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PSEUDODIFFERENTIAL MULTI-PRODUCT REPRESENTATION OF THE SOLUTION OPERATOR OF A PARABOLIC EQUATION

HIROSHI ISOZAKI AND JÉRÔME LE ROUSSEAU

Abstract. By using a time slicing procedure, we represent the solution operator of a second-order parabolic pseudodifferential equation on $\mathbf{R}^n$ as an infinite product of zero-order pseudodifferential operators. A similar representation formula is proven for parabolic differential equations on a compact Riemannian manifold. Each operator in the multi-product is given by a simple explicit Ansatz. The proof is based on an effective use of the Weyl calculus and the Fefferman-Phong inequality.

Keywords: Parabolic equation; Pseudodifferential initial value problem; Weyl quantization; Infinite product of operators; Compact manifold.

AMS 2000 subject classification: 35K15, 35S10, 47G30, 58J35, 58J40.

1. Introduction and notation

We begin with recalling standard notation for the calculus of pseudodifferential operators (ψDOs). Throughout the article, we shall most often use spaces of global symbols; a function $a \in \mathscr{C}^\infty(\mathbf{R}^n \times \mathbf{R}^p)$ is in $S^m(\mathbf{R}^n \times \mathbf{R}^p)$ if for all multi-indices α, β there exists $C_{\alpha\beta} > 0$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha\beta} \langle\xi\rangle^{m-|\beta|}, \quad x \in \mathbf{R}^n,\ \xi \in \mathbf{R}^p, \quad \langle\xi\rangle := (1+|\xi|^2)^{1/2}. \tag{1.1}$$

We write $S^m = S^m(\mathbf{R}^n \times \mathbf{R}^n)$. ψDOs of order m, in Weyl quantization, are formally given by (see [Hör79] or [Hör85, Chapter 18.5])

$$\mathrm{Op}^w(a)\, u(x) = a^w(x, D_x)u(x) = (2\pi)^{-n} \iint e^{i\langle x-y,\xi\rangle} a((x+y)/2,\xi)\, u(y)\, dy\, d\xi, \quad u \in \mathscr{S}'(\mathbf{R}^n).$$

We denote by $\Psi^m(\mathbf{R}^n)$, or simply by Ψ^m, the space of such ψDOs of order m.

We consider a second-order ψDO defined by the Weyl quantization of $q(x,\xi)$. Assuming uniform ellipticity and positivity for $q(x,\xi)$, we study the following parabolic Cauchy problem

$$\partial_t u + q^w(t,x,D_x)u = 0, \quad 0 < t \le T, \tag{1.2}$$

$$u\,|_{t=0} = u_0, \tag{1.3}$$

for u_0 in $L^2(\mathbf{R}^n)$ or in some Sobolev space. The solution operator of this Cauchy problem is denoted by $U(t',t)$, $0 \le t \le t' \le T$. Here, we are interested in providing a representation of $U(t',t)$ in the form of a multi-product of ψDOs.

Such a representation is motivated by the results of the second author in the case of hyperbolic equations [Le 06, Le 07]. If the symbol q is only a function of ξ, the solution of (1.2)–(1.3) is simply given by means

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of Fourier transformations as

$$u(t,x) = (2\pi)^{-n} \iint e^{i\langle x-y,\xi\rangle} e^{-tq(\xi)}\, u(y)\, dy\, d\xi.$$

Following [Le 06], we then hope to have a good approximation of $u(t,x)$, for small t, in the case where q depends also on both t and x:

$$u(t,x) \simeq (2\pi)^{-n} \iint e^{i\langle x-y,\xi\rangle} e^{-tq(0,(x+y)/2,\xi)}\, u(y)\, dy\, d\xi = p^w_{(t,0)}(x,D_x)u(x),$$

where $p_{(t'',t')}(x,\xi) := e^{-(t''-t')q(t',x,\xi)}$, $0 \le t' \le t'' \le T$, which is in S^0. The infinitesimal approximation we introduce is thus of pseudodifferential nature (as opposed to Fourier integral operators in the hyperbolic case [Le 06]).

With such an infinitesimal operator, by iterations, we are then led to introducing the following multi-product of ψDOs to approximate the solution operator $U(t',t)$ of the Cauchy problem (1.2)–(1.3):

$$\mathcal{W}_{\mathfrak{P},t} := \begin{cases} p^w_{(t,0)}(x,D_x) & \text{if } 0 \le t \le t^{(1)}, \\ p^w_{(t,t^{(k)})}(x,D_x) \displaystyle\prod_{i=k}^{1} p^w_{(t^{(i)},t^{(i-1)})}(x,D_x) & \text{if } t^{(k)} \le t \le t^{(k+1)}. \end{cases}$$

where $\mathfrak{P} = \{t^{(0)}, t^{(1)}, \dots, t^{(N)}\}$ is a subdivision of $[0,T]$ with $0 = t^{(0)} < t^{(1)} < \cdots < t^{(N)} = T$. It should be noted that in general $p^w_{(t'',t')}(x,D_x)$, $t' \le t''$, does not have semi-group properties.

In [Le 06], for the hyperbolic case, the standard quantization is used for the equation and for the approximation Ansatz. However, in the present parabolic case this approach fails (see Remark 2.7 below). Instead, the choice of Weyl quantization yields convergence results of the Ansatz $\mathcal{W}_{\mathfrak{P},t}$ comparable to those in [Le 06, Le 07]. The convergence of $\mathcal{W}_{\mathfrak{P},t}$ to the solution operator $U(t,0)$ is shown in operator norm with an estimate of the convergence rate depending on the (Hölder) regularity of $q(t,x,\xi)$ w.r.t. the evolution parameter t. See Theorem 3.8 in Section 3 below for a precise statement.

Such a convergence result thus yields a representation of the solution operator of the Cauchy problem (1.2)–(1.3) by an infinite multi-product of ψDOs. The result relies (i) on the proof of the stability of the multi-product $\mathcal{W}_{P,t}$ as $N = |\mathfrak{P}|$ grows to ∞ (Proposition 3.1) and (ii) on a consistency estimate that measures the infinitesimal error made by replacing $U(t',t)$ by $p^w_{(t',t)}(x,D_x)$ (Proposition 3.6). The stability in fact follows from a sharp Sobolev-norm estimate for $p^w_{(t',t)}(x,D_x)$ (see Theorem 2.2): for $s \in \mathbf{R}$, there exists $C \ge 0$ such that

$$\|p^w_{(t',t)}(x,D_x)\|_{(H^s,H^s)} \le 1 + C(t'-t). \tag{1.4}$$

The Fefferman-Phong inequality plays an important role here.

The representation of the solution operator by multi-products of ψDO follows from the exact convergence of the Ansatz $\mathcal{W}_{\mathfrak{P},t}$ in some operator norm. We emphasize that the convergence we obtain is not up to a regularizing operator. A further interesting aspect of this result is that each constituting operator of the multi-product is given explicitly. With such a product representation, we have in mind the development of numerical schemes for practical applications. Once the problem is discretized in space, the use of fast Fourier transformations (FFT) can yield numerical methods with low computational complexity,

with possibly microlocal approximations of the symbols in question as is sometimes done in the case of hyperbolic equations (see for instance [dHLW00, LdH01a, LdH01b, LdH03]). We also anticipate that our representation procedure can be used in theoretical purposes.

As described above, the first part of this article is devoted to the parabolic Cauchy problem on $\mathbf{R}^n$ and to the study of the properties of the approximation Ansatz $\mathcal{W}_{\mathfrak{P},t}$. In the second part, we shall consider a parabolic problem on a compact Riemannian manifold without boundaries. In this case, the operator $q^w(x, D_x)$ is only considered of differential type, in particular for its full symbol to be known exactly. In each local chart we can define an infinitesimal approximation of the solution operator as is done in $\mathbf{R}^n$ and we combine these local ψDOs together with the help of a partition of unity. This yields a counterpart of $p^w_{(t',t)}(x, D_x)$ for the manifold case (see Section 4.1) denoted by $P_{(t',t)}$. In fact, the sharp estimate (1.4) still holds in L^2 ($s = 0$) for this infinitesimal approximation (Theorem 4.4). The proof of a consistency estimate (Proposition 4.6) requires the analysis of the effect of changes of variables for Weyl symbols of the form of $e^{-(t'-t)q}$. The choice we have made for the definition of $P_{(t',t)}$ is invariant through such changes of variables up to a first-order precision w.r.t. the small parameter $h = t' - t$, which is compatible with the kind of results we are aiming at. With stability and consistency at hand, the convergence result then follows as in the case of $\mathbf{R}^n$.

In the manifold case, the constituting ψDOs of the multi-product are given explicitly in each local chart. We observe moreover that the computation of the action of these local operators can be essentialy performed as in the case of $\mathbf{R}^n$, which is appealing for practical implementations.

Another approach to representation of the solution operator $U(t', t)$ can be found in the work of C. Iwasaki (see [Tsu74, Iwa77, Iwa84]). Her work encompasses the case of degenerate parabolic operators, utilizes multi-product of ψDOs and analyses the symbol of the resulting operator, using the work of Kumano-go [Kg81]. However, the symbol of the solution operator $U(t', t)$ is finally obtained by solving a Volterra equation. Such integral equations also appear in related works on the solution operator of parabolic equations (see e.g. [Gre71, ST84]). The alternative method we present here will be more suitable for applications because of the explicit aspect of the representations. The step of the integral equation in the above works makes the representation formula less explicit. However, the reader will note that the technique we use in our approach here do not apply to the case of degenerate parabolic equations like those treated in [Tsu74, Iwa77, Iwa84]. The question of the extension of the convergence and representation results we present here to the case of degenerate parabolic equations appears to us an interesting question.

Let us further recall some standard notions. We denote by $\sigma(.,.)$ the symplectic 2-form on the vector space $T^*(\mathbf{R}^n)$:

$$\sigma((x, \xi), (y, \eta)) = \langle \xi, y\rangle - \langle \eta, x\rangle, \tag{1.5}$$

and we denote by $\{f, g\}$ the Poisson bracket of two functions, i.e.

$$\{f, g\} = \sum_{j=1}^{n} \partial_{\xi_j} f\, \partial_{x_j} g - \partial_{x_j} f\, \partial_{\xi_j} g.$$

We shall use the notation $\#^w$ to denote the composition of symbols in Weyl quantization,i.e., $a^w(x, D_x) \circ b^w(x, D_x) = (a \#^w b)^w(x, D_x)$. The following result is classical.

Proposition 1.1. *Let $a \in S^m$, $b \in S^{m'}$. Then $a \#^w b \in S^{m+m'}$ and*

(1.6)

$$(a \#^w b)(x,\xi) = \sum_{j=0}^{k} \frac{1}{j!}\left(\frac{i}{2}\sigma((D_x, D_\xi),(D_y, D_\eta))\right)^j a(x,\xi)b(y,\eta)\Big|_{\substack{y=x\\ \eta=\xi}}$$

$$+ \pi^{-2n}\int_0^1 \frac{(1-r)^k}{k!}\int_{④} e^{i\Sigma(z,\zeta,t,\tau,\xi)}\left(\frac{i}{2}\sigma((D_x, D_\zeta),(D_y, D_\tau))\right)^{k+1} a(x+rz,\zeta)b(y+rt,\tau)\, dr\, dz\, d\zeta\, dt\, d\tau\Big|_{y=x},$$

where $\Sigma(z,\zeta,t,\tau,\xi) = 2(\langle\tau-\xi, z\rangle - \langle\zeta-\xi, t\rangle)$.

The result of Proposition 1.1 is to be understood in the sense of oscillatory integrals (see e.g. [Hör90, Chapter 7.8], [AG91], [GS94] or [Kg81]). For the sake of concision we have introduced

$$\int_{ⓝ} := \underbrace{\int\cdots\int}_{n \text{ times}}, \quad \text{for } n \geq 3,\ n \in \mathbf{N}.$$

For the exposition to be self contained, we prove Proposition 1.1 in Appendix A.

We sometimes use the notion of multiple symbols. A function $a(x,\xi,y,\eta) \in C^\infty(\mathbf{R}^{q_1}\times\mathbf{R}^{p_1}\times\mathbf{R}^{q_2}\times\mathbf{R}^{p_2})$ is in $S^{m,m'}(\mathbf{R}^{q_1}\times\mathbf{R}^{p_1}\times\mathbf{R}^{q_2}\times\mathbf{R}^{p_2})$, if for all multi-indices $\alpha_1, \beta_1, \alpha_2, \beta_2$, there exists $C^{\beta_1\beta_2}_{\alpha_1\alpha_2} > 0$ such that

$$|\partial_x^{\alpha_1}\partial_\xi^{\beta_1}\partial_y^{\alpha_2}\partial_\eta^{\beta_2} a(x,\xi,y,\eta)| \leq C^{\beta_1\beta_2}_{\alpha_1\alpha_2}\langle\xi\rangle^{m-|\beta_1|}\langle\eta\rangle^{m'-|\beta_2|}, \tag{1.7}$$

$x \in \mathbf{R}^{q_1}$, $y \in \mathbf{R}^{q_2}$, $\xi \in \mathbf{R}^{p_1}$, $\eta \in \mathbf{R}^{p_2}$ (see for instance [Kg81, Chapter 2]).

For $s \in \mathbf{R}$. We set $E^{(s)} := \langle D_x\rangle^s = \mathrm{Op}(\langle\xi\rangle^s)$, which realizes an isometry from $H^r(\mathbf{R}^n)$ onto $H^{r-s}(\mathbf{R}^n)$ for any $r \in \mathbf{R}$. We denote by $(.,.)$ and $\|.\|$ the inner product and the norm of $L^2(\mathbf{R}^n)$, respectively and $\|\cdot\|_{H^s}$ for the norm on $H^s(\mathbf{R}^n)$, $s \in \mathbf{R}$. For two Hilbert spaces K and L, we use $\|\cdot\|_{(K,L)}$ to denote the norm in $\mathcal{L}(K,L)$, the set of bounded operators from K into L.

Our basic strategy is to obtain a bound for ψDOs involving a small parameter $h \geq 0$. In the following, we say that an inequality holds uniformly in h if it is the case when h varies in $[0, h_{\max}]$ for some $h_{\max} > 0$. In the sequel, C will denote a generic constant independent of h, whose value may change from line to line. The semi-norms

$$p_{\alpha\beta}(a) := \sup_{(x,\xi)\in\mathbf{R}^n\times\mathbf{R}^p}\langle\xi\rangle^{-m+|\beta|}|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)| \tag{1.8}$$

endow a Fréchet space structure to $S^m(\mathbf{R}^n\times\mathbf{R}^p)$. In the case of a symbol a_h that depends on the parameter h we shall say that a_h is in $S^m_{\rho,\delta}$ uniformly in h if for all α, β the semi-norm $p_{\alpha\beta}(a_h)$ is uniformly bounded in h. Similarly, we shall say that an operator A is in Ψ^m uniformly in h if its (Weyl) symbol is itself in S^m uniformly in h.

