysis ([12], [14]).

The other thing to do is to study subgroups of  $\mathcal{Q}_n$ . This was started independently by the authors in [2], [3] and [7], [8], and continued in the joint paper [4]. The notion that emerges from this work is that of quantum automorphism group of a vertex-transitive graph.

In this paper we describe quantum automorphism groups of vertex-transitive graphs having  $n \leq 11$  vertices, with one graph left. This enhances previous classification work from [2], [3], [4], where we have  $n \leq 9$ , also with one graph left.

Needless to say, in this classification project the value of  $n$  is there only for showing how far our techniques go.

The four main features of the present work are:

(1) Product operations. We have general decomposition results for Cartesian and lexicographic products. These are motivated by the graphs  $\text{Pr}(C_5)$ ,  $\text{Pr}(K_5)$  and  $C_{10}(4)$ , which appear at  $n = 10$ .

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(2) The discrete torus. Here  $n = 9$ . We prove that its quantum group is equal to its classical group, namely  $S_3 \wr \mathbb{Z}_2$ . This answers a question left open in [3], [4], and provides the first example of a graph having a usual wreath product as quantum symmetry group.

(3) Circulant graphs. It is known from [2] that the  $n$ -cycle with  $n \neq 4$  has quantum symmetry group  $D_n$ . This is extended in [3] to a bigger class of circulant graphs. Here we further enlarge the list of such graphs, with an ad-hoc proof for  $C_{10}(2)$ , which appears at  $n = 10$ .

(4) The Petersen graph. This appears at  $n = 10$ , and the corresponding quantum group seems to be different from the known ones. We prove that it is not isomorphic with  $\mathcal{Q}_5$ . In other words, we might have here a “new” quantum group. However, we don’t have a proof, and the question is left open.

As a conclusion, we have two questions:

(I) First is to decide whether the Petersen graph produces or not a new quantum group. In case it is so, this would probably change a bit the landscape: in the big table at the end, based on work since Wang’s paper [15], all quantum groups can be obtained from  $\mathbb{Z}_n, S_n, \mathcal{Q}_n$ .

(II) A good question is to try to characterize graphs  $X$  having no quantum symmetry. This paper provides many new examples, and we have found some more by working on the subject, but so far we were unable to find a conceptual result here.

The paper is organized as follows. Sections 1, 2 are quite detailed preliminary sections, the whole paper, or at least the ideas involved in it, being intended to be accessible to non-specialists. Sections 3, 4, 5, 6 deal with different kind of graphs, once again in a quite self-contained way. In Section 7 we present the classification result, in the form of a big, independent table. In the last section we present a technical result about the quantum group of the Petersen graph.

## 1. QUANTUM PERMUTATION GROUPS

Let  $A$  be a  $\mathbb{C}^*$ -algebra with unit, given with morphisms

$$\begin{aligned} \Delta & : A \rightarrow A \otimes A \\ \varepsilon & : A \rightarrow \mathbb{C} \\ S & : A \rightarrow A^{op} \end{aligned}$$

satisfying the axioms for a comultiplication, counit and antipode, along with the extra condition  $S^2 = id$ . For axioms see Woronowicz ([16]).

**Definition 1.1.** *Such an algebra  $A$  is called a Hopf  $\mathbb{C}^*$ -algebra.*

The first example is with a compact group  $G$ . We can consider the algebra of continuous functions  $A = \mathbb{C}(G)$ , with operations

$$\begin{aligned} \Delta(f) & = (g, h) \rightarrow f(gh) \\ \varepsilon(f) & = f(1) \\ S(f) & = g \rightarrow f(g^{-1}) \end{aligned}$$

where we use the canonical identification  $A \otimes A = \mathbb{C}(G \times G)$ .

The second example is with a discrete group  $\Gamma$ . We have here the algebra  $A = \mathbb{C}^*(\Gamma)$ , obtained from the usual group algebra  $\mathbb{C}[\Gamma]$  by a standard completion procedure, with operations

$$\begin{aligned}\Delta(g) &= g \otimes g \\ \varepsilon(g) &= 1 \\ S(g) &= g^{-1}\end{aligned}$$

where we use the canonical embedding  $\Gamma \subset A$ .

In general, associated to an arbitrary Hopf  $\mathbb{C}^*$ -algebra  $A$  are a compact quantum group  $G$  and a discrete quantum group  $\Gamma$ , according to the following heuristic formulae:

$$\begin{aligned}A &= \mathbb{C}(G) = \mathbb{C}^*(\Gamma) \\ G &= \widehat{\Gamma} \\ \Gamma &= \widehat{G}\end{aligned}$$

These formulae can be made in fact into precise statements, by using categories, dual categories, and functors. See Woronowicz ([16]).

A square matrix  $u = (u_{ij}) \in M_n(A)$  is said to be multiplicative if

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \quad \text{and} \quad \varepsilon(u_{ij}) = \delta_{ij}$$

Multiplicative matrices correspond to corepresentations of the Hopf  $\mathbb{C}^*$ -algebra  $A$ , that is, to representations of the compact quantum group  $G$  with  $A = \mathbb{C}(G)$ . Such a multiplicative matrix  $u$  will also be interpreted as a linear map  $\mathbb{C}^n \rightarrow \mathbb{C}^n \otimes A$ .

In this paper we are essentially interested in the following special type of multiplicative matrices.

**Definition 1.2.** *A magic unitary is a square matrix, all whose entries are projections and all whose rows and columns are partitions of unity.*

Here we say that a finite family of projections is a partition of unity if these projections are pairwise orthogonal and if their sum equals 1.

As a first example, consider a finite group  $G$  acting on a finite set  $X$ . The characteristic functions

$$p_{ij} = \chi\{\sigma \in G \mid \sigma(j) = i\}$$

form a magic unitary, because the corresponding sets form partitions of  $G$ , when  $i$  or  $j$  varies. We have the following formulae for  $\mathbb{C}(G)$ :

$$\begin{aligned}\Delta(p_{ij}) &= \sum_k p_{ik} \otimes p_{kj} \\ \varepsilon(p_{ij}) &= \delta_{ij} \\ S(p_{ij}) &= p_{ji}\end{aligned}$$

and therefore  $p = (p_{ij})$  is a multiplicative matrix.

In the particular case of the symmetric group  $S_n$  acting on  $\{1, \dots, n\}$ , the Stone-Weierstrass theorem shows that entries of  $p$  generate  $\mathbb{C}(S_n)$ . This suggests the following construction, due to Wang ([15]).

