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Claire Debord, Jean-Marie Lescure

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***K*-DUALITY FOR STRATIFIED PSEUDOMANIFOLDS**

CLAIRE DEBORD AND JEAN-MARIE LESCURE

ABSTRACT. This paper continues the project started in [13] where Poincaré duality in  $K$ -theory was studied for singular manifolds with isolated conical singularities. Here, we extend the study and the results to general stratified pseudomanifolds. We review the axiomatic definition of a smooth stratification  $S$  of a topological space  $X$  and we define a groupoid  $T^S X$ , called the  $S$ -tangent space. This groupoid is made of different pieces encoding the tangent spaces of strata, and these pieces are glued into the smooth noncommutative groupoid  $T^S X$  using the familiar procedure introduced by A. Connes for the tangent groupoid of a manifold. The main result is that  $C^*(T^S X)$  is Poincaré dual to  $C(X)$ , in other words, the  $S$ -tangent space plays the role in  $K$ -theory of a tangent space for  $X$ .

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## INTRODUCTION

This paper takes place in a longstanding project aiming to study index theory and related questions on stratified pseudomanifolds using tools and concepts from noncommutative geometry.

The key observation at the beginning of this project is that in  $K$ -theoretic form, the Atiyah-Singer index theorem [2] involves ingredients that should survive to the singularities allowed in a stratified pseudomanifold. This is possible, from our opinion, as soon as one accepts reasonable generalizations and new presentation of certain classical objects on smooth manifolds, making sense on stratified pseudomanifolds.

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The first instance of these classical objects that need to be adapted to singularities is the notion of *tangent space*. Since index maps in [2] are defined on the  $K$ -theory of the tangent spaces of smooth manifolds, one must have a similar space adapted to stratified pseudomanifolds. Moreover, such a space should satisfy natural attempts. It should coincide with the usual notion on the regular part of the pseudomanifold and incorporate in some way copies of usual tangent spaces of strata, while keeping enough smoothness to allow interesting computations. Moreover, it should be Poincaré dual in  $K$ -theory (shortly,  $K$ -dual) to the pseudomanifold itself. This  $K$ -theoretic property involves bivariant  $K$ -theory and was proved between smooth manifolds and their tangent spaces by G. Kasparov [17] and A. Connes-G. Skandalis [10].

In [13], we introduced a candidate to be the tangent space of a pseudomanifold with isolated conical singularities. It appeared to be a smooth groupoid, leading to a noncommutative  $C^*$ -algebra, and we proved that it fulfills the expected  $K$ -duality.

In [19], the second author interpreted the duality proved in [13] as a principal symbol map, thus recovering the classical picture of Poincaré duality in  $K$ -theory for smooth manifolds. This interpretation used a notion of noncommutative elliptic symbols, which appeared to be the cycles of the  $K$ -theory of the noncommutative tangent space.

In [5], the noncommutative tangent space together with other deformation groupoids was used to construct analytical and topological index maps, and their equality was proved. As expected, these index maps are straight generalizations of those of [2] for manifolds.

The present paper is devoted to the construction of the noncommutative tangent space for a general stratified pseudomanifold and the proof of the  $K$ -duality. It is thus a sequel of [13], but can be read independently. At first glance, one should have expected that the technics of [13] iterate easily to give the general result. In fact, if the definition of the groupoid giving the noncommutative tangent space itself is natural and intuitive in the general case, its smoothness is quite intricate and brings issues that did not exist in the conical case. We have given here a detailed treatment of this point, since we believe that this material will be useful in further studies about the geometry of stratified spaces. Another difference with [13] is that we have given up the explicit construction of a dual Dirac element. Instead, we use an easily defined Dirac element and then prove the Poincaré duality by an induction, based on a operation called unfolding which consists in removing the minimal strata in a pseudomanifold and then “doubling” it to get a new pseudomanifold, less singular. The difficulty in this approach is moved to the proof of the commutativity of certain diagrams in  $K$ -theory, necessary to apply the five lemma and to continue the induction.

The interpretation of this  $K$ -duality in terms of noncommutative symbols and pseudodifferential operators, as well as the construction of index maps together with the statement of an index theorem, is postponed to forthcoming papers.

This approach of index theory on singular spaces in the framework of noncommutative geometry takes place in a long history of past and present research works. But the specific issues about Poincaré duality, topological index maps and statement of Atiyah-Singer like theorems are quite recent and attract an increasing interest [26, 23, 30, 27].

## 1. BASIC DEFINITIONS

**1.1. Around Lie groupoids.** We refer to [29, 7, 20, 12] for the classical definitions and constructions related to groupoids, their Lie algebroids and groupoids  $C^*$ -algebras. In this section, we fix the notations and recall the less classical definitions and results needed in the sequel. Some material presented here is already in [13, 5].

**1.1.1. Pull back groupoids.**

Let  $G \rightrightarrows M$  be a locally compact Hausdorff groupoid with source  $s$  and range  $r$ . If  $f : N \rightarrow M$  is a surjective map, the *pull back* groupoid  ${}^*f^*(G) \rightrightarrows N$  of  $G$  by  $f$  is by definition the set

$${}^*f^*(G) := \{(x, \gamma, y) \in N \times G \times N \mid r(\gamma) = f(x), s(\gamma) = f(y)\}$$

with the structural morphisms given by

- (1) the unit map  $x \mapsto (x, f(x), x)$ ,
- (2) the source map  $(x, \gamma, y) \mapsto y$  and range map  $(x, \gamma, y) \mapsto x$ ,
- (3) the product  $(x, \gamma, y)(y, \eta, z) = (x, \gamma\eta, z)$  and inverse  $(x, \gamma, y)^{-1} = (y, \gamma^{-1}, x)$ .

The results of [28] apply to show that the groupoids  $G$  and  ${}^*f^*(G)$  are Morita equivalent when  $f$  is surjective, open and proper.

Let us assume for the rest of this subsection that  $G$  is a smooth groupoid and that  $f$  is a surjective proper submersion, then  ${}^*f^*(G)$  is also a Lie groupoid. Let  $(\mathcal{A}(G), q, [\cdot, \cdot])$  be the Lie algebroid of  $G$ . Recall that  $q : \mathcal{A}(G) \rightarrow TM$  is the anchor map. Let  $(\mathcal{A}({}^*f^*(G)), p, [\cdot, \cdot])$  be the Lie algebroid of  ${}^*f^*(G)$  and  $Tf : TN \rightarrow TM$  be the differential of  $f$ . Then we claim that there exists an isomorphism

$$\mathcal{A}({}^*f^*(G)) \simeq \{(V, U) \in TN \times \mathcal{A}(G) \mid Tf(V) = q(U) \in TM\}$$

under which the anchor map  $p : \mathcal{A}({}^*f^*(G)) \rightarrow TN$  identifies with the projection  $TN \times \mathcal{A}(G) \rightarrow TN$ . (In particular, if  $(V, U) \in \mathcal{A}({}^*f^*(G))$  with  $V \in T_x N$  and  $U \in \mathcal{A}_y(G)$ , then  $y = f(x)$ .)

**1.1.2. Subalgebras and exact sequences of groupoid  $C^*$ -algebras.**

To any smooth groupoid  $G$  is associated two  $C^*$ -algebras corresponding more or less to two different completions of the involutive convolution algebra  $C_c^\infty(G)$ , namely the reduced and maximal  $C^*$ -algebras [6, 8, 29]. We will denote respectively these  $C^*$ -algebras by  $C_r^*(G)$  and  $C^*(G)$ . Recall that the identity on  $C_c^\infty(G)$  induces a surjective morphism from  $C^*(G)$  onto  $C_r^*(G)$  which is an isomorphism if the groupoid  $G$  is amenable. Moreover in this case the  $C^*$  algebra of  $G$  is nuclear [1].

We will use the following usual notations:

Let  $G \begin{smallmatrix} \xrightarrow{s} \\ \xrightarrow{r} \end{smallmatrix} G^{(0)}$  be a smooth groupoid with source  $s$  and range  $r$ . If  $U$  is any subset of  $G^{(0)}$ , we let:

$$G_U := s^{-1}(U), \quad G^U := r^{-1}(U) \quad \text{and} \quad G_U^U = G|_U := G_U \cap G^U.$$

To an open subset  $O$  of  $G^{(0)}$  corresponds an inclusion  $i_O$  of  $C_c^\infty(G|_O)$  into  $C_c^\infty(G)$  which induces an injective morphism, again denoted by  $i_O$ , from  $C^*(G|_O)$  into  $C^*(G)$ .

When  $O$  is saturated,  $C^*(G|_O)$  is an ideal of  $C^*(G)$ . In this case,  $F := G^{(0)} \setminus O$  is a saturated closed subset of  $G^{(0)}$  and the restriction of functions induces a surjective

morphism  $r_F$  from  $C^*(G)$  to  $C^*(G|_F)$ . Moreover, according to [15], the following sequence of  $C^*$ -algebras is exact:

$$0 \longrightarrow C^*(G|_O) \xrightarrow{i_O} C^*(G) \xrightarrow{r_F} C^*(G|_F) \longrightarrow 0.$$

### 1.1.3. $KK$ -elements associated to deformation groupoids.

A smooth groupoid  $G$  is called a *deformation groupoid* if:

$$G = G_1 \times \{0\} \cup G_2 \times ]0, 1] \rightrightarrows G^{(0)} = M \times [0, 1],$$

where  $G_1$  and  $G_2$  are smooth groupoids with unit space  $M$ . That is,  $G$  is obtained by gluing  $G_2 \times ]0, 1] \rightrightarrows M \times ]0, 1]$ , which is the cartesian product of the groupoid  $G_2 \rightrightarrows M$  with the space  $]0, 1]$ , with the groupoid  $G_1 \times \{0\} \rightrightarrows M \times \{0\}$ .

In this situation one can consider the saturated open subset  $M \times ]0, 1]$  of  $G^{(0)}$ . Using the isomorphisms  $C^*(G|_{M \times ]0, 1]}) \simeq C^*(G_2) \otimes C_0(]0, 1])$  and  $C^*(G|_{M \times \{0\}}) \simeq C^*(G_1)$ , we obtain the following exact sequence of  $C^*$ -algebras:

$$0 \longrightarrow C^*(G_2) \otimes C_0(]0, 1]) \xrightarrow{i_{M \times ]0, 1]}} C^*(G) \xrightarrow{ev_0} C^*(G_1) \longrightarrow 0$$

where  $i_{M \times ]0, 1]}$  is the inclusion map and  $ev_0$  is the *evaluation map* at 0, that is  $ev_0$  is the map coming from the restriction of functions to  $G|_{M \times \{0\}}$ .

We assume now that  $C^*(G_1)$  is nuclear. Since the  $C^*$ -algebra  $C^*(G_2) \otimes C_0(]0, 1])$  is contractible, the long exact sequence in  $KK$ -theory shows that the group homomorphism  $(ev_0)_* = \cdot \otimes [ev_0] : KK(A, C^*(G)) \rightarrow KK(A, C^*(G_1))$  is an isomorphism for each  $C^*$ -algebra  $A$  [10].

In particular with  $A = C^*(G)$  we get that  $[ev_0]$  is invertible in  $KK$ -theory: there is an element  $[ev_0]^{-1}$  in  $KK(C^*(G_1), C^*(G))$  such that  $[ev_0] \otimes [ev_0]^{-1} = 1_{C^*(G)}$  and  $[ev_0]^{-1} \otimes [ev_0] = 1_{C^*(G_1)}$ .

Let  $ev_1 : C^*(G) \rightarrow C^*(G_2)$  be the evaluation map at 1 and  $[ev_1]$  the corresponding element of  $KK(C^*(G), C^*(G_2))$ .

The  $KK$ -element associated to the deformation groupoid  $G$  is defined by:

$$\delta = [ev_0]^{-1} \otimes [ev_1] \in KK(C^*(G_1), C^*(G_2)).$$

One can find examples of such elements related to index theory in [8, 15, 13, 5, 12].

**1.2. Generalities about  $K$ -duality.** We give in this paragraph some general facts about Poincaré duality in bivariant  $K$ -theory. Most of them are well known and proofs are only added when no self contained demonstration could be found in the literature. All  $C^*$ -algebras are assumed to be separable and  $\sigma$ -unital.

Let us first recall what means the Poincaré duality in  $K$ -theory [18, 10, 8]:

**Definition 1.** Let  $A, B$  be two  $C^*$ -algebras. One says that  $A$  and  $B$  are Poincaré dual, or shortly  $K$ -dual, when there exists  $\alpha \in K^0(A \otimes B) = KK(A \otimes B, \mathbb{C})$  and  $\beta \in KK(\mathbb{C}, A \otimes B) \simeq K_0(A \otimes B)$  such that

$$\beta \otimes_B \alpha = 1 \in KK(A, A) \text{ and } \beta \otimes_A \alpha = 1 \in KK(B, B)$$

Such elements are then called Dirac and dual-Dirac elements.

It follows that for  $A, B$  two  $K$ -dual  $C^*$ -algebras and for any  $C^*$ -algebras  $C, D$ , the following isomorphisms hold:

$$\beta \otimes_B \cdot : KK(B \otimes C, D) \longrightarrow KK(C, A \otimes D);$$

$$\beta \otimes_A \cdot : KK(A \otimes C, D) \longrightarrow KK(C, B \otimes D);$$

with inverses given respectively by  $\cdot \otimes_A \alpha$  and  $\cdot \otimes_B \alpha$ .

**Example 1.** A basic example is  $A = C(V)$  and  $B = C_0(T^*V)$  where  $V$  is a closed smooth manifold ([18, 10], see also [13] for a description of the Dirac element in terms of groupoids). This duality allows to recover that the usual quantification and principal symbol maps are mutually inverse isomorphisms in  $K$ -theory:

$$\Delta_V = (\cdot \otimes_{C_0(T^*V)} \alpha) : K_0(C_0(T^*V)) \xrightarrow{\cong} K^0(C(V))$$

$$\Sigma_V = (\beta \cdot \otimes_{C(V)} \cdot) : K^0(C(V)) \xrightarrow{\cong} K_0(C_0(T^*V))$$

We observe that:

**Lemma 2.** Let  $A, B$  be two  $C^*$ -algebras. Assume that there exists  $\alpha \in KK(A \otimes B, \mathbb{C})$  and  $\beta, \beta' \in KK(\mathbb{C}, A \otimes B)$  satisfying

$$\beta \otimes_B \alpha = 1 \in KK(A, A) \text{ and } \beta' \otimes_A \alpha = 1 \in KK(B, B)$$

Then  $\beta = \beta'$  so  $A, B$  are  $K$ -dual.

*Proof.* A simple calculation shows that for all  $x \in KK(C, A \otimes D)$  we have:

$$\beta \otimes_B (x \otimes_A \alpha) = x \otimes_{A \otimes B} (\beta \otimes_B \alpha).$$

Applying this to  $C = \mathbb{C}$ ,  $D = A$  and  $x = \beta'$  we get:

$$\beta' = \beta \otimes_B (\beta' \otimes_A \alpha) = \beta \otimes_B 1 = \beta$$

□

This implies in particular that given two  $K$ -dual  $C^*$ -algebras and a Dirac element  $\alpha$ , the dual-Dirac element  $\beta$  satisfying the definition 1 is unique. The lemma implies also that if there exists  $\alpha \in KK(A \otimes B, \mathbb{C})$  such that

$$\cdot \otimes_B \alpha : KK(\mathbb{C}, A \otimes B) \longrightarrow KK(A, A) \text{ and } \cdot \otimes_A \alpha : KK(\mathbb{C}, A \otimes B) \longrightarrow KK(B, B)$$

are onto, then  $A, B$  are  $K$ -dual.

## 2. STRATIFIED PSEUDOMANIFOLDS

We are interested in studying stratified pseudomanifolds [32, 21, 14]. We will use the notations and equivalent descriptions given by A. Verona in [31] or used by J.P. Brasselet, G. Hector and M. Saralegi in [4]. The reader should also look at [16] for a hepfull survey of the subject.

### 2.0.1. Definitions.

Let  $X$  be a locally compact separable metrizable space.

