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**MOTIVIC AND QUANTUM INVARIANCE  
UNDER  
STRATIFIED MUKAI FLOPS**

BAOHUA FU AND CHIN-LUNG WANG

ABSTRACT. For stratified Mukai flops of type  $A_{n,k}$ ,  $D_{2k+1}$  and  $E_{6,l}$ , it is shown the fiber product induces isomorphisms between Chow motives, cohomology rings and full Gromov-Witten theory.

1. INTRODUCTION

Two smooth projective varieties over  $\mathbb{C}$  are *K equivalent* if there are birational morphisms  $\phi : Y \rightarrow X$  and  $\phi' : Y \rightarrow X'$  such that  $\phi^*K_X = \phi'^*K_{X'}$ . This basic equivalence relation had caught considerable attention in recent years through its appearance in minimal model theory, crepant resolutions, as well as other related fields.

The conjectural behavior of *K equivalence* has been formulated in [W]. A *canonical correspondence*  $\mathcal{F} \in A^*(X \times X')$  should exist and gives an isomorphism of Chow motives  $[X] \cong [X']$ . Under  $\mathcal{F}$ ,  $X$  and  $X'$  should have isomorphic *B-models* (complex moduli with Hodge theory on it) as well as *A-models* (quantum cohomology ring up to analytic continuations over the extended Kähler moduli space).

Basic examples of *K equivalence* are *flops* (with exceptional loci  $Z, Z', S$ )

$$\begin{array}{ccc} (X, Z) & \xrightarrow{\quad f \quad} & (X', Z') \\ \psi \searrow & & \swarrow \psi' \\ & (\tilde{X}, S) & \end{array}$$

Among them the *ordinary flops* had been studied in [LLW] where the equivalence of motives and A-models was proved. In that case  $\mathcal{F}$  is the graph closure  $\mathcal{F}_0 := \bar{\Gamma}_f$ . In general,  $\mathcal{F}$  must contain *degenerate correspondences*.

The typical examples are *Mukai flops*. They had been extensively studied in the literature in hyper-Kähler geometry. Over a general base  $S$ , they had also been studied in [LLW], where the invariance of Gromov-Witten theory was proved. In that case  $\mathcal{F} = X \times_{\tilde{X}} X' = \mathcal{F}_0 + \mathcal{F}_1$  with  $\mathcal{F}_1 = Z \times_S Z'$ . We expect that for flops  $\mathcal{F}$  should be basically  $X \times_{\tilde{X}} X'$ .

To understand the general picture we are led to study flops with  $\mathcal{F}$  consisting of many components. The *stratified Mukai flops* provide such examples. They appear naturally in the study of symplectic resolutions [Fu]

[Na2] and they should play important roles in higher dimensional birational geometry of hyper-Kähler manifolds. See for example [Mar].

In this paper, we study general stratified Mukai flops without any assumptions on the global structure of  $X$  and  $X'$ . By way of it, we hope to develop tools with perspective on future studies.

Fix two natural numbers  $n, k$  such that  $2k < n + 1$ . Consider two smooth projective varieties  $X$  and  $X'$ . Let  $F_k \subset F_{k-1} \subset \cdots \subset F_1 \subset X$  and  $F'_k \subset F'_{k-1} \subset \cdots \subset F'_1 \subset X'$  be two collections of closed subvarieties. Assume that there exist two birational morphisms  $X \xrightarrow{\psi} \bar{X} \xleftarrow{\psi'} X'$ . The induced birational map  $f : X \dashrightarrow X'$  is called a Mukai flop of type  $A_{n,k}$  over  $\bar{X}$  if the following conditions are satisfied:

- (i) The map  $f$  induces an isomorphism  $X \setminus F_1 \xrightarrow{\sim} X' \setminus F'_1$ ;
- (ii)  $\psi(F_j) = \psi'(F'_j) =: S_j$  for  $1 \leq j \leq k$ ;
- (iii)  $S_k$  is smooth and there exists a vector bundle  $V$  of rank  $n + 1$  over it such that  $F_k$  is isomorphic to the relative Grassmanian  $G_{S_k}(k, V)$  of  $k$ -planes over  $S_k$  and the restriction  $\psi|_{F_k} : F_k \rightarrow S_k$  is the natural projection. Furthermore, the normal bundle  $N_{F_k/X}$  is isomorphic to the relative cotangent bundle  $T_{F_k/S_k}^*$ . The analogue property holds for  $F'_k$  and  $\psi'$  with  $V$  replaced by its dual  $V^*$ ;
- (iv) If  $k = 1$ , we require that  $f$  is a usual Mukai flop along  $F_k$ . When  $k \geq 2$ , let  $Y$  (resp.  $Y', \bar{Y}$ ) be the blow-up of  $X$  (resp.  $X', \bar{X}$ ) along  $F_k$  (resp.  $F'_k, S_k$ ). By the universal property of the blow-ups, we obtain morphisms  $Y \rightarrow \bar{Y} \leftarrow Y'$ . The proper transforms of  $F_j, F'_j$  give collections of subvarieties on  $Y, Y'$ . We require that the birational map  $Y \dashrightarrow Y'$  is a Mukai flop of type  $A_{n-2,k-1}$ .

We define a Mukai flop of type  $D_k$  (with  $k \geq 3$  being an odd integer) in a similar way with the following changes: (1) one requires that  $S_k$  is simply connected; (2) the vector bundle  $V$  is of rank  $2k$  with a fiber-wise non-degenerate symmetric 2-form. Then the relative Grassmanians of  $k$ -dimensional isotropic subspaces of  $V$  over  $S_k$  has two components  $G_{iso}^+$  and  $G_{iso}^-$ . We require that  $F_k$  (resp.  $F'_k$ ) is isomorphic to  $G_{iso}^+$  (resp.  $G_{iso}^-$ ); (3) when  $k = 3$ ,  $f$  is a usual Mukai flop.

Similarly one can define a Mukai flop of type  $E_{6,I}$  by taking  $k = 2$  with  $V$  being a vector bundle of rank 27 over  $S_2$  whose fibers are endowed with a Jordan algebra structure of the octonions and  $F_2$  is the relative  $E_6/P_1$ -bundle over  $S_2$  in  $\mathbb{P}_S(V)$ . The dual variety  $F'_2$  is given by the relative  $E_6/P_6$ -bundle in  $\mathbb{P}_S(V^*)$ , where  $P_1, P_6$  are maximal standard parabolic subgroup in  $E_6$  corresponding to the simple roots  $\alpha_1, \alpha_6$  respectively.

Our main objective of this work is to prove the following theorem.

**Theorem 1.1.** *Let  $f : X \dashrightarrow X'$  be a Mukai flop of type  $A_{n,k}, D_{2k+1}$  or  $E_{6,I}$  over  $\bar{X}$ . Let  $\mathcal{F}$  be the correspondence  $X \times_{\bar{X}} X'$ . Then  $X$  and  $X'$  have isomorphic Chow motives, cohomology rings and full Gromov-Witten theory under  $\mathcal{F}$ .*

Note that the flops of type  $A_{n,1}$  and of type  $D_3$  are the usual Mukai flops, and in these cases the theorem has been proven in [LLW]. This will be the base of our induction. Our proof uses an induction on  $k$  (for all  $n$ ) via our assumption (iv). We shall give details of the proof for  $A_{n,k}$  flops, while omitting the proof of the other two types, since the argument is essentially the same.

Here is an outline of the contents. In Section 2, the existence of  $A_{n,k}$  flops in the projective category is proved via the cone theorem. In Section 3, a general criterion on equivalence of Chow motives *via graph closure* is established for *strictly semi-small* flops. While a given flop may not be so, generic deformations of it may sometimes do. When this works, we then restrict the graph closure of the one parameter deformation back to the central fiber to get the correspondence, which is necessarily the fiber product.

It is thus crucial to study deformations of flops. Global deformations are usually obstructed, so instead we study in Section 4 the *deformations of local models* of  $A_{n,k}$  flops and show that local  $A_{n,k}$  flops is deformed into simpler, though non-local,  $A_{n-2,k-1}$  flops. This deformation is good enough for applying the equivalence criterion of motives.