The outline of the article is as follows. Sections 2 and 3 are devoted to the multi-product representation of solutions on $\mathbf{R}^n$. In Section 2 we prove the sharp Sobolev norm estimate (1.4), which leads in Section 3 to the

stability of the multi-product representation. We then prove convergence of the multi-product representation in Section 3. Some of the results of these two sections make use of composition-like formulae, whose proofs are provided in Appendix A. In Section 4, we address the multi-product representation of solutions of a second-order differential parabolic problem on a compact Riemannian manifold. As in the previous sections we prove stability (in the L^2 case) through a sharp operator norm estimate and we prove convergence of the multi-product representation. The convergence proof requires an analysis of the effect of a change of variables on symbols of the form $e^{-hq(x,\xi)}$, from one local chart to another, which we present in Appendix B.

2. A sharp H^s bound

We first make precise the assumption on the symbol $q(x,\xi)$ mentioned in the introduction.

Assumption 2.1. *The symbol q is of the form $q = q_2 + q_1$, where $q_j \in S^j$, $j = 1, 2$, $q_2(x,\xi)$ is real-valued and for some $C \geq 0$ we have*

$$q_2(x,\xi) \geq C|\xi|^2, \quad x \in \mathbf{R}^n, \quad \xi \in \mathbf{R}^n, \ |\xi| \ \textit{sufficiently large.}$$

Consequently, for some $C \geq 0$, we have

$$q_2(x,\xi) + \operatorname{Re} q_1(x,\xi) \geq C|\xi|^2, \quad x \in \mathbf{R}^n, \quad \xi \in \mathbf{R}^n, \ |\xi| \ \text{sufficiently large, say } |\xi| \geq \vartheta > 0. \tag{2.1}$$

As is stated in the introduction, our main aim is to deal with the operator $p_h^w(x, D_x)$ where

$$p_h(x,\xi) = e^{-hq(x,\xi)}.$$

It is well-known that the ψDO $p_h^w(x, D_x)$ is uniformly H^s-bounded in h, $s \in \mathbf{R}$. Actually we have the following sharper estimate.

Theorem 2.2. *Let $s \in \mathbf{R}$. There exists a constant $C \geq 0$ such that*

$$\|p_h^w(x, D_x)\|_{(H^s,H^s)} \leq 1 + Ch,$$

holds for all $h \geq 0$.

To prove Theorem 2.1 we shall need some preliminary results.

Lemma 2.3. (i) *Let $l \geq 0$ and $r \in S^l$. Then $h^{l/2} r\, p_h$ is in S^0 uniformly in h.*
(ii) *Let α and β be multi-indices such that $|\alpha + \beta| \geq 1$. Then, for any $0 \leq m \leq 1$, we have $\partial_x^\alpha \partial_\xi^\beta p_h = h^m \tilde{p}_h^{m\alpha\beta}$, where $\tilde{p}_h^{m\alpha\beta}$ is in $S^{2m-|\beta|}$ uniformly in h.*

Proof. We have

$$(h\langle\xi^2\rangle)^j e^{-h \operatorname{Re} q(x,\xi)} \leq C_j, \quad x \in \mathbf{R}^n, \quad \xi \in \mathbf{R}^n, \quad h \geq 0,$$

for all $j \in \mathbf{N}$ by (2.1), hence

$$h^{l/2}|r(x,\xi)p_h(x,\xi)| \leq C, \quad x \in \mathbf{R}^n, \quad \xi \in \mathbf{R}^n, \quad h \geq 0.$$

For multi-indices α and β we observe that $h^{l/2}\partial_x^\alpha\partial_\xi^\beta(r(x,\xi)p_h(x,\xi))$ is a linear combination of terms of the form

$$h^{k+l/2}(\partial_x^{\alpha_0}\partial_\xi^{\beta_0}r)(x,\xi)(\partial_x^{\alpha_1}\partial_\xi^{\beta_1}q)(x,\xi)\cdots(\partial_x^{\alpha_k}\partial_\xi^{\beta_k}q)(x,\xi)p_h(x,\xi),$$

for $k\geq 0$, $\alpha_0+\alpha_1+\cdots+\alpha_k=\alpha$, $\beta_0+\beta_1+\cdots+\beta_k=\beta$ and the absolute value of this term can be estimated by

$$h^{k+l/2}\langle\xi\rangle^{l+2k-|\beta|}|p_h(x,\xi)|\leq C\langle\xi\rangle^{-|\beta|},\quad x\in\mathbf{R}^n,\quad \xi\in\mathbf{R}^n,\quad h\geq 0,$$

by (2.1), which concludes the proof of (i). For $|\alpha+\beta|\geq 1$, $\partial_x^\alpha\partial_\xi^\beta p_h(x,\xi)$ is a linear combination of terms of the form

$$h^k(\partial_x^{\alpha_1}\partial_\xi^{\beta_1}q)(x,\xi)\cdots(\partial_x^{\alpha_k}\partial_\xi^{\beta_k}q)(x,\xi)p_h(x,\xi),$$

for $k\geq 1$, $\alpha_1+\cdots+\alpha_k=\alpha$, $\beta_1+\cdots+\beta_k=\beta$, which can be rewritten as $h^m\lambda_h(x,\xi)$, where

$$\lambda_h(x,\xi)=(\partial_x^{\alpha_1}\partial_\xi^{\beta_1}q)(x,\xi)\cdots(\partial_x^{\alpha_k}\partial_\xi^{\beta_k}q)(x,\xi)\langle\xi\rangle^{2m-2k}\left(h\langle\xi\rangle^2\right)^{k-m}p_h(x,\xi).$$

Since $(\partial_x^{\alpha_1}\partial_\xi^{\beta_1}q)\cdots(\partial_x^{\alpha_k}\partial_\xi^{\beta_k}q)\in S^{2k-|\beta|}$, we see that $\lambda_h\in S^{2m-|\beta|}$ uniformly in h by using (i). ■

From Weyl Calculus and the previous lemma we have the following composition results for the symbol p_h.

Proposition 2.4. *Let r_h be bounded in S^l, $l\in\mathbf{R}$, uniformly in h. We then have*

$$r_h\,\#^w p_h=r_hp_h+h^{\frac12}\lambda_h^{(0)}=r_hp_h+h\lambda_h^{(1)}=r_hp_h+\frac{1}{2i}\{r_h,p_h\}+h\tilde\lambda_h^{(0)},\tag{2.2}$$

$$p_h\,\#^w r_h=r_hp_h+h^{\frac12}\mu_h^{(0)}=r_hp_h+h\mu_h^{(1)}=r_hp_h+\frac{1}{2i}\{p_h,r_h\}+h\tilde\mu_h^{(0)},\tag{2.3}$$

where $\lambda_h^{(0)}$, $\mu_h^{(0)}$, $\tilde\lambda_h^{(0)}$, and $\tilde\mu_h^{(0)}$ are in S^l uniformly in h and $\lambda_h^{(1)}$ and $\mu_h^{(1)}$ are in S^{l+1} uniformly in h.

To ease the reading of the article, the proof of Proposition 2.4 has been placed in Appendix A. We apply the result of Proposition 2.4 to prove the following lemma.

Lemma 2.5. *We have $\overline{p_h}\,\#^w\langle\xi\rangle^{2s}\,\#^w p_h-\langle\xi\rangle^{2s}|p_h|^2=hk_h$ with k_h in S^{2s} uniformly in h.*

Proof. By Proposition 2.4 we have

$$\langle\xi\rangle^{2s}\,\#^w p_h=\langle\xi\rangle^{2s}\,p_h+\frac{1}{2i}\left\{\langle\xi\rangle^{2s},p_h\right\}+h\lambda_{1,h}=\langle\xi\rangle^{2s}\,p_h+\frac{1}{2i}\sum_{j=1}^n(\partial_{\xi_j}\langle\xi\rangle^{2s})\,\partial_{x_j}p_h+h\lambda_{1,h},$$

with $\lambda_{1,h}$ in S^{2s} uniformly in h. We then obtain

$$\overline{p_h}\,\#^w\langle\xi\rangle^{2s}\,\#^w p_h=\overline{p_h}\,\#^w\left(\langle\xi\rangle^{2s}\,p_h+\frac{1}{2i}\sum_{j=1}^n(\partial_{\xi_j}\langle\xi\rangle^{2s})\,\partial_{x_j}p_h\right)+h\lambda_{2,h},$$

with $\lambda_{2,h}$ in S^{2s} uniformly in h. By Proposition 2.4 we have

$$\begin{aligned}\overline{p_h}\,\#^w\left(\langle\xi\rangle^{2s}\,p_h\right) &= \langle\xi\rangle^{2s}|p_h|^2 + \frac{1}{2i}\left\{\overline{p_h},\langle\xi\rangle^{2s}p_h\right\} + h\lambda_{3,h}\\ &= \langle\xi\rangle^{2s}|p_h|^2 + \frac{1}{2i}\sum_{j=1}^{n}\left((\partial_{\xi_j}\overline{p_h})\langle\xi\rangle^{2s}\partial_{x_j}p_h - (\partial_{x_j}\overline{p_h})\,\partial_{\xi_j}(\langle\xi\rangle^{2s}p_h)\right) + h\lambda_{3,h},\end{aligned}$$

with $\lambda_{3,h}$ in S^{2s} uniformly in h. We also have

$$\overline{p_h}\,\#^w\frac{1}{2i}\sum_{j=1}^{n}(\partial_{\xi_j}\langle\xi\rangle^{2s})\,\partial_{x_j}p_h = \frac{1}{2i}\sum_{j=1}^{n}\overline{p_h}\,(\partial_{\xi_j}\langle\xi\rangle^{2s})\partial_{x_j}p_h + h\lambda_{4,h},$$

with $\lambda_{4,h}$ in S^{2s}, uniformly in h, by Proposition 2.4. We have thus obtained

$$\overline{p_h}\,\#^w\,\langle\xi\rangle^{2s}\,\#^w p_h = \langle\xi\rangle^{2s}|p_h|^2 + \frac{1}{2i}l_h^{(1)} + \frac{1}{2i}\sum_{j=1}^{n}l_{h,j}^{(2)} + h\lambda_{5,h}, \tag{2.4}$$

with $\lambda_{5,h}$ in S^{2s} uniformly in h, and with

$$l_h^{(1)} = \{\overline{p_h},p_h\}\langle\xi\rangle^{2s} = \sum_{j=1}^{n}(\partial_{\xi_j}\overline{p_h}\,\partial_{x_j}p_h - \partial_{x_j}\overline{p_h}\,\partial_{\xi_j}p_h)\langle\xi\rangle^{2s}$$

and

$$l_{h,j}^{(2)} = (\overline{p_h}\,\partial_{x_j}p_h - p_h\,\partial_{x_j}\overline{p_h})\,\partial_{\xi_j}\langle\xi\rangle^{2s}.$$

We introduce $\alpha := q_2 + \operatorname{Re} q_1$ and $\beta := \operatorname{Im} q_1$. We have

$$l_h^{(1)} = 2ih^2|p_h|^2\langle\xi\rangle^{2s}\,\{\alpha,\beta\} \tag{2.5}$$

and

$$l_{h,j}^{(2)} = -2ih|p_h|^2(\partial_{x_j}\beta)\,\partial_{\xi_j}\langle\xi\rangle^{2s}.$$

Since $\alpha\in S^2$ and $\beta\in S^1$ we then have $\{\alpha,\beta\}\in S^2$. From Lemma 2.3, we thus obtain $l_h^{(1)} = hk_h^{(1)}$, with $k_h^{(1)}$ in S^{2s} uniformly in h. We also have that $l_{h,j}^{(2)} = hk_{h,j}^{(2)}$ with $k_{h,j}^{(2)}$ in S^{2s} uniformly in h, which from (2.4) concludes the proof. ■

We shall also need the following lemma.

Lemma 2.6. *We have $\langle\xi\rangle^s\,\#^w\,|p_h|^2\,\#^w\,\langle\xi\rangle^s - \langle\xi\rangle^{2s}|p_h|^2 = hk_h$ with k_h in S^{2s} uniformly in h.*

Proof. We set $\rho_h = \langle\xi\rangle^s\,\#^w\,|p_h|^2\,\#^w\,\langle\xi\rangle^s$. From Weyl calculus we have

$$\rho_h(x,\xi) = \pi^{-2n}\int_{④} e^{i\Sigma(z,t,\zeta,\tau,\xi)}\langle\zeta\rangle^s\langle\tau\rangle^s\,|p_h|^2(x+z+t,\xi)\,dz\,d\zeta\,dt\,d\tau$$

where $\Sigma(z,t,\zeta,\tau,\xi) = 2(\langle z,\tau-\xi\rangle - \langle t,\zeta-\xi\rangle)$. Arguing as in the proof of Proposition 1.1 in Appendix A we write

$$\rho_h(x,\xi) - \langle\xi\rangle^{2s}|p_h|^2(x,\xi) = \pi^{-2n}\int_{④} e^{i\Sigma(z,t,\zeta,\tau,\xi)}\langle\zeta\rangle^s\langle\tau\rangle^s\,(|p_h|^2(x+z+t,\xi) - |p_h|^2(x,\xi))\,dz\,d\zeta\,dt\,d\tau$$

$$= \frac{i}{2}\pi^{-2n}\sum_{j=1}^{n}\int_0^1\int_{④} e^{i\Sigma(z,t,\zeta,\tau,\xi)}(\partial_{\tau_j}-\partial_{\zeta_j})(\langle\zeta\rangle^s\langle\tau\rangle^s)\,(\partial_{x_j}|p_h|^2)(x+r(z+t),\xi)\,dr\,dz\,d\zeta\,dt\,d\tau, \tag{2.6}$$

by a first-order Taylor formula and integrations by parts w.r.t. ζ and τ. Observing that we have

$$0 = \frac{i}{2}\sum_{j=1}^{n}(\partial_{\tau_j}-\partial_{\zeta_j})(\langle\zeta\rangle^s\langle\tau\rangle^s)\,(\partial_{x_j}|p_h|^2)(x,\xi)\,\Big|_{\zeta=\tau=\xi}$$

$$= \frac{i}{2}\pi^{-2n}\sum_{j=1}^{n}\int_{④} e^{i\Sigma(z,t,\zeta,\tau,\xi)}(\partial_{\tau_j}-\partial_{\zeta_j})(\langle\zeta\rangle^s\langle\tau\rangle^s)\,(\partial_{x_j}|p_h|^2)(x,\xi)\,dz\,d\zeta\,dt\,d\tau,$$

we can proceed as in the proof of Proposition 1.1 (integration by parts w.r.t. r in (2.6) and further integrations by parts w.r.t. ζ and τ) and conclude after noting that $|p_h|^2$ satisfies the properties listed in Lemma 2.3 like p_h. ■

Remark 2.7. Note that the use of the Weyl quantization is crucial in the proofs of Lemmata 2.5 and 2.6. The use of the standard (left) quantization would only yield a result of the form $\overline{p_h}\,\#\,\langle\xi\rangle^{2s}\,\#\,p_h - \langle\xi\rangle^{2s}|p_h|^2 = h^{\frac{1}{2}}k_h$ with k_h in S^{2s} uniformly in h. Such a result would yield a $h^{\frac{1}{2}}$ term in the statement of Theorem 2.2 and the subsequent analysis would not carry through.