**Definition 1.** A  $C^\infty$ -stratification of  $X$  is a pair  $(S, N)$  such that:

- (1)  $S = \{s_i\}$  is a locally finite partition of  $X$  into locally closed subsets of  $X$ , called the strata, which are smooth manifolds and which satisfies:

$$s_0 \cap \bar{s}_1 \neq \emptyset \text{ if and only if } s_0 \subset \bar{s}_1.$$

In that case we will write  $s_0 \leq s_1$  and  $s_0 < s_1$  if moreover  $s_0 \neq s_1$ .

- (2)  $N = \{\mathcal{N}_s, \pi_s, \rho_s\}_{s \in \mathcal{S}}$  is the set of control data or tube system:  
 $\mathcal{N}_s$  is an open neighborhood of  $s$  in  $X$ ,  $\pi_s : \mathcal{N}_s \rightarrow s$  is a continuous retraction and  $\rho_s : \mathcal{N}_s \rightarrow [0, +\infty[$  is a continuous map such that  $s = \rho_s^{-1}(0)$ . The map  $\rho_s$  is either surjective or constant equal to 0.  
 Moreover if  $\mathcal{N}_{s_0} \cap \mathcal{N}_{s_1} \neq \emptyset$  then the map

$$(\pi_{s_0}, \rho_{s_0}) : \mathcal{N}_{s_0} \cap \mathcal{N}_{s_1} \rightarrow s_0 \times ]0, +\infty[$$

is a smooth submersion.

- (3) For any strata  $s, t$  such that  $s < t$ , the inclusion  $\pi_t(\mathcal{N}_s \cap \mathcal{N}_t) \subset \mathcal{N}_s$  is true and the equalities:

$$\pi_s \circ \pi_t = \pi_s \quad \text{and} \quad \rho_s \circ \pi_t = \rho_s$$

hold on  $\mathcal{N}_s \cap \mathcal{N}_t$ .

- (4) For any two strata  $s_0$  and  $s_1$  the following equivalences hold:

$$s_0 \cap \bar{s}_1 \neq \emptyset \quad \text{if and only if} \quad \mathcal{N}_{s_0} \cap \mathcal{N}_{s_1} \neq \emptyset,$$

$$\mathcal{N}_{s_0} \cap \mathcal{N}_{s_1} \neq \emptyset \quad \text{if and only if} \quad s_0 \subset \bar{s}_1, \quad s_0 = s_1 \quad \text{or} \quad s_1 \subset \bar{s}_0.$$

A stratification gives rise to a filtration: let  $X_j$  be the union of strata of dimension  $\leq j$ , then:

$$\emptyset \subset X_0 \subset \cdots \subset X_n = X.$$

We call  $n$  the *dimension* of  $X$  and  $X^\circ := X \setminus X_{n-1}$  the *regular* part of  $X$ . The strata included in  $X^\circ$  are called *regular* while strata included in  $X \setminus X^\circ$  are called *singular*. The set of singular (resp. regular) strata is denoted  $\mathcal{S}_{sing}$  (resp.  $\mathcal{S}_{reg}$ ). For any subset  $A$  of  $X$ ,  $A^\circ$  will denote  $A \cap X^\circ$ .

A crucial notion for our purpose will be the notion of *depth*. Observe that the binary relation  $s_0 \leq s_1$  is a partial ordering on  $\mathcal{S}$ .

**Definition 2.** The *depth*  $d(s)$  of a stratum  $s$  is the biggest  $k$  such that one can find  $k$  different strata  $s_0, \dots, s_{k-1}$  such that

$$s_0 < s_1 < \cdots < s_{k-1} < s_k := s.$$

The *depth* of the stratification  $(\mathcal{S}, N)$  of  $X$  is:

$$d(X) := \sup\{d(s), s \in \mathcal{S}\}.$$

A stratum whose depth is 0 will be called *minimal*.

We have followed the terminology of [4], but remark that the opposite convention for the depth also exists [31].

Finally we can define stratified pseudomanifolds:

**Definition 3.** A *stratified pseudomanifold* is a triple  $(X, \mathcal{S}, N)$  where  $X$  is a locally compact separable metrizable space,  $(\mathcal{S}, N)$  is a  $C^\infty$ -stratification on  $X$  and the regular part  $X^\circ$  is a dense open subset of  $X$ .

If  $(X, \mathcal{S}_X, N_X)$  and  $(Y, \mathcal{S}_Y, N_Y)$  are two stratified pseudomanifolds an homeomorphism  $f : X \rightarrow Y$  is an isomorphism of stratified pseudomanifold if:

- (1)  $\mathcal{S}_Y = \{f(s), s \in \mathcal{S}_X\}$  and the restriction of  $f$  to each stratum is a diffeomorphism onto its image.
- (2)  $\pi_{f(s)} \circ f = f \circ \pi_s$  and  $\rho_s = \rho_{f(s)} \circ f$  for any stratum  $s$  of  $X$ .

Let us make some basic remark on the previous definitions.

- Remark 1.** (1) *At a first sight, the definition of a stratification given here seems more restrictive than the usual one. In fact according to [31] these definitions are equivalent.*
- (2) *Usually, for example in [14], the extra assumption  $X_{n-1} = X_{n-2}$  is required in the definition of stratified pseudomanifold. Our constructions remain without this extra assumption.*
- (3) *A stratum  $s$  is regular if and only if  $\mathcal{N}_s = s$  and then  $\rho_s = 0$ .*
- (4) *Pseudomanifolds of depth 0 are smooth manifolds, and the strata are then union of connected components.*

The following simple consequence of the axioms will be usefull enough in the sequel to be pointed out:

**Proposition 1.** *Let  $(X, \mathbf{S}, N)$  be a stratified pseudomanifold. Any subset  $\{s_i\}_I$  of distinct elements of  $\mathbf{S}$  is totally ordered by  $<$  as soon as the intersection  $\bigcap_{i \in I} \mathcal{N}_{s_i}$  is non empty. In particular if the strata  $s_0$  and  $s_1$  are such that  $\mathcal{N}_{s_0} \cap \mathcal{N}_{s_1} \neq \emptyset$  then  $d(s_0) \neq d(s_1)$  or  $s_0 = s_1$ .*

By a slight abuse of language we will sometime talk about a stratified pseudomanifold  $X$  while we only have a partition  $\mathbf{S}$  on the space  $X$ . This means that one can find at least one control data  $N$  such that  $(X, \mathbf{S}, N)$  is a stratified pseudomanifold in the sense of our definition 3.

### 2.0.2. Examples.

- (1) Smooth manifolds are, without other mention, pseudomanifolds of depth 0 and with a single stratum.
- (2) Stratified pseudomanifolds of depth one are *wedges* and are obtained as follows. Take  $M$  to be a manifold with a compact boundary  $L$  and let  $\pi$  be a surjective submersion of  $L$  onto a manifold  $s$ . Consider the *mapping cone* of  $(L, \pi)$  :

$$c_\pi L := L \times [0, 1] / \sim_\pi$$

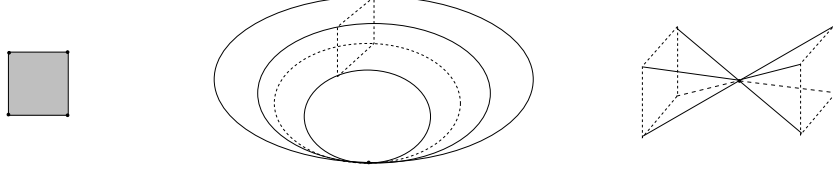
where  $(z, t) \sim_\pi (z', t')$  if and only if  $(z, t) = (z', t')$  or  $t = t' = 0$  and  $\pi(z) = \pi(z')$ . The image of  $L \times \{0\}$  identifies with  $s$  and by a slight abuse of notation we will denote it  $s$ . Now glue  $c_\pi L$  and  $M$  along their boundary in order to get  $X$ . The space  $X$  with the partition  $\{s, X \setminus s\}$  is a stratified pseudomanifold.

Two extreme examples are obtained by considering  $\pi$  either equal to identity, with  $s = L$  or equal to the projection on one point  $c$ . In the first case  $X$  is a manifold with boundary  $L$  isomorphic to  $M$  and the stratification corresponds to the partition of  $X$  by  $\{L, X \setminus L\}$ . In the second case  $X$  is a *conical manifold* and the stratification corresponds to the partition of  $X$  by  $\{c, X \setminus c\}$ , where  $c$  is the singular point.

- (3) Manifolds with corners with their partition into faces are stratified pseudomanifolds [22, 24].
- (4) If  $(X, \mathbf{S}, N)$  is a pseudomanifold and  $M$  is a smooth manifold then  $X \times M$  is naturally endowed with a structure of pseudomanifold of same depth as  $X$  whose strata are  $\{s \times M, s \in \mathbf{S}\}$ .
- (5) If  $(X, \mathbf{S}, N)$  is a pseudomanifold of depth  $k$  then  $CrX := X \times S^1 / X \times \{p\}$  is naturally endowed with a structure of pseudomanifold of depth  $k + 1$ , whose strata are  $\{s \times ]0, 1[, s \in \mathbf{S}\} \cup \{[p]\}$ . Here we have identified  $S^1 \setminus \{p\}$  with  $]0, 1[$  and we have denoted by  $[p]$  the image of  $X \times \{p\}$  in  $CrX$ .



For example, if  $X$  is the square we get the following picture:



### 2.0.3. The unfolding process.

Let  $(X, \mathbf{S}, N)$  be a stratified pseudomanifold. If  $s$  is a singular stratum, we let  $L_s := \rho_s^{-1}(1)$ . Then  $L_s$  inherits from  $X$  a structure of stratified pseudomanifold.

One can then define the *open mapping cone* of  $(L_s, \pi_s)$ :

$$c_{\pi_s} L_s := L_s \times [0, +\infty[ / \sim_{\pi_s}$$

where  $\sim_{\pi_s}$  is as before.

According to [31], see also [4] the open mapping cone is naturally endowed with a structure of stratified pseudomanifold whose strata are  $\{(t \cap L_s) \times ]0, +\infty[, t \in \mathbf{S}\} \cup \{s\}$ . Here we identify  $s$  with the image of  $L_s \times \{0\}$  in  $c_{\pi_s} L_s$ . Moreover, up to isomorphism, the control data on  $X$  can be chosen such that one can find a continuous retraction  $f_s : \mathcal{N}_s \setminus s \rightarrow L_s$  for which the map

$$(2.1) \quad \Psi_s : \mathcal{N}_s \rightarrow \begin{cases} c_{\pi_s} L_s & \\ [f_s(z), \rho_s(z)] & \text{if } z \notin s \\ z & \text{elsewhere} \end{cases}$$

is an isomorphism of stratified pseudomanifolds. Here  $[y, t]$  denotes the class in  $c_{\pi_s} L_s$  of  $(y, t) \in L_s \times [0, +\infty[$ .

This result of local triviality around strata will be crucial for our purpose. In particular it enables one to make the unfolding process [4] which consists in replacing each minimal stratum  $s$  by  $L_s$ . Precisely suppose that  $d(X) = k > 0$  and let  $\mathbf{S}_0$  be the set of strata of depth 0. Define  $O_0 := \cup_{s \in \mathbf{S}_0} \{z \in X \mid \rho_s(z) < 1\}$ ,  $X_b = X \setminus O_0$  and  $L := \cup_{s \in \mathbf{S}_0} \{z \in X \mid \rho_s(z) = 1\} \subset X_b$ . Notice that it follows from remark 1 that the  $L_s$ 's where  $s \in \mathbf{S}_0$  are disjoint and thus  $L = \sqcup_{s \in \mathbf{S}_0} L_s$ . We let

$$2X = X_b^- \cup L \times [-1, 1] \cup X_b^+$$

where  $X_b^\pm = X_b$  and  $X_b^-$  (respectively  $X_b^+$ ) is glued along  $L$  with  $L \times \{-1\} \subset L \times [-1, 1]$  (respectively  $L \times \{1\} \subset L \times [-1, 1]$ ).

Let  $s$  be a stratum of  $X$  which is not minimal and which encounter  $O_0$ . We define the following subset of  $2X$ :

$$\tilde{s} := (s \cap X_b^-) \cup (s \cap L) \times [-1, 1] \cup (s \cap X_b^+)$$

We then define

$$\mathbf{S}_{2X} := \{\tilde{s}; s \in \mathbf{S} \text{ and } s \cap O_0 \neq \emptyset\} \cup \{s^-, s^+; s^\pm = s \in \mathbf{S} \text{ and } s \cap O_0 = \emptyset\}.$$

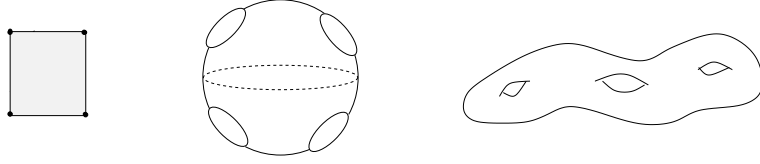
The space  $2X$  inherits from  $X$  a structure of stratified pseudomanifold of depth  $k - 1$  whose set of strata is  $\mathbf{S}_{2X}$ .

Notice that there is a natural map  $p$  from  $2X$  onto  $X$ . The restriction  $p$  to any copy of  $X_b$  is identity and for  $(z, t) \in L_s \times [-1, 1]$ ,  $p(z, t) = \Psi_s^{-1}([z, |t|])$ . The strata of  $2X$  are the connected component of the preimage by  $p$  of the strata of  $X$ .

The interested reader can find all the details related to the unfolding process in [4] and [31] where it is called decomposition. In particular starting with a compact

pseudomanifold  $X$  of depth  $k$ , one can iterate this process  $k$  times and obtain a compact smooth manifold  $2^k X$  together with a continuous surjective map  $\pi : 2^k X \rightarrow X$  whose restriction to  $\pi^{-1}(X^\circ)$  is a trivial  $2^k$ -fold covering.

**Example 2.** Look at the square  $C$  with stratification given by its vertices, edges and its interior. It can be endowed with a structure of stratified pseudomanifold of depth 2. Applying once the unfolding process gives a sphere with 4 holes:  $S := S^2 \setminus \{D_1, D_2, D_3, D_4\}$  where the  $D_i$ 's are disjoint and homeomorphic to open disks. The set of strata of  $S^2$  is then  $\{\overset{\circ}{S}, S_1, S_2, S_3, S_4\}$  where  $S_i$  is the boundary of  $D_i$  and  $\overset{\circ}{S}$  the interior of  $S$ . Applying the unfolding process once more gives the torus with three holes.



### 3. THE TANGENT GROUPOID AND $\mathcal{S}$ -TANGENT SPACE OF A COMPACT STRATIFIED PSEUDO-MANIFOLD

**3.1. The set construction.** We begin by the description at the set level of the  $\mathcal{S}$ -tangent groupoid and the  $\mathcal{S}$ -tangent space of a compact stratified pseudo-manifold.

We keep the notation of the previous section:  $X$  is a compact stratified pseudo-manifold,  $\mathcal{S}$  the set of strata,  $X^\circ$  the regular part and  $N = \{\mathcal{N}_s, \pi_s, \rho_s\}_{s \in \mathcal{S}}$  the set of control data.

For each  $s \in \mathcal{S}$  we let

$$O_s := \{z \in \mathcal{N}_s \mid \rho_s(z) < 1\} \text{ and } F_s := O_s \setminus \bigcup_{s_0 < s} O_{s_0} .$$

Note that  $F_s = O_s$  if and only if  $s$  is a minimal stratum and  $O_s = s$  when  $s$  is regular.

**Lemma 3.** The set  $\{F_s\}_{s \in \mathcal{S}}$  form a partition of  $X$ .

*Proof.* If  $z$  belongs to  $X$ , let  $R_z := \{s \in \mathcal{S} \mid \rho_s(z) < 1\}$ . It follows from proposition 1 that  $R_z$  is a countable set totally ordered by  $<$ . Let  $s_0^z$  be the minimal element of  $R_z$ . Then  $z$  belongs to  $F_{s_0^z}$ . Thus  $X \subset \bigcup_{s \in \mathcal{S}} F_s$ .