**Definition 1.3.** *The  $\mathbb{C}^*$ -algebra  $A_s(n)$  is the universal  $\mathbb{C}^*$ -algebra generated by  $n^2$  elements  $u_{ij}$ , with relations making  $u$  a magic unitary matrix, and with maps*

$$\begin{aligned}\Delta(u_{ij}) &= \sum_k u_{ik} \otimes u_{kj} \\ \varepsilon(u_{ij}) &= \delta_{ij} \\ S(u_{ij}) &= u_{ji}\end{aligned}$$

*as comultiplication, counit and antipode, making it into a Hopf  $\mathbb{C}^*$ -algebra.*

This Hopf  $\mathbb{C}^*$ -algebra was discovered by Wang [15]. The corresponding compact quantum group is denoted  $\mathcal{Q}_n$  and we call it the quantum permutation group or quantum symmetric group. This is motivated by the fact that the algebra  $A_s(n)$  is the biggest Hopf  $\mathbb{C}^*$ -algebra coacting on the algebra  $\mathbb{C}^n$ , which is to say that the quantum group  $\mathcal{Q}_n$  is the biggest one acting on  $\{1, \dots, n\}$ . The coaction  $u : \mathbb{C}^n \rightarrow \mathbb{C}^n \otimes A_s(n)$  is defined on Dirac masses by

$$u(\delta_i) = \sum_j \delta_j \otimes u_{ji}$$

and verification of axioms of coactions, as well as proof of universality, is by direct computation. See [15].

We have a surjective morphism of Hopf  $\mathbb{C}^*$ -algebras

$$A_s(n) \rightarrow \mathbb{C}(S_n)$$

mapping  $u_{ij} \rightarrow p_{ij}$  for any  $i, j$ . This morphism expresses the fact that the compact quantum group corresponding to  $A_s(n)$  contains  $S_n$ .

This map is an isomorphism for  $n = 2, 3$ , as known from [3], [15], and explained in section 3 below. At  $n = 4$  we have Wang's matrix

$$u = \begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{pmatrix}$$

with  $p, q$  free projections, which shows that  $A_s(n)$  is not commutative and infinite dimensional. The same remains true for any  $n \geq 4$ .

## 2. QUANTUM AUTOMORPHISM GROUPS OF GRAPHS

Consider a finite graph  $X$ . In this paper this means that we have a finite set of vertices, and certain pairs of distinct vertices are connected by unoriented edges.

It is convenient to assume that the vertex set is  $\{1, \dots, n\}$ .

**Definition 2.1.** *The adjacency matrix of  $X$  is the matrix*

$$d \in M_n(0, 1)$$

*given by  $d_{ij} = 1$  if  $i, j$  are connected by an edge, and  $d_{ij} = 0$  if not.*

The adjacency matrix is symmetric, and has 0 on the diagonal. In fact, graphs having vertex set  $\{1, \dots, n\}$  are in one-to-one correspondence with  $n \times n$  symmetric 0–1 matrices having 0 on the diagonal.

The quantum automorphism group of  $X$  is obtained as an appropriate subgroup of the quantum permutation group of  $\{1, \dots, n\}$ . At level of Hopf algebras, this means taking an appropriate quotient of  $A_s(n)$ .

**Definition 2.2.** *Associated to a finite graph  $X$  is the algebra*

$$A(X) = A_s(n) / \langle du = ud \rangle$$

where  $n$  is the number of vertices, and  $d$  is the adjacency matrix.

Since a permutation of the set  $X$  is a graph automorphism if and only if the corresponding permutation matrix commutes with the adjacency matrix, it is reasonable to say that the quantum group corresponding to  $A(X)$  is the quantum automorphism group of  $X$ . In this way we have a commutative diagram of Hopf  $\mathbb{C}^*$ -algebras

$$\begin{array}{ccc} A_s(n) & \rightarrow & A(X) \\ \downarrow & & \downarrow \\ \mathbb{C}(S_n) & \rightarrow & \mathbb{C}(G) \end{array}$$

where  $G = G(X)$  is the usual automorphism group of  $X$ , with the kernel of the right arrow being the commutator ideal of  $A(X)$ . Moreover, for a graph without edges we get indeed  $A_s(n)$ , and we have the formula

$$A(X) = A(X^c)$$

where  $X^c$  is the complement of  $X$ . See [3], [4] for details.

The defining equations  $ud = du$  of  $A(X)$  means that  $d$ , considered as a linear map  $\mathbb{C}^n \rightarrow \mathbb{C}^n$ , is a morphism in the category of corepresentations of  $A(X)$ , i.e. a morphism in the category of representations of the quantum group dual to  $A(X)$ . General properties of the representation category of a compact quantum group (see e.g. [16]) now ensure that the spectral projections occurring in the spectral decomposition of  $d$  are corepresentations morphisms, and hence the corresponding eigensubspaces are subcorepresentations. This key fact will be used freely in the paper.

The following notion will play a central role in this paper.

**Definition 2.3.** *We say that  $X$  has no quantum symmetry if*

$$A(X) = \mathbb{C}(G)$$

where  $G = G(X)$  is the usual automorphism group of  $X$ .

This is the same as saying that  $A(X)$  is commutative, because by the above considerations,  $\mathbb{C}(G)$  is its biggest commutative quotient.

We are particularly interested in the case of graphs  $X$  having the property that  $G$  acts transitively on the set of vertices. These graphs were called homogeneous in previous work [3], [4], but we use here the following more traditional terminology.

**Definition 2.4.** *The graph  $X$  is called vertex-transitive if for any two vertices  $i, j$  there is  $\sigma \in G(X)$  such that  $\sigma(i) = j$ .*

Each section of the paper ends with a small table, gathering information about vertex-transitive graphs having  $\leq 11$  vertices. These small tables are to be put together in a single big table, at the end.

What we know so far is that we have

$$A(K_n) = A_s(n)$$

where  $K_n$  is the complete graph having  $n$  vertices. Moreover, we already mentioned that for  $n = 2, 3$  the arrow

$$A_s(n) \rightarrow \mathbb{C}(S_n)$$

is an isomorphism, and for  $n \geq 4$  it is not.

This information is summarized in the following table.

Order	Graph	Classical group	Quantum group
2	$K_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
3	$K_3$	$S_3$	$S_3$
4	$K_4$	$S_4$	$\mathcal{Q}_4$
...	...	...	...
11	$K_{11}$	$S_{11}$	$\mathcal{Q}_{11}$

Here in the right column  $\mathcal{Q}_n$  with  $n \geq 4$  is the compact quantum group associated to  $A_s(n)$ .

### 3. CIRCULANT GRAPHS

A graph with  $n$  vertices is called circulant if its automorphism group contains a cycle of length  $n$  (and hence in particular a copy of the cyclic group  $\mathbb{Z}_n$ ). We are particularly interested in connected circulant graphs, which are the cycles with chords.