We now define the symbol $\nu_h(x,\xi) = \frac{1-|p_h|^2(x,\xi)}{h}$, for $h>0$, and prove the following lemma.

Lemma 2.8. *The symbol ν_h is in S^2 uniformly in h.*

Proof. We write $\nu_h(x,\xi) = 2\,\mathrm{Re}\,q(x,\xi)\int_0^1 e^{-2rh\,\mathrm{Re}\,q(x,\xi)}\,dr$. The integrand is in S^0 uniformly in $r\in[0,1]$ and h by Lemma 2.3 and $\mathrm{Re}\,q\in S^2$. ■

Lemma 2.9. *The symbol p_h is such that*

$$\big((|p_h|^2)^w(x,D_x)u,u\big) \le (1+Ch)\|u\|^2, \quad u\in L^2(\mathbf{R}^n),$$

for $C\ge 0$, uniformly in $h\ge 0$.

Proof. Let $\chi\in\mathscr{C}_c^\infty(\mathbf{R}^n)$, $0\le\chi\le 1$, be such that $\chi(\xi)=1$ if $|\xi|\le\vartheta$. Then we write

$$|p_h|^2(x,\xi) = e^{-2h\,\mathrm{Re}\,q(x,\xi)} = e^{-2h\chi(\xi)\,\mathrm{Re}\,q(x,\xi)}e^{-2h(1-\chi(\xi))\,\mathrm{Re}\,q(x,\xi)} = (1+h\mu(x,\xi))e^{-2h(1-\chi(\xi))\,\mathrm{Re}\,q(x,\xi)},$$

where $\mu(x,\xi) = -2\chi(\xi)\,\mathrm{Re}\,q(x,\xi)\int_0^1 e^{-2rh\chi(\xi)\,\mathrm{Re}\,q(x,\xi)}dr$. From [Hör85, 18.1.10], the symbol $e^{-2rh\chi(\xi)\,\mathrm{Re}\,q(x,\xi)}$ is in S^0 uniformly in r and h; hence the symbol $\mu(x,\xi)$ is in S^0 uniformly in h. From [Hör85, Theorem 18.6.3]

and Lemma 2.3, it is thus sufficient to prove the result with $p_h(x,\xi)$ replaced by $\tilde{p}_h(x,\xi) = e^{-h(1-\chi(\xi))q(x,\xi)}$. We set

$$\tilde{\nu}_h(x,\xi) := \frac{1}{h}\left(1 - |\tilde{p}_h|^2(x,\xi)\right),$$

for $h > 0$. From (the proof of) Lemma 2.8 we find that $\tilde{\nu}_h(x,\xi)$ is in S^2 uniformly in h. Since $1-\chi(\xi) = 0$ if $|\xi| \le \vartheta$ and $\operatorname{Re} q(x,\xi) \ge 0$ if $|\xi| \ge \vartheta$, we observe that $\tilde{\nu}_h(x,\xi) \ge 0$. Then the Fefferman-Phong inequality reads ([FP78], [Hör85, Corollary 18.6.11])

$$\left(\tilde{\nu}_h^w(x,D_x)u, u\right) \ge -C\|u\|_{L^2}^2, \quad u \in L^2(\mathbf{R}^n),$$

for some non-negative constant C that can be chosen uniformly in h. This yields

$$\|u\|_{L^2}^2 - \left((|\tilde{p}_h|^2)^w(x,D_x)u, u\right) \ge -Ch\|u\|_{L^2}^2, \quad u \in L^2(\mathbf{R}^n),$$

which concludes the proof. ■

We are now ready to prove Theorem 2.2.

Proof of Theorem 2.2. We use the following commutative diagram,

$$\begin{array}{ccc} H^s & \xrightarrow{\;p_h^w(x,D_x)\;} & H^s \\ {\scriptstyle E^{(s)}}\downarrow & & \downarrow{\scriptstyle E^{(s)}} \\ L^2 & \xrightarrow{\;T_h\;} & L^2 \end{array}$$

and prove that the operator T_h satisfies $\|T_h\|_{(L^2,L^2)} \le 1 + Ch$.

The Weyl symbol of $T_h^* \circ T_h$ is given by

$$\sigma_h = \langle\xi\rangle^{-s} \#^w \overline{p_h} \#^w \langle\xi\rangle^{2s} \#^w p_h \#^w \langle\xi\rangle^{-s}.$$

By Lemmata 2.5 and 2.6, we have $\sigma_h = |p_h|^2 + hk_h$, with k_h in S^0 uniformly in h. We note that $k_h(x,\xi)$ is real valued. To estimate the L^2 operator norm of T_h we write

$$\begin{aligned} \|T_h u\|^2 = \left(T_h^* \circ T_h u, u\right) &= \left((|p_h|^2)^w(x,D_x)u, u\right) + h\left(k_h^w(x,D_x)u, u\right) \\ &\le \left((|p_h|^2)^w(x,D_x)u, u\right) + Ch\|u\|^2, \end{aligned}$$

for $C \ge 0$ [Hör85, Theorem 18.6.3]. The result of Theorem 2.2 thus follows from Lemma 2.9. ■

Let $m(x)$ be a smooth function that satisfies

$$0 < m_{\min} \le m(x) \le m_{\max} < \infty, \tag{2.7}$$

along with all its derivatives. With such a function m, we define the following norm on $L^2(\mathbf{R}^n)$

$$\|f\|_{L^2(\mathbf{R}^n, m\,dx)}^2 = \int_{\mathbf{R}^n} f^2(x)\, m(x) dx,$$

which is equivalent to the classical L^2 norm. We shall need the following result in Section 4.

Proposition 2.10. *There exists a constant $C \geq 0$ such that*

$$\|p_h^w(x, D_x)u\|_{L^2(\mathbf{R}^n, m\,dx)} \leq (1 + Ch)\,\|u\|_{L^2(\mathbf{R}^n, m\,dx)}, \qquad u \in L^2(\mathbf{R}^n),$$

holds for all $h \geq 0$.

Proof. We follow the proof of Lemma 2.9 and use $\tilde{p}_h(x, \xi)$ in place of $p_h(x, \xi)$. We then set

$$\tilde{\nu}_h(x, \xi) := \frac{m(x)}{h}\left(1 - |\tilde{p}_h|^2(x, \xi)\right),$$

Then $\tilde{\nu}_h(x, \xi) \geq 0$ is in S^2 uniformly in h. The Fefferman-Phong inequality yields

$$\left(\tilde{\nu}_h^w(x, D_x)u, u\right) \geq -C\|u\|_{L^2}^2 \geq -C'\|u\|_{L^2(\mathbf{R}^n, m\,dx)}^2, \quad u \in L^2(\mathbf{R}^n).$$

This yields

$$\|u\|_{L^2(\mathbf{R}^n, m\,dx)}^2 - \left((m|\tilde{p}_h|^2)^w(x, D_x)u, u\right) \geq -C'h\|u\|_{L^2(\mathbf{R}^n, m\,dx)}^2, \quad u \in L^2(\mathbf{R}^n),$$

By Lemma 2.11 just below, we have

$$\left((m|\tilde{p}_h|^2)^w(x, D_x)u, u\right) = \left(\overline{\tilde{p}_h}^w(x, D_x) \circ m \circ (\tilde{p}_h^w(x, D_x)u, u\right) + h(\lambda^w(x, D_x)u, u),$$

with $\lambda(x, \xi)$ in S^0 uniformly in h and where m stands for the associated multiplication operator here. We conclude since $\overline{\tilde{p}_h}^w(x, D_x) = (\tilde{p}_h^w(x, D_x))^*$. ■

Lemma 2.11. *Let $f \in \mathscr{C}^\infty(\mathbf{R}^n)$ be bounded along with all its derivatives. We have*

$$\overline{p_h} \#^w f \#^w p_h - f|p_h|^2 = h\lambda_h,$$

with λ_h in S^0 uniformly in h.

Proof. From Proposition 2.4 we have

$$f \#^w p_h = f p_h - \frac{1}{2i}\sum_{j=1}^n (\partial_{x_j} f)\,\partial_{\xi_j} p_h + h\lambda_{1,h},$$

with $\lambda_{1,h}$ in S^0 uniformly in h. By Proposition 2.4 we also have

$$\begin{aligned}\overline{p_h} \#^w (f p_h) &= f|p_h|^2 + \frac{1}{2i}\{\overline{p_h}, f p_h\} + h\lambda_{2,h} \\ &= f|p_h|^2 + \frac{1}{2i}\sum_{j=1}^n \left((\partial_{\xi_j}\overline{p_h})(\partial_{x_j} f)p_h + (\partial_{\xi_j}\overline{p_h})f(\partial_{x_j} p_h) - (\partial_{x_j}\overline{p_h})f(\partial_{\xi_j} p_h)\right) + h\lambda_{2,h},\end{aligned}$$

with $\lambda_{2,h}$ in S^0 uniformly in h, and

$$\overline{p_h} \#^w \left((\partial_{x_j} f)\partial_{\xi_j} p_h\right) = \overline{p_h}(\partial_{x_j} f)\partial_{\xi_j} p_h + h\mu_{j,h}, \quad j = 1, \ldots, n,$$

with $\mu_{j,h}$ in S^0 uniformly in h. It follows that

$$\overline{p_h} \#^w f \#^w p_h = f|p_h|^2 - \frac{1}{2i}\sum_{j=1}^n (\partial_{x_j} f)(\overline{p_h}\,\partial_{\xi_j} p_h - p_h\,\partial_{\xi_j}\overline{p_h}) + \frac{1}{2i}\,f\,\{\overline{p_h}, p_h\} + h\lambda_{4,h},$$

with $\lambda_{4,h}$ in S^0 uniformly in h. With the notation of the proof of Lemma 2.5 (see expression (2.5)) we have

$$\{\overline{p_h}, p_h\} = 2ih^2|p_h|^2\,\{\alpha, \beta\} = hk_h^{(1)},$$

with $k_h^{(1)}$ in S^0 uniformly in h by Lemma 2.3, since β is in S^1 and $\alpha \in S^2$. We also have

$$\sum_{j=1}^{n} \partial_{x_j} f(\overline{p_h}\, \partial_{\xi_j} p_h - p_h \partial_{\xi_j} \overline{p_h}) = h k_h^{(2)},$$

with $k_h^{(2)}$ in S^0 uniformly in h by Lemma 2.3. ■

3. Multi-product representation: stability and convergence

We are interested in a representation of the solution operator for the following parabolic Cauchy problem

$$\partial_t u + q^w(t, x, D_x)u = 0, \quad 0 < t \le T, \tag{3.1}$$

$$u\,|_{t=0} = u_0 \in H^s(\mathbf{R}^n). \tag{3.2}$$

Here the symbol $q(t, x, \xi)$ is assumed to satisfy Assumption 2.1 uniformly w.r.t. the evolution parameter t and to remain in a bounded domain in S^2 as t varies. We then note that the result of the previous section remains valid in this case, i.e., the constant C obtained in Theorem 2.2 is uniform w.r.t. t. We denote by $U(t', t)$ the solution operator to the evolution problem (3.1).

Following [Le 06], we introduce the following approximation of $U(t, 0)$. With $\mathfrak{P} = \{t^{(0)}, t^{(1)}, \dots, t^{(N)}\}$, a subdivision of $[0, T]$ with $0 = t^{(0)} < t^{(1)} < \cdots < t^{(N)} = T$, we define the following multi-product

$$\mathcal{W}_{\mathfrak{P},t} := \begin{cases} P_{(t,0)} & \text{if } \ 0 \le t \le t^{(1)}, \\ P_{(t,t^{(k)})} \displaystyle\prod_{i=k}^{1} P_{(t^{(i)},t^{(i-1)})} & \text{if } \ t^{(k)} \le t \le t^{(k+1)}. \end{cases} \tag{3.3}$$

where $P_{(t',t)}$ is the ψDO with Weyl symbol $p_{(t',t)}$ given by $p_{(t',t)} := e^{-(t'-t)q(t,x,\xi)}$ for $t' \ge t$:

$$P_{(t',t)}v(x) = p^w_{(t',t)}(x, D_x)v(x) = (2\pi)^{-n} \iint e^{i\langle x-y,\xi\rangle} e^{-(t'-t)q(t,(x+y)/2,\xi)} v(y)\, dy\, d\xi.$$

We shall prove the convergence of $\mathcal{W}_{\mathfrak{P},t}$ to $U(t, 0)$ in some operator norms as well as its strong convergence.

3.1. Stability.

As a consequence of the estimate proven in Theorem 2.2 we have the following proposition.

Proposition 3.1. *Let $s \in \mathbf{R}$. There exists $K \ge 0$ such that for every subdivision $\mathfrak{P}$ of $[0, T]$, we have*

$$\forall t \in [0, T], \ \ \|\mathcal{W}_{\mathfrak{P},t}\|_{(H^s,H^s)} \le e^{KT}.$$

Proof. By Theorem 2.2, there exists $C \ge 0$ such that we have $\|P_{(t',t)}\|_{(H^s,H^s)} \le 1 + C(t'-t)$ for all $t', t \in [0, T]$, $t' > t$; we then obtain

$$\|\mathcal{W}_{\mathfrak{P},t}\|_{(H^s,H^s)} \le \prod_{i=0}^{N-1} (1 + C(t^{(i+1)} - t^{(i)})).$$

Setting $U_{\mathfrak{P}} = \ln\left(\prod_{i=0}^{N-1}(1 + C(t^{(i+1)} - t^{(i)}))\right)$, we then have $U_{\mathfrak{P}} \le \sum_{i=0}^{N-1} C(t^{(i+1)} - t^{(i)}) = CT$. We thus obtain $\|\mathcal{W}_{\mathfrak{P},t}\|_{(H^s,H^s)} \le e^{CT}$. ■

3.2. **Convergence.** To obtain a convergence result we shall need the following assumption on the regularity of the symbol $q(t,x,\xi)$ w.r.t. the evolution parameter t.