If  $z \in F_s \cap F_{s'}$  then  $s < s'$ ,  $s = s'$  or  $s' < s$ . Suppose that  $s' < s$ , since  $z \in F_{s'} \subset O_{s'}$  we have  $z \in \bigcup_{s_0 < s'} O_{s_0}$  which is incompatible with  $z \in F_s$ . Thus  $F_s \cap F_{s'} = \emptyset$  or  $s = s'$ .  $\square$

Recall that  $O_s^\circ = O_s \cap X^\circ$ . We denote again by  $\pi_s : O_s^\circ \rightarrow s$  the projection. When  $s$  is a stratum,  $\pi_s$  is a proper submersion and one can consider the pull-back groupoid  ${}^* \pi_s^*(Ts) \rightrightarrows O_s^\circ$  of the usual tangent space  $Ts \rightrightarrows s$  by  $\pi_s$ . It is naturally endowed with a structure of smooth groupoid. When  $s$  is a regular stratum,  $s = O_s = O_s^\circ$  and  $\pi_s$  is the identity map, thus  ${}^* \pi_s^*(Ts) \simeq TO_s^\circ$  in a canonical way.

At the set level, the  $\mathcal{S}$ -tangent space of  $X$  is the groupoid:

$$T^{\mathcal{S}}X = \bigcup_{s \in \mathcal{S}} {}^* \pi_s^*(Ts)|_{F_s^\circ} \rightrightarrows X^\circ$$

where  $F_s^\circ = F_s \cap X^\circ$ . Following the cases of smooth manifolds [8] and isolated conical singularities [13], the *S-tangent groupoid* of  $X$  is defined to be a deformation of the pair groupoid of the regular part of  $X$  onto its S-tangent space:

$$\mathcal{G}_X^t := T^S X \times \{0\} \cup X^\circ \times X^\circ \times ]0, 1] \rightrightarrows X^\circ \times [0, 1].$$

- Examples 4.** (1) When  $X$  has depth 0, we recover the usual tangent space and tangent groupoid.  
(2) Suppose that  $X$  is a trivial wedge (see example 2.0.2):

$$X = c_\pi L \cup M$$

where  $M$  is a manifold with boundary  $L$  and  $L$  is the product of two manifolds  $L = s \times Q$  with  $\pi : L \rightarrow s$  being the first projection. We have denoted by  $c_\pi L = L \times [0, 1] / \sim_\pi$  the mapping cone of  $(L, \pi)$ . In other word  $c_\pi L = s \times cQ$  where  $cQ := Q \times [0, 1] / Q \times \{0\}$  is the cone over  $Q$ . We denote again by  $s$  the image of  $L \times \{0\}$  in  $X$ . Then  $X$  admits two strata:  $s$  and  $X^\circ = X \setminus s$ ,  $F_s = O_s = L \times ]0, 1[$  and  $F_{X^\circ} = X^\circ \setminus O_s = M$ . The tangent space is

$$T^S X = Ts \times (Q \times ]0, 1]) \times (Q \times ]0, 1]) \sqcup TM \rightrightarrows X^\circ$$

where  $Ts \times (Q \times ]0, 1]) \times (Q \times ]0, 1])$  is the product of the tangent space  $Ts \rightrightarrows s$  with the pair groupoid over  $Q \times ]0, 1[$  and  $TM$  denotes the restriction of the usual tangent bundle  $TX^\circ$  to the sub-manifold with boundary  $M$ .

**Remark 5.** For any stratum  $s$ , the restriction of  $\mathcal{G}_X^t$  to  $F_s^\circ$  is equal to

$$*\pi_s^*(TS)|_{F_s^\circ} \times \{0\} \cup F_s^\circ \times F_s^\circ \times ]0, 1] \rightrightarrows F_s^\circ \times [0, 1]$$

which is also the restriction to  $F_s^\circ$  of  $*(\pi_s \times \text{Id})^*(\mathcal{G}_s^t)$ , the pull-back by  $\pi_s \times \text{Id} : O_s^\circ \times [0, 1] \rightarrow S \times [0, 1]$  of the (usual) tangent groupoid of  $s$ :

$$\mathcal{G}_s^t = Ts \times \{0\} \cup s \times s \times ]0, 1] \rightrightarrows s \times [0, 1].$$

In the following, we will denote by  $\mathcal{A}_{\pi_s \times \text{Id}}^t$  the Lie algebroid of  $*(\pi_s \times \text{Id})^*(\mathcal{G}_s^t)$ .

**3.2. The Recursive construction.** Thanks to the unfolding process described in 2.0.3, one can also construct the S-tangent spaces of stratified pseudomanifolds by an induction on the depth.

If  $X$  is of depth 0, it is a smooth manifold and the S-tangent space is the usual tangent space  $TX$  viewed as a groupoid on  $X$ .

Let  $k$  be an integer and assume that the S-tangent space of any pseudomanifold of depth smaller than  $k$  is defined. Let  $X$  be a stratified pseudomanifold of depth  $k+1$  and take  $2X$  be the stratified pseudomanifold of depth  $k$  obtained from  $X$  by applying 2.0.3. With the notations of 2.0.3 we define

$$T^S X = T^S 2X|_{2X^\circ \cap X_b^+} \cup_{s \in S_0} *\pi_s^*(Ts)|_{O_s^\circ} \rightrightarrows X^\circ$$

where  $T^S 2X$  is the S-tangent space of the stratified pseudomanifold  $2X$ . Here we have identified  $2X^\circ \cap X_b^+$  with the subset  $X^\circ \setminus O_0 = X_b \cap X^\circ$  of  $X^\circ$ . It is a simple exercise to see that this construction leads to the same definition of S-tangent space as the previous one.

**3.3. The smooth structure.** In this subsection we prove that the  $\mathbf{S}$ -tangent space of a stratified pseudomanifold, as well as its  $\mathbf{S}$ -tangent groupoid, can be endowed with a smooth structure which reflects the local structure of the pseudomanifold itself.

Let  $(X, \mathbf{S}, N)$  be a stratified pseudomanifold. The smooth structure of  $T^{\mathbf{S}}X$  will depend on the stratification and a smooth, decreasing, positive function  $\tau : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\tau([0, +\infty[) = [0, 1]$ ,  $\tau^{-1}(0) = [1, +\infty[$  and  $\tau'$  does not vanish on  $]0, 1[$ . The function  $\tau$  will be called a *gluing function* and can be considered as a part of the stratification. we will also use functions associated with  $\tau$  and defined on  $\mathcal{N}_s$  for any singular stratum  $s$  by: for each singular stratum:

$$\tau_s = \tau \circ \rho_s$$

Observe that  $\tau_s = 0$  outside  $O_s^\circ$ .

Before coming into the details of the smooth structure of  $T^{\mathbf{S}}X$ , let us describe its consequences for the convergence of sequences:

A sequence  $(x_n, V_n, y_n) \in {}^*\pi_{s_n}^*(Ts_n)|_{F_{s_n}}$  where  $n$  belongs to  $\mathbb{N}$ , goes to  $(x, V, y) \in {}^*\pi_s^*(Ts)|_{F_s}$  if and only if:

$$(3.1) \quad x_n \rightarrow x, \quad y_n \rightarrow y, \quad V_n + \frac{\pi_s(x_n) - \pi_s(y_n)}{\tau_{s_n}(x_n)} \rightarrow V$$

The first two convergences have an obvious meaning, and they imply that for  $n$  big enough,  $s_n \leq s$ . The third one needs some explanations. Let us note  $z = \pi_s(x) = \pi_s(y)$  and  $z_n = \pi_{s_n}(x_n) = \pi_{s_n}(y_n)$ . Since  $\pi_s(x_n)$  and  $\pi_s(y_n)$  become close to  $z$ , we can interpret  $w_n = \pi_s(x_n) - \pi_s(y_n)$  as a vector in  $T_{\pi_s(y_n)}s$  (use any local chart of  $s$  around  $z$ ). Moreover, using  $\pi_{s_n} \circ \pi_s = \pi_{s_n}$ , we see that this vector  $w_n$  is vertical for  $\pi_{s_n}$ , that is, belongs to the kernel  $\mathbf{K}_n$  of the differential of  $\pi_{s_n}$  (suitably restricted to  $s_n$ ). Now, the meaning of last convergence in (3.1) is  $T\pi_{s_n}(V - w_n/\tau_{s_n}(x_n)) - V_n \rightarrow 0$  which has to be interpreted for each subsequences of  $(x_n, V_n, y_n)_n$  with  $s_n = s_{n_0}$  for all  $n \geq n_0$  big enough.

The smooth structure of  $T^{\mathbf{S}}X$  will be obtained by an induction on the depth of the stratification, and a concrete atlas will be given. For the sake of completeness, we also explicit a Lie algebroid whose integration gives the tangent groupoid  $\mathcal{G}_X^t$ . We begin by describing the local structure of  $X^\circ$  around its strata, then we will prove inductively the existence of a smooth structure on the  $\mathbf{S}$ -tangent space. Next, an atlas of the resulting smooth structure is given by brut computations. A similar construction is easy to guess for the tangent groupoid  $\mathcal{G}_X^t$ . In the last part the previous smooth structure is recovered in a more abstract approach using an integrable Lie algebroid.

These parts are quite technical and can be left out as soon as you believe that the tangent space and the tangent groupoid can be endowed with a smooth structure compatible with the topology described above.

**3.3.1. The local structure of  $X^\circ$ .** We are describing now local charts of  $X^\circ$  adapted to the stratification, called *distinguished charts*.

Let  $z \in X^\circ$  and consider the set

$$\mathbf{S}_z := \{s \in \mathbf{S} \mid \rho_s(z) \leq 1\}$$

It is a non empty finite set, totally ordered according to proposition 1, thus we can write

$$\mathcal{S}_z = \{s_0, \dots, s_\kappa\}, \quad s_0 < s_1 < \dots < s_\kappa$$

where  $s_\kappa \subset X^\circ$  must be regular. Let  $n_i$  be the dimension of  $s_i$ ,  $i \in \{0, 1, \dots, \kappa\}$  and  $n = n_\kappa = \dim X^\circ$ .

Let  $U_z$  be an open neighborhood of  $z$  in  $X^\circ$  such that the following hold:

$$(3.2) \quad U_z \subset \bigcap_{s \in \mathcal{S}_z} \mathcal{N}_s \quad \text{and} \quad \forall s \in \mathcal{S}_{\text{sing}}, U_z \cap O_s \neq \emptyset \Leftrightarrow s \in \mathcal{S}_z$$

In particular, the following hold on  $U_z$ :

$$(3.3) \quad \text{for } 0 \leq i \leq j \leq \kappa : \pi_{s_i} \circ \pi_{s_j} = \pi_{s_i} \text{ and } \rho_{s_i} \circ \pi_{s_j} = \rho_{s_i} .$$

Without loss of generality, we can also assume that  $U_z$  is the domain of a local chart of  $X^\circ$ .

If  $\kappa = 0$ , any local chart of  $X^\circ$  with domain  $U_z$  will be called distinguished. When  $\kappa \geq 1$ , we can take successively canonical forms of the submersions  $\pi_{s_0}, \pi_{s_1}, \dots, \pi_{s_\kappa}$  available on a possibly smaller  $U_z$ , that is, one can shrink  $U_z$  enough and find diffeomorphisms:

$$(3.4) \quad \phi_i : \pi_{s_i}(U_z) \rightarrow \mathbb{R}^{n_i} \text{ for all } i \in \{0, 1, \dots, \kappa\}$$

such that the diagram:

$$(3.5) \quad \begin{array}{ccc} \pi_{s_i}(U_z) & \xrightarrow{\phi_i} & \mathbb{R}^{n_i} \\ \pi_{s_j} \downarrow & & \downarrow \sigma_{n_j} \\ \pi_{s_j}(U_z) & \xrightarrow{\phi_j} & \mathbb{R}^{n_j} \end{array}$$

commutes for all  $i, j \in \{0, 1, \dots, \kappa\}$  such that  $i \geq j$ . Above, for any integers  $p \geq d$ , the map  $\sigma_d : \mathbb{R}^p \rightarrow \mathbb{R}^d$  denotes the canonical projection onto the last  $d^{\text{th}}$  coordinates.

Remember that  $s_\kappa$  is regular so  $\pi_{s_\kappa}$  is the identity map and  $\phi := \phi_\kappa$  is a local chart around  $z$  of  $X^\circ$ . Now we set:

**Definition 6.** A *distinguished chart* of  $X^\circ$  around  $z \in X^\circ$  is a local chart  $(U_z, \phi)$  around  $z$  such that  $U_z$  satisfies (3.2) together with diffeomorphisms (3.4) satisfying (3.5) and  $\phi = \phi_\kappa$ .

From now on, a riemannian metric is chosen on  $X^\circ$  (any adapted metric in the sense of [3] is suitable for our purpose). Recall that for any stratum  $s$ , the map  $\pi_s : \mathcal{N}_s^\circ \rightarrow s$  is a smooth submersion. Thus, if  $K_s \subset T\mathcal{N}_s^\circ$  denotes the kernel of the differential map  $T\pi_s$  and  $q_s : T\mathcal{N}_s^\circ \rightarrow T\mathcal{N}_s^\circ$  the orthogonal projection on  $K_s$ , the map

$$(3.6) \quad (q_s, T\pi_s) : T\mathcal{N}_s^\circ \rightarrow K_s \oplus \pi_s^*(Ts)$$

is an isomorphism and the vector bundle  $\pi_s^*(Ts)$  can be identified with the orthogonal complement of  $K_s$  into  $T\mathcal{N}_s = TX^\circ|_{\mathcal{N}_s}$ .

Now, let  $(U_z, \phi)$  be a distinguished chart around some  $z \in X^\circ$ . Set  $\mathcal{S}_z = \{s_0, s_1, \dots, s_\kappa\}$  with  $s_0 < s_1 < \dots < s_\kappa$ , and set  $K_i = K_{s_i}|_{U_z}$ ,  $U_i = \pi_{s_i}(U_z)$  for all  $i = 0, 1, \dots, \kappa$ .

By 3.3 we have:

$$(3.7) \quad U_z \times \{0\} = K_\kappa \subset K_{\kappa-1} \subset \dots \subset K_1 \subset K_0 \subset TU_z.$$

Rewriting the diagram (3.5) for the differential maps and  $i = \kappa$ , we get for all  $j \leq \kappa$ :

$$(3.8) \quad \begin{array}{ccc} TU_z & \xrightarrow{T\phi} & \mathbb{R}^n \times \mathbb{R}^n \\ T\pi_{s_j} \downarrow & & \downarrow \sigma_{n_j} \times \sigma_{n_j} \\ TU_j & \xrightarrow{T\phi_j} & \mathbb{R}^{n_j} \times \mathbb{R}^{n_j} \end{array}$$

and we see that  $T\phi$  sends the filtration (3.7) to the following filtration:

$$(3.9) \quad \mathbb{R}^n \times \{0\} \subset \mathbb{R}^n \times \mathbb{R}^{n-n_{\kappa-1}} \subset \dots \subset \mathbb{R}^n \times \mathbb{R}^{n-n_0} \subset \mathbb{R}^n \times \mathbb{R}^n,$$

where  $\mathbb{R}^{n-n_i}$  is included in  $\mathbb{R}^n$  by the map  $v \mapsto (v, 0) \in \mathbb{R}^{n-n_i} \times \mathbb{R}^{n_i} \simeq \mathbb{R}^n$ . This property can be reformulated in terms of natural graduations associated with (3.7) and (3.9) (and will be used in this latter form). Indeed, let  $T^i$  be the orthogonal complement of  $K_i$  into  $K_{i-1}$  for all  $i = 0, \dots, \kappa$  (with the convention  $K_{-1} = TU_z$ ). Moreover, on the euclidean side, let us embed  $\mathbb{R}^{n_i-n_{i-1}}$  into  $\mathbb{R}^n$  by the map:

$$v \in \mathbb{R}^{n_i-n_{i-1}} \longmapsto (0, v, 0) \in \mathbb{R}^{n-n_i} \times \mathbb{R}^{n_i-n_{i-1}} \times \mathbb{R}^{n_{i-1}} \simeq \mathbb{R}^n$$

for all  $i = 0, 1, \dots, \kappa$  (by convention  $n_{-1} = 0$ ). With these notations and conventions, the filtrations (3.7) and (3.9) give rise to the following decompositions:

$$(3.10) \quad TU_z = T^\kappa \oplus T^{\kappa-1} \oplus \dots \oplus T^0$$

and

$$(3.11) \quad \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^n \times (\mathbb{R}^{n-n_\kappa} \oplus \mathbb{R}^{n_\kappa-n_{\kappa-1}} \oplus \dots \oplus \mathbb{R}^{n_1-n_0} \oplus \mathbb{R}^{n_0})$$

Now, that  $T\phi$  respects the filtrations (3.7) and (3.9) means that for all  $x \in U_z$  the linear map  $T\phi_x$  is upper triangular with respect to the decompositions (3.10) and (3.11).