**Definition 3.1.** *The graph  $C_n(k_1, \dots, k_r)$ , where*

$$1 < k_1 < \dots < k_r \leq [n/2]$$

*are integers, is obtained by drawing the  $n$ -cycle  $C_n$ , then connecting all pairs of vertices at distance  $k_i$ , for any  $i$ .*

As basic examples, we have the  $n$ -cycle  $C_n$ , corresponding to the value  $r = 0$ , and the  $2n$ -cycle with diagonals,  $C_n^+ = C_{2n}(n)$ .

Observe that  $K_n$  is a cycle with chords as well.

The adjacency matrix of a cycle with chords, denoted as usual  $d$ , is a circulant matrix. We use the following basic fact.

**Proposition 3.1.** *We have  $d(\xi^s) = 2f(s)\xi^s$ , where*

$$f(s) = \sum_{i=0}^r \cos\left(\frac{2k_i s \pi}{n}\right) \quad (\text{with } k_0 = 1)$$

and  $\xi$  is the vector whose coordinates are  $1, \omega, \dots, \omega^{n-1}$  in the canonical basis of  $\mathbb{C}^n$ , with  $\omega = e^{\frac{2i\pi}{n}}$ .

This tells us that we have the following eigenspaces for  $d$ :

$$\begin{aligned} V_0 &= \mathbb{C}1 \\ V_1 &= \mathbb{C}\xi + \mathbb{C}\xi^{n-1} \\ V_2 &= \mathbb{C}\xi^2 + \mathbb{C}\xi^{n-2} \\ \dots &= \dots \\ V_m &= \mathbb{C}\xi^m + \mathbb{C}\xi^{n-m} \end{aligned}$$

where  $m = \lfloor n/2 \rfloor$  and all sums are direct, except maybe for the last one, which depends on the parity of  $n$ .

The fact that these eigenspaces correspond or not to different eigenvalues depends of course on  $f$ .

We use the following result from [3], whose proof is briefly explained, because several versions of it will appear throughout the paper.

**Theorem 3.1.** *If  $n \neq 4$  and the associated function*

$$f : \{1, 2, \dots, \lfloor n/2 \rfloor\} \rightarrow \mathbb{R}$$

*is injective, then  $C_n(k_1, \dots, k_r)$  has no quantum symmetry.*

*Proof.* Since  $\mathbb{C}\xi \oplus \mathbb{C}\xi^{n-1}$  is invariant, the coaction can be written as

$$v(\xi) = \xi \otimes a + \xi^{n-1} \otimes b$$

for some  $a, b$ . By taking powers and using  $n \neq 4$  we get by induction

$$v(\xi^s) = \xi^s \otimes a^s + \xi^{n-s} \otimes b^s$$

for any  $s$ , along with the relations  $ab = -ba$  and  $ab^2 = ba^2 = 0$ .

Now from  $v(\xi^*) = v(\xi)^*$  we get  $b^* = b^{n-1}$ , so  $(ab)(ab)^* = ab^n a^* = 0$ . Thus  $ab = ba = 0$ , so  $A(X)$  is commutative and we are done.  $\square$

**Corollary 3.1.** *The following graphs have no quantum symmetry:*

- (1) *The cycles  $C_n$  with  $n \neq 4$ .*
- (2) *The cycles with diagonals  $C_8^+, C_{10}^+$ .*
- (3) *The cycles with chords  $C_9(3), C_{11}(2), C_{11}(3)$ .*

*Proof.* (1) follows from the fact that  $f$  is decreasing, hence injective. As for (2) and (3), the corresponding 5 functions are given by

$$\begin{aligned} C_8^+ &: -0.29, 1, -1.7, 0 \\ C_{10}^+ &: -0.19, 1.3, -1.3, 0.19, -2 \end{aligned}$$



$$\begin{aligned}
C_9(3) & : 0.26, -0.32, 0.5, -1.43 \\
C_{11}(2) & : 1.25, -0.23, -1.10, -0.79, -0.11 \\
C_{11}(3) & : 0.69, -0.54, 0.27, 0.18, -1.61
\end{aligned}$$

with 0.01 error, so they are injective, and Theorem 3.1 applies.  $\square$

The graphs in Corollary 3.1 have usual symmetry group  $D_n$ , where  $n$  is the number of vertices. We don't know if  $G = D_n$  with  $n \neq 4$  implies that the graph has no quantum symmetry. However, we are able to prove this for  $n \leq 11$ : graphs satisfying  $G = D_n$  are those in Corollary 3.1, plus the cycle with chords  $C_{10}(2)$ , discussed below.

**Theorem 3.2.** *The graph  $C_{10}(2)$  has no quantum symmetry.*

*Proof.* The function  $f$  is given by

$$f(s) = \cos\left(\frac{s\pi}{5}\right) + \cos\left(\frac{2s\pi}{5}\right)$$

and we have  $f(1) = -f(3) \simeq 1.11$ ,  $f(2) = f(4) = -0.5$  and  $f(5) = 0$ . Thus the list of eigenspaces is:

$$\begin{aligned}
V_0 & = \mathbb{C}1 \\
V_1 & = \mathbb{C}\xi \oplus \mathbb{C}\xi^{-1} \\
V_2 & = \mathbb{C}\xi^2 \oplus \mathbb{C}\xi^4 \oplus \mathbb{C}\xi^{-4} \oplus \mathbb{C}\xi^{-2} \\
V_3 & = \mathbb{C}\xi^3 \oplus \mathbb{C}\xi^{-3} \\
V_5 & = \mathbb{C}\xi^5
\end{aligned}$$

Since coactions preserve eigenspaces, we can write

$$v(\xi) = \xi \otimes a + \xi^{-1} \otimes b$$

for some  $a, b$ . Taking the square of this relation gives

$$v(\xi^2) = \xi^2 \otimes a^2 + \xi^{-2} \otimes b^2 + 1 \otimes (ab + ba)$$

and once again since  $v$  preserves eigenspaces, we get  $ab = -ba$ . Taking now the cube of the above relation gives

$$\begin{aligned}
v(\xi^3) & = \xi^3 \otimes a^3 + \xi^{-3} \otimes b^3 \\
& \quad + \xi \otimes (aba + aab + baa) \\
& \quad + \xi^{-1} \otimes (abb + bab + bba)
\end{aligned}$$

and once again since  $v$  preserves eigenspaces, we get:

$$ab^2 = ab^2 + b(ab + ba) = abb + bab + bba = 0$$

$$ba^2 = a(ba + ab) + ba^2 = aba + aab + baa = 0$$

With the relations  $ab = -ba$  and  $ab^2 = ba^2 = 0$  in hand, we get by induction the formula

$$v(\xi^s) = \xi^s \otimes a^s + \xi^{-s} \otimes b^s$$

and we can conclude by using adjoints, as in proof of Theorem 3.1.  $\square$

For graphs having  $n \leq 11$  vertices, results in this section are summarized in the following table.