Assumption 3.2. *The symbol $q(t,x,\xi)$ is in $\mathscr{C}^{0,\alpha}([0,T],S^2(\mathbf{R}^n\times\mathbf{R}^n))$, i.e., Hölder continuous w.r.t. t with values in S^2, in the sense that, for some $0<\alpha\le 1$,*

$$q(t',x,\xi)-q(t,x,\xi)=(t'-t)^\alpha\,\tilde q(t',t,x,\xi),\quad 0\le t\le t'\le T,$$

with $\tilde q(t',t,x,\xi)$ in S^2 uniformly in t' and t.

We now give some regularity properties for the approximation Ansatz $\mathcal{W}_{\mathfrak{P},t}$ we have introduced.

Lemma 3.3. *Let $s\in\mathbf{R}$ and $t'',t\in[0,T]$, with $t<t''$. The map $t'\mapsto P_{(t',t)}$, for $t'\in[t,t'']$, is Lipschitz continuous with values in $\mathcal{L}(H^s(\mathbf{R}^n),H^{s-2}(\mathbf{R}^n))$. More precisely there exists $C>0$ such that, for all $v\in H^s(\mathbf{R}^n)$ and $t^{(1)},t^{(2)}\in[t,t'']$,*

$$\left\|\left(P_{(t^{(2)},t)}-P_{(t^{(1)},t)}\right)(v)\right\|_{H^{s-2}}\le C\left|t^{(2)}-t^{(1)}\right|\,\|v\|_{H^s}.$$

Proof. We simply write

$$(P_{(t^{(2)},t)}-P_{(t^{(1)},t)})(v)(x)=-(2\pi)^{-n}\int_{t^{(1)}}^{t^{(2)}}\iint e^{i\langle x-y,\xi\rangle}e^{-(t'-t)q(t,(x+y)/2,\xi)}\,q(t,(x+y)/2,\xi)\;v(y)\,dy\,d\xi\,dt'.$$

We thus obtain a ψDO whose Weyl symbol is in S^2 uniformly w.r.t. $t^{(2)}$ and $t^{(1)}$ and conclude with Theorem 18.6.3 in [Hör85]. ∎

Lemma 3.4. *Let $s\in\mathbf{R}$, $t'',t\in[0,T]$, with $t<t''$, and let $v\in H^s(\mathbf{R}^n)$. Then the map $t'\mapsto P_{(t',t)}(v)$ is in $\mathscr{C}^0([t,t''],H^s(\mathbf{R}^n))\cap\mathscr{C}^1([t,t''],H^{s-2}(\mathbf{R}^n))$.*

Proof. Let $t^{(1)}\in[t,t'']$ and let $\varepsilon>0$. Choose $v_1\in H^{s+2}(\mathbf{R}^n)$ such that $\|v-v_1\|_{H^s}\le\varepsilon$. Then for $t^{(2)}\in[t,t'']$

$$\begin{aligned}\left\|P_{(t^{(2)},t)}(v)-P_{(t^{(1)},t)}(v)\right\|_{H^s}&\le\left\|P_{(t^{(2)},t)}(v-v_1)\right\|_{H^s}+\left\|P_{(t^{(1)},t)}(v-v_1)\right\|_{H^s}+\left\|P_{(t^{(2)},t)}(v_1)-P_{(t^{(1)},t)}(v_1)\right\|_{H^s}\\&\le 2(1+C(t''-t))\varepsilon+C\left|t^{(2)}-t^{(1)}\right|\,\|v_1\|_{H^{s+2}}.\end{aligned}\tag{3.4}$$

The continuity of the map follows. Differentiating $P_{(t',t)}(v)$ w.r.t. t', we can prove that the resulting map $t'\mapsto\partial_{t'}P_{(t',t)}(v)$ is Lipschitz continuous with values in $\mathcal{L}(H^{s+2}(\mathbf{R}^n),H^{s-2}(\mathbf{R}^n))$ following the proof of Lemma 3.3: there exists $C>0$ such that for all $w\in H^{s+2}(\mathbf{R}^n)$

$$\left\|(\partial_{t'}P_{(t^{(2)},t)}-\partial_{t'}P_{(t^{(1)},t)})(w)\right\|_{H^{s-2}}\le C\left|t^{(2)}-t^{(1)}\right|\,\|w\|_{H^{s+2}}.$$

Here $\partial_{t'}P_{(t^{(i)},t)}$ means $\partial_{t'}P_{(t',t)}\big|_{t'=t^{(i)}}$. We also see that the map $v\mapsto\partial_{t'}P_{(t',t)}(v)$ is continuous from $H^s(\mathbf{R}^n)$ into $H^{s-2}(\mathbf{R}^n)$ with bounded continuity module: with $v\in H^s(\mathbf{R}^n)$, we make a similar choice as above for $v_1\in H^{s+2}(\mathbf{R}^n)$ and obtain an estimate for

$$\left\|\partial_{t'}P_{(t^{(2)},t)}(v)-\partial_{t'}P_{(t^{(1)},t)}(v)\right\|_{H^{s-2}}$$

of the same form as in (3.4). ∎

Gathering the results of the previous lemmata we obtain the following regularity result for the Ansatz $\mathcal{W}_{\mathfrak{P},t}$.

Proposition 3.5. *Let $s \in \mathbf{R}$, let $u_0 \in H^s(\mathbf{R}^n)$. Then the map $\mathcal{W}_{\mathfrak{P},t}(u_0)$ is in $\mathscr{C}^0([0,T], H^s(\mathbf{R}^n))$ and piecewise $\mathscr{C}^1([0,T], H^{s-2}(\mathbf{R}^n))$.*

The following energy estimate holds for a function $f(t)$ that is in $\mathscr{C}^0([0,T], H^s(\mathbf{R}^n))$ and piecewise $\mathscr{C}^1([0,T], H^{s-2}(\mathbf{R}^n))$ (by adapting the proof the energy estimate in Section 6.5 in [CP82]):

$$\|f(t)\|^2_{H^{s-1}} + \int_0^T \|f(t')\|^2_{H^s}\, dt' \le C\left[\, \|f(0)\|^2_{H^{s-1}} + \int_0^T \|\left(\partial_{t'} + q^w(t',x,D_x)\right) f(t')\|^2_{H^{s-2}}\, dt' \,\right], \tag{3.5}$$

for all $t \in [0,T]$. Once applied to $(U(t,0) - \mathcal{W}_{\mathfrak{P},t})(u_0)$ with $u_0 \in H^s(\mathbf{R}^n)$ we obtain

$$\begin{aligned}
\|(U(t,0) - \mathcal{W}_{\mathfrak{P},t})(u_0)\|^2_{H^{s-1}} + \int_0^T \|(U(t,0) - \mathcal{W}_{\mathfrak{P},t})(u_0)\|^2_{H^s}\, dt' &\le C \int_0^T \|\left(\partial_{t'} + q^w(t',x,D_x)\right) \mathcal{W}_{\mathfrak{P},t'}(u_0)\|^2_{H^{s-2}}\, dt' \\
&= C \sum_{j=0}^{N-1} \int_{t^{(j)}}^{t^{(j+1)}} \|\left(\partial_{t'} + q^w(t',x,D_x)\right) P_{(t',t^{(j)})} \mathcal{W}_{\mathfrak{P},t^{(j)}}(u_0)\|^2_{H^{s-2}}\, dt' \\
&\le C \sum_{j=0}^{N-1} \int_{t^{(j)}}^{t^{(j+1)}} \|\left(\partial_{t'} + q^w(t',x,D_x)\right) P_{(t',t^{(j)})}\|^2_{(H^s,H^{s-2})}\, dt'\; e^{KT} \|u_0\|^2_{H^s},
\end{aligned} \tag{3.6}$$

where we have used the stability result of Proposition 3.1. It remains to estimate the Sobolev operator norm of $(\partial_{t'} + q^w(t',x,D_x))\, P_{(t',t)}$, for $t' > t$, which can be understood as estimating the *consistency* of the proposed approximation Ansatz. This is the object of the following proposition.

Proposition 3.6. *Let $s \in \mathbf{R}$. There exists $C > 0$ such that $\|(\partial_{t'} + q^w(t',x,D_x))P_{(t',t)}\|_{(H^s,H^{s-2})} \le C(t'-t)^\alpha$, for $0 \le t \le t' \le T$.*

Proof. We have

$$\partial_{t'} P_{(t',t)} u(x) = -(2\pi)^{-n} \iint e^{i\langle x-y,\xi\rangle} q(t,(x+y)/2,\xi) e^{-(t'-t)q(t,(x+y)/2,\xi)}\, u(y)\, dy\, d\xi, \tag{3.7}$$

and thus the operator $(\partial_{t'} + q^w(t',x,D_x))P_{(t',t)}$ admits

$$\sigma_{(t',t)} = q(t',.,.)\, \#^w p_{(t',t)} - q(t,.,.) p_{(t',t)}$$

for its Weyl symbol. Since by Assumption 3.2 we have

$$q(t',x,\xi) - q(t,x,\xi) = (t'-t)^\alpha\, \tilde{q}(t',t,x,\xi), \quad 0 \le t \le t' \le T,$$

with $\tilde{q}(t',t,x,\xi)$ in S^2 uniformly in t' and t. We can thus conclude with the following lemma. ■

Lemma 3.7. *We have $q\, \#^w p_h - q p_h = h\lambda_h$ with λ_h in S^2 uniformly in h.*

Proof. As in Section 2, we ignore the evolution parameter t in the notation. The result is however uniform w.r.t. t. By Proposition 2.4 we have

$$q \#^w p_h = qp_h + \frac{1}{2i}\{q, p_h\} + h\lambda_h,$$

with λ_h in S^2 uniformly in h. We note however that $\{q, p_h\} = 0$. ■

The result of Proposition 3.6 and estimate (3.6) yield

$$\|(U(t,0) - \mathcal{W}_{\mathfrak{P},t})(u_0)\|_{H^{s-1}} + \Big(\int_0^T \|(U(t,0) - \mathcal{W}_{\mathfrak{P},t})(u_0)\|^2_{H^s}\, dt'\Big)^{\frac{1}{2}} \le CTe^{CT}\, \Delta^{\alpha}_{\mathfrak{P}}\, \|u_0\|_{H^s}, \tag{3.8}$$

where $\Delta_{\mathfrak{P}} = \max_{0\le j\le N-1}(t^{j+1} - t^{(j)})$. This error estimate implies the following convergence results which provides a representation of $U(t,0)$ by an infinite multi-product of ψDOs: $U(t,0) = \lim_{\Delta_{\mathfrak{P}}\to 0} \mathcal{W}_{\mathfrak{P},t}$. We now state our main theorem.

Theorem 3.8. *Assume that $q(t,x,\xi)$ satisfies Assumptions 2.1 and 3.2. Then the approximation Ansatz $\mathcal{W}_{\mathfrak{P},t}$ converges to the solution operator $U(t,0)$ of the Cauchy problem* (3.1)–(3.2) *in $\mathcal{L}(H^s(\mathbf{R}^n), H^{s-1+r}(\mathbf{R}^n))$ uniformly w.r.t. t as $\Delta_{\mathfrak{P}} = \max_{0\le j\le N-1}(t^{j+1} - t^{(j)})$ goes to 0 with a convergence rate of order $\alpha(1-r)$:*

$$\|\mathcal{W}_{\mathfrak{P},t} - U(t,0)\|_{(H^s,H^{s-1+r})} \le C\Delta^{\alpha(1-r)}_{\mathfrak{P}}, \quad t\in[0,T], \quad 0\le r<1.$$

The operator $\mathcal{W}_{\mathfrak{P},t}$ also converges to $U(t,0)$ in $L^2(0,T,\mathcal{L}(H^s(\mathbf{R}^n), H^s(\mathbf{R}^n)))$ with a convergence rate of order α:

$$\Big(\int_0^T \|\mathcal{W}_{\mathfrak{P},t} - U(t,0)\|^2_{(H^s,H^s)} dt\Big)^{\frac{1}{2}} \le C\Delta^{\alpha}_{\mathfrak{P}}.$$

Furthermore $\mathcal{W}_{\mathfrak{P},t}$ strongly converges to $U(t,0)$ in $\mathcal{L}(H^s(\mathbf{R}^n), H^s(\mathbf{R}^n))$ uniformly w.r.t. $t\in[0,T]$.

Proof. The first two results are consequences of (3.8). The proof of the first result for $r \neq 0$ follows by interpolation between Sobolev Spaces [LM68].

Let $u_0 \in H^s(\mathbf{R}^n)$ and let $\varepsilon > 0$. For the strong convergence in $H^s(\mathbf{R}^n)$ we choose $u_1 \in H^{s+1}(\mathbf{R}^n)$ such that $\|u_0 - u_1\|_{H^s} \le \varepsilon$. We then write

$$\begin{aligned}\|\mathcal{W}_{\mathfrak{P},t}(u_0) - U(t,0)(u_0)\|_{H^s} &\le \|\mathcal{W}_{\mathfrak{P},t}(u_0 - u_1)\|_{H^s} + \|\mathcal{W}_{\mathfrak{P},t}(u_1) - U(t,0)(u_1)\|_{H^s} \\ &\quad + \|U(t,0)(u_0 - u_1)\|_{H^s} \le C\varepsilon + C\Delta^{\alpha}_{\mathfrak{P}}\|u_1\|_{H^{s+1}},\end{aligned}$$

from the case $r = 0$ of the first part of the theorem and from the stability of $\mathcal{W}_{\mathfrak{P},t}$ (Proposition 3.1). This last estimate is uniform w.r.t. $t\in[0,T]$ and yields the result. ■

4. Multi-product representation on a compact manifold

4.1. **Notation and setting.** We shall now consider the case of a parabolic equation on an n-dimensional compact $\mathscr{C}^\infty$-Riemannian manifold $(\mathcal{M}, g)$, where g is a smooth Riemannian metric. We let A be a second-order elliptic differential operator on $\mathcal{M}$ whose principal part, A_2, is given by the Laplace-Beltrami operator

on $\mathcal{M}$, which reads

$$A_2 = -\mathsf{g}^{-\frac{1}{2}}\partial_i\left(\mathsf{g}^{\frac{1}{2}}g^{ij}\partial_j\right),$$

in local coordinates, where $\mathsf{g} = \det(g_{ij})$. Other uniformly-elliptic operators can be considered by changing the metric. We choose here to focus on the differential case instead of the pseudodifferential case because the full symbol of the operator can then be completely defined on the manifold $\mathcal{M}$.

We allow the operator A to depend on an evolution parameter t. We shall thus assume that the metric is itself time-dependent, yet continuous w.r.t. t, $g = g(t, x)$, and satisfies

$$0 < c \leq g(t, x) \leq C < \infty, \qquad t \in [0, T], \quad x \in \mathcal{M}. \tag{4.1}$$

For the L^2 norm on $\mathcal{M}$, we shall use the metric $g(0, x)$ as a reference metric. We set $\mathsf{g}_0(x) = \mathsf{g}(0, x)$. We then denote by dv the volume form which is given by $dv = \mathsf{g}_0^{1/2}(x)\,dx$ in local coordinates. The L^2-inner product is then given by $(u, w) = \int_{\mathcal{M}} u\,\overline{w}\,dv$ [Heb96].