The *diagonal blocks* of  $T\phi$  are the maps:

$$(3.12) \quad \delta^j \phi : T^j \longrightarrow \mathbb{R}^n \times \mathbb{R}^{n_j-n_{j-1}}; \quad j = 0, 1, \dots, \kappa,$$

obtained by composing  $T\phi$  on the left and on the right respectively by the projections:

$$TU_z = T^\kappa \oplus T^{\kappa-1} \oplus \dots \oplus T^0 \longrightarrow T^j$$

and

$$\mathbb{R}^n \times (\mathbb{R}^{n-n_\kappa} \oplus \mathbb{R}^{n_\kappa-n_{\kappa-1}} \oplus \dots \oplus \mathbb{R}^{n_1-n_0} \oplus \mathbb{R}^{n_0}) \longrightarrow \mathbb{R}^n \times \mathbb{R}^{n_j-n_{j-1}}.$$

The *diagonal part* of  $T\phi$  will be defined by  $\Delta\phi = (\delta^\kappa\phi, \delta^{\kappa-1}\phi, \dots, \delta^0\phi)$ . Of course, the inverse of  $T\phi$  is also upper triangular with diagonal blocks given by  $(\delta^j\phi)^{-1}$ ,  $j = 0, 1, \dots, \kappa$ .

We have similar properties for all the underlying maps  $\phi_i$ ,  $i = 0, 1, \dots, \kappa - 1$  coming with the distinguished chart. To fix notations and for future references, let  $U_i$  denotes  $\pi_{s_i}(U_z)$ , and  $T_i^j$  denotes  $T\pi_{s_i}(T^j)$  for all  $j \leq i < \kappa$ . Applying now  $T\pi_{s_i}$  to (3.10) yields:

$$(3.13) \quad TU_i = T_i^i \oplus T_i^{i-1} \oplus \dots \oplus T_i^0,$$

It follows that the differential maps:

$$(3.14) \quad T\phi_i : T_i^i \oplus T_i^{i-1} \oplus \dots \oplus T_i^0 \longrightarrow \mathbb{R}^{n_i} \times (\mathbb{R}^{n_i - n_{i-1}} \oplus \dots \oplus \mathbb{R}^{n_1 - n_0} \oplus \mathbb{R}^{n_0})$$

for all  $i = 0, 1, \dots, \kappa - 1$  are upper triangular with diagonal blocks  $\delta^j \phi_i$  defined as above. Note that for all  $j \leq i \leq k \leq \kappa$ ,  $(T\pi_{s_i})(T_k^j) = T_i^j$  and that applying the correct restrictions and projections in (3.8) gives the following commutative diagram:

$$(3.15) \quad \begin{array}{ccc} T^j & \xrightarrow{\delta^j \phi_k} & \mathbb{R}^{n_k} \times \mathbb{R}^{n_j - n_{j-1}} \\ T\pi_{s_i} \downarrow & & \downarrow \sigma_{n_i} \times \text{Id} \\ T_i^j & \xrightarrow{\delta^j \phi_i} & \mathbb{R}^{n_i} \times \mathbb{R}^{n_j - n_{j-1}} \end{array}$$

### 3.3.2. The smooth structure by induction.

We show that  $T^S X$  can be provided with a smooth structure by a simple recursive argument.

Let us first introduce the *s-exponential maps*. Let  $s$  be a stratum. Recall that the map  $\pi_s : \mathcal{N}_s^\circ \rightarrow s$  is a smooth submersion,  $K_s \subset T\mathcal{N}_s^\circ$  denotes the kernel of the differential map  $T\pi_s$  and  $q_s : T\mathcal{N}_s^\circ \rightarrow T\mathcal{N}_s^\circ$  the orthogonal projection on  $K_s$ . The subbundle  $K_s$  of  $TX^\circ$  inherits from  $TX^\circ$  a riemannian metric whose associated riemannian connection is  $\nabla^s := q_s \circ \nabla$ , where  $\nabla$  is the riemannian connection of the metric on  $X^\circ$ . The associated exponential map

$$\text{Exp}^s : V_s \subset K_s \rightarrow \mathcal{N}_s^\circ$$

is smooth and defined on an open subset  $V_s$  of the zero section of  $K_s$ . Moreover it satisfies:

- $\pi_s \circ \text{Exp}^s = \pi_s$ .
- For any fiber  $L^s$  of  $\pi_s$ , the restriction of  $\text{Exp}^s$  to  $L^s$  is the usual exponential map for the submanifold  $L^s$  of  $X^\circ$  with the induced riemannian structure.

If  $X$  is a stratified pseudomanifold of depth 0 it is smooth and its S-tangent space is the usual tangent space  $TX$  equipped with its usual smooth structure.

Suppose that the S-tangent space of any stratified pseudomanifold of depth strictly smaller than  $k$  is equipped with a smooth structure for some integer  $k > 0$ . Let  $X$  be a stratified pseudomanifold of depth  $k$  and take  $2X$  be the stratified pseudomanifold of depth  $k - 1$  obtained from  $X$  by the unfolding process 2.0.3. According to 3, with the notations of 2.0.3 we have

$$T^S X = T^S 2X|_{2X^\circ \cap X_b^+} \cup_{s \in \mathcal{S}_0} {}^* \pi_s^*(Ts)|_{O_s^\circ} \rightrightarrows X^\circ .$$

Let  $L^\circ$  be the boundary of  $2X^\circ \cap X_b^+$  in  $X^\circ$ . We equip the restriction of  $T^S X$  to  $2X^\circ \cap X_b^+ \setminus L^\circ$  with the smooth structure coming from  $T^S 2X$  and its restriction to any  $O_{s_0}$ ,  $s_0 \in \mathcal{S}_0$ , with the usual smooth structure. It remains to describe the gluing over  $L^\circ$ . One can find an open subset  $W$  of  $T^S 2X$  which contains the restriction of  $T^S 2X$  to  $L^\circ$  such that the following map is defined:

$$\Theta : \quad W \quad \longrightarrow \quad T^S X \\ (x, u, y) \quad \longmapsto \quad \begin{cases} (x, T\pi_{s_0}(u), \text{Exp}^{s_0}(y, -\tau_{s_0}(x)q_{s_0}(u))) & \text{if } x \in O_{s_0}^\circ, s_0 \in \mathcal{S}_0 \\ (x, u, y) & \text{elsewhere} \end{cases}$$

Here, if  $s$  denotes the unique stratum such that  $x, y \in F_s$ , the vector bundle **!!**

$\pi_s^*(Ts)$  is identified with the orthogonal complement of  $K_s$  into  $T\mathcal{N}_s^\circ$ , in other words  $q_{s_0}(y, u) = q_{s_0}(W - q_s(W))$  where  $W \in T_y X^\circ$  satisfies  $T\pi_s(W) = u$ .

Then, we equip  $T^S X$  with the unique smooth structure compatible with the one previously defined on  $T^S X|_{X^\circ \setminus L^\circ}$  and such that the map  $\Theta$  is a smooth diffeomorphism onto its image. The non trivial point is to check that the restriction of the map  $\Theta$  over  $O_{s_0}^\circ$  is a diffeomorphism onto its image for any  $s_0 \in S_0$ . This will follow from the following lemma.

**Lemma 1.** *If  $s_0 < s$ , for any  $x_0 \in s_0$  and  $x \in s$  with  $\pi_{s_0}(x) = x_0$ . The following assertion hold:*

- (1)  $E := q_{s_0}(\pi_s^*(Ts))$  is a sub-bundle of  $K_{s_0}$  of dimension  $\dim(s) - \dim(s_0)$ .
- (2) Let  $E^x := q_{s_0}(\pi_s^*(Ts))|_{\pi_{s_0}^{-1}(x)}$  be the restriction of  $E$  to the submanifold  $\pi_{s_0}^{-1}(x)$ . There exists a neighborhood  $W$  of the zero section of  $E^x$  such that the restriction of  $Exp^{s_0}$  to  $W$  is a diffeomorphism onto a neighborhood of  $\pi_{s_0}^{-1}(x)$  in  $\pi_{s_0}^{-1}(x_0)$ .

*Proof.* 1. The first assertion follows from the inclusion:  $\pi_{s_0}^*(Ts_0) = K_{s_0}^\perp \subset K_s^\perp = \pi_s^*(Ts)$  which ensures that the dimension of the fibers of  $q_{s_0}(\pi_s^*(Ts))$  is constant equal to  $\dim(s) - \dim(s_0)$ .

The same argument shows that  $K_{s_0} = K_s \oplus E$ .

2. If  $\Psi$  denotes the restriction of  $Exp^{s_0}$  to  $E^x$  then  $T\Psi(z, 0)(U, V) = U + V$  where  $(z, U) \in K_s$  and  $V \in E_z$ . Since  $K_s \cap E$  is the trivial bundle we get that  $T\Psi$  is injective and since  $E^x$  and  $\pi_{s_0}^{-1}(x_0)$  have same dimension, it is bijective. We conclude with the local inversion theorem.  $\square$

### 3.3.3. An atlas for $T^S X$ .

The atlas will contain two kinds of local charts. The kind of these charts will depend on the fact that their domains meet or not a gluing between the different pieces composing the tangent space  $T^S X$ , that is the boundary of some  $F_s$ .

The first kind of charts, called *regular charts* are charts whose domain is contained in  $T^S X|_{F_s^\circ}$  for a given stratum  $s$  of the stratification. We observe that  $T^S X|_{F_s^\circ}$  is a smooth groupoid as an open subgroupoid of  ${}^*\pi_s^*(Ts) \rightrightarrows \mathcal{N}_s^\circ$ . Thus, regular charts have domains contained in

$$\sqcup_{s \in S} \overset{\circ}{F}_s$$

and coincide with the usual local charts of the (disjoint) union of the smooth groupoids  $\sqcup_{s \in S} {}^*\pi_s^*(Ts)$ .

The second kind of charts, called *deformation charts* (adapted to a stratum  $s$ ), are charts whose domain meets  $T^S X|_{\partial F_s}$  for a given stratum  $s$ , that is, charts around points in

$$\bigcup_{s \in S} T^S X|_{\partial F_s}.$$

Their description is more involved. Let  $(p, u, q) \in T^S X$ , let  $s$  be the stratum such that  $p \in F_s$  and assume that  $p \in \partial F_s$ . This means that  $\rho_s(p) < 1$ , that  $\rho_t(p) \geq 1$  for all strata  $t < s$  and that the set of strata  $t$  such that  $t < s$  and  $\rho_t(p) = 1$  is not empty. Using again the axioms of the stratification, we see that this set is totally ordered and we denote  $s_0, s_1, \dots, s_{l-1}$  its elements listed by increasing order. We also set  $s_l = s$ . Observe that:

$$(3.16) \quad \{s_0, s_1, \dots, s_l\} = S_p \cap \{t \in S \mid t \leq s\}$$



and that, thanks to the compatibility conditions (3.3), this set only depends on  $\pi_s(p)$  and thus is equal with the corresponding set associated with  $q$ .

Let us take distinguished charts  $\phi : U_p \rightarrow \mathbb{R}^n$  around  $p$  and  $\phi' : U_q \rightarrow \mathbb{R}^n$  around  $q$ . Since  $\pi_s(p) = \pi_s(q)$ , we can also assume without loss of generality that:

$$(3.17) \quad \pi_{s_i}(U_p) = \pi_{s_i}(U_q) \text{ and } \phi_i = \phi'_i \text{ for } i = 0, \dots, l.$$

We will use the same notations than in paragraph 3.3.1:  $n_i = \dim s_i$ ,  $U_i = \pi_{s_i}(U_p)$ ,  $K_i = \ker(T\pi_{s_i})|_{U_p}$ ,  $T^i = K_i^{\perp K_{i-1}}$  for all  $i = 0, 1, \dots, l$  (here again  $K_{-1} = TU_p$ ). The main difference with the settings of the previous paragraph is that we forget the strata bigger than  $s$  in  $S_p$  and  $S_q$  to concentrate on the lower (and common) strata in  $S_p$  and  $S_q$ . It amounts to forget the tail of the filtration (3.7) up to the term  $K_l$ :

$$(3.18) \quad K_l \subset K_{l-1} \subset \dots \subset K_0 \subset TU_p$$

and this leads to a less finer graduation:

$$(3.19) \quad TU_p = K_l \oplus T^l \oplus T^{l-1} \oplus \dots \oplus T^0$$

Let us also introduce the positive smooth functions:

$$t_i = \sum_{j=0}^i \tau \circ \rho_{s_j}, \quad i = 0, 1, \dots, l; \quad \theta_i = \prod_{j=i-1}^l t_j, \quad i = 1, \dots, l$$

Note that  $t_j$  (resp.  $\theta_j$ ) is strictly positive on  $F_{s_i}$  if  $j \geq i$  (resp.  $j > i$ ) and vanishes identically if  $j < i$  (resp.  $j \leq i$ ).

Finally we will write:

$$\forall x \in U_p, \quad \phi(x) = (x^{l+1}, x^l, \dots, x^1, x^0) \in \mathbb{R}^{n-n_l} \times \mathbb{R}^{n_l-n_{l-1}} \times \dots \times \mathbb{R}^{n_1-n_0} \times \mathbb{R}^{n_0},$$

and for all  $j = 0, 1, \dots, l$ ,

$$\pi_{s_j}(x) = x_j,$$

thus  $\phi_j(x_j) = (x^j, x^{j-1}, \dots, x^0) \in \mathbb{R}^{n_j}$ ; and we adopt similar notations for  $\phi'$  and  $y \in U_q$ .

We are ready to define a deformation chart around the point  $(p, u, q)$ . The domain will be:

$$(3.20) \quad \tilde{U} = T^S X|_{U_q}^{U_p}$$

and the chart itself:

$$(3.21) \quad \tilde{\phi} : \tilde{U} \rightarrow \mathbb{R}^{2n}$$

is defined as follows. Up to a shrinking of  $U_p$  and  $U_q$ , the following is true: for all  $(x, v, y) \in \tilde{U}$ , there exists a unique  $i \in \{0, 1, \dots, l\}$  such that  $x \in F_{s_i}$ . Then  $(x, v) \in \pi_{s_i}^*(TU_i)$ , and we set:

$$(3.22) \quad \tilde{\phi}(x, v, y) = \left( \phi(x), \frac{x^{l+1} - y^{l+1}}{\theta_{l+1}(x)}, \dots, \frac{x^{i+1} - y^{i+1}}{\theta_{i+1}(x)}, \Delta\phi_i(x_i, v) \right)$$

The map  $\tilde{\phi}$  is clearly injective with inverse defined as follows. For  $(\mathbf{x}, \mathbf{w}) \in \tilde{\phi}(\tilde{U})$  and  $i$  such that  $\phi^{-1}(\mathbf{x}) \in F_{s_i}$ :

$$\tilde{\phi}^{-1}(\mathbf{x}, \mathbf{w}) = \left( \phi^{-1}(\mathbf{x}), (\Delta\phi_i)^{-1}(\mathbf{x}_i, \mathbf{w}), \phi'^{-1}(\mathbf{x} - \Theta^{[i+1]}(\phi^{-1}(\mathbf{x})) \cdot \mathbf{w}) \right)$$

where  $\mathbf{x}_i = \sigma_{n_i}(\mathbf{x})$  and, using the decomposition

$$\mathbf{w} = (\mathbf{w}^{l+1}, \mathbf{w}^l, \dots, \mathbf{w}^0) \in \mathbb{R}^{n-n_l} \times \mathbb{R}^{n_l-n_{l-1}} \times \dots \times \mathbb{R}^{n_1-n_0} \times \mathbb{R}^{n_0},$$

we have set

$$\Theta^{[i+1]}(x) \cdot \mathbf{w} = \theta_{l+1}(x)\mathbf{w}^{l+1} + \dots + \theta_{i+1}(x)\mathbf{w}^{i+1} \in \mathbb{R}^{n-n_i} \times \{0\} \subset \mathbb{R}^n.$$