Order	Graph	Classical group	Quantum group
5	$C_5$	$D_5$	$D_5$
6	$C_6$	$D_6$	$D_6$
7	$C_7$	$D_7$	$D_7$
8	$C_8, C_8^+$	$D_8$	$D_8$
9	$C_9, C_9(3)$	$D_9$	$D_9$
10	$C_{10}, C_{10}(2), C_{10}^+$	$D_{10}$	$D_{10}$
11	$C_{11}, C_{11}(2), C_{11}(3)$	$D_{11}$	$D_{11}$

As already mentioned, we don't know if these computations are particular cases of some general result.

#### 4. PRODUCTS OF GRAPHS

For a finite graph  $X$ , it is convenient to use the notation

$$X = (X, \sim)$$

where the  $X$  on the right is the set of vertices, and where we write  $i \sim j$  when two vertices  $i, j$  are connected by an edge.

**Definition 4.1.** *Let  $X, Y$  be two finite graphs.*

(1) *The direct product  $X \times Y$  has vertex set  $X \times Y$ , and edges*

$$(i, \alpha) \sim (j, \beta) \iff i \sim j, \alpha \sim \beta.$$

(2) *The Cartesian product  $X \square Y$  has vertex set  $X \times Y$ , and edges*

$$(i, \alpha) \sim (j, \beta) \iff i = j, \alpha \sim \beta \text{ or } i \sim j, \alpha = \beta.$$

The direct product is the usual one in a categorical sense. As for the Cartesian product, this is a natural one from a geometric viewpoint: for instance a product by a segment gives a prism.

**Definition 4.2.** *The prism having basis  $X$  is  $\text{Pr}(X) = K_2 \square X$ .*

We have embeddings of usual symmetry groups

$$G(X) \times G(Y) \subset G(X \times Y)$$

$$G(X) \times G(Y) \subset G(X \square Y)$$

which have the following quantum analogues.

**Proposition 4.1.** *We have surjective morphisms of Hopf  $\mathbb{C}^*$ -algebras*

$$A(X \times Y) \longrightarrow A(X) \otimes A(Y)$$

$$A(X \square Y) \longrightarrow A(X) \otimes A(Y).$$

*Proof.* We use the canonical identification

$$\mathbb{C}(X \times Y) = \mathbb{C}(X) \otimes \mathbb{C}(Y)$$

given by  $\delta_{(i,\alpha)} = \delta_i \otimes \delta_\alpha$ . The adjacency matrices are given by

$$d_{X \times Y} = d_X \otimes d_Y$$

$$d_{X \square Y} = d_X \otimes 1 + 1 \otimes d_Y$$

so if  $u$  commutes with  $d_X$  and  $v$  commutes with  $d_Y$ , the matrix

$$u \otimes v = (u_{ij}v_{\alpha\beta})_{(i\alpha,j\beta)}$$

is a magic unitary that commutes with both  $d_{X \times Y}$  and  $d_{X \square Y}$ . This gives morphisms as in the statement, and surjectivity follows by summing over  $i$  and  $\beta$ .  $\square$

**Theorem 4.1.** *Let  $X$  and  $Y$  be finite connected regular graphs. If their spectra  $\{\lambda\}$  and  $\{\mu\}$  do not contain 0 and satisfy*

$$\{\lambda_i/\lambda_j\} \cap \{\mu_k/\mu_l\} = \{1\}$$

*then  $A(X \times Y) = A(X) \otimes A(Y)$ . Also, if their spectra satisfy*

$$\{\lambda_i - \lambda_j\} \cap \{\mu_k - \mu_l\} = \{0\}$$

*then  $A(X \square Y) = A(X) \otimes A(Y)$ .*

*Proof.* We follow [3], where the first statement is proved. Let  $\lambda_1$  be the valence of  $X$ . Since  $X$  is regular we have  $\lambda_1 \in \text{Sp}(X)$ , with 1 as eigenvector, and since  $X$  is connected  $\lambda_1$  has multiplicity one. Hence if  $P_1$  is the orthogonal projection onto  $\mathbb{C}1$ , the spectral decomposition of  $d_X$  is of the following form:

$$d_X = \lambda_1 P_1 + \sum_{i \neq 1} \lambda_i P_i$$

We have a similar formula for  $d_Y$ :

$$d_Y = \mu_1 Q_1 + \sum_{j \neq 1} \mu_j Q_j$$

This gives the following formulae for products:

$$d_{X \times Y} = \sum_{ij} (\lambda_i \mu_j) P_i \otimes Q_j$$

$$d_{X \square Y} = \sum_{i,j} (\lambda_i + \mu_i) P_i \otimes Q_j$$

Here projections form partitions of unity, and the scalar are distinct, so these are spectral decomposition formulae. We can conclude as in [3]. The universal coactions will commute with any of the spectral projections, and hence with both  $P_1 \otimes 1$  and  $1 \otimes Q_1$ . In both cases the universal coaction  $v$  is the tensor product of its restrictions to the images of  $P_1 \otimes 1$  (i.e.  $1 \otimes \mathbb{C}(Y)$ ) and of  $1 \otimes Q_1$  (i.e.  $\mathbb{C}(X) \otimes 1$ ).  $\square$

**Corollary 4.1.**

- (1) We have  $A(K_m \times K_n) = A(K_m) \otimes A(K_n)$  for  $m \neq n$ .
- (2) We have  $A(\text{Pr}(K_n)) = \mathbb{C}(\mathbb{Z}_2) \otimes A_s(n)$ , for any  $n$ .
- (3) We have  $A(\text{Pr}(C_n)) = \mathbb{C}(D_{2n})$ , for  $n$  odd.
- (4) We have  $A(\text{Pr}(C_4)) = \mathbb{C}(\mathbb{Z}_2) \otimes A_s(4)$ .

*Proof.* The spectra of graphs involved are  $\text{Sp}(K_2) = \{-1, 1\}$  and

$$\begin{aligned} \text{Sp}(K_n) &= \{-1, n-1\} \\ \text{Sp}(C_n) &= \{2 \cos(2k\pi/n) \mid k = 1, \dots, n\} \end{aligned}$$

so the first three assertions follow from Theorem 4.1. We have

$$\text{Pr}(C_4) = K_2 \times K_4$$

(this graph is the cube) and the fourth assertion follows from the first one.  $\square$

We get the following table, the product operation  $\times$  on quantum groups being the one dual to the tensor product of Hopf  $\mathbb{C}^*$ -algebras.