Since we are going to consider an infinite product of ψDOs, a little attention should be paid to a finite atlas. We shall use an atlas $\mathscr{A} = (\theta_i, \psi_i)_{i\in\mathcal{I}}$ of $\mathcal{M}$, $|\mathcal{I}| < \infty$, with $\psi_i : \theta_i \to \tilde{\theta}_i$, where $\tilde{\theta}_i$ is a smooth bounded open subset of $\mathbf{R}^n$. For $i \in \mathcal{I}$, we set

$$\mathcal{J}_i := \{j \in \mathcal{I};\ \theta_i \cap \theta_j \neq \emptyset\}, \quad \mathcal{J}_i^{(2)} := \{l \in \mathcal{J}_j;\ j \in \mathcal{J}_i\},$$

which lists the neighboring charts and the "second"-neighboring charts for the chart (θ_i, ψ_i). For technical reasons, we shall assume that there exists a coarser finite atlas $\mathscr{B} = (\Theta_k, \Psi_k)_{k\in\mathcal{K}}$ of $\mathcal{M}$, $\Psi_k : \Theta_k \to \tilde{\Theta}_k \subset \mathbf{R}^n$, such that for each chart $(\theta_i, \psi_i) \in \mathscr{A}$ there exists a chart $(\Theta_{k(i)}, \Psi_{k(i)}) \in \mathscr{B}$, such that

$$\bigcup_{l\in\mathcal{J}_i^{(2)}} \theta_l \Subset \Theta_{k(i)},$$

i.e., $\Theta_{k(i)}$ contains all the "second"-neighbors of θ_i. This is always possible by choosing the atlas $\mathscr{A}$ sufficiently fine. We shall denote by $a_i(t)$, $i \in \mathcal{I}$, the Weyl symbol of $A(t)$ in each local chart (θ_i, ψ_i).

We set $(\varphi_i)_{i\in\mathcal{I}}$ as a family of $\mathscr{C}^\infty$ real-valued functions defined on $\mathcal{M}$ such that the functions $(\varphi_i^2)_{i\in\mathcal{I}}$ form a partition of unity subordinated to the open covering $(\theta_i)_{i\in\mathcal{I}}$, i.e.,

$$\operatorname{supp}(\varphi_i) \subset \theta_i, \quad 0 \leq \varphi_i \leq 1, \quad i \in \mathcal{I}, \quad \text{and} \quad \sum_{i\in\mathcal{I}} \varphi_i^2 = 1.$$

We denote

$$\tilde{\varphi}_i = (\psi_i^{-1})^*\varphi_i = \varphi_i \circ \psi_i^{-1},$$

and similarly, for $l \in \mathcal{J}_i^{(2)}$, we shall set

$$\hat{\varphi}_l = (\Psi_{k(i)}^{-1})^*\varphi_l,$$

with $\Psi_{k(i)}$ as above, when there is no possible confusion on $k(i)$.

We set $Q(t)$ as the elliptic operator on $\mathcal{M}$ defined through

$$A(t) = \sum_{i\in\mathcal{I}} \varphi_i \circ Q(t) \circ \varphi_i.$$

The construction of Q can be done recursively: we write $Q = Q_2 + Q_1 + Q_0$, with Q_l a differential operator of order l, $l = 0, 1, 2$ and obtain

$$Q_2 = A, \quad Q_1 = -\sum_{i\in\mathcal{I}}[\varphi_i, Q_2]\circ\varphi_i, \quad Q_0 = -\sum_{i\in\mathcal{I}}[\varphi_i, Q_1]\circ\varphi_i.$$

The recursion stops after two iterations since we consider differential operators here.

In each local chart (θ_i, ψ_i), $i \in \mathcal{I}$, we denote by $q_i(t, x, \xi)$ the Weyl symbol of $Q(t)$, i.e.,

$$\forall u \in \mathscr{C}_c^\infty(\theta_i), \quad Q(t)u = \psi_i^*\left(q_i^w(t, x, D_x)((\psi_i^{-1})^* u)\right),$$

or equivalently

$$\forall \tilde{u} \in \mathscr{C}_c^\infty(\tilde{\theta}_i), \quad q_i^w(t, x, D_x)\tilde{u} = (\psi_i^{-1})^*\left(Q(t)(\psi_i^*\tilde{u})\right).$$

The symbol $q_i(t, x, \xi)$ is uniquely defined since $Q(t)$ is a differential operator. We also let $\hat{q}_k(t, x, \xi)$ be the Weyl symbol of $Q(t)$ in the chart (Θ_k, Ψ_k), $k \in \mathcal{K}$. From (4.1) we then have

Lemma 4.1. *In each chart the symbol of $Q(t)$ satisfies the properties of Assumption 2.1.*

We set

$$p_{i,(t',t)}(x, \xi) = e^{-(t'-t)q_i(t,x,\xi)}, \quad i \in \mathcal{I}, \quad 0 \le t \le t' \le T, \quad x \in \tilde{\theta}_i,\ \xi \in \mathbf{R}^n.$$

With these symbols in $S^0(\tilde{\theta}_i \times \mathbf{R}^n)$, we define the following ψDOs on $\mathcal{M}$:

$$P_{i,(t',t)}u := \varphi_i \circ \psi_i^* \circ p^w_{i,(t',t)}(x, D_x) \circ (\psi_i^{-1})^* \circ \varphi_i = \psi_i^* \circ (\tilde{\varphi}_i \circ p^w_{i,(t',t)}(x, D_x) \circ \tilde{\varphi}_i) \circ (\psi_i^{-1})^*, \tag{4.2}$$

$$P_{(t',t)} := \sum_{i\in\mathcal{I}} P_{i,(t',t)}, \tag{4.3}$$

where φ_i and $\tilde{\varphi}_i$ are understood here as multiplication operators. The operator $P_{(t',t)}$ is the counterpart of the operator $p^w_{(t',t)}(x, D_x)$ introduced in Sections 2 and 3. We shall compose such operators in the form of a multi-product as is done in Section 3 to obtain a representation of the solution operator to the following well-posed parabolic Cauchy problem on $\mathcal{M}$

$$\partial_t u + A(t)u = 0, \quad 0 < t \le T, \tag{4.4}$$

$$u\,|_{t=0} = u_0 \in H^s(\mathcal{M}). \tag{4.5}$$

We denote by $U(t', t)$ the solution operator of (4.4)–(4.5) and we define the multi-product operator $\mathcal{W}_{\mathfrak{P},t}$ as in (3.3) for a subdivision $\mathfrak{P} = \{t^{(0)}, t^{(1)}, \dots, t^{(N)}\}$ of $[0, T]$:

$$\mathcal{W}_{\mathfrak{P},t} := \begin{cases} P_{(t,0)} & \text{if } 0 \le t \le t^{(1)}, \\ P_{(t,t^{(k)})}\displaystyle\prod_{i=k}^{1} P_{(t^{(i)},t^{(i-1)})} & \text{if } t^{(k)} \le t \le t^{(k+1)}. \end{cases} \tag{4.6}$$

We shall make the following regularity assumption on the operator $A(t)$, which is equivalent to that made in Section 3 (Assumption 3.2).

Assumption 4.2. *The symbol of $A(t)$ is Hölder continuous of order α, $0 < \alpha \leq 1$, w.r.t. t with values in S^2: for each chart (θ_i, ψ_i) we have $a_i \in \mathscr{C}^{0,\alpha}([0,T], S^2(\mathbf{R}^n \times \mathbf{R}^n))$, in the sense that,*

$$a_i(t', x, \xi) - a_i(t, x, \xi) = (t' - t)^\alpha \, \tilde{a}_i(t', t, x, \xi), \quad 0 \leq t \leq t' \leq T,$$

with $\tilde{a}_i(t', t, x, \xi)$ in S^2 uniformly in t' and t. Note that the same property then holds for the symbol of $A(t)$ in any chart.

This property naturally translates to the symbols $q_i(t)$, $i \in \mathcal{I}$.

Remark 4.3. The form we have chosen for the operator $P_{(t',t)}$ can be motivated at this point. First, a natural requirement is that $P_{(t,t)} = \mathrm{Id}$, which is achieved since $\sum_{i\in\mathcal{I}} \varphi_i^2 = 1$. Second, the consistency analysis of Proposition 3.6 gears towards having $\left(\partial_t' P_{(t',t)} - A(t') \circ P_{(t',t)}\right)\big|_{t'=t} = 0$, which is achieved here thanks to the form we have chosen for the differential operator $Q(t)$.

As in Section 3, we first need to address the stability of the multi-product. Here, we shall only consider the L^2 case.

4.2. **L^2 Stability.** As in Section 2, we find a sharp estimate of the L^2-norm of the operator $P_{(t',t)}$ over $\mathcal{M}$.

Theorem 4.4. *There exists a constant $C \geq 0$ such that*

$$\|P_{(t',t)}\|_{(L^2(\mathcal{M}),L^2(\mathcal{M}))} \leq 1 + C(t' - t),$$

holds for all $0 \leq t \leq t' \leq T$.

Therefore, as in Section 3, we obtain the following stability result for $\mathcal{W}_{\mathfrak{P},t}$.

Corollary 4.5. *There exists $K \geq 0$ such that for every subdivision $\mathfrak{P}$ of $[0,T]$, we have*

$$\forall t \in [0,T], \quad \|\mathcal{W}_{\mathfrak{P},t}\|_{(L^2(\mathcal{M}),L^2(\mathcal{M}))} \leq e^{KT}.$$

Proof of Theorem 4.4. We let $u, w \in L^2(\mathcal{M})$. We have

$$(P_{(t'-t)}u, w) = \sum_{i\in\mathcal{I}} \int_{\mathcal{M}} \varphi_i \overline{w} \, \psi_i^* (p^w_{i,(t',t)}(x, D_x)((\psi_i^{-1})^*(\varphi_i u))) \, dv = \sum_{i\in\mathcal{I}} \int_{\tilde{\theta}_i} (\overline{\tilde{\varphi}_i \tilde{w}_i}) \, p^w_{i,(t',t)}(x, D_x)(\tilde{\varphi}_i \tilde{u}_i) \, \mathsf{g}_0^{1/2}(x) \, dx,$$

where $\tilde{u}_i$ (resp. $\tilde{w}_i$) is the pullback of $u|_{\theta_i}$ (resp. $w|_{\theta_i}$) by ψ_i^{-1}. We now extend the symbol $q_i(t, .)$ to $\mathbf{R}^n \times \mathbf{R}^n$ to obtain a symbol satisfying Assumption 2.1 like its counterpart in Section 2. We still denote by $q_i(t, .)$ this extended symbol. Then, by Proposition 2.10, for all $i \in \mathcal{I}$, there exists $C_i \geq 0$ such that

$$\|p^w_{i,(t',t)}(x, D_x)\|_{(L^2(\mathbf{R}^n, \mathsf{g}_0^{1/2}\, dx), L^2(\mathbf{R}^n, \mathsf{g}_0^{1/2}\, dx))} \leq 1 + C_i(t' - t),$$

where g_0 is also extended from $\tilde{\theta}_i$ to $\mathbf{R}^n$, yet still preserving Property (2.7). With $C = \max_{i\in\mathcal{I}} C_i$ (recall that $\mathcal{I}$ is finite) we thus obtain

$$|\,(P_{(t'-t)}u, w)\,| \leq (1 + C(t' - t)) \sum_{i\in\mathcal{I}} \|\tilde{\varphi}_i \tilde{w}_i\|_{L^2(\tilde{\theta}_i, \mathsf{g}_0^{1/2}\, dx)} \, \|\tilde{\varphi}_i \tilde{u}_i\|_{L^2(\tilde{\theta}_i, \mathsf{g}_0^{1/2}\, dx)}.$$

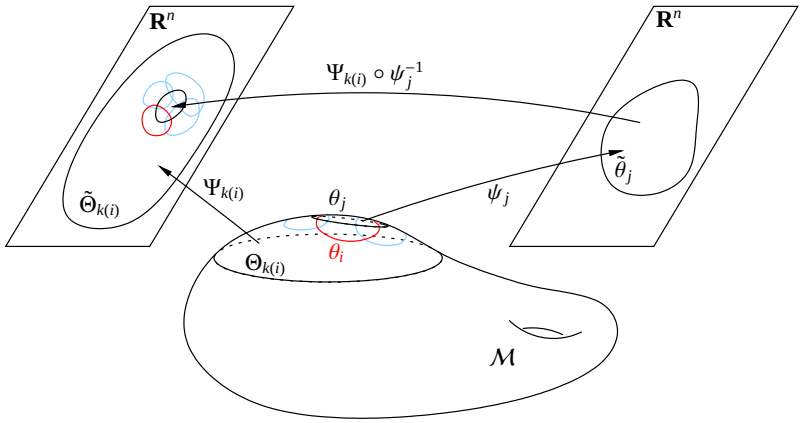

FIGURE 1. Change of variables bringing the analysis to the chart $(\Theta_{k(i)}, \Psi_{k(i)})$ for the charts (θ_i, ψ_i) and (θ_j, ψ_j), $j \in \mathcal{J}_i$, and their neighboring charts.

A Cauchy-Schwarz inequality then yields

$$|(P_{(t'-t)}u, w)| \le (1 + C(t'-t)) \Big(\sum_{i\in\mathcal{I}} \|\tilde{\varphi}_i \tilde{w}_i\|^2_{L^2(\tilde{\theta}_i, \mathrm{g}_0^{1/2}\,dx)} \Big)^{\frac{1}{2}} \Big(\sum_{i\in\mathcal{I}} \|\tilde{\varphi}_i \tilde{u}_i\|^2_{L^2(\tilde{\theta}_i, \mathrm{g}_0^{1/2}\,dx)} \Big)^{\frac{1}{2}}.$$

Observing that

$$\sum_{i\in\mathcal{I}} \|\tilde{\varphi}_i \tilde{u}_i\|^2_{L^2(\tilde{\theta}_i, \mathrm{g}_0^{1/2}\,dx)} = \sum_{i\in\mathcal{I}} \int_{\tilde{\theta}_i} \tilde{\varphi}_i^2 \tilde{u}_i^2\, \mathrm{g}_0^{1/2}(x)\,dx = \sum_{i\in\mathcal{I}} \int_{\mathcal{M}} \varphi_i^2 u^2\,dv = \|u\|^2_{L^2(\mathcal{M})},$$

since $\sum_{i\in\mathcal{I}} \varphi_i^2 = 1$, we find

$$|(P_{(t'-t)}u, w)| \le (1 + C(t'-t))\, \|w\|_{L^2(\mathcal{M})}\, \|u\|_{L^2(\mathcal{M})},$$

which concludes the proof. ■

4.3. **Consistency estimate.** As in Section 3, Proposition 3.6, for the case of $\mathbf{R}^n$, we shall now analyze the symbol of the operator $(\partial_{t'} + A(t'))P_{(t',t)}$ and prove the following proposition that corresponds to a consistency estimate.