To ensure that  $(\tilde{\phi}, \tilde{U})$  is a local chart, it remains to check that  $\tilde{\phi}(\tilde{U})$  is an open subset of  $\mathbb{R}^{2n}$ . It is easy to see that  $\tilde{\phi}(F_{s_i}^\circ)$  is open for every  $i \in \{0, \dots, l\}$  so we consider  $(p, u, q) \in \tilde{U}$  such that  $p \in \partial F_{s_i}$  for some integer  $i$ . Let  $J = \{i_0, \dots, i_k\} \subset \{0, 1, \dots, i-1\}$  such that:

$$\forall j \in J, \quad \rho_{s_j}(p) = 1.$$

Thus we have:

$$(3.23) \quad \rho_{s_i}(p) < 1; \quad \forall j \in J, \quad \rho_{s_j}(p) = 1; \quad \forall j \notin J \text{ and } j < i, \quad \rho_{s_j}(p) > 1$$

by construction,  $q$  satisfies the same relations. Set  $\tilde{\phi}(p, u, q) = (\mathbf{x}_0, \mathbf{v}_0)$ . Using the Taylor formula and the fact that  $\theta_{j+1}$  is negligible with respect to  $1 - \rho_{s_j}$  at the region  $\rho_{s_j} = 1$ , noting also the invariance of  $\rho_{s_k}$  with respect to perturbations of points along the fibers of  $\pi_{s_{k+1}}, \pi_{s_{k+2}}, \dots$ ; we prove that there exist an open ball  $B_1$  of  $\mathbb{R}^n$  centered at  $\mathbf{x}_0$  and an open ball  $B_2$  of  $\mathbb{R}^n$  centered at 0 and containing  $\mathbf{v}_0$  such that for all  $(\mathbf{x}, \mathbf{v}) \in B_1 \times B_2$ , if

$$x = \phi^{-1}(\mathbf{x}) \in F_{s_j} \text{ for } j \in J \text{ or } j = i, \text{ then } y = \phi'^{-1}(\mathbf{x} - \Theta^{[j+1]}(x) \cdot \mathbf{v}) \in F_{s_j}.$$

This proves that  $(\mathbf{x}, \mathbf{v}) \in \text{Im } \tilde{\phi}$ , thus

$$\tilde{\phi}(p, u, q) \in B_1 \times B_2 \subset \text{Im } \tilde{\phi}$$

and the required assertion is proved. We end with:

**Theorem 1.** *The collection of regular and deformation charts provides  $T^S X$  with a structure of smooth groupoid.*

*Proof.* The compatibility between a regular and a deformation chart contains no issue and is omitted. We need only to check the compatibility between a deformation chart adapted to a stratum  $s$  and a deformation chart adapted to a stratum  $t$ , when their domains overlap, which implies automatically that  $s < t$  or  $s > t$  or  $s = t$ .

Let us work out only the case  $s = t$ , since the other case is similar. We have here to compare two charts  $\tilde{\phi}$  and  $\tilde{\psi}$  with common domain  $\tilde{U}$  and involving the same chain of strata  $s = s_l > s_{l-1} > \dots > s_0$ . The whole notations are as before and  $\psi, \psi'$  are the underlying charts of  $X^\circ$  allowing the definition of  $\tilde{\psi}$ . We note, for the sake of concision,  $u^k$  (resp.  $u'^k$ ),  $k = l+1, \dots, 0$ , the coordinate functions of  $u := \psi \circ \phi^{-1}$  (resp.  $\psi' \circ \phi'^{-1}$ ) with respect to the decomposition (3.11) of  $\mathbb{R}^n$ . Observe, thanks to the particular assumptions made on  $\phi, \phi', \psi, \psi'$  (cf. (3.5), (3.17)), that  $u^k(\mathbf{x})$  only depends on  $\mathbf{x}_k := (\mathbf{x}^k, \mathbf{x}^{k-1}, \dots, \mathbf{x}^0) \in \mathbb{R}^{n_k}$  and that  $u^k = u'^k$  for all  $k < l+1$ . Let  $(\mathbf{x}, \mathbf{v}) \in \text{Im } \tilde{\phi}$  and  $i$  such that  $x = \phi^{-1}(\mathbf{x}) \in F_{s_i}$ . Then:

$$(3.24) \quad \tilde{\psi} \circ \tilde{\phi}^{-1}(\mathbf{x}, \mathbf{v}) = \left( u(\mathbf{x}), \frac{u^{l+1}(\mathbf{x}) - u'^{l+1}(\mathbf{x} - \Theta^{[i+1]} \cdot \mathbf{v})}{\theta^{l+1}}, \frac{u^l(\mathbf{x}) - u'^l(\mathbf{x} - \Theta^{[i+1]} \cdot \mathbf{v})}{\theta^l}, \dots \right. \\ \left. \dots, \frac{u^{i+1}(\mathbf{x}) - u'^{i+1}(\mathbf{x} - \Theta^{[i+1]} \cdot \mathbf{v})}{\theta^{i+1}}, (\Delta\psi_i) \circ (\Delta\phi_i)^{-1}(\mathbf{v}) \right)$$

We need to check that the above expression matches smoothly with the corresponding expression for an integer  $k \in [i, l]$  when  $\theta_k(x)$  (and thus  $\theta_{k-1}, \dots, \theta_{i+1}$ ) goes to zero. For that, the Taylor formula applied to  $u^r$ ,  $k \geq r \geq i+1$ , shows that the map defined below is smooth in  $(\mathbf{x}, \mathbf{v}, t)$  where  $(\mathbf{x}, \mathbf{v})$  are as before and  $t = (t_l, t_{l-1}, \dots, t_0) \in \mathbb{R}^{l+1}$  is this time an arbitrary  $(l+1)$ -uple close to 0:

$$\begin{cases} \frac{u^r(\mathbf{x}) - u^r(\mathbf{x} - \Theta^{[i+1]} \cdot \mathbf{v})}{\theta^r} & \text{if } \theta_r = \prod_{r-1}^l t_j \neq 0 \\ d(u^r)_{\mathbf{x}}(\mathbf{v}^r + t_{r-2}\mathbf{v}^{r-1} + \dots + t_i\mathbf{v}^{i+1}) & \text{if } \exists j \in \{r-1, r, \dots, l\} \text{ such that } t_j = 0, \end{cases}$$

In our case,  $t_j = t_j(x)$  and  $t_{k-1}, \dots, t_i$  go to zero, so the second line in the previous expression is just:

$$d(u^r)_{\mathbf{x}}(\mathbf{v}^r)$$

and for obvious matricial reasons:

$$d(u^r)_{\mathbf{x}}(\mathbf{v}^r) = (\Delta\psi_k) \circ (\Delta\phi_k)^{-1}(\mathbf{v}^r)$$

Summing up these relations for  $r = i+1, \dots, k$ , we arrive at the desired identity.

Thus,  $T^5X$  is endowed with a structure of smooth manifold. Changing the riemannian metric on  $X^\circ$  modifies the choices of the  $T_j^i$ 's, but gives rise to compatible charts. Moreover, the smoothness of all algebraic operations associated with this groupoid is easy to check in these local charts.  $\square$

### 3.3.4. The Lie algebroid of the tangent groupoid.

We define

$$Q_s : TX^\circ \longrightarrow TX^\circ \\ (z, V) \longmapsto \begin{cases} (z, \tau_s(z)q_s(z, V)) & \text{if } z \in \mathcal{N}_s^\circ \\ 0 & \text{elsewhere} \end{cases}$$

By a slight abuse of notation, we will keep the notations  $q_s$  and  $Q_s$  for the corresponding maps induced on the set of local tangent vector fields on  $X^\circ$ .

Let  $\mathcal{A}$  be the smooth vector bundle  $\mathcal{A} := TX^\circ \times [0, 1]$  over  $X^\circ \times [0, 1]$ . We define the following morphism of vector bundle :

$$\Phi : \mathcal{A} = TX^\circ \times [0, 1] \longrightarrow TX^\circ \times T[0, 1] \\ (z, V, t) \longmapsto (z, tV + \sum_{s \in \mathcal{S}_{sing}} Q_s(z, V); t, 0)$$

In the sequel we will give an idea of how one can show that there is a unique structure of Lie algebroid on  $\mathcal{A}$  such that  $\Phi$  is its anchor map. The Lie algebroid  $\mathcal{A}$  is almost injective and so it is integrable, moreover we will see that at a set level  $\mathcal{G}_X^t$  must be a groupoid which integrates it [9, 11]. In particular  $\mathcal{G}_X^t$  can be equipped with a unique smooth structure such that it integrates the Lie algebroid  $\mathcal{A}$ .

Now we can state the following:

**Theorem 2.** *There exists a unique structure of Lie algebroid on the smooth vector bundle  $\mathcal{A} = TX^\circ \times [0, 1]$  over  $X^\circ \times [0, 1]$  with  $\Phi$  as anchor.*

To prove this theorem we will need several lemmas:

**Lemma 2.** *Let  $s_0$  and  $s_1$  be two strata such that  $d(s_0) \leq d(s_1)$ .*

- (1) *For any tangent vector field  $W$  on  $X^\circ$ ,  $Q_{s_1}(W)(\tau_{s_0}) = 0$ .*
- (2) *For any  $(z, V) \in TX^\circ$ , the following equality holds:*

$$Q_{s_1} \circ Q_{s_0}(z, V) = Q_{s_0} \circ Q_{s_1}(z, V) = \tau_{s_0}(z)Q_{s_1}(z, V) .$$

*Proof.* First notice that outside  $O_{s_0} \cap O_{s_1}$  either  $Q_{s_1}$  hence  $Q_{s_1}(W)$  or  $\tau_{s_0}$  and  $Q_{s_0}$  vanish thus the equalities of (1) and (2) are simply  $0 = 0$ .

(1) According to the compatibility conditions 3.3 we have  $\rho_{s_0} \circ \pi_{s_1} = \rho_{s_0}$  on  $O_{s_0} \cap O_{s_1}$ . Thus  $\rho_{s_0}$  is constant on the fibers of  $\pi_{s_1}$  and since  $\tau_{s_0} = \tau \circ \rho_{s_0}$ ,  $\tau_{s_0}$  is also constant on the fibers of  $\pi_{s_1}$ . For any tangent vector field  $W$ , and any  $z \in O_{s_1}^\circ$  the vector  $Q_{s_1}(W)(z)$  is tangent to the fibers of  $\pi_{s_1}$  thus  $Q_{s_1}(V)(\tau_{s_0}) = 0$  on  $O_{s_0} \cap O_{s_1}$ .

(2) The result follows from the first remark and the equality 3.7 of the part above.  $\square$

The next lemma ensures that  $\Phi$  is almost injective, in particular it is injective in restriction to  $X^\circ \times ]0, 1]$ . A simple calculation shows the following:

**Lemma 7.** *For any  $t \in ]0, 1]$  the bundle map  $\Phi_t$  is bijective, moreover*

$$\Phi_t^{-1}(z) = \frac{1}{t}V - \sum_{s \in \mathcal{S}_{sing}} \frac{1}{(t + t_s(z)) \cdot (t + t_s(z) - \tau_s(z))} Q_s(z, V)$$

where for any singular stratum  $s$  the map  $t_s$  is defined as follows:

$$t_s : X^\circ \rightarrow \mathbb{R}, \quad t_s(z) = \sum_{s_0 \leq s} \tau_{s_0}(z).$$

Thus in order to prove the theorem 2 it is enough to show that locally the image of the map induced by  $\Phi$  from the set of smooth local sections of  $\mathcal{A}$  to the set of smooth local tangent vector fields on  $X^\circ \times [0, 1]$  is stable under the Lie bracket.

*Idea of the proof of Theorem 2.* First notice that outside the closure of  $\cup_{s_i \in \mathcal{S}_{sing}} O_{s_i}^\circ$  the image under  $\Phi$  of local tangent vector fields is clearly stable under Lie Bracket. Thus using decomposition of the form 3.10 described in the last part and standard arguments it remains to show that if  $s_a$  and  $s_b$  are strata of depth respectively  $a$  and  $b$  with  $s_a \leq s_b$ , if  $U$  is an open subset of  $X^\circ$ , as small as we want contained in  $\mathcal{N}_{s_a} \cap \mathcal{N}_{s_b}$ , and if  $W^\perp, V^\perp, V_a$  and  $W_b$  are tangent vector fields on  $U$ , satisfying:

$$\begin{aligned} & V^\perp \text{ and } V_a \text{ can be projected by } \pi_{s_b}, \\ & Q_s(W^\perp) = Q_s(V^\perp) = 0 \text{ for any } s \in \mathcal{S}, \\ & Q_s(V_a) = \begin{cases} \tau_s V_a & \text{when } s \leq s_a \\ 0 & \text{elsewhere} \end{cases} \quad \text{and} \quad Q_s(W_b) = \begin{cases} \tau_s W_b & \text{when } s \leq s_b \\ 0 & \text{elsewhere} \end{cases}, \end{aligned}$$

then  $[\Phi(W^\perp + W_b), \Phi(V^\perp)]$  and  $[\Phi(W_b), \Phi(V_a)]$  are in the image of  $\Phi$ . In other word, we have to show that the maps  $(z, t) \in X^\circ \times ]0, 1] \mapsto (\Phi_t^{-1}([\Phi(W^\perp + W_b), \Phi(V^\perp)](z)), t)$  and  $(z, t) \in X^\circ \times ]0, 1] \mapsto (\Phi_t^{-1}([\Phi(W_b), \Phi(V_a)](z)), t)$  can be extended into smooth local section of  $\mathcal{A}$ . The result follows from our preceding lemmas and usual calculations.  $\square$

Now we can state:

**Theorem 8.** *The groupoid  $\mathcal{G}_X^t$  can be equipped with a smooth structure such that its Lie algebroid is  $\mathcal{A}$  with  $\Phi$  as anchor.*

*Proof.* According to proposition 2 and lemma 7, the Lie algebroid  $\mathcal{A}$  is almost injective. Thus according to [11] there is a unique quasi-graphoid  $\mathcal{G}(\mathcal{A}) \rightrightarrows X^\circ \times [0, 1]$  which integrates  $\mathcal{A}$ . Suppose for simplicity that for each stratum  $s$ ,  $O_s$  is connected (which will ensure that  $\mathcal{G}_X^t|_{F_s^\circ \times [0, 1]}$  is a quasi-graphoid).

Moreover the map  $\Phi$  satisfies:

- (i)  $\Phi$  induces an isomorphism from  $\mathcal{A}|_{]0,1]} := \mathcal{A}|_{X^\circ \times ]0,1]}$  to  $TX^\circ \times ]0,1]$ ,
- (ii) for any stratum  $s$ , the Lie algebroid  $\mathcal{A}$  restricted over  $F_s^\circ \times [0,1]$  to a Lie algebroid  $\mathcal{A}_s := \mathcal{A}|_{F_s^\circ \times [0,1]}$  which is isomorphic to the restriction of  $\mathcal{A}_{\pi_s \times \text{Id}}^t$  over  $F_s^\circ \times [0,1]$ .

Thus, again by using the uniqueness of quasi-graphoid integrating a given almost injective Lie algebroid, we obtain:

- (i) the restriction of the  $\mathcal{G}(\mathcal{A})$  over  $X^\circ \times ]0,1]$  is isomorphic to  $X^\circ \times X^\circ \times ]0,1] \rightrightarrows X^\circ \times ]0,1]$ , the pair groupoid on  $X^\circ$  parametrized by  $]0,1]$ ,
- (ii) for each stratum  $s$  the restriction over  $F_s^\circ \times [0,1]$  is equal to  $\mathcal{G}_X^t|_{F_s^\circ \times [0,1]}$ .

Finally  $\mathcal{G}(\mathcal{A}) = \mathcal{G}_X^t$  and there is a unique smooth structure on  $\mathcal{G}_X^t$  such that  $\mathcal{A}$  is its Lie algebroid.

If some  $O_s$  is not connected, replace in the construction of the tangent space the groupoid  $(\pi_s)_*(Ts)|_{F_s}$  by its  $s$ -connected component. Let  $CT^S X$  and  $C\mathcal{G}_X^t$  be the corresponding groupoids. Then the previous proof applied to equip  $C\mathcal{G}_X^t$  with a unique smooth structure such that  $\mathcal{A}$  is its Lie algebroid. One can then show that there is a unique smooth structure on  $\mathcal{G}_X^t$  such that  $C\mathcal{G}_X^t$  is its  $s$ -connected component.  $\square$

Thus  $T^S X$  which is the restriction of  $\mathcal{G}_X^t$  to the saturated set  $X^\circ \times \{0\}$  can be also endowed with a smooth structure.