Order	Graph	Classical group	Quantum group
8	$\text{Pr}(C_4)$	$S_4 \times \mathbb{Z}_2$	$\mathcal{Q}_4 \times \mathbb{Z}_2$
10	$\text{Pr}(C_5)$	$D_{10}$	$D_{10}$
10	$\text{Pr}(K_5)$	$S_5 \times \mathbb{Z}_2$	$\mathcal{Q}_5 \times \mathbb{Z}_2$

5. THE TORUS GRAPH

Theorem 4.1 doesn't apply to the case  $X = Y$ , and the problem of computing the algebras  $A(X \times X)$  and  $A(X \square X)$  appears. At level of classical symmetry groups, there is no simple formula describing  $G(X \times X)$  and  $G(X \square X)$ . Thus we have reasons to believe that the above problem doesn't have a simple solution either.

A simpler question is to characterize graphs  $X$  such that  $X \times X$  or  $X \square X$  has no quantum symmetry. We don't have a general result here, but we are able however to deal with the case  $X = K_3$ .

**Definition 5.1.** *The graph Torus is the graph  $K_3 \times K_3 = K_3 \square K_3$ .*

The result below answers a question asked in [3], [4]. It also provides the first example of graph having a classical wreath product as quantum symmetry group.

**Theorem 5.1.** *The graph Torus has no quantum symmetry.*

*Proof.* The spectrum of  $K_3$  is known to be

$$\text{Sp}(K_3) = \{-1, 2\}$$

with corresponding eigenspaces given by

$$\begin{aligned} F_2 &= \mathbb{C}1 \\ F_{-1} &= \mathbb{C}\xi \oplus \mathbb{C}\xi^2 \end{aligned}$$

where  $\xi$  is the vector formed by third roots of unity.

Tensoring the adjacency matrix of  $K_3$  with itself gives

$$\mathrm{Sp}(\mathrm{Torus}) = \{-2, 1, 4\}$$

with corresponding eigenspaces given by

$$\begin{aligned} E_4 &= \mathbb{C}\xi_{00} \\ E_{-2} &= \mathbb{C}\xi_{10} \oplus \mathbb{C}\xi_{01} \oplus \mathbb{C}\xi_{20} \oplus \mathbb{C}\xi_{02} \\ E_1 &= \mathbb{C}\xi_{11} \oplus \mathbb{C}\xi_{12} \oplus \mathbb{C}\xi_{21} \oplus \mathbb{C}\xi_{22} \end{aligned}$$

where we use the notation  $\xi_{ij} = \xi^i \otimes \xi^j$ .

The universal coaction  $v$  preserves eigenspaces, so we have

$$\begin{aligned} v(\xi_{10}) &= \xi_{10} \otimes a + \xi_{01} \otimes b + \xi_{20} \otimes c + \xi_{02} \otimes d \\ v(\xi_{01}) &= \xi_{10} \otimes \alpha + \xi_{01} \otimes \beta + \xi_{20} \otimes \gamma + \xi_{02} \otimes \delta \end{aligned}$$

for some  $a, b, c, d, \alpha, \beta, \gamma, \delta \in A$ . Taking the square of  $v(\xi_{10})$  gives

$$v(\xi_{20}) = \xi_{20} \otimes a^2 + \xi_{02} \otimes b^2 + \xi_{10} \otimes c^2 + \xi_{01} \otimes d^2$$

along with relations coming from eigenspace preservation:

$$\begin{aligned} ab &= -ba, \quad ad = -da, \quad bc = -cb, \quad cd = -dc \\ ac + ca &= -(bd + db) \end{aligned}$$

Now since  $a, b$  anticommute, their squares have to commute:

$$a^2b^2 = aabb = -abab = -baba = bbaa = b^2a^2$$

On the other hand, by applying  $v$  to the equality  $\xi_{10}^* = \xi_{20}$ , we get the following formulae for adjoints:

$$a^* = a^2, \quad b^* = b^2, \quad c^* = c^2, \quad d^* = d^2$$

The commutation relation  $a^2b^2 = b^2a^2$  reads now  $a^*b^* = b^*a^*$ , and by taking adjoints we get  $ba = ab$ . Together with  $ab = -ba$  this gives:

$$ab = ba = 0$$

The same method applies to  $ad, bc, cd$ , and we end up with:

$$ab = ba = 0, \quad ad = da = 0, \quad bc = cb = 0, \quad cd = dc = 0$$

We apply now  $v$  to the equality  $1 = \xi_{10}\xi_{20}$ . We get that 1 is the sum of 16 terms, all of them of the form  $\xi_{ij} \otimes P$ , where  $P$  are products between  $a, b, c, d$  and their squares. Due to the above formulae 8 terms vanish, and the 8 remaining ones produce the formula

$$1 = a^3 + b^3 + c^3 + d^3$$

along with relations coming from eigenspace preservation:

$$ac^2 = ca^2 = bd^2 = db^2 = 0$$

Now from  $ac^2 = 0$  we get  $a^2c^2 = 0$ , and by taking adjoints this gives  $ca = 0$ . The same method applies to  $ac, bd, db$ , and we end up with:

$$ac = ca = 0, \quad bd = db = 0$$

In the same way one shows that  $\alpha, \beta, \gamma, \delta$  pairwise commute:

$$\alpha\beta = \beta\alpha = \dots = \gamma\delta = \delta\gamma = 0$$

It remains to show that  $a, b, c, d$  commute with  $\alpha, \beta, \gamma, \delta$ . For, we apply  $v$  to the following equality:

$$\xi_{10}\xi_{01} = \xi_{01}\xi_{10}$$

We get an equality between two sums having 16 terms each, and by using again eigenspace preservation we get the following formulae relating the corresponding 32 products  $a\alpha, a\alpha$  etc.:

$$\begin{aligned} a\alpha = 0 = \alpha a, \quad b\beta = 0 = \beta b, \quad c\gamma = 0 = \gamma c, \quad d\delta = 0 = \delta d, \\ a\gamma + c\alpha + b\delta + d\beta = 0 = \alpha c + \gamma a + \beta d + \delta b, \\ a\beta + b\alpha = \alpha b + \beta a, \quad b\gamma + c\beta = \beta c + \gamma b, \\ c\delta + d\gamma = \gamma d + \delta c, \quad a\delta + d\alpha = \alpha d + \delta a \end{aligned}$$

Multiplying the first equality in the second row on the left by  $a$  and on the right by  $\gamma$  gives  $a^2\gamma^2 = 0$ , and by taking adjoints we get  $\gamma a = 0$ . The same method applies to the other 7 products involved in the second row, so all 8 products involved in the second row vanish:

$$a\gamma = c\alpha = b\delta = d\beta = \alpha c = \gamma a = \beta d = \delta b = 0$$

We use now the first equality in the third row. Multiplying it on the left by  $a$  gives  $a^2\beta = a\beta a$ , and multiplying it on the right by  $a$  gives  $a\beta a = \beta a^2$ . Thus we get the commutation relation  $a^2\beta = \beta a^2$ .