Proposition 4.6. *Let $0 \le t \le t' \le T$. We have*

$$(\partial_{t'} + A(t')) \circ P_{(t',t)} = (t'-t)^\alpha L_{(t',t)}, \qquad \text{with} \quad L_{(t',t)} \in \Psi^2(\mathcal{M}),$$

and for all $s \in \mathbf{R}$, there exists $C \ge 0$ such that

$$\|L_{(t',t)}\|_{(H^s(\mathcal{M}), H^{s-2}(\mathcal{M}))} \le C, \tag{4.7}$$

uniformly in t' and t.

Proof. For $u \in \mathscr{C}^\infty(\mathcal{M})$ we have $u = \sum_{i\in\mathcal{I}} \varphi_i^2 u$. It thus suffices to take $u_i \in \mathscr{C}^\infty(\mathcal{M})$, with $\mathrm{supp}(u_i) \subset \theta_i$, for some $i \in \mathcal{I}$, and to prove that we have

$$(\partial_{t'} + A(t'))P_{(t',t)}(u_i) = (t'-t)L_{i,(t',t)}(u_i), \qquad L_{i,(t',t)}, \in \Psi^2(\mathcal{M}),$$

and that $L_{i,(t',t)}$ satisfies (4.7) uniformly in t' and t.

For concision we write $\hat{q}$ for $\hat{q}_{k(i)}$ here. Let us recall that $\hat{q}_k$ is the Weyl symbol of $Q(t)$ in the chart (Θ_k, Ψ_k), $k \in K$. We set $\hat{p}_{(t',t)}(x,\xi) := e^{-(t'-t)\hat{q}(t,x,\xi)}$. Making use of the assumption made on the chart $(\Theta_{k(i)}, \Psi_{k(i)})$, we consider the action of the change of variables $\kappa = \Psi_{k(i)} \circ \psi_j^{-1}$ on the operators $\tilde{\varphi}_j \circ p^w_{j,(t',t)}(x, D_x) \circ \tilde{\varphi}_j \in \Psi^0(\tilde{\theta}_j)$ for $j \in \mathcal{J}_i$ (see Figure 1). By Lemma B.2, we obtain

$$P_{(t',t)}u_i = \Psi^*_{k(i)} \circ \Big(\sum_{j\in\mathcal{J}_i} \hat{\varphi}_j \circ \hat{p}^w_{(t',t)}(x, D_x) \circ \hat{\varphi}_j \Big) \circ (\Psi^{-1}_{k(i)})^* u_i + (t'-t) R^{(0)}_{(t',t)} u_i,$$

with $R^{(0)}_{(t',t)}$ in $\Psi^0(\mathcal{M})$ uniformly in t' and t. We then have

$$\begin{aligned} A(t') \circ P_{(t',t)}u_i = \sum_{j\in\mathcal{J}_i} \sum_{l\in\mathcal{J}_i^{(2)}} \Psi^*_{k(i)} \circ \Big(\hat{\varphi}_l \circ \hat{q}^w(t', x, D_x) \circ \hat{\varphi}_l \hat{\varphi}_j \circ \hat{p}^w_{(t',t)}(x, D_x) \circ \hat{\varphi}_j \Big) \circ (\Psi^{-1}_{k(i)})^* u_i \\ + (t'-t) R^{(1)}_{(t',t)} u_i, \end{aligned} \tag{4.8}$$

where $R^{(1)}_{(t',t)}$ is in $\Psi^2(\mathcal{M})$ uniformly in t' and t.

From (4.2) we have

$$\partial_{t'} P_{j,(t',t)} u_i = \psi_j^* \circ (\tilde{\varphi}_j \circ (q_j(t,.) p_{j,(t',t)})^w(x, D_x) \circ \tilde{\varphi}_j) \circ (\psi_j^{-1})^* u_i, \qquad j \in \mathcal{J}_i,$$

which we may write

$$\begin{aligned} \partial_{t'} P_{j,(t',t)} u_i =& \psi_j^* \circ (\tilde{\varphi}_j \circ q^w_j(t, x, D_x) \circ p^w_{j,(t',t)}(x, D_x) \circ \tilde{\varphi}_j) \circ (\psi_j^{-1})^* u_i \\ &+ (t'-t) \psi_j^* (\tilde{\varphi}_j \circ \tilde{R}^{(2)}_{j,(t'-t)} \circ \tilde{\varphi}_j) \circ (\psi_j^{-1})^* u_i, \qquad j \in \mathcal{J}_i, \end{aligned}$$

where $\tilde{R}^{(2)}_{j,(t',t)}$ is in $\Psi^2(\tilde{\theta}_j)$ uniformly in t' and t by Lemma 3.7. We choose $\chi_j \in \mathscr{C}_c^\infty(\tilde{\theta}_j)$ such that χ_j is equal to one on $\mathrm{supp}(\tilde{\varphi}_j)$. We then have

$$\begin{aligned} \partial_{t'} P_{j,(t',t)} u_i =& \psi_j^* \circ (\tilde{\varphi}_j \circ q^w_j(t, x, D_x) \circ \chi_j \circ p^w_{j,(t',t)}(x, D_x) \circ \chi_j \circ \tilde{\varphi}_j) \circ (\psi_j^{-1})^* u_i \\ &+ (t'-t) \psi_j^* (\tilde{\varphi}_j \circ \tilde{R}^{(2)}_{j,(t'-t)} \circ \tilde{\varphi}_j) \circ (\psi_j^{-1})^* u_i, \qquad j \in \mathcal{J}_i, \end{aligned}$$

recalling that $q^w_j(t, x, D_x)$ is a differential operator, hence a local operator. Applying Lemma B.2 to $\chi_j \circ p^w_{j,(t',t)}(x, D_x) \circ \chi_j$ we obtain

$$\partial_{t'} P_{j,(t',t)} u_i = \Psi^*_{k(i)} \circ (\hat{\varphi}_j \circ \hat{q}^w(t, x, D_x) \circ \hat{\chi}_j \circ \hat{p}^w_{(t',t)}(x, D_x) \circ \hat{\chi}_j \circ \hat{\varphi}_j) \circ (\Psi^{-1}_{k(i)})^* u_i + (t'-t) R^{(3)}_{j,(t'-t)} u_i, \quad j \in \mathcal{J}_i,$$

where $R^{(3)}_{j,(t',t)}$ is in $\Psi^2(\mathcal{M})$ uniformly in t' and t and $\hat{\chi}_j = \left(\psi_j \circ \Psi^{-1}_{k(i)}\right)^* \chi_j$. Using again that $\hat{q}^w(t, x, D_x)$ is a differential operator, we finally obtain

$$\partial_{t'} P_{(t',t)} u_i = \sum_{j\in\mathcal{J}_i} \Psi^*_{k(i)} \circ (\hat{\varphi}_j \circ \hat{q}^w(t, x, D_x) \circ \hat{p}^w_{(t',t)}(x, D_x) \circ \hat{\varphi}_j) \circ (\Psi^{-1}_{k(i)})^* u_i + (t'-t) R^{(3)}_{(t'-t)} u_i, \tag{4.9}$$

where $R^{(3)}_{(t',t)}$ is in $\Psi^2(\mathcal{M})$ uniformly in t' and t.

The operators $R^{(1)}_{(t',t)}$ in (4.8) and $R^{(3)}_{(t',t)}$ in (4.9) will contribute to the operator $L_{i,(t',t)}$ and we discard them from the subsequent analysis. Observe that we may change the sums over $j \in \mathcal{J}_i$ to sums over $j \in \mathcal{J}_i^{(2)}$ (4.8) and in (4.9) since we only consider the action of the two operators on u_i.

Now that we have brought the analysis to the open set $\tilde{\Theta}_{k(i)}$, we shall consider and analyze the following symbol, $\sigma_{(t',t)}$, which corresponds to the operator $(\Psi_{k(i)}^{-1})^* \circ (\partial_t + A(t')) \circ P_{(t',t)} \circ \Psi_{k(i)}^*$ ignoring the operators $R^{(1)}_{(t',t)}$ and $R^{(3)}_{(t',t)}$ as explained above:

$$\sigma_{(t',t)}(x,\xi) = -\sum_{j\in\mathcal{J}_i^{(2)}} \hat{\varphi}_j \#^w \hat{q}(t,.) \#^w \hat{p}_{(t',t)} \#^w \hat{\varphi}_j + \underbrace{\sum_{j,l\in\mathcal{J}_i^{(2)}} \hat{\varphi}_l \#^w \hat{q}(t',.) \#^w \hat{\varphi}_l \hat{\varphi}_j \#^w \hat{p}_{(t',t)} \#^w \hat{\varphi}_j}_{=:\sigma^{(1)}_{(t',t)}},$$

keeping in mind that we only consider the action of the associated operator on $(\Psi_{k(i)}^{-1})^* u_i$ whose support is compact and contained in $\Psi_{k(i)}(\theta_i)$. We extend the symbol $\hat{q}(t,.)$ to $\mathbf{R}^n \times \mathbf{R}^n$ to obtain a symbol satisfying Assumption 2.1 like its counterpart in Section 2. We still denote by $\hat{q}(t,.)$ this extended symbol. We may then use global symbols in $\mathbf{R}^n$. As in the proof of Proposition 3.6, we may replace $\hat{q}(t,.)$ by $\hat{q}(t',.)$ by Assumption 4.2. For the symbol $\sigma_{(t',t)}(x,\xi)$, this yields an error term of the form $(t'-t)^\alpha \lambda^{(0)}_{(t',t)}(x,\xi)$, with $\lambda^{(0)}_{(t',t)}(x,\xi)$ in S^2 uniformly in t' and t, that will contribute to the operator $L_{i,(t',t)}$. We thus discard this term in the subsequent analysis and we still denote by $\sigma_{(t',t)}(x,\xi)$ the modified symbol.

We now set $\chi = 1 - \sum_{j\in\mathcal{J}_i^{(2)}} \hat{\varphi}_j^2$ and we write

$$\begin{aligned}
\sigma^{(1)}_{(t',t)} &= \sum_{j\in\mathcal{J}_i^{(2)}} \hat{\varphi}_j \#^w \hat{q}(t',.) \#^w \hat{\varphi}_j^2 \#^w \hat{p}_{(t',t)} \#^w \hat{\varphi}_j + \sum_{\substack{j,\,l\,\in\,\mathcal{J}_i^{(2)}\\ l\,\neq\, j}} \hat{\varphi}_l \#^w \hat{q}(t',.) \#^w \hat{\varphi}_l \hat{\varphi}_j \#^w \hat{p}_{(t',t)} \#^w \hat{\varphi}_j \\
&= \sum_{j\in\mathcal{J}_i^{(2)}} \hat{\varphi}_j \#^w \hat{q}(t',.) \#^w (1-\chi) \#^w \hat{p}_{(t',t)} \#^w \hat{\varphi}_j + \sum_{\substack{l,\,j\,\in\,\mathcal{J}_i^{(2)}\\ l\,\neq\, j}} \hat{\varphi}_l \#^w \hat{q}(t',.) \#^w \hat{\varphi}_l \hat{\varphi}_j \#^w \hat{p}_{(t',t)} \#^w \hat{\varphi}_j \\
&\quad - \sum_{\substack{l,\,j\,\in\,\mathcal{J}_i^{(2)}\\ l\,\neq\, j}} \hat{\varphi}_j \#^w \hat{q}(t',.) \#^w \hat{\varphi}_l^2 \#^w \hat{p}_{(t',t)} \#^w \hat{\varphi}_j,
\end{aligned}$$

which yields

$$\sigma_{(t',t)}(x,\xi) = -\sum_{j\in\mathcal{J}_i^{(2)}} \hat{\varphi}_j \#^w \hat{q}(t,.) \#^w \chi \# \hat{p}_{(t',t)} \#^w \hat{\varphi}_j \tag{4.10}$$

$$+ \sum_{\substack{l,\,j\,\in\,\mathcal{J}_i^{(2)}\\ l\,\neq\, j}} \hat{\varphi}_l \#^w \hat{q}(t',.) \#^w \Big(\underbrace{\hat{\varphi}_j \hat{\varphi}_l \#^w \hat{p}_{(t',t)} \#^w \hat{\varphi}_j - \hat{\varphi}_j^2 \#^w \hat{p}_{(t',t)} \#^w \hat{\varphi}_l}_{=:\sigma^{(2)}_{(t',t)}(x,\xi)} \Big) \tag{4.11}$$

From Weyl calculus [Hör85], since $\mathrm{supp}(\chi) \cap \mathrm{supp}((\Psi_{k(i)}^{-1})^* u_i) = \emptyset$, we find that the first term in the r.h.s. of (4.10) can be written in the form $(t'-t)\lambda^{(1)}(x,\xi)$, with $\lambda^{(1)}$ in S^2 uniformly in t' and t, making use of the composition formula (1.6) and Lemma 2.3.

Applying Proposition A.1 (with $k = 1$), we find

$$\sigma^{(2)}_{(t',t)}(x,\xi) = \sum_{m=1}^{n} \frac{i}{2} \Big(2(\partial_{x_m}\hat{\varphi}_l)\hat{\varphi}_j^2 - \hat{\varphi}_l \partial_{x_m}(\hat{\varphi}_j^2) \Big) \partial_{\xi_m} \hat{p}_{(t',t)} + (t'-t)\lambda^{(2)}(x,\xi),$$

with $\lambda^{(2)}$ in S^0 uniformly in t' and t arguing as in the proof of Proposition 2.4 in Appendix A (using Lemma 2.3 and Theorem 2.2.5 in [Kg81]). Therefore, we are now left with computing

$$\begin{aligned}\sigma^{(3)}_{(t',t)}(x,\xi) &= \frac{i}{2}\sum_{m=1}^{n}\sum_{\substack{l,\,j\,\in\,\mathcal{J}_i^{(2)}\\ l\neq j}} \hat{\varphi}_l \#^w \hat{q}(t',.)\#^w\Big[\Big(2(\partial_{x_m}\hat{\varphi}_l)\hat{\varphi}_j^2 - \hat{\varphi}_l\partial_{x_m}(\hat{\varphi}_j^2)\Big)\partial_{\xi_m}\hat{p}_{(t',t)}\Big]\\ &= \frac{i}{2}\sum_{m=1}^{n}\sum_{\substack{l,\,j\,\in\,\mathcal{J}_i^{(2)}\\ l\neq j}} \Big((\partial_{x_m}(\hat{\varphi}_l^2))\hat{\varphi}_j^2 - \hat{\varphi}_l^2\partial_{x_m}(\hat{\varphi}_j^2)\Big)\hat{q}(t',x,\xi)\partial_{\xi_m}\hat{p}_{(t',t)} + (t'-t)\lambda^{(3)}(x,\xi),\end{aligned}$$

with $\lambda^{(3)}$ in S^2 uniformly in t' and t by the composition formula (1.6) and Lemma 2.3. Observing that the first term just obtained in fact vanishes, we finally have $\sigma_{(t',t)}(x,\xi) = (t'-t)\lambda(x,\xi)$ with $\lambda(x,\xi)$ in S^2 uniformly in t' and t. This concludes the proof. ■

4.4. **Convergence and representation theorem.** We observe that the energy estimate (3.5) also holds for the differential operator $A(t)$ on $\mathcal{M}$ (since the proof relies on the Gårding inequality which holds for positive elliptic operators on $\mathcal{M}$). Combined with the (L^2) stability result of Corollary 4.5 and the consistency estimate of Proposition 4.6, the energy estimate yields, as in Section 3, the following representation theorem through the convergence of $\mathcal{W}_{\mathfrak{P},t}$ to $U(t,0)$, the solution operator of the parabolic Cauchy problem (4.4)–(4.5): $U(t,0) = \lim_{\Delta_{\mathfrak{P}}\to 0}\mathcal{W}_{\mathfrak{P},t}$ in the following sense.