**3.3.5. Standard projection from the tangent space onto the space.** The space of orbits of  $X^\circ/T^S X$  is equivalent to  $X$  in the sense that there is a canonical isomorphism  $C_0(X^\circ/T^S X) \simeq C(X)$ .

**Definition 9.** Let  $r, s : T^S X \rightarrow X^\circ$  be the target and source maps of the  $S$ -tangent space of  $X$ . A continuous map  $p : X \rightarrow X$  is a *standard projection* for  $T^S X$  on  $X$  if:

- (1)  $p \circ r = p \circ s$ .
- (2)  $p$  is homotopic to the identity map of  $X$ .

A standard projection  $p$  for  $T^S X$  on  $X$  is *surjective* if  $p|_{X^\circ} : X^\circ \rightarrow X$  is onto.

This definition leads to the following:

**Lemma 10.** (1) *There exists a standard surjective projection for  $T^S X$  on  $X$ .*  
(2) *Two standard projections are homotopic and the homotopy can be done within the set of standard projections.*

*Proof.* 1) If  $X$  has depth 0,  $X^\circ = X$  and we just take  $p = \text{id}$ . Let consider  $X$  with depth  $k > 0$ . Choose a smooth non decreasing function  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $f([0,1]) = 0$  and  $f|_{[2,+\infty[} = \text{Id}$ . Recall that there exists for each singular stratum  $s$  an isomorphism 2.1:

$$\Psi_s : \mathcal{N}_s \rightarrow c_{\pi_s} L_s = L_s \times [0, +\infty[ / \sim_s .$$

we define the map

$$p_s : \mathcal{N}_s \longrightarrow \mathcal{N}_s$$

by the formula:

$$\Psi_s \circ p_s \circ \Psi_s^{-1}[x, t] = [x, f(t)].$$

For each integer  $i \in [0, k-1]$ , we define a continuous map:

$$p_i : X \longrightarrow X$$

by setting  $p_i(z) = p_s(z)$  if  $z$  belongs to  $\mathcal{N}_s$  for some singular stratum of depth  $i$  and  $p_i(z) = z$  elsewhere. In particular,  $p_i|_{O_s} = \pi_s$  for every stratum  $s$  of depth  $i$ . Finally we set:

$$p = p_0 \circ p_1 \circ \cdots \circ p_{k-1}.$$

This is the map we looked for. Indeed:

Let  $\gamma \in T^S X$ . There exists a unique stratum  $s$  such that  $\gamma \in {}^* \pi_s^*(TS)$ . If  $s$  is regular, then  $r(\gamma) = s(\gamma)$  so the result is trivial here. Let us assume that  $s$  is singular and let  $i < k$  its depth. By definition,  $r(\gamma)$  and  $s(\gamma)$  belong to  $O_s$ . For each stratum  $t \geq s$  of depth  $j \geq i$ , we have everywhere it makes sense:

$$\pi_t \circ p_t = \pi_t, \quad \pi_s \circ \pi_t = \pi_s, \quad \rho_s \circ \pi_t = \rho_s$$

thus:

$$\rho_s \circ p_t = \rho_s \circ \pi_t \circ p_t = \rho_s \circ \pi_t = \rho_s$$

which proves that  $p_j(O_s) = O_s$ , and moreover:

$$\pi_s \circ p_t = \pi_s \circ \pi_t \circ p_t = \pi_s \circ \pi_t = \pi_s$$

Recalling that  $p_i|_{O_s} = \pi_s|_{O_s}$ , this last relation implies:

$$p_i \circ \cdots \circ p_{k-1}|_{O_s} = \pi_s \circ p_{i+1} \circ \cdots \circ p_{k-1}|_{O_s} = \pi_s|_{O_s}$$

Since by definition we also have  $\pi_s(r(\gamma)) = \pi_s(s(\gamma))$ , we conclude that:

$$p(r(\gamma)) = p_0 \circ \cdots \circ p_{i-1} \circ \pi_s(r(\gamma)) = p_0 \circ \cdots \circ p_{i-1} \circ \pi_s(s(\gamma)) = p(s(\gamma))$$

If in the definition of  $p$ , we replace the function  $f$  by  $t \text{Id}_{\mathbb{R}_+} + (1-t)f$ , we get a homotopy between  $p$  and  $\text{Id}_X$ .

Finally,  $p$  has the required surjectivity property:  $p_{k-1}(X^\circ) = X^\circ \cup_{d(s)=k-1} s$  and for all  $j$   $p_{j-1}(X^\circ \cup \bigcup_{d(s) \geq j} s) = X^\circ \cup \bigcup_{d(s) \geq j-1} s$ .

2) Let  $q$  be a standard projection and  $p$  be the standard projection built in 1). Let also  $q_t$  be a homotopy between  $q$  and  $\text{Id}_X$  and  $p_t$  the homotopy built in 1) between  $p$  and  $\text{Id}_X$ . Observe that  $q_t \circ p$  is a standard projection, providing a path of standard projections between  $q \circ p$  and  $p$ . Moreover, by construction of  $p_t$ , the inclusion  $\text{Im}(p_t \circ r, p_t \circ s) \subset \text{Im}(r, s)$  holds for any  $1 \geq t > 0$ , thus  $q \circ p_t$  is a standard projection, providing a path of standard projections between  $q \circ p$  and  $q$ . Thus, any standard projection  $q$  is homotopic to  $p$  within the set of standard projections and the result is proved.  $\square$

**Remark 11.** Let  $p$  be the surjective standard projection built in the proof of the last proposition. The map  $p \circ r : T^S X \rightarrow X$  provides  $T^S X$  with a structure of continuous field of groupoids. Following the arguments of ([13], remark 5), it can be shown that each fiber of this field is amenable, thus  $T^S X$  is amenable and  $C^*(T^S X) = C_r^*(T^S X)$  is nuclear. The same holds for  $\mathcal{G}_X^t$  and all other deformation groupoids used below.

#### 4. POINCARÉ DUALITY FOR STRATIFIED PSEUDO-MANIFOLDS

Let  $X$  be a compact stratified pseudomanifold of depth  $k \geq 0$ .

The tangent groupoid  $\mathcal{G}_X^t$  is a deformation groupoid, thus it provides us with a  $K$ -homology class, called a *pre-Dirac* element:

$$(4.1) \quad \delta_X = [e_0]^{-1} \otimes [e_1] \in KK(C^*(T^S X), \mathbb{C}).$$

Here  $e_0 : C^*(\mathcal{G}_X^t) \rightarrow C^*(T^S X)$  and  $e_1 : C^*(\mathcal{G}_X^t) \rightarrow \mathcal{K}(L^2(X^\circ))$  are the usual evaluation homomorphisms.

We are going to pull-back this pre-Dirac element by a homomorphism:

$$\Psi_X : C^*(T^S X) \otimes C(X) \rightarrow C^*(T^S X),$$

whose construction depends on a standard projection for  $T^S X$  and existence is guaranteed by the following easy lemma:

**Lemma 12.** *Let  $p : X^\circ \rightarrow X$  be a standard projection for  $T^S X$ . The following formula:*

$$\forall a \in C^*(T^S X), b \in C(X), \gamma \in T^S X, \quad \Psi_X(a \otimes b)(\gamma) = b(p \circ r(\gamma)).a(\gamma)$$

defines a homomorphism:

$$\Psi_X : C^*(T^S X) \otimes C(X) \rightarrow C^*(T^S X)$$

The class  $[\Psi_X] \in KK(C^*(T^S X) \otimes C(X), C^*(T^S X))$  does not depend on the choice of the standard projection  $p$ .

The last assertion uses the lemma 10. Note that if  $k = 0$ ,  $X$  is smooth and we can choose  $p = \text{Id}$ , thus:

$$(4.2) \quad \Psi_X(a \otimes b)(V) = b(x).a(x, V)$$

for all  $V \in T_x X$ ,  $x \in X$ ,  $a \in C(X)$  and  $b \in C^*(TX)$ .

In the sequel, we pick a surjective standard projection to define  $\Psi_X$  and we set

$$(4.3) \quad D_X = \Psi_X^*(\delta_X) = [\Psi_X] \otimes \delta_X \in KK(C^*(T^S X) \otimes C(X), \mathbb{C}).$$

This section is devoted to the proof of the main theorem:

**Theorem 3.** *Let  $X$  be a compact stratified pseudomanifold. The  $K$ -homology class  $D_X$  is a Dirac element, that is, it provides a Poincaré duality between the algebras  $C^*(T^S X)$  and  $C(X)$ .*

This will be proved by an induction on the depth of the stratification, and the unfolding process will be used to reduce this depth.

If  $\text{depth}(X) = 0$  the content of the theorem is well known, and that  $D_X$  is a Dirac element is a consequence of [13]. Let  $k \geq 0$ , assume that the theorem 3 holds for all compact stratified pseudomanifolds with  $\text{depth} \leq k$  and let  $X$  be a compact stratified pseudomanifold of depth  $k + 1$ . The proof of the induction is divided into three steps which are briefly described here before going into the details.

Let us first recall the notations of 2.0.3. We let  $S_0$  be the set of minimal strata and

$$\begin{aligned} O_0 &:= \cup_{s \in S_0} \{z \in X \mid \rho_s(z) < 1\}, \quad X_b = X \setminus O_0, \\ L &:= \cup_{s \in S_0} \{z \in X \mid \rho_s(z) = 1\} \subset X_b \text{ and } O_b = X_b \setminus L. \end{aligned}$$

(1) First step: We define a natural ‘‘restriction’’ of  $D_X$  into a  $K$ -homology class of  $C^*(T^S X_b) \otimes C_0(O_b)$  denoted  $D_{O_b}$ , where  $T^S X_b$  is just the restriction  $T^S X|_{X_b}$ . Moreover, the restriction  $T^S X|_{O_0}$  is Morita equivalent to  $\bigcup_{s \in S_0} Ts$  and  $C(\overline{O_0})$  is homotopically equivalent to  $C(\bigcup_{s \in S_0} s)$ . Since  $C(\bigcup_{s \in S_0} s)$  and  $C^*(\bigcup_{s \in S_0} Ts)$  are Poincaré dual, so are  $C^*(T^S X|_{O_0})$  and  $C(\overline{O_0})$ .

Next, considering the decompositions  $X = \overline{O_0} \cup O_b$  and  $T^S X = T^S X|_{O_0} \cup T^S X_b$ , we state the proposition 2, which shows that  $D_X$  is a Dirac element if and only if  $D_{O_b}$  is.

(2) Second step: Let  $2X$  be the stratified pseudomanifold of depth  $k$  obtained by unfolding  $X$ :

$$2X = X_b^- \cup L \times [-1, 1] \cup X_b^+.$$

Since  $L$  and  $2X$  are pseudomanifolds with depth  $\leq k$ , the  $K$ -homology class  $D_{2X}$  and  $D_L$  are Dirac elements. The decompositions  $2X = O_b \cup L \times ]-1, 1[ \cup O_b$  and  $T^S 2X = T^S X_b^- \cup T^S 2X|_{L \times [-1, 1]} \cup T^S X_b^+$  leads to enunciate the proposition 4. The latter proposition proves that the condition:  $D_{2X}$  and  $D_L$  are Dirac elements, is sufficient for  $D_{O_b} \oplus D_{O_b}$ , hence  $D_{O_b}$ , to be a Dirac element.

(3) Third step: we check the technical propositions (2) and (4).

4.0.6. *The first step.* We will first modify a little the notations and consider

$$W = \bigcup_{s \in \mathcal{S}_0} \rho_s^{-1}([0, 2[), \quad \overline{W} = \bigcup_{s \in \mathcal{S}_0} \rho_s^{-1}([0, 2]),$$

$$X_b = X \setminus W, \quad L := \bigcup_{s \in \mathcal{S}_0} \rho_s^{-1}(2) \quad \text{and} \quad O_b = X_b \setminus L.$$

The stratification of  $X$  induces on  $L$  a structure of pseudomanifold of depth  $k$ .

Now we set:

$$T^S X_b = T^S X|_{X_b^\circ} \quad \text{and} \quad T^S W = T^S X|_{W^\circ}.$$

We are going to define Dirac elements associated to  $O_b$  and  $\overline{W}$ .

The inclusion

$$i' : T^S O_b \hookrightarrow T^S X$$

provides a  $K$ -homological element  $\delta_{O_b} = i'^*(\delta_X) \in K^0(C^*(T^S O_b))$  which coincides with the  $K$ -class coming with the deformation groupoid:

$$\mathcal{G}_{O_b}^t = \mathcal{G}_X^t|_{O_b^\circ \times [0, 1]} \xrightarrow{i'} \mathcal{G}_X^t$$

Next, if  $a \in C^*(T^S X_b)$  and  $b \in C_0(O_b)$ , then  $\Psi_X(\tilde{a} \otimes \tilde{b})$ , where  $f(\tilde{a}) = a$  and  $\tilde{b} \in C(X)$  is the extension by 0 of  $b$ , does not depend on the choice of  $\tilde{a}$  and lies in the ideal  $C^*(T^S O_b)$ . In other words,  $\Psi_X$  induces a homomorphism:

$$(4.4) \quad \Psi_{O_b} : C^*(T^S X_b) \otimes C_0(O_b) \longrightarrow C^*(T^S O_b)$$

We thus get:

$$(4.5) \quad D_{O_b} = [\Psi_{O_b}] \otimes \delta_{O_b} \in KK(C^*(T^S X_b) \otimes C_0(O_b), \mathbb{C}).$$

We can do similar things with  $T^S W$  and  $\overline{W}$ . Indeed the inclusions  $i : T^S W \hookrightarrow T^S X$  and  $\mathcal{G}_W^t = \mathcal{G}_X^t|_{W^\circ \times [0, 1]} \hookrightarrow \mathcal{G}_X^t$  show that  $i^*\delta_X$  coincides with the element  $\delta_W$  associated with the deformation groupoid  $\mathcal{G}_W^t$ . Moreover,  $\Psi_X$  induces as before a homomorphism:

$$(4.6) \quad \Psi_{\overline{W}} : C^*(T^S W) \otimes C(\overline{W}) \longrightarrow C^*(T^S W)$$

So we can set:

$$(4.7) \quad D_{\overline{W}} = [\Psi_{\overline{W}}] \otimes \delta_W \in KK(C^*(T^S W) \otimes C(\overline{W}), \mathbb{C}).$$

Since  $X_b^\circ$  is a closed saturated subspace of  $X^\circ$ , we have the following exact sequence of  $C^*$ -algebras:

$$(4.8) \quad 0 \longrightarrow C^*(T^S W) \xrightarrow{i} C^*(T^S X) \xrightarrow{f} C^*(T^S X_b) \longrightarrow 0$$

We also have

$$(4.9) \quad 0 \longrightarrow C_0(O_b) \xrightarrow{j} C(X) \xrightarrow{g} C(\overline{W}) \longrightarrow 0$$



We are ready to state:

**Proposition 2.** *The elements  $D_{O_b} \in KK(C^*(T^S X_b) \otimes C_0(O_b), \mathbb{C})$  and  $D_{\overline{W}} \in KK(C^*(T^S W) \otimes C(\overline{W}), \mathbb{C})$  satisfy:*

1) *The following equalities hold:*

$$(4.10) \quad [i]_{C^*(T^S X)} \otimes D_X = [g]_{C(\overline{W})} \otimes D_{\overline{W}}$$

$$(4.11) \quad [f]_{C^*(T^S X_b)} \otimes D_{O_b} = [j]_{C(X)} \otimes D_X$$

2) *If  $\partial_f \in KK(C_0(\mathbb{R}) \otimes C^*(T^S X_b), C^*(T^S W))$  and  $\partial_g \in KK(C_0(\mathbb{R}) \otimes C(\overline{W}), C(O_b))$  denote the boundary elements in  $KK$ -theory associated respectively with (4.8) and (4.9), then:*

$$(4.12) \quad \partial_g_{C^*(T^S X_b)} \otimes D_{O_b} = \partial_f_{C(\overline{W})} \otimes D_{\overline{W}}$$

This proposition has the following obvious corollary:

**Corollary 1.** *The following diagrams commute for all  $C^*$ -algebras  $C$  and  $D$ :*

$$(4.13) \quad \begin{array}{ccccccc} \cdots \rightarrow KK_i(C, D \otimes C_0(O_b)) & \longrightarrow & KK_i(C, D \otimes C(X)) & \longrightarrow & KK_i(C, D \otimes C(\overline{W})) & \longrightarrow & KK_{i+1}(C, D \otimes C_0(O_b)) \rightarrow \cdots \\ \downarrow \cdot_{C_0(O_b)} \otimes D_{O_b} & & \downarrow \cdot_{C(X)} \otimes D_X & & \downarrow \cdot_{C(\overline{W})} \otimes D_{\overline{W}} & & \downarrow \cdot_{C_0(O_b)} \otimes D_{O_b} \\ \cdots \rightarrow KK_i(C \otimes C^*(T^S X_b), D) & \longrightarrow & KK_i(C \otimes C^*(T^S X), D) & \longrightarrow & KK_i(C \otimes C^*(T^S W), D) & \longrightarrow & KK_{i+1}(C \otimes C^*(T^S X_b), D) \rightarrow \cdots \end{array}$$

$$(4.14) \quad \begin{array}{ccccccc} \cdots \rightarrow KK_i(C, D \otimes C^*(T^S W)) & \longrightarrow & KK_i(C, D \otimes C^*(T^S X)) & \longrightarrow & KK_i(C, D \otimes C^*(T^S X_b)) & \longrightarrow & KK_{i+1}(C, D \otimes C^*(T^S W)) \rightarrow \cdots \\ \downarrow \cdot_{C^*(T^S W)} \otimes D_{\overline{W}} & & \downarrow \cdot_{C^*(T^S X)} \otimes D_X & & \downarrow \cdot_{C^*(T^S X_b)} \otimes D_{O_b} & & \downarrow \cdot_{C^*(T^S W)} \otimes D_{\overline{W}} \\ \cdots \rightarrow KK_i(C \otimes C(\overline{W}), D) & \longrightarrow & KK_i(C \otimes C(X), D) & \longrightarrow & KK_i(C \otimes C_0(O_b), D) & \longrightarrow & KK_{i+1}(C \otimes C(\overline{W}), D) \rightarrow \cdots \end{array}$$

The proposition (2) will be proved in the third step.