To handle global situations, we consider *degenerations to the normal cone* to reduce problems on  $A_{n,k}$  flops to problems on  $A_{n-2,k-1}$  flops and on local models of  $A_{n,k}$  flops. This is carried out for correspondences in Section 5. This makes inductive argument work since local models are already well handled. We also carry out this for the *classical cup product* by proving an *orthogonal decomposition* under degenerations to the normal cone.

In the final section we prove the invariance of full Gromov-Witten theory under stratified Mukai flops. We need to combine the *degeneration formula* for GW invariants [LR] [Li] to run the induction. The degeneration formula splits the *absolute* GW invariants into the *relative* ones. After degenerations, the flop is deformed into *two simpler ones*. We then use a result in [MP] to get invariance of relative invariants from the result on absolute ones (for simpler flops). This then completes the proof of Theorem 1.1.

At the end, a differential-geometrical heuristic is discussed.

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## 2. EXISTENCE OF (TWISTED) $A_{n,k}$ FLOPS

Given  $k, n \in \mathbb{N}$ ,  $2k < n + 1$ , a flopping contraction  $\psi : (X, F) \rightarrow (\bar{X}, S)$  is of type  $A_{n,k}$  if it admits the following inductive structure: There is a filtration  $F = F_1 \supset \cdots \supset F_k$  with induced filtration  $S = S_1 \supset \cdots \supset S_k$ ,  $S_j := \psi(F_j)$  such that  $\psi_{S_k} : F_k \cong G_{S_k}(k, V) \rightarrow S_k$  is a  $G(k, n + 1)$  bundle for some vector bundle  $V \rightarrow S_k$  of rank  $n + 1$  with

$$N_{F_k/X} \cong T_{F_k/S_k}^* \otimes \psi_{S_k}^* L_k \quad \text{for some } L_k \in \text{Pic } S_k.$$

Moreover, the blow-up maps  $\phi, \bar{\phi}$  fit into a cartesian diagram

$$\begin{array}{ccc}
 Y = \mathrm{Bl}_{F_k} X \supset E & & \\
 \phi \downarrow & \searrow \bar{\psi} & \\
 X & & \bar{Y} = \mathrm{Bl}_{S_k} \bar{X} \supset \bar{E} \\
 & \searrow \psi & \downarrow \bar{\phi} \\
 & & \bar{X}
 \end{array}$$

such that the induced contractions  $\bar{\psi} : (Y, \tilde{F}) \rightarrow (\bar{Y}, \tilde{S})$  with filtrations  $\tilde{F} = \tilde{F}_1 \supset \cdots \supset \tilde{F}_{k-1}$ ,  $\tilde{F}_j := \phi_*^{-1}(F_j)$ ,  $\tilde{S} = \tilde{S}_1 \supset \cdots \supset \tilde{S}_{k-1}$ ,  $\tilde{S}_j = \bar{\phi}_*^{-1}(S_j)$ ,  $1 \leq j \leq k-1$  is of type  $A_{n-2, k-1}$ . Here we use the convention that an  $A_{n,0}$  contraction is an isomorphism. By definition,  $A_{n,1}$  contractions are twisted Mukai contractions.

The main results of this paper are all concerned with the (untwisted) stratified Mukai flops, namely  $L_k \cong \mathcal{O}_{S_k}$ . The starting basic existence theorem of flops does however hold for the twisted case too.

**Proposition 2.1.** *Given any  $A_{n,k}$  contraction  $\psi$ , the corresponding  $A_{n,k}$  flop*

$$\begin{array}{ccc}
 (X, F) & \xrightarrow{\quad f \quad} & (X', F') \\
 \psi \searrow & & \swarrow \psi' \\
 & (\bar{X}, S) &
 \end{array}$$

exists with  $\psi'$  being an  $A_{n,k}$  contraction.