Theorem 4.7. *Assume that $A(t)$ satisfies Assumption 4.2. Then the approximation Ansatz $\mathcal{W}_{\mathfrak{P},t}$ converges to the solution operator $U(t,0)$ of the Cauchy problem* (4.4)–(4.5) *in $\mathcal{L}(L^2(\mathcal{M}), H^{-1+r}(\mathcal{M}))$ uniformly w.r.t. t as $\Delta_{\mathfrak{P}} = \max_{0\le j\le N-1}(t^{j+1}-t^{(j)})$ goes to 0 with a convergence rate of order $\alpha(1-r)$:*

$$\|\mathcal{W}_{\mathfrak{P},t} - U(t,0)\|_{(L^2,H^{-1+r})} \le C\Delta_{\mathfrak{P}}^{\alpha(1-r)}, \quad t\in[0,T], \quad 0\le r<1.$$

The operator $\mathcal{W}_{\mathfrak{P},t}$ also converges to $U(t,0)$ in $L^2(0,T,\mathcal{L}(L^2(\mathcal{M}),L^2(\mathcal{M})))$ with a convergence rate of order α:

$$\Big(\int_0^T \|\mathcal{W}_{\mathfrak{P},t} - U(t,0)\|^2_{(L^2,L^2)}dt\Big)^{\frac{1}{2}} \le C\Delta_{\mathfrak{P}}^{\alpha}.$$

Furthermore $\mathcal{W}_{\mathfrak{P},t}$ strongly converges to $U(t,0)$ in $\mathcal{L}(L^2(\mathcal{M}),L^2(\mathcal{M}))$ uniformly w.r.t. $t\in[0,T]$.

Appendix A. Proofs of composition-like formulae

We prove Proposition 1.1 and derive composition results for the symbol $p_h(x,\xi) = e^{-hq(x,\xi)}$.

A.1. **Proof of Proposition 1.1.** From Weyl Calculus we have

$$(a\#^w b)(x,\xi) = \pi^{-2n}\int_{④} e^{i\Sigma(z,\zeta,t,\tau,\xi)}a(x+z,\zeta)b(x+t,\tau)\,dz\,d\zeta\,dt\,d\tau,$$

where $\Sigma(z,\zeta,t,\tau,\xi) = 2(\langle\tau-\xi,z\rangle - \langle\zeta-\xi,t\rangle)$ (see [Hör85], p. 152). This yields

$$(a\,\#^w b)(x,\xi) - (ab)(x,\xi) = \pi^{-2n}\int_{④} e^{i\Sigma(z,\zeta,t,\tau,\xi)}\Big(a(x+z,\zeta)b(x+t,\tau) - a(x,\zeta)b(x,\tau)\Big)\,dz\,d\zeta\,dt\,d\tau$$

$$= \sum_{j=1}^{n}\pi^{-2n}\int_0^1\int_{④} e^{i\Sigma(z,\zeta,t,\tau,\xi)}\Big(z_j\partial_{x_j}a(x+rz,\zeta)b(x+rt,\tau) + t_j a(x+rz,\zeta)\partial_{x_j}b(x+rt,\tau)\Big)\,dr\,dz\,d\zeta\,dt\,d\tau$$

by a first-order Taylor formula. In the first (resp. second) term that we have obtained, we write

$$z_j e^{i\Sigma(z,\zeta,t,\tau,\xi)} = -\frac{i}{2}\partial_{\tau_j}e^{i\Sigma(z,\zeta,t,\tau,\xi)} \quad (\text{resp. } t_j e^{i\Sigma(z,\zeta,t,\tau,\xi)} = \frac{i}{2}\partial_{\zeta_j}e^{i\Sigma(z,\zeta,t,\tau,\xi)}).$$

Integration by parts w.r.t. τ and ζ in the oscillatory integral yields

$$(a\,\#^w b)(x,\xi) - (ab)(x,\xi) = \sum_{j=1}^{n}\frac{\pi^{-2n}}{2i}\int_0^1\int_{④} e^{i\Sigma(z,\zeta,t,\tau,\xi)}\Big(\partial_{\xi_j}a(x+rz,\zeta)\partial_{x_j}b(x+rt,\tau)$$
$$- \partial_{x_j}a(x+rz,\zeta)\partial_{\xi_j}b(x+rt,\tau)\Big)\,dr\,dz\,d\zeta\,dt\,d\tau,$$

$$= \frac{i\pi^{-2n}}{2}\int_0^1\int_{④} e^{i\Sigma(z,\zeta,t,\tau,\xi)}\sigma((D_x,D_\zeta),(D_y,D_\tau))\Big(a(x+rz,\zeta)b(y+rt,\tau)\Big)\,dr\,dz\,d\zeta\,dt\,d\tau\Big|_{y=x},$$

which gives the result of Proposition 1.1 for $k=0$.To proceed further we integrate by parts w.r.t. r and obtain

$$(a\,\#^w b)(x,\xi) - (ab)(x,\xi) = \frac{i\pi^{-2n}}{2}\int_{④} e^{i\Sigma(z,\zeta,t,\tau,\xi)}\sigma((D_x,D_\zeta),(D_y,D_\tau))\Big(a(x,\zeta)b(y,\tau)\Big)\,dz\,d\zeta\,dt\,d\tau\Big|_{y=x}$$

$$+ \sum_{j=1}^{n}\frac{i\pi^{-2n}}{2}\int_0^1(1-r)\int_{④} e^{i\Sigma(z,\zeta,t,\tau,\xi)}\sigma((D_x,D_\zeta),(D_y,D_\tau))\Big(z_j\partial_{x_j}a(x+rz,\zeta)b(y+rt,\tau)$$
$$+ t_j a(x+rz,\zeta)\partial_{x_j}b(y+rt,\tau)\Big)\,dr\,dz\,d\zeta\,dt\,d\tau\Big|_{y=x}$$

$$= \frac{i}{2}\sigma((D_x,D_\xi),(D_y,D_\eta))\Big(a(x,\xi)b(y,\eta)\Big)\Big|_{\substack{y=x\\ \eta=\xi}}$$

$$+ \pi^{-2n}\left(\frac{i}{2}\right)^2\int_0^1(1-r)\int_{④} e^{i\Sigma(z,\zeta,t,\tau,\xi)}\Big(\sigma((D_x,D_\zeta),(D_y,D_\tau))\Big)^2\Big(a(x+rz,\zeta)b(y+rt,\tau)\Big)\,dr\,dz\,d\zeta\,dt\,d\tau\Big|_{y=x},$$

which gives the result for $k=1$. Formula 1.6 then follows from induction by integration par parts w.r.t. r each time. ■

A.2. **From amplitudes to symbols.** Here we give a formula of the form of (1.6) to compute the Weyl symbol of a ψDO starting from an arbitrary amplitude.

Proposition A.1. *Let $a(x,y,\xi)\in S^m(\mathbf{R}^n\times\mathbf{R}^n\times\mathbf{R}^n)$ be the amplitude of a ψDO A, i.e.,*

$$Au(x) = (2\pi)^{-n}\iint e^{i\langle x-y,\xi\rangle}a(x,y,\xi)\,u(y)\,dy\,d\xi.$$

The Weyl symbol b of A, i.e. $A = b^w(x, D_x)$, is then given by

$$\begin{aligned} b(x,\xi) &= e^{\frac{i}{2}(\langle D_y, D_\xi\rangle - \langle D_x, D_\xi\rangle)} a(x,y,\xi)\Big|_{y=x} = \pi^{-n} \iint e^{2i\langle z,\zeta-\xi\rangle} a(x+z, x-z, \zeta)\, dz\, d\zeta \\ &= \sum_{j=0}^{k} \frac{1}{j!}\left(\frac{i}{2}\langle \partial_x - \partial_y, \partial_\xi\rangle\right)^j a(x,y,\xi)\Big|_{y=x} \\ &\quad + \pi^{-n} \int_0^1 \frac{(1-r)^k}{k!} \iint e^{2i\langle z,\zeta-\xi\rangle} \left(\frac{i}{2}\langle \partial_x - \partial_y, \partial_\xi\rangle\right)^{k+1} a(x+rz, y-rz, \zeta)\, dr\, dz\, d\zeta\Big|_{y=x}. \end{aligned}$$

The proof is analogous to that of Proposition 1.1 given above.

A.3. **Proof of Proposition 2.4.** We prove the results for $r_h \#^w p_h$. The results for $p_h \#^w r_h$ follow similarly. We first use Proposition 1.1 for $k = 0$:

(A.1)

$$\begin{aligned} r_h \#^w p_h(x,\xi) = r_h(x,\xi)p_h(x,\xi) + \frac{\pi^{-2n}}{2i} \sum_{j=1}^{n} \int_0^1 \int_{④} e^{i\Sigma(z,\zeta,t,\tau,\xi)} \Big(\partial_{\xi_j} r_h(x+rz,\zeta)\, \partial_{x_j} p_h(x+rt,\tau) \\ - \partial_{x_j} r_h(x+rz,\zeta)\, \partial_{\xi_j} p_h(x+rt,\tau) \Big)\, dr\, dz\, d\zeta\, dt\, d\tau. \end{aligned}$$

By Lemma 2.3, we have

$$\partial_{x_j} p_h = h\nu_{h(j)}, \quad \partial_{\xi_j} p_h = h\nu_h^{(j)},$$

with $\nu_{h(j)}$ in S^2 and $\nu_h^{(j)}$ in S^1 uniformly in h. We thus observe that the last term in (A.1) can be written as a linear combination of terms of the form

$$h \int_0^1 \int_{④} e^{i\Sigma(z,\zeta,t,\tau,\xi)} \nu_{1,h}(x+rz,\zeta)\nu_{2,h}(x+rt,\tau)\, dr\, dz\, d\zeta\, dt\, d\tau, \tag{A.2}$$

where $\nu_{1,h}$ and $\nu_{2,h}$ are respectively in S^{m_1} and S^{m_2} uniformly in h with $m_1 + m_2 = l + 1$. Setting

$$\nu_h(x,\tilde{x},y,\tilde{y},\xi,\eta) = \int_0^1 \nu_{1,h}(rx + (1-r)\tilde{x}, \xi)\, \nu_{2,h}(ry + (1-r)\tilde{y}, \eta)\, dr,$$

we see that it is a multiple symbol in $S^{m_1,m_2}(\mathbf{R}^{2n} \times \mathbf{R}^n \times \mathbf{R}^{2n} \times \mathbf{R}^n)$ and the term in (A.2) can be written as

$$h \int_{④} e^{i\Sigma(z,\zeta,t,\tau,\xi)} \nu_h(x+z, \tilde{x}, y+t, \tilde{y}, \zeta, \tau)\, dz\, d\zeta\, dt\, d\tau \Big|_{\tilde{x}=\tilde{y}=y=x},$$

Applying Theorem 2.2.5 in [Kg81] twice (once for the integrations w.r.t. z and τ, a second time for the integrations w.r.t. t and ζ, recalling that $\Sigma(z,\zeta,t,\tau,\xi) = 2(\langle \tau - \xi, z\rangle - \langle \zeta - \xi, t\rangle)$) we obtain that the last term in (A.1) is of the form $h\lambda_h^{(1)}(x,\xi)$, where $\lambda_h^{(1)}(x,\xi)$ is in S^{l+1} uniformly in h.

Similarly, by Lemma 2.3, we write

$$\partial_{x_j} p_h = h^{\frac{1}{2}} \tilde{\nu}_{h(j)}, \quad \partial_{\xi_j} p_h = h^{\frac{1}{2}} \tilde{\nu}_h^{(j)},$$

with $\tilde{\nu}_{h(j)}$ in S^1 and $\tilde{\nu}_h^{(j)}$ in S^0 uniformly in h. The same reasoning as above yields the last term in (A.1) is of the form $h^{\frac{1}{2}}\lambda_h^{(0)}(x,\xi)$, where $\lambda_h^{(0)}(x,\xi)$ is in S^l uniformly in h.

To now treat the last equality in (2.2) we use Proposition 1.1 for $k = 1$:

$$\text{(A.3)}\quad r_h \#^w p_h(x,\xi) = (r_h p_h)(x,\xi) + \frac{1}{2i}\{r_h, p_h\}(x,\xi)$$
$$+ \frac{\pi^{-2n}}{(2i)^2}\int_0^1 (1-r)\sum_{1\le j,k\le n}\int_{④} e^{i\Sigma(z,\zeta,t,\tau,\xi)}\Big(\partial^2_{\xi_j\xi_k} r_h(x+rz,\zeta)\partial^2_{x_jx_k} p_h(x+rt,\tau)$$
$$- 2\partial^2_{x_j\xi_k} r_h(x+rz,\zeta)\partial^2_{\xi_jx_k} p_h(x+rt,\tau) + \partial^2_{x_jx_k} r_h(x+rz,\zeta)\partial^2_{\xi_j\xi_k} p_h(x+rt,\tau)\Big)\, dr\, dz\, d\zeta\, dt\, d\tau.$$

Here, by Lemma 2.3, we write

$$\partial^2_{x_jx_k} p_h = h\nu_{h(j,k)}, \quad \partial^2_{\xi_jx_k} p_h = h\nu^{(j)}_{h(k)}, \quad \partial^2_{\xi_j\xi_k} p_h = h\nu^{(j,k)}_h,$$

where $\nu_{h(j,k)}$, $\nu^{(j)}_{h(k)}$, and $\nu^{(j,k)}_h$ are respectively in S^2, S^1, and S^0 uniformly in h. We can then conclude as above with Theorem 2.2.5 in [Kg81] and find the last term in (A.3) of the form $h\tilde\lambda^{(0)}_h$ with $\tilde\lambda^{(0)}_h$ in S^l uniformly in h. ■

Appendix B. Effect of a change of variables

B.1. **Pseudodifferential calculus results.** We shall be interested in transformation formulae for Weyl symbols under a change of variables and apply them to the particular symbols we consider in Section 4. We let X and $\tilde X$ be two open subsets of $\mathbf{R}^n$ and let $\kappa : X \to \tilde X$ be a diffeomorphism. We shall study the effect of the change of variables $x \mapsto \kappa(x)$ on the symbol $\chi \#^w p_h \#^w \chi$ in the Weyl quantization, where $p_h = e^{-hq}$, with q satisfying the assumptions made in Section 4 above and $\chi \in \mathscr{C}^\infty_c(X)$.