Moreover, for any minimal stratum  $s$  the inclusion of  $C^*$ -algebras  $C^*(T^S X|_{\rho_s^{-1}([0,2])}) \subset C^*(T^S X|_{\rho_s^{-1}([0,1])})$  is a  $KK$ -equivalence and since, as already mentioned, the groupoid  $T^S X|_{\rho_s^{-1}([0,1])} = T^S X|_{O_s} = * \pi_s^*(T_s)$  is Morita equivalent to  $T_s$ , we have:

**Proposition 3.** *There is a  $KK$ -equivalence  $I = C^*(T^S W) \sim C^*(T^S S)$  under which the element  $D_{\overline{W}}$  corresponds to the Dirac element  $D_S$  of the disjoint union of closed smooth manifolds  $S = \cup_{s \in \mathcal{S}_{min}} s$ . In particular,  $D_{\overline{W}}$  is a Dirac element.*

Now, the commutativity of the diagrams (4.13) and (4.14), the lemma 2 and the fact that  $D_{\overline{W}}$  is a Dirac element show that, thanks to the five lemma,  $D_X$  is a Dirac element if and only if  $D_{O_b}$  is.

4.0.7. *The second step:  $D_{O_b}$  is a Dirac element.* Let us go back to the compact pseudomanifold of depth  $k$  coming from the unfolding process:  $2X$ . Recall from 2.0.3 that:  $2X = X_+ \cup L_1 \times [-1, 1] \cup X_+$  where  $X_+ = X \setminus \{z \in X \mid \exists s \in \mathcal{S}_0, \rho_s(z) < 1\}$  and  $L_1 := \cup_{s \in \mathcal{S}_0} \{z \in X \mid \rho_s(z) = 1\} \subset X_+$ . Notes that  $X_+ \simeq X_b$  and  $L_1 \simeq L$  thus under obvious identification, we will set

$$2X = X_b^- \cup L \times [-1, 1] \cup X_b^+$$

where  $X_b^\pm$  are two disjoint copies of  $X_b$ . We consider the exact sequences of  $C^*$ -algebras:

$$(4.15) \quad 0 \longrightarrow C^*(T^S 2X|_{L \times [-1, 1]}) \xrightarrow{i} C^*(T^S 2X) \xrightarrow{f} C^*(T^S 2X|_{X_b^- \sqcup X_b^+}) \longrightarrow 0$$

$$(4.16) \quad 0 \longrightarrow C_0(O_b^- \sqcup O_b^+) \xrightarrow{j} C(2X) \xrightarrow{g} C(L \times [-1, 1]) \longrightarrow 0$$

Let us define three  $K$ -homology classes related to these algebras and allowing commutative diagrams analogous to (4.13), (4.14).

Since  $2X$  is a compact stratified pseudomanifold of depth  $k$ , we know that  $D_{2X}$  is a Dirac element, and it will be our first  $K$ -homology class. The homomorphism  $\Psi_{2X}$  used in  $D_{2X}$  induces a homomorphism:

$$(4.17) \quad \Psi_{L \times [-1, 1]} : C^*(T^S 2X|_{L \times [-1, 1]}) \otimes C(L \times [-1, 1]) \longrightarrow C^*(T^S 2X|_{L \times [-1, 1]})$$

while the subgroupoid:

$$(4.18) \quad \mathcal{G}_{L \times [-1, 1]}^t := \mathcal{G}_{2X|_{L \times [-1, 1] \cap (2X)^\circ}^t$$

of the tangent groupoid  $\mathcal{G}_{2X}^t$  gives rise to an element  $\delta_{L \times [-1, 1]} \in KK(C^*(T^S 2X|_{L \times [-1, 1]}), \mathbb{C})$ . The second  $K$ -homology class will be:

$$(4.19) \quad D_{L \times [-1, 1]} := [\Psi_{L \times [-1, 1]}] \otimes \delta_{L \times [-1, 1]} \in KK(C^*(T^S 2X|_{L \times [-1, 1]}) \otimes C(L \times [-1, 1]), \mathbb{C})$$

The third  $K$ -homology class is built by doubling  $D_{O_b}$ . Denoting  $J^\pm = C_0(O_b^\pm)$  and  $P^\pm = C^*(T^S 2X|_{X_b^\pm})$ , we can define:

$$D_{O_b}^\pm := D_{O_b} \in K^0(C^*(T^S X_b)^\pm \otimes J^\pm),$$

and we set:

$$(4.20) \quad D_{2O_b} = D_{O_b}^- \oplus D_{O_b}^+ \in K^0(P^- \otimes J^-) \oplus K^0(P^+ \otimes J^+) \subset K^0(C^*(T^S 2X|_{X_b^- \sqcup X_b^+}) \otimes C_0(O_b^- \sqcup O_b^+))$$

**Proposition 4.**

- 1) The element  $D_{L \times [-1, 1]}$  is a Dirac element.
- 2) The following equalities hold:

$$(4.21) \quad [i] \otimes_{C^*(T^S 2X)} D_{2X} = [g] \otimes_{C(L \times [-1, 1])} D_{L \times [-1, 1]}$$

$$(4.22) \quad [f] \otimes_{C^*(T^S 2X|_{X_b^- \sqcup X_b^+})} D_{2O_b} = [j] \otimes_{C(2X)} D_{2X}$$

- 3) If  $\partial_f \in KK(C_0(\mathbb{R}) \otimes C^*(T^S 2X|_{X_b^- \sqcup X_b^+}), C^*(T^S 2X|_{L \times [-1, 1]})$  and  $\partial_g \in KK(C_0(\mathbb{R}) \otimes C(L \times [-1, 1]), C(O_b^- \sqcup O_b^+))$  denote the boundary elements in  $KK$ -theory associated respectively with (4.15) and (4.16), then:

$$(4.23) \quad \partial_g \otimes_{C^*(T^S 2X|_{X_b^- \sqcup X_b^+})} D_{2O_b} = \partial_f \otimes_{C(L \times [-1, 1])} D_{L \times [-1, 1]}$$

As previously, we get immediately:

**Corollary 2.** *The following two diagrams commute for all  $C^*$ -algebras  $C$  and  $D$ :*

$$(4.24) \quad \begin{array}{ccc} \cdots \rightarrow KK_i(C, D \otimes C_0(O_b^- \sqcup O_b^+)) & \longrightarrow & KK_i(C, D \otimes C(2X)) \rightarrow \cdots \\ & \downarrow \cdot \otimes_{C_0(O_b^- \sqcup O_b^+)}^{D_{2O_b}} & \downarrow \cdot \otimes_{C(2X)}^{D_{2X}} \quad \cdots \\ \cdots \rightarrow KK_i(C \otimes C^*(T^S 2X|_{X_b^- \sqcup X_b^+}), D) & \longrightarrow & KK_i(C \otimes C^*(T^S 2X, D)) \rightarrow \cdots \end{array}$$

$$\begin{array}{ccc} \cdots \rightarrow KK_i(C, D \otimes C(L \times [-1, 1])) & \longrightarrow & KK_{i+1}(C, D \otimes C_0(O_b^- \sqcup O_b^+)) \rightarrow \cdots \\ & \downarrow \cdot \otimes_{C(L \times [-1, 1])}^{D_{L \times [-1, 1]}} & \downarrow \cdot \otimes_{C_0(O_b^- \sqcup O_b^+)}^{D_{2O_b}} \\ \cdots \rightarrow KK_i(C \otimes C^*(T^S 2X|_{L \times [-1, 1]}), D) & \longrightarrow & KK_{i+1}(C \otimes C^*(T^S 2X|_{X_b^- \sqcup X_b^+}), D) \rightarrow \cdots \end{array}$$

and:

$$(4.25) \quad \begin{array}{ccc} \cdots \rightarrow KK_i(C, D \otimes C^*(T^S 2X|_{L \times [-1, 1]})) & \longrightarrow & KK_i(C, D \otimes C^*(T^S 2X)) \rightarrow \cdots \\ & \downarrow \cdot \otimes_{C^*(T^S 2X|_{L \times [-1, 1]})}^{D_{L \times [-1, 1]}} & \downarrow \cdot \otimes_{C^*(T^S 2X)}^{D_{2X}} \quad \cdots \\ \cdots \rightarrow KK_i(C \otimes C(L \times [-1, 1]), D) & \longrightarrow & KK_i(C \otimes C(2X), D) \rightarrow \cdots \end{array}$$

$$\begin{array}{ccc} \cdots \rightarrow KK_i(C, D \otimes C^*(T^S 2X|_{X_b^- \sqcup X_b^+})) & \longrightarrow & KK_{i+1}(C, D \otimes C^*(T^S 2X|_{L \times [-1, 1]})) \rightarrow \cdots \\ & \downarrow \cdot \otimes_{C^*(T^S 2X|_{X_b^- \sqcup X_b^+})}^{D_{2O_b}} & \downarrow \cdot \otimes_{C^*(T^S 2X|_{L \times [-1, 1]})}^{D_{L \times [-1, 1]}} \\ \cdots \rightarrow KK_i(C \otimes C_0(O_b^- \sqcup O_b^+), D) & \longrightarrow & KK_{i+1}(C \otimes C(L \times [-1, 1]), D) \rightarrow \cdots \end{array}$$

Thus, since  $D_{2X}$  and  $D_{L \times [-1, 1]}$  are Dirac elements, the five lemma yields that  $D_{2O_b}$  is a Dirac element, and this implies that  $D_{O_b}^+ = D_{O_b}$  is also a Dirac element. This result and step 1 end the proof of the theorem 3.

4.0.8. *Proof of propositions 2 and 4.* The only tricky point is the commutativity with the boundary maps, that is assertion 2 of proposition 2 and assertion 3 of proposition 4.

Let us start with the proof of proposition 2.

1) We check only (4.10), since (4.11) is proved similarly. We have:

$$\begin{aligned} [i] \otimes_{C^*(T^S X)} D_X &= [(i \otimes \text{Id}_X) \circ \Psi_X] \otimes \delta_X \\ &= [(\text{Id}_{T^S X} \otimes g) \circ \Psi_{\overline{W}} \circ i] \otimes \delta_X \\ &= [g] \otimes_{C(\overline{W})} ([\Psi_{\overline{W}}] \otimes [i] \otimes \delta_X) \end{aligned}$$

and  $[i] \otimes \delta_X = \delta_{\overline{W}}$  as noted before.

2) Recall that  $\partial \overline{W} = \partial X_b = L$  and insert the exact sequence (4.8) into the following commutative diagram of  $C^*$ -algebras, where the new morphisms are made

with obvious inclusions of open subgroupoids and restrictions onto closed saturated subgroupoids:

$$(4.26) \quad \begin{array}{ccccccc} 0 & \longrightarrow & C^*(T^S W) & \longrightarrow & C^*(T^S X) & \xrightarrow{f} & C^*(T^S X_b) \longrightarrow 0 \\ & & \downarrow = & & \downarrow p' & & \downarrow p \\ 0 & \longrightarrow & C^*(T^S \overline{W}) & \longrightarrow & C^*(T^S \overline{W}) & \xrightarrow{f'} & C^*(T^S L \times \mathbb{R}) \longrightarrow 0 \end{array}$$

Let us call  $\partial_{f'}$  the boundary element in  $KK$ -theory associated with the bottom exact sequence. By functoriality, we have:

$$p^*(\partial_{f'}) = [p] \otimes \partial_{f'} = \partial_f$$

Similarly we may consider:

$$(4.27) \quad \begin{array}{ccccccc} 0 & \longrightarrow & C_0(O_b) & \longrightarrow & C(X) & \xrightarrow{g} & C(\overline{W}) \longrightarrow 0 \\ & & \downarrow = & & \downarrow q' & & \downarrow q \\ 0 & \longrightarrow & C_0(O_b) & \longrightarrow & C(X_b) & \xrightarrow{g'} & C(L) \longrightarrow 0 \end{array}$$

which leads to the relation:

$$q^*(\partial_{g'}) = [q] \otimes \partial_{g'} = \partial_g$$

Thus, the following diagram commutes:

$$(4.28) \quad \begin{array}{ccccc} & & K^0(C^*(T^S X_b) \otimes C_0(O_b)) & \xrightarrow{\tau_{C^*(T^S X_b)}(\partial_{g'})} & K^1(C^*(T^S X_b) \otimes C(L)) \\ & & \downarrow \tau_{C^*(T^S X_b)}(\partial_g) & \swarrow (1 \otimes q)^* & \uparrow (p \otimes 1)^* \\ K^0(C^*(T^S W) \otimes C(\overline{W})) & \xrightarrow{\tau_{C(\overline{W})}(\partial_f)} & K^1(C^*(T^S X_b) \otimes C(\overline{W})) & & \\ \downarrow \tau_{C(\overline{W})}(\partial_{f'}) & \swarrow (p \otimes 1)^* & & \swarrow (p \otimes q)^* & \\ K^1(C^*(T^S L \times \mathbb{R}) \otimes C(\overline{W})) & \xleftarrow{(1 \otimes q)^*} & & & K^1(C^*(T^S L \times \mathbb{R}) \otimes C(L)) \end{array}$$

The proof of the assertion 2) in the proposition 2 is thus reduced to the next lemma.

**Lemma 13.** *There exists  $\alpha_L \in K^1(C^*(T^S L \times \mathbb{R}) \otimes C(L))$  such that*

$$(4.29) \quad \tau_{C(\overline{W})}(\partial_{f'}) \otimes D_{\overline{W}} = (1 \otimes q)^*(\alpha_L)$$

and

$$(4.30) \quad \tau_{C^*(T^S X_b)}(\partial_{g'}) \otimes D_{O_b} = (p \otimes 1)^*(\alpha_L)$$

Moreover,  $\alpha_L$  can be chosen to be the image of the Dirac element  $D_L$  under the Bott periodicity isomorphism  $K^0(C^*(T^S L) \otimes C(L)) \simeq K^1(C^*(T^S L \times \mathbb{R}) \otimes C(L))$ .