*Proof.* We construct the flop by induction on  $k$ . The case  $k = 1$  has been done in [LLW, Section 6], so we let  $k \geq 2$  and  $n+1 > 2k$ . By induction we have a diagram

$$\begin{array}{ccccc}
 Y & \xrightarrow{\quad g \quad} & Y' & & \\
 \phi \downarrow & \searrow \bar{\psi} & \swarrow \bar{\psi}' & \searrow \phi' & \\
 X & & \bar{Y} & & X' \\
 & \searrow \psi & \downarrow \bar{\phi} & & \\
 & & \bar{X} & &
 \end{array}$$

where  $g : (Y, \tilde{F}) \dashrightarrow (Y', \tilde{F}')$  is an  $A_{n-2, k-1}$  flop and  $\bar{\psi}' : (Y', \tilde{F}') \rightarrow (\bar{Y}, \tilde{S})$  is an  $A_{n-2, k-1}$  contraction.

Let  $C \subset E$  be a  $\phi$ -exceptional curve and  $C' = g_*C$  be its proper transform in  $E' = g_*E$ . We shall construct a blow-down map  $\phi' : Y' \rightarrow X'$  for  $C'$ . Let  $\gamma$  (resp.  $\gamma'$ ) be the flopping curve for  $\bar{\psi}$  (resp.  $\bar{\psi}'$ ).

Since the Poincaré pairing is preserved by the graph correspondence  $\mathcal{F}_0$  of  $g$  in the divisor/curve level (c.f. [LLW] Remark 2.4), and  $\mathcal{F}_0 C = C' + a\gamma'$

for some  $a \in \mathbb{N}$  (in fact  $a = 1$ ), we compute

$$(K_{Y'} \cdot C') = (K_{Y'} \cdot \mathcal{F}_0 C) = (K_Y \cdot C) < 0.$$

To show that  $C'$  is an Mori (negative) extremal curve, it is thus sufficient to find a supporting divisor for it.

Let  $\bar{L}$  be a supporting divisor for  $\bar{C} = \bar{\psi}(C)$  in  $\bar{Y}$ . Then  $\bar{\psi}'^* \bar{L}$  is a supporting divisor for the extremal face spanned by  $C'$  and  $\gamma'$ . The idea is to perturb it to make it positive along  $\gamma'$  while keeping it vanishing along  $C'$ .

Let  $D$  be a supporting divisor for  $C$  in  $Y$  with  $\lambda := (D \cdot \gamma) > 0$ . Let  $D' = g_* D = \mathcal{F}_0 D$ . Since  $\mathcal{F}_0 \gamma = -\gamma'$ , we compute

$$(D' \cdot \gamma') = -(D, \gamma) = -\lambda < 0,$$

$$(D' \cdot C') = (D' \cdot \mathcal{F}_0 C) - a(D', \gamma') = (D \cdot C) + a\lambda = a\lambda.$$

Let  $H'$  be a supporting divisor for  $\gamma'$  in  $Y'$  with  $c' := (H' \cdot C') > 0$ . Then

$$W := a\lambda H' - c' D'$$

has the property that  $(W \cdot \gamma') > 0$  and  $(W \cdot C') = 0$ .

Now for  $m$  large enough, the perturbation

$$L' := m\bar{\psi}'^* \bar{L} + W$$

is a supporting divisor for  $C'$ . Indeed,  $L'$  takes the same values as  $W$  on  $\gamma'$  and  $C'$ , while  $(L' \cdot \beta') > 0$  for other curve classes  $\beta'$  in  $Y'$ . That is,  $L'$  is big and nef which vanishes precisely on the ray  $\mathbb{Z}^+[C']$ .

By the (relative) cone theorem applying to  $\bar{\phi} \circ \bar{\psi}' : Y' \rightarrow \bar{X}$ , we complete the diagram and achieve the flop  $f : X \dashrightarrow X'$ :

$$\begin{array}{ccccc}
 Y & \xrightarrow{\quad g \quad} & Y' & & \\
 \phi \downarrow & \searrow \bar{\psi} & \swarrow \bar{\psi}' & \downarrow \phi' & \\
 X & \xrightarrow{\quad \quad} & \bar{Y} & \xrightarrow{\quad \quad} & X' \\
 & \searrow \psi & \downarrow \bar{\phi} & \swarrow \psi' & \\
 & & \bar{X} & & 
 \end{array}$$

It remains to show that the contraction  $\psi' : X' \rightarrow \bar{X}$  is of type  $A_{n,k}$ . By construction, it amounts to analyze the local structure of  $F'_k := \phi'(E')$ . Since the flop  $f$  is unique and local with respect to  $\bar{X}$ , it is enough to determine its structure in a neighborhood of  $S_k$ . This can be achieved by explicit constructions.