On the other hand from  $a^3 + b^3 + c^3 + d^3 = 1$  we get  $a^4 = a$ , so:

$$a\beta = a^4\beta = a^2a^2\beta = \beta a^2a^2 = \beta a$$

One shows in a similar manner that the missing commutation formulae  $a\delta = \delta a$  etc. hold as well. Thus  $A$  is commutative.  $\square$

Order	Graph	Classical group	Quantum group
9	Torus	$S_3 \wr \mathbb{Z}_2$	$S_3 \wr \mathbb{Z}_2$

## 6. LEXICOGRAPHIC PRODUCTS

Let  $X$  and  $Y$  be two finite graphs. Their lexicographic product is obtained by putting a copy of  $X$  at each vertex of  $Y$ :

**Definition 6.1.** *The lexicographic product  $X \circ Y$  has vertex set  $X \times Y$ , and edges are given by*

$$(i, \alpha) \sim (j, \beta) \iff \alpha \sim \beta \text{ or } \alpha = \beta, i \sim j.$$

The terminology comes from a certain similarity with the ordering of usual words, which is transparent when iterating  $\circ$ .

The simplest example is with the graph  $Y = X_n$ , having  $n$  vertices and no edges at all: we get  $n$  copies of  $X$ .

**Definition 6.2.**  $nX$  is the disjoint union of  $n$  copies of  $X$ .

When  $X$  is connected, we have an isomorphism

$$G(nX) = G(X) \wr S_n$$

where  $\wr$  is a wreath product. In other words, we have:

$$G(X \circ X_n) = G(X) \wr G(X_n)$$

In the general case, we have the following embedding of usual symmetry groups:

$$G(X) \wr G(Y) \subset G(X \circ Y)$$

The quantum analogues of these results use the notion of free wreath product from [8, 4]. In the following definition, a pair  $(A, u)$  is what we call a quantum permutation group in [4]:  $A$  is a Hopf  $\mathbb{C}^*$ -algebra and  $u$  is a multiplicative magic unitary matrix.

**Definition 6.3.** The free wreath product of  $(A, u)$  and  $(B, v)$  is

$$A *_w B = (A^{*n} * B) / \langle [u_{ij}^{(a)}, v_{ab}] = 0 \rangle$$

where  $n$  is the size of  $v$ , with magic unitary matrix  $w_{ia,jb} = u_{ij}^{(a)} v_{ab}$ .

In other words,  $A *_w B$  is the algebra generated by  $n$  copies of  $A$  and a copy of  $B$ , with the  $a$ -th copy of  $A$  commuting with the  $a$ -th row of  $v$ , for any  $a$ . The Hopf  $\mathbb{C}^*$ -algebra structure on  $A *_w B$  is the unique one making  $w$  into a multiplicative matrix. With this definition, we have the following result ([4]).

**Theorem 6.1.** If  $X$  is connected we have  $A(nX) = A(X) *_w A_s(n)$ .

Note that the embedding  $A(X)^{*n} \hookrightarrow A(X) *_w A_s(n)$  ensures that  $A(nX)$  is an infinite-dimensional algebra whenever  $n \geq 2$  and  $G(X)$  is non trivial.

In the general case, we have the following quantum analogue of the embedding result for  $G(X) \wr G(Y)$ .

**Proposition 6.1.** We have a surjective morphism of Hopf  $\mathbb{C}^*$ -algebras

$$A(X \circ Y) \longrightarrow A(X) *_w A(Y).$$

*Proof.* We use the canonical identification

$$\mathbb{C}(X \times Y) = \mathbb{C}(X) \otimes \mathbb{C}(Y)$$

given by  $\delta_{(i,\alpha)} = \delta_i \otimes \delta_\alpha$ . The adjacency matrix of  $X \circ Y$  is

$$d_{X \circ Y} = d_X \otimes 1 + \mathbb{I} \otimes d_Y$$

where  $\mathbb{I}$  is the square matrix filled with 1's.

Let  $u, v$  be the magic unitary matrices of  $A(X), A(Y)$ . The magic unitary matrix of  $A(X) *_w A(Y)$  is given by

$$w_{ia,jb} = u_{ij}^{(a)} v_{ab}$$

and from the fact that  $u$  commutes with  $d_X$  (and  $\mathbb{I}$ ) and  $v$  commutes with  $d_Y$ , we get that  $w$  commutes with  $d_{X \circ Y}$ . This gives a morphism as in the statement, and surjectivity follows by summing over  $i$  and  $b$ .  $\square$

**Theorem 6.2.** *Let  $X, Y$  be regular graphs, with  $X$  connected. If their spectra  $\{\lambda_i\}$  and  $\{\mu_j\}$  satisfy the condition*

$$\{\lambda_1 - \lambda_i \mid i \neq 1\} \cap \{-n\mu_j\} = \emptyset$$

where  $n$  and  $\lambda_1$  are the order and valence of  $X$ , then  $A(X \circ Y) = A(X) *_w A(Y)$ .

*Proof.* We denote by  $P_i, Q_j$  the spectral projections corresponding to  $\lambda_i, \mu_j$ . Since  $X$  is connected we have  $P_1 = \frac{1}{n} \mathbb{1}$ , and we get:

$$\begin{aligned} d_{X \circ Y} &= d_X \otimes 1 + \mathbb{1} \otimes d_Y \\ &= \left( \sum_i \lambda_i P_i \right) \otimes \left( \sum_j Q_j \right) + (nP_1) \otimes \left( \sum_i \mu_j Q_j \right) \\ &= \sum_j (\lambda_1 + n\mu_j)(P_1 \otimes Q_j) + \sum_{i \neq 1} \lambda_i (P_i \otimes 1) \end{aligned}$$

In this formula projections form a partition of unity and scalars are distinct, so this is the spectral decomposition of  $d_{X \circ Y}$ .