*Proof of the lemma.* Firstly, the morphism  $\Psi_X$  induces by restriction the morphisms:

$$(4.31) \quad \Psi_L : C^*(T^S L) \otimes C(L) \longrightarrow C^*(T^S L),$$

and

$$(4.32) \quad \Psi'_{\overline{W}} : C^*(T^S \overline{W}) \otimes C(\overline{W}) \longrightarrow C^*(T^S \overline{W}).$$

Of course  $\Psi'_{\overline{W}}$  extends  $\Psi_{\overline{W}}$ . These morphisms fit into the following commutative diagram:  
(4.33)

$$\begin{array}{ccccccc} 0 & \longrightarrow & C^*(T^S W) \otimes C(\overline{W}) & \longrightarrow & C^*(T^S \overline{W}) \otimes C(\overline{W}) & \xrightarrow{f' \otimes 1} & C^*(T^S L \times \mathbb{R}) \otimes C(\overline{W}) \longrightarrow 0 \\ & & \downarrow \Psi_{\overline{W}} & & \downarrow \Psi'_{\overline{W}} & & \downarrow \Psi_L \circ (1 \otimes q) \\ 0 & \longrightarrow & C^*(T^S W) & \longrightarrow & C^*(T^S \overline{W}) & \xrightarrow{f'} & C^*(T^S L \times \mathbb{R}) \longrightarrow 0 \end{array}$$

We note that  $\tau_{C(\overline{W})}(\partial f')$  is precisely the boundary element  $\partial_{f' \otimes 1}$  of the top exact sequence above and it follows by functoriality again that the left hand side of (4.29) can be rewritten as follows:

$$\begin{aligned} \tau_{C(\overline{W})}(\partial f') \otimes D_{\overline{W}} &= \partial_{f' \otimes 1} \otimes [\Psi_{\overline{W}}] \otimes \delta_W \\ &= (\Psi_L \circ (1 \otimes q))^*(\partial_{f'} \otimes \delta_W) \\ &= (1 \otimes q)^* \circ \Psi_L^*(\partial_{f'} \otimes \delta_W) \end{aligned}$$

Similarly,  
(4.34)

$$\begin{array}{ccccccc} 0 & \longrightarrow & C^*(T^S X_b) \otimes C_0(O_b) & \longrightarrow & C^*(T^S X_b) \otimes C(X_b) & \xrightarrow{1 \otimes g'} & C^*(T^S X_b) \otimes C(L) \longrightarrow 0 \\ & & \downarrow \Psi_{O_b} & & \downarrow \Psi_{X_b} & & \downarrow \Psi_L \circ (p \otimes 1) \\ 0 & \longrightarrow & C^*(T^S O_b) & \longrightarrow & C^*(T^S X_b) & \xrightarrow{p} & C^*(T^S L \times \mathbb{R}) \longrightarrow 0 \end{array}$$

This gives by functoriality for the left hand side of (4.30):

$$\begin{aligned} \tau_{C^*(T^S X_b)}(\partial_{g'}) \otimes D_{O_b} &= \partial_{g' \otimes 1} \otimes [\Psi_{O_b}] \otimes \delta_{O_b} \\ &= (p \otimes 1)^* \circ \Psi_L^*(\partial_p \otimes \delta_{O_b}) \end{aligned}$$

We are now going to check that

$$(4.35) \quad \partial_{f'} \otimes \delta_W = \partial_p \otimes \delta_{O_b}$$

by computing explicitly these elements. To understand the left hand side above, we glue the groupoids  $\mathcal{G}_W^t$  and  $\mathcal{G}_L^t \times \mathbb{R}$  into a new smooth groupoid:

$$\mathcal{G}_{\overline{W}}^t := \mathcal{G}_W^t \sqcup \mathcal{G}_L^t \times \mathbb{R},$$

Let us explain this gluing. Consider the following deformation groupoid associated with the locally trivial bundle  $\pi_L : L \rightarrow S$ :

$$\tilde{\mathcal{G}}_{\pi_L}^t = ]-\infty, 0[ \times {}^* \pi_L^*(TS)|_{L^\circ} \sqcup [0, +\infty[ \times T^S L$$

It is an extended version (because of the presence of positive values of the deformation parameter) of a *fibred tangent groupoid*, or Thom groupoid [5], associated with the bundle  $\pi_L$ . This groupoid is provided with a smooth structure, compatible with those of  ${}^* \pi_L^*(TS)|_{L^\circ}$  and  $T^S L$ , using the function  $\tau$  already used in the previous gluings. On the other hand,  $\tau$  gives us an action  $\phi$  of  $\mathbb{R}$  onto itself by the (complete) flow of the vector field  $\tau(h)\partial_h$ . This gives a smooth isomorphism of groupoids:

$$\mathbb{R} \rtimes_\phi \mathbb{R} \simeq ]-\infty, 0[ \times ]-\infty, 0[ \sqcup [0, +\infty[ \times \mathbb{R} = T^S \mathbb{R}_+$$

where  $\mathbb{R}_+$  is regarded as a conical pseudomanifold with vertex 0. This action makes sense on the deformation parameter in  $\tilde{\mathcal{G}}_{\pi_L}^t$ , it is trivial when the deformation parameter is positive and thus can be arbitrarily restricted in this part, leading to a new presentation of  $T^S \overline{W}$ . Indeed, there is a smooth isomorphism:

$$\tilde{\mathcal{G}}_{\pi_L}^t |_{]-\infty, 1]} \rtimes_\phi \mathbb{R} \simeq {}^* \pi^*(TS)|_{O_S} \sqcup T^S L \times [1, 2] \times \mathbb{R} \simeq T^S \overline{W}$$

where  $O_S = \cup_{s \in \mathcal{S}_{min}} O_s$ ,  $\pi$  is the projection map  $O_S \rightarrow S$  coinciding with  $\pi_s$  on each  $O_s$  and a reparametrization  $] -\infty, 1] \simeq ]0, 2]$  is implicitly used. In fact we can describe  $\mathcal{G}_{\overline{W}}^t$  in the same way by replacing  $\tilde{\mathcal{G}}_{\pi_L}^t$  by a *two deformation variables* fibered tangent groupoid:

$$\tilde{\mathcal{G}}_{\pi_L}^T = (\mathbb{R} \times [0, 1]) \setminus \{(h, 0) \mid h \geq 0\} \times {}^* \pi_L^*(TS)|_{L^\circ} \sqcup \{(h, 0) \mid h \geq 0\} \times T^S L$$

We replace  $\tau : \mathbb{R} \rightarrow [0, 1]$  by a smooth function  $T : \mathbb{R} \times [0, 1] \rightarrow [0, 1]$  in such a way that  $T(h, 0) = \tau(h)$ ,  $T(h, k) > 0$  if  $k > 0$  and  $h < 1$ ,  $T(h, k) = 0$  if  $h \geq 1$ , and  $T(h, k) = 1$  as soon as  $k \geq \varepsilon$  and  $h \leq 1 - \varepsilon$  for some small fixed positive number  $\varepsilon$ . We can also ask that  $T(h, k)$  does not depend on  $k$  when  $k \geq \varepsilon$  and that  $T(h, \varepsilon) = \tau(h - 1)$ .

We then replace  $\phi$  by the (complete) flow  $\Phi$  of the vector field  $T(h, k)\partial_h$ , this gives us an action of  $\mathbb{R}$  onto  $\tilde{\mathcal{G}}_{\pi_L}^T$  and an isomorphism:

$$\tilde{\mathcal{G}}_{\pi_L}^T \rtimes_{\Phi} \mathbb{R} |_{]-\infty, 1] \times [0, 1]} \simeq \mathcal{G}_{\overline{W}}^t$$

which allows us to provide  $\mathcal{G}_{\overline{W}}^t$  with the smooth structure of  $\tilde{\mathcal{G}}_{\pi_L}^T \rtimes_{\Phi} \mathbb{R}$ . Now, the restriction at the value 1 of the remaining deformation parameter  $k$  in  $\mathcal{G}_{\overline{W}}^t$  gives, using the identification  $W^\circ \simeq ]0, +2[ \times L^\circ$  coming from the stratification data:

$$\mathcal{G}_{\overline{W}}^t|_{t=1} = W^\circ \times W^\circ \sqcup L^\circ \times L^\circ \times \mathbb{R} \simeq (L^\circ \times L^\circ) \times (]-\infty, 0]) \rtimes_{\phi} \mathbb{R} =: G_b(\overline{W})$$

The groupoid  $G_b(\overline{W})$  is analogous to the defined in [25] for manifolds with boundary. In particular, the arguments in [24] prove that  $G_b(\overline{W})$  has vanishing  $K$ -theory. This implies that the boundary element

$$\partial_{f''} \in KK(C_0(\mathbb{R}) \otimes C^*(L^\circ \times L^\circ \times \mathbb{R}), C^*(W^\circ \times W^\circ))$$

associated with the exact sequence:

$$(4.36) \quad 0 \longrightarrow C^*(W^\circ \times W^\circ) \longrightarrow C^*(G_b(\overline{W})) \xrightarrow{f''} C^*(L^\circ \times L^\circ \times \mathbb{R}) \longrightarrow 0$$

is an isomorphism and it is easy to check that it coincides with the inverse of the Bott projector  $b$  after the obvious identifications:

$$KK(C_0(\mathbb{R}) \otimes C^*(L^\circ \times L^\circ \times \mathbb{R}), C^*(W^\circ \times W^\circ)) \simeq KK(C_0(\mathbb{R}^2), \mathbb{C})$$

We invoke now the commutative diagram of exact sequences:

$$(4.37) \quad \begin{array}{ccccccc} 0 & \longrightarrow & C^*(T^S W) & \longrightarrow & C^*(T^S \overline{W}) & \xrightarrow{f'} & C^*(T^S L \times \mathbb{R}) \longrightarrow 0 \\ & & \uparrow e_0^W & & \uparrow e_0^{\overline{W}} & & \uparrow e_0^L \otimes 1 \\ 0 & \longrightarrow & C^*(\mathcal{G}_{\overline{W}}^t) & \longrightarrow & C^*(\mathcal{G}_{\overline{W}}^t) & \longrightarrow & C^*(\mathcal{G}_L^t \times \mathbb{R}) \longrightarrow 0 \\ & & \downarrow e_1^W & & \downarrow e_1^{\overline{W}} & & \downarrow e_1^L \otimes 1 \\ 0 & \longrightarrow & C^*(W^\circ \times W^\circ) & \longrightarrow & C^*(G_b(\overline{W})) & \xrightarrow{f''} & C^*(L^\circ \times L^\circ \times \mathbb{R}) \longrightarrow 0 \end{array}$$

which gives by functoriality, using the identification  $C_0(\mathbb{R}) \otimes C^*(L^\circ \times L^\circ \times \mathbb{R}) \simeq C_0(\mathbb{R}^2) \otimes C^*(L^\circ \times L^\circ)$ :

$$\partial_{f'} \otimes [e_0^W]^{-1} \otimes [e_1^W] = \tau_{C_0(\mathbb{R}^2)}([e_0^L]^{-1} \otimes [e_1^L]) \otimes \partial_{f''}$$

and, after inserting the Morita equivalences  $C^*(W^\circ \times W^\circ) \sim \mathbb{C}$ ,  $C^*(L^\circ \times L^\circ) \sim \mathbb{C}$ :

$$\partial_{f'} \otimes \delta_W = \delta_L \otimes b^{-1}$$

We proceed in the same way to manage the right hand side  $\partial_p \otimes \delta_{O_b}$  of (4.35): we consider the tangent groupoid

$$\mathcal{G}_{X_b}^t = \mathcal{G}_{O_b}^t \sqcup \mathcal{G}_L^t \times \mathbb{R}$$

and the same arguments as above lead to

$$\partial_p \otimes \delta_{O_b} = \delta_L \otimes b^{-1}$$

which completes the proof of the lemma.  $\square$

*Proof of proposition 4.* (1) The groupoid isomorphism

$$T^S 2X|_{L \times ]-1, 1[} \simeq T^S L \times T] - 1, 1[$$

yields an isomorphism:

$$(4.38) \quad C^*(T^S 2X|_{L \times ]-1, 1[}) \simeq C^*(T^S L) \otimes C^*(T] - 1, 1[)$$

Under this isomorphism, we see that:

$$(4.39) \quad [\Psi_{L \times ]-1, 1[}] \otimes \delta_{L \times ]-1, 1[} = ([\Psi_L] \otimes \delta_L) \otimes_{\mathbb{C}} ([\Psi_{]-1, 1[}] \otimes \delta_{]-1, 1[})$$

where  $\delta_L \in KK(C^*(T^S L), \mathbb{C})$  comes from the tangent groupoid of  $L$  while  $\delta_{]-1, 1[} \in KK(C^*(T] - 1, 1[), \mathbb{C})$  comes from the tangent groupoid of  $] - 1, 1[$ . One easily see that  $\delta_{]-1, 1[}$  corresponds to the inverse of the Bott projector  $b \in K_0(C_0(\mathbb{R}^2))$  via the isomorphisms  $C^*(T] - 1, 1[) \simeq C^*(T\mathbb{R}) \simeq C_0(\mathbb{R}^2)$ . In particular, using the  $KK$ -equivalence  $C(]-1, 1]) \sim \mathbb{C}$ , one gets that:

$$D_{]-1, 1[} := [\Psi_{]-1, 1[}] \otimes \delta_{]-1, 1[} \in KK(C^*(T] - 1, 1[) \otimes C(]-1, 1]), \mathbb{C})$$

is a Dirac element. Since  $L$  is a stratified pseudomanifold of depth  $k$ , we know that  $D_L$  is a Dirac element, so is its product over  $\mathbb{C}$  with  $D_{]-1, 1[}$ , which proves using (4.39) that  $D_{L \times ]-1, 1[}$  defined by (4.19) is a Dirac element.

The proofs for the assertions in 2) and 3) of the proposition 4 are similar to the proofs of assertions 1) and 2) of the proposition 2.  $\square$

4.0.9. *Stratified pseudomanifold with boundary.* As a byproduct of this proof, we have proved that Poincaré duality also holds for compact stratified pseudomanifolds with boundary. Precisely a stratified pseudomanifold with boundary is  $(X_b, L, S_b, N_b)$  where:

- (1)  $X_b$  is a compact separable metrizable space and  $L$  is a compact subspace of  $X_b$ .
- (2)  $S_b = \{s_i\}$  is a finite partition of  $X_b$  into locally closed subset of  $X_b$ , which are smooth manifolds possibly with boundary. Moreover for each  $s_i$  we have

$$s_i \cap L = \partial s_i .$$

- (3)  $N_b = \{\mathcal{N}_s, \pi_s, \rho_s\}_{s \in S_b}$ , where  $\mathcal{N}_s$  is an open neighborhood of  $s$  in  $X$ ,  $\pi_s : \mathcal{N}_s \rightarrow s$  is a continuous retraction and  $\rho_s : \mathcal{N}_s \rightarrow [0, +\infty[$  is a continuous map such that  $s = \rho_s^{-1}(0)$ .
- (4) The *double*:

$$X = X_b \cup_L X_b$$

obtained by gluing two copies of  $X_b$  along  $L$  together with the partition  $S := \{s_i \mid \partial s_i = \emptyset\} \cup \{s_i \cup_{\partial s_i} s_i\} \cup \{s_i \mid \partial s_i = \emptyset\}$  and the set of control data

$N = \{\tilde{\mathcal{N}}_s, \tilde{\pi}_s, \tilde{\rho}_s\}_{s \in S}$  where

$$\mathcal{N}_s = \mathcal{N}_{s_i}, \pi_s = \pi_{s_i}, \rho_s = \rho_{s_i} \text{ if } s = s_i \text{ with } \partial s_i = \emptyset$$

and

$$\mathcal{N}_s = \mathcal{N}_{s_i} \cup_{\mathcal{N}_{s_i} \cap L} \mathcal{N}_{s_i}, \pi_s|_{\mathcal{N}_{s_i} \setminus L} = \pi_{s_i}, \rho_s|_{\mathcal{N}_{s_i} \setminus L} = \rho_{s_i} \text{ elsewhere}$$

is a stratified pseudomanifold.

We let  $O_b := X_b \setminus L$ . According to the previous work, one can define the tangent spaces:

$$T^S X_b := T^S X|_{X_b} \text{ and } T^S O_b := T^S X|_{O_b}$$

We deduce the following:

**Theorem 4.** *The  $C^*$ -algebras  $C^*(T^S X_b)$  and  $C_0(O_b)$  are Poincaré Dual as well as the  $C^*$ -algebras  $C^*(T^S O_b)$  and  $C(X_b)$ .*

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