Suppose that  $F_k = G_{S_k}(k, V)$ . We consider the pair of spaces  $(\tilde{X}', \tilde{F}'_k)$  defined by duality. Namely  $\tilde{F}'_k := G_{S_k}(k, V^*)$  and

$$\tilde{X}' \text{ is the total space of } T_{\tilde{F}'_k/S_k}^* \otimes \psi_{S_k}^* L_k.$$

It is well-known that, in a neighborhood of  $S_k$ ,  $X \dashrightarrow X'_0$  is an  $A_{n,k}$  flop. Thus the local structure of  $(X', F'_k)$  must agree with  $(\tilde{X}', \tilde{F}'_k)$ . The proof is complete.  $\square$

*Remark 2.2.* In the definition of  $A_{n,k}$  contractions, the restriction to exceptional divisors  $\tilde{\psi}|_E : (E, \tilde{F}|_E) \rightarrow (\bar{E}, \tilde{S}|_{\bar{E}})$  is also an  $A_{n-2,k-1}$  contraction. Moreover, in the proposition the restriction

$$\begin{array}{ccc} (E, \tilde{F}|_E) & \xrightarrow{\mathcal{S}|_E} & (E', \tilde{F}'|_{E'}) \\ & \searrow \tilde{\psi}|_E & \swarrow \tilde{\psi}'|_{E'} \\ & (\bar{E}, \tilde{S}|_{\bar{E}}) & \end{array}$$

is also an  $A_{n-2,k-1}$  flop.

### 3. EQUIVALENCE CRITERIA OF MOTIVES

Let  $X \xrightarrow{\psi} \bar{X} \xleftarrow{\psi'} X'$  be two projective resolutions of a quasi-projective normal variety  $\bar{X}$ , and  $f : X \dashrightarrow X'$  the induced birational map. Consider the graph closure  $\Gamma$  of  $f$  and  $X \xleftarrow{\phi} \Gamma \xrightarrow{\phi'} X'$  the two graph projections. Then we obtain a morphism between Chow groups:

$$\mathcal{F} := \phi'_* \phi^* : A^*(X) \rightarrow A^*(X').$$

For any  $i$ , we will consider the closed subvariety

$$E_i = \{x \in X \mid \dim_x \psi^{-1}(\psi(x)) \geq i\}.$$

In a similar way we define the subvariety  $E'_i$  on  $X'$ . By Zariski's main theorem,  $\psi$  is an isomorphism over  $X \setminus E_1$ , thus  $\psi(E_1) = \psi'(E'_1) = \bar{X}_{\text{sing}}$ .

The following criterion generalizes the one for ordinary flops in [LLW]:

**Proposition 3.1.** *If for any irreducible component  $D, D'$  of  $E_i$  and  $E'_i$  respectively, we have*

$$2 \operatorname{codim} D > \operatorname{codim} \psi(D), \text{ and } 2 \operatorname{codim} D' > \operatorname{codim} \psi'(D'),$$

*then  $\mathcal{F}$  is an isomorphism on Chow groups which preserves the Poincaré pairing on cohomology groups.*

*Moreover, the correspondence  $[\Gamma]$  induces an isomorphism between Chow motives:  $[X] \simeq [X']$ .*

*Proof.* For any smooth  $T$ ,  $f \times \operatorname{id}_T : X \times T \dashrightarrow X' \times T$  is also a birational map with the same condition. Thus by the identity principle we only need to prove the equivalence of Chow groups under  $\mathcal{F}$ .

For any  $\alpha \in A_k(X)$ , up to replacing by an equivalent cycle, we may assume that  $\alpha$  intersects  