Let  $W$  be the universal coaction on  $X \circ Y$ . Then  $W$  must commute with all spectral projections, and in particular:

$$[W, P_1 \otimes Q_j] = 0$$

Summing over  $j$  gives  $[W, P_1 \otimes 1] = 0$ , so  $1 \otimes \mathbb{C}(Y)$  is invariant under the coaction. The corresponding restriction of  $W$  gives a coaction of  $A(X \circ Y)$  on  $1 \otimes \mathbb{C}(Y)$ , say

$$W(1 \otimes e_a) = \sum_b 1 \otimes e_b \otimes y_{ba}$$

where  $y$  is a magic unitary. On the other hand we can write

$$W(e_i \otimes 1) = \sum_{jb} e_j \otimes e_b \otimes x_{ji}^b$$

and by multiplying by the previous relation we get:

$$\begin{aligned} W(e_i \otimes e_a) &= \sum_{jb} e_j \otimes e_b \otimes y_{ba} x_{ji}^b \\ &= \sum_{jb} e_j \otimes e_b \otimes x_{ji}^b y_{ba} \end{aligned}$$

This shows that coefficients of  $W$  have the following form:

$$W_{jb,ia} = y_{ba} x_{ji}^b = x_{ji}^b y_{ba}$$

Consider now the matrix  $x^b = (x_{ij}^b)$ . Since  $W$  is a morphism of algebras, each row of  $x^b$  is a partition of unity. Also using the antipode, we have

$$S \left( \sum_j x_{ji}^b \right) = S \left( \sum_{ja} x_{ji}^b y_{ba} \right)$$



$$\begin{aligned}
&= S \left( \sum_{ja} W_{jb,ia} \right) \\
&= \sum_{ja} W_{ia,jb} \\
&= \sum_{ja} x_{ij}^a y_{ab} \\
&= \sum_a y_{ab} \\
&= 1
\end{aligned}$$

so we conclude that  $x^b$  is a magic unitary.

We check now that  $x^a, y$  commute with  $d_X, d_Y$ . We have

$$(d_{X \circ Y})_{ia,jb} = (d_X)_{ij} \delta_{ab} + (d_Y)_{ab}$$

so the two products between  $W$  and  $d_{X \circ Y}$  are given by:

$$\begin{aligned}
(W d_{X \circ Y})_{ia,kc} &= \sum_j W_{ia,jc} (d_X)_{jk} + \sum_{jb} W_{ia,jb} (d_Y)_{ab} \\
(d_{X \circ Y} W)_{ia,kc} &= \sum_j (d_X)_{ij} W_{ja,kc} + \sum_{jb} (d_Y)_{ab} W_{jb,kc}
\end{aligned}$$

Now since  $W$  commutes with  $d_{X \circ Y}$ , the terms on the right are equal, and by summing over  $c$  we get:

$$\sum_j x_{ij}^a (d_X)_{jk} + \sum_{cb} y_{ab} (d_Y)_{bc} = \sum_j (d_X)_{ij} x_{jk}^a + \sum_{cb} (d_Y)_{ab} y_{bc}$$

The graph  $Y$  being regular, the second sums in both terms are equal to the valency of  $Y$ , so we get  $[x^a, d_X] = 0$ .

Now once again from the formula coming from commutation of  $W$  with  $d_{X \circ Y}$ , we get  $[y, d_Y] = 0$ .

Summing up, the coefficients of  $W$  are of the form

$$W_{jb,ia} = x_{ji}^b y_{ba}$$

where  $x^b$  are magic unitaries commuting with  $d_X$ , and  $y$  is a magic unitary commuting with  $d_Y$ . This gives a morphism

$$A(X) *_w A(Y) \longrightarrow A(X \circ Y)$$

mapping  $u_{ji}^{(b)} \rightarrow x_{ji}^b$  and  $v_{ba} \rightarrow y_{ba}$ , which is inverse to the morphism in the previous proposition.  $\square$

**Corollary 6.1.** *We have  $A(C_{10}(4)) = \mathbb{C}(\mathbb{Z}_2) *_w \mathbb{C}(D_5)$ .*

*Proof.* We have isomorphisms

$$C_{10}(4) = C_{10}(4, 5)^c = K_2 \circ C_5$$

and Theorem 6.2 applies to the product on the right.  $\square$

Together with Theorem 6.1, this corollary gives the following table, where  $\wr_*$  is defined by  $\mathbb{C}(G \wr_* H) = \mathbb{C}(G) *_w \mathbb{C}(H)$ .

Order	Graph	Classical group	Quantum group
4	$2K_2$	$\mathbb{Z}_2 \wr \mathbb{Z}_2$	$\mathbb{Z}_2 \wr_* \mathbb{Z}_2$
6	$2K_3$	$S_3 \wr \mathbb{Z}_2$	$S_3 \wr_* \mathbb{Z}_2$
6	$3K_2$	$\mathbb{Z}_2 \wr S_3$	$\mathbb{Z}_2 \wr_* S_3$
8	$2K_4$	$S_4 \wr \mathbb{Z}_2$	$\mathcal{Q}_4 \wr_* \mathbb{Z}_2$
8	$2C_4$	$(\mathbb{Z}_2 \wr \mathbb{Z}_2) \wr \mathbb{Z}_2$	$(\mathbb{Z}_2 \wr_* \mathbb{Z}_2) \wr_* \mathbb{Z}_2$
8	$4K_2$	$\mathbb{Z}_2 \wr S_4$	$\mathbb{Z}_2 \wr_* \mathcal{Q}_4$
9	$3K_3$	$S_3 \wr S_3$	$S_3 \wr_* S_3$
10	$2C_5$	$D_5 \wr \mathbb{Z}_2$	$D_5 \wr_* \mathbb{Z}_2$
10	$2K_5$	$S_5 \wr \mathbb{Z}_2$	$\mathcal{Q}_5 \wr_* \mathbb{Z}_2$
10	$5K_2$	$\mathbb{Z}_2 \wr S_5$	$\mathbb{Z}_2 \wr_* \mathcal{Q}_5$
10	$C_{10}(4)$	$\mathbb{Z}_2 \wr D_5$	$\mathbb{Z}_2 \wr_* D_5$

### 7. CLASSIFICATION TABLE

We are now in position of writing down a big table. We first recall the graph notations used in the paper.

**Definition 7.1.** *We use the following notations.*

- (1) *Basic graphs:* - the complete graph having  $n$  vertices is denoted  $K_n$ .  
 - the disjoint union of  $n$  copies of  $X$  is denoted  $nX$ .  
 - the prism having basis  $X$  is denoted  $\text{Pr}(X)$ .
- (2) *Circulant graphs:*  
 - the  $n$ -cycle is denoted  $C_n$ .  
 - the  $2n$ -cycle with diagonals is denoted  $C_{2n}^+$ .  
 - the  $n$ -cycle with chords of length  $k$  is denoted  $C_n(k)$ .
- (3) *Special graphs:*  
 - the triangle times itself is denoted **Torus**.  
 - the Petersen graph is denoted **Petersen**.