We first consider general amplitudes before specializing to Weyl symbols. Let $a(x,y,\xi)$ be the amplitude in $S^m(X\times X\times \mathbf{R}^n)$ of $A \in \Psi^m(X)$ whose kernel is compactly supported. In particular, below, we shall consider $a(x,y,\xi)$ to be of the form $\chi(x)\chi(y)\,\tilde a(x,y,\xi)$, with $\tilde a \in S^m(X\times X\times\mathbf{R}^n)$. With $\zeta \in \mathscr{C}^\infty_c(\mathbf{R}^n)$ equal to 1 in a neighborhood of 0 we set

$$a_0(x,y,\xi) = \zeta(x-y)\,a(x,y,\xi), \quad \text{and} \quad a_\infty(x,y,\xi) = (1-\zeta(x-y))\,a(x,y,\xi).$$

If we set $A_\kappa = (\kappa^{-1})^*\circ A\circ\kappa^*$, then $A_\kappa \in \Psi^m(\tilde X)$. In fact, for $\mathrm{supp}(\zeta)$ sufficiently small, $A_\kappa = A_{0,\kappa} + A_{\infty,\kappa}$, with $A_{\infty,\kappa} \in \Psi^{-\infty}(\tilde X)$, and an amplitude of $A_{0,\kappa}$ is given by [GS94]

$$\text{(B.1)}\qquad a_{0,\kappa}(x,y,\xi) = a_0(\kappa^{-1}(x), \kappa^{-1}(y), {}^t(\widetilde{\kappa^{-1}}(x,y))^{-1}\xi)\,|\det(\kappa^{-1})'(y)|\,|\det\widetilde{\kappa^{-1}}(x,y)|^{-1},$$

where $\widetilde{\kappa^{-1}}(x,y) = (\widetilde{\kappa^{-1}_{kl}}(x,y))_{1\le k,l\le n}$ is defined through

$$\kappa^{-1}_k(x) - \kappa^{-1}_k(y) = \sum_{l=1}^n \widetilde{\kappa^{-1}_{kl}}(x,y)(x_l - y_l).$$

Note that $\widetilde{\kappa^{-1}}(x,x) = (\kappa^{-1})'(x)$ which implies that $\widetilde{\kappa^{-1}}(x,y)$ is indeed invertible in the support of a_0 when $\mathrm{supp}(\zeta)$ is sufficiently small. Note also that

$$\text{(B.2)}\qquad \partial_{x_j}\widetilde{\kappa^{-1}}(x,y)|_{y=x} = \partial_{y_j}\widetilde{\kappa^{-1}}(x,y)|_{y=x} = \frac{1}{2}\partial_{x_j}(\kappa^{-1})'(x), \quad j = 1,\dots,n.$$

Note that, for the operator A_∞, we can regularize its kernel by integration by parts and use the amplitude

$$a_\infty^{(k)}(x,y,\xi) = L^k a_\infty(x,y,\xi), \quad \text{with } L = \frac{i}{|x-y|^2}\sum_{i=1}^n (x_i - y_i)\partial_{\xi_i}, \quad k \in \mathbf{N}, \tag{B.3}$$

in place of $a_\infty(x,y,\xi)$.

By Proposition A.1 (with $k = 1$), the Weyl symbol of $A_{0,\kappa}$ is given by

$$\begin{aligned}\alpha_\kappa(x,\xi) &= e^{\frac{i}{2}(\langle D_y, D_\xi\rangle - \langle D_x, D_\xi\rangle)} a_{0,\kappa}(x,y,\xi)\Big|_{y=x} = \pi^{-n}\iint e^{2i\langle z,\zeta-\xi\rangle} a_{0,\kappa}(x+z, x-z, \zeta)\, dz\, d\zeta \\ &= \underbrace{a_{0,\kappa}(x,x,\xi)}_{=:\alpha_{\kappa,0}(x,\xi)} + \underbrace{\frac{i}{2}\,\langle \partial_x - \partial_y, \partial_\xi\rangle\, a_{0,\kappa}(x,y,\xi)\Big|_{y=x}}_{=:\alpha_{\kappa,1}(x,\xi)} \\ &\quad + \pi^{-n}\left(\frac{i}{2}\right)^2 \int_0^1 \iint e^{2i\langle z,\zeta-\xi\rangle}(1-r)(\langle \partial_x - \partial_y, \partial_\zeta\rangle^2 a_{0,\kappa})(x+rz, y-rz, \zeta)\, dr\, dz\, d\zeta\Big|_{y=x}.\end{aligned} \tag{B.4}$$

We now specialize to an amplitude $\tilde{a}(x,y,\xi)$ given by the Weyl quantization, i.e.,

$$a(x,y,\xi) = \chi(x)\chi(y)\, b((x+y)/2, \xi).$$

To simplify some notation we set $L = \kappa^{-1}$. The symbol $\alpha_{\kappa,0}(x,\xi)$ is then given by

$$\alpha_{\kappa,0}(x,\xi) = \chi(L(x))^2\, b(L(x), {}^t\kappa'(L(x))\xi). \tag{B.5}$$

Lemma B.1. *The symbol $\alpha_{\kappa,1}(x,\xi)$ is given by*

$$\alpha_{\kappa,1}(x,\xi) = \frac{i}{2}\chi(L(x))^2 \sum_{k=1}^n f_k(x)\,(\partial_{\xi_k} b)(L(x), {}^t\kappa'(L(x))\xi),$$

where

$$f_k(x) = \sum_{l=1}^n \partial_{x_l}(\kappa'_{lk}(L(x)) = \sum_{1\le m,l\le n} (\partial^2_{x_k,x_l}\kappa_m)(L(x))\,(\partial_{x_m} L_l)(x).$$

Proof. From the definition of $\alpha_{\kappa,1}$ in (B.4), and (B.1) we have

$$\begin{aligned}\alpha_{\kappa,1}(x,\xi) = \frac{i}{2}\langle \partial_x - \partial_y, \partial_\xi\rangle \Big(&\chi(L(x))\chi(L(y))\, b((L(x)+L(y))/2, {}^t(\widetilde{\kappa^{-1}}(x,y))^{-1}\xi) \\ &\times |\det(L)'(y)|\,|\det \widetilde{\kappa^{-1}}(x,y)|^{-1}\Big)\Big|_{y=x},\end{aligned}$$

where we have used that ζ is equal to one in a neighborhood of the origin. From (B.2), we see that we need not take into account the spatial differentiations acting on the terms ${}^t(\widetilde{\kappa^{-1}}(x,y))^{-1}$. Similarly the spatial differentiations acting on the cut-off functions $\chi(L(x))$ and $\chi(L(y))$ cancel each other, and so do the spatial differentiations acting on the first variable of the symbol b. Note also that the absolute values for the last two terms can be removed before differentiation since their product yields 1 in the case $y = x$. To simplify the notation we set $M = \widetilde{\kappa^{-1}}(x,x)$. We thus obtain

$$\alpha_{\kappa,1}(x,\xi) = -\frac{i}{2}\sum_{1\le j,k\le n} \chi(L(x))^2\,(\partial_{\xi_k} b)(L(x), {}^t\kappa'(L(x))\xi)\,(M^{-1})_{jk}\,(\partial_{x_j}\det(L'(x)))\,(\det M)^{-1}.$$

From the multi-linearity of the determinant we find that

$$(\partial_{x_j} \det(L'(x)))\,(\det M)^{-1} = \sum_{1\le p,l\le n} \partial_{x_j} L'_{pl}(x)\,\kappa'_{lp}(L(x)),$$

which yields

$$\begin{aligned} f_k(x) &= -\sum_{1\le j,p,l\le n} \kappa'_{jk}(L(x))\,\left(\partial_{x_j} L'_{pl}(x)\right)\,\kappa'_{lp}(L(x)) = -\sum_{1\le j,p,l\le n} \kappa'_{lp}(L(x))\,\left(\partial_{x_l} L'_{pj}(x)\right)\,\kappa'_{jk}(L(x)) \\ &= \sum_{l=1}^{n} \partial_{x_l}(\kappa'_{lk}(L(x))), \end{aligned}$$

since $\kappa'(L(x))\,L'(x) = \mathrm{Id}_{\tilde{X}}$. ■

B.2. **Application to the operator $\chi \circ p_h^w(x, D_x) \circ \chi$.** We use the notation introduced above. In the case $b = e^{-hq} = p_h$ then $a(x, y, \xi) = \chi(x)\chi(y)\,e^{-hq((x+y)/2,\xi)}$ is an amplitude for the operator $A = \chi \circ p_h^w(x, D_x) \circ \chi$ with Weyl symbol $\alpha = \chi \#^w p_h \#^w \chi$. Making use of the form of the amplitude $a_\infty^{(k)}$ in (B.3), we see that $A_{\infty,\kappa} = h\tilde{A}_{\infty,\kappa}$ with $\tilde{A}_{\infty,\kappa}$ in $\Psi^0(\tilde{X})$ uniformly in h, using Lemma 2.3.

We now focus on the operators A_0 and $A_{0,\kappa}$. From (B.5) and Lemma B.1, the expression of the remainder term in (B.4) and using Lemma 2.3 we obtain

$$\alpha_\kappa(x,\xi) = \chi(L(x))^2 p_h(L(x), {}^t\kappa'(L(x))\xi)\left(1 - h\frac{i}{2}\sum_{k=1}^{n} f_k(x)\,(\partial_{\xi_k} q)(L(x), {}^t\kappa'(L(x))\xi)\right) + h\tilde{\alpha}_\kappa, \tag{B.6}$$

with $\tilde{\alpha}_\kappa$ in S^0 uniformly in h. Similarly, if we denote by q_κ the Weyl symbol of $(\kappa^{-1})^* \circ q^w(x, D_x) \circ \kappa^*$, we have

$$q_\kappa(x,\xi) = q(L(x), {}^t\kappa'(L(x))\xi) + \frac{i}{2}\sum_{k=1}^{n} f_k(x)\,(\partial_{\xi_k} q)(L(x), {}^t\kappa'(L(x))\xi) + \tilde{q}_\kappa, \tag{B.7}$$

where $\tilde{q}_\kappa \in S^0$. We now prove that after the change of variables $x \mapsto \kappa(x)$, for the operator $\chi \circ p_h^w(x, D_x) \circ \chi = \alpha^w(x, D_x)$, we may use the symbol $\chi(L(x)) \#^w e^{-hq_\kappa(x,\xi)} \#^w \chi(L(x))$ in place of $\alpha_\kappa(x,\xi)$, the pullback of α in the Weyl quantization, yet remaining within a first-order precision w.r.t. to the small parameter h.

Lemma B.2. *We set $\hat{p}_h(x,\xi) = e^{-hq_\kappa(x,\xi)}$. We have*

$$\left(((\kappa^{-1})^*\chi) \#^w \hat{p}_h \#^w ((\kappa^{-1})^*\chi)\right)(x,\xi) - \alpha_\kappa(x,\xi) = h\lambda_h(x,\xi),$$

where λ_h is in S^0 uniformly in h.

Proof. We set $\nu(x,\xi) = \frac{i}{2}\sum_{k=1}^{n} f_k(x)(\partial_{\xi_k} q)(L(x), {}^t\kappa'(L(x))\xi)$. Making use of (B.7), we write

$$\begin{aligned} \hat{p}_h(x,\xi) &= p_h(L(x), {}^t\kappa'(L(x))\xi) e^{-h\nu(x,\xi)} e^{-h\tilde{q}_\kappa(x,\xi)} \\ &= p_h(L(x), {}^t\kappa'(L(x))\xi)\left(1 - h\nu(x,\xi) + (h\nu(x,\xi))^2 \int_0^1 e^{-rh\nu(x,\xi)}(1-r)\,dr\right)(1 + h\mu_1(x,\xi)), \end{aligned}$$

by two Taylor formulae, where μ_1 is in S^0 uniformly in h. From Lemmata 4.1 and 2.3 we obtain that

$$p_h(L(x), {}^t\kappa'(L(x))\xi)(h\nu(x,\xi))^2 \int_0^1 e^{-rh\nu(x,\xi)}(1-r)\,dr = h\mu_2(x,\xi),$$

with μ_2 in S^0 uniformly in h. From (B.6) we hence obtain

$$\alpha_\kappa(x,\xi) - \chi(L(x))^2\hat{p}_h(x,\xi) = h\mu_3,$$

with μ_3 in S^0 uniformly in h. We conclude the proof with the following lemma since $\hat{p}_h$ and p_h are of the same nature. ■

Lemma B.3. *Let $\phi \in \mathscr{C}_c^\infty(X)$. We then have*

$$\phi \#^w p_h \#^w \phi - \phi^2 p_h = h\lambda_h,$$

where λ_h is in S^0 uniformly in h.

Proof. Since $\phi(x)\phi(y)p_h((x+y)/2,\xi)$ is an amplitude for the operator with Weyl symbol $\phi \#^w p_h \#^w \phi$, by (B.4) we obtain

$$\begin{aligned}(\phi \#^w p_h \#^w \phi)(x,\xi) &= \pi^{-n} \iint e^{2i\langle z,\xi-\zeta\rangle}\phi(x-z)\,\phi(x+z)\,p_h(x,\zeta)\,dz\,d\zeta \\ &= \phi^2(x)p_h(x,\xi) - \frac{1}{4}\pi^{-n}\sum_{1\le j,k\le n}\int_0^1\iint (1-r)e^{2i\langle z,\xi-\zeta\rangle}\Big(-\phi(x+rz)\,\partial^2_{x_j,x_k}\phi(x-rz) \\ &\quad - 2\partial_{x_j}\phi(x+rz)\,\partial_{x_k}\phi(x-rz) + \partial^2_{x_j,x_k}\phi(x+rz)\,\phi(x-rz)\Big)\,\partial^2_{\xi_j,\xi_k}p_h(x,\zeta)\;dr\,dz\,d\zeta.\end{aligned}$$

We then conclude as in the proof of Proposition 2.4 in Appendix A by using Lemma 2.3 and Theorem 2.2.5 in [Kg81]. ■

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