As for quantum group notations, these have to be taken with care, because quantum groups do not really exist etc. Here they are.

**Definition 7.2.** *We use the following notations.*

- $\mathbb{Z}_n, D_n, S_n$  are the cyclic, dihedral and symmetric groups.
- $\mathcal{Q}_n$  is the quantum permutation group.
- $\times, \wr, \wr_*$  are the product, wreath product and free wreath product.

The vertex-transitive graphs of order less than 11, modulo complementation, are given by the following table.

Order	Graph	Classical group	Quantum group
2	$K_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
3	$K_3$	$S_3$	$S_3$
4	$2K_2$	$\mathbb{Z}_2 \wr \mathbb{Z}_2$	$\mathbb{Z}_2 \wr_* \mathbb{Z}_2$
4	$K_4$	$S_4$	$\mathcal{Q}_4$
5	$C_5$	$D_5$	$D_5$
5	$K_5$	$S_5$	$\mathcal{Q}_5$
6	$C_6$	$D_6$	$D_6$
6	$2K_3$	$S_3 \wr \mathbb{Z}_2$	$S_3 \wr_* \mathbb{Z}_2$
6	$3K_2$	$\mathbb{Z}_2 \wr S_3$	$\mathbb{Z}_2 \wr_* S_3$
6	$K_6$	$S_6$	$\mathcal{Q}_6$
7	$C_7$	$D_7$	$D_7$
7	$K_7$	$S_7$	$\mathcal{Q}_7$
8	$C_8, C_8^+$	$D_8$	$D_8$
8	$\text{Pr}(C_4)$	$S_4 \times \mathbb{Z}_2$	$\mathcal{Q}_4 \times \mathbb{Z}_2$
8	$2K_4$	$S_4 \wr \mathbb{Z}_2$	$\mathcal{Q}_4 \wr_* \mathbb{Z}_2$
8	$2C_4$	$(\mathbb{Z}_2 \wr \mathbb{Z}_2) \wr \mathbb{Z}_2$	$(\mathbb{Z}_2 \wr_* \mathbb{Z}_2) \wr_* \mathbb{Z}_2$
8	$4K_2$	$\mathbb{Z}_2 \wr S_4$	$\mathbb{Z}_2 \wr_* \mathcal{Q}_4$
8	$K_8$	$S_8$	$\mathcal{Q}_8$
9	$C_9, C_9(3)$	$D_9$	$D_9$
9	<b>Torus</b>	$S_3 \wr \mathbb{Z}_2$	$S_3 \wr \mathbb{Z}_2$
9	$3K_3$	$S_3 \wr S_3$	$S_3 \wr_* S_3$
9	$K_9$	$S_9$	$\mathcal{Q}_9$
10	$C_{10}, C_{10}(2), C_{10}^+, \text{Pr}(C_5)$	$D_{10}$	$D_{10}$
10	<b>Petersen</b>	$S_5$	?
10	$\text{Pr}(K_5)$	$S_5 \times \mathbb{Z}_2$	$\mathcal{Q}_5 \times \mathbb{Z}_2$
10	$C_{10}(4)$	$\mathbb{Z}_2 \wr D_5$	$\mathbb{Z}_2 \wr_* D_5$
10	$2C_5$	$D_5 \wr \mathbb{Z}_2$	$D_5 \wr_* \mathbb{Z}_2$
10	$2K_5$	$S_5 \wr \mathbb{Z}_2$	$\mathcal{Q}_5 \wr_* \mathbb{Z}_2$
10	$5K_2$	$\mathbb{Z}_2 \wr S_5$	$\mathbb{Z}_2 \wr_* \mathcal{Q}_5$
10	$K_{10}$	$S_{10}$	$\mathcal{Q}_{10}$
11	$C_{11}, C_{11}(2), C_{11}(3)$	$D_{11}$	$D_{11}$
11	$K_{11}$	$S_{11}$	$\mathcal{Q}_{11}$

Here the first three columns are well-known, and can be found in various books or websites. The last one collects results in this paper.

By using the equality  $D_n = \mathbb{Z}_n \rtimes \mathbb{Z}_2$ , we reach the conclusion in the abstract: with one possible exception, all quantum groups in the right column can be obtained from  $\mathbb{Z}_n, S_n, Q_n$  by using the operations  $\times, \rtimes, \wr, \wr_*$ .

The exceptional situation is that of the Petersen graph, which might give a new quantum group. We discuss it in the next section.

## 8. THE PETERSEN GRAPH

The usual symmetry group of the Petersen graph is  $S_5$ . In view of the results in our classification table, we have at least two natural candidates for the quantum symmetry group of the Petersen graph:  $S_5$  and  $Q_5$ .

**Theorem 8.1.** *The quantum automorphism group of the Petersen graph has an irreducible 5-dimensional representation. In particular it is not isomorphic to the quantum symmetric group  $Q_5$ .*

*Proof.* Let  $G$  be the quantum automorphism group of the Petersen graph, denoted here  $P$ . We have an inclusion  $S_5 \subset G$ . It is well-known that

$$\mathrm{Sp}(P) = \{4, -2, 1\}$$

and that the corresponding eigenspaces have dimensions 1, 4, 5. These eigenspaces furnish representations of  $G$  and of  $S_5$ . It is straightforward to compute the character of the permutation representation of  $S_5$  on  $\mathbb{C}(P)$ , and then using the character table of  $S_5$  (see e.g. [10]), we see that  $\mathbb{C}(P)$  is the direct sum of 3 irreducible representations of  $S_5$ . These have to be the previous eigenspaces, and in particular the 5-dimensional one is an irreducible representation of  $S_5$ , and of  $G$ . On the other hand, it is known from [1] that  $Q_5$  has no irreducible 5-dimensional representation. Thus the quantum groups  $G$  and  $Q_5$  are not isomorphic.  $\square$

The question now is: does the Petersen graph have quantum symmetry? In other words, is  $A(P)$  commutative? The above result seems to indicate that if  $A(P)$  is not commutative, we probably will have a new quantum permutation group.

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DEPARTEMENT OF MATHEMATICS, UNIVERSITE PAUL SABATIER, 118 ROUTE DE NARBONNE,  
31062 TOULOUSE, FRANCE

*E-mail address:* [banica@picard.ups-tlse.fr](mailto:banica@picard.ups-tlse.fr)

LABORATOIRE DE MATHEMATIQUES APPLIQUEES, UNIVERSITE DE PAU ET DES PAYS DE L'ADOUR,  
IPRA, AVENUE DE L'UNIVERSITE, 64000 PAU, FRANCE

*E-mail address:* [bichon@univ-pau.fr](mailto:bichon@univ-pau.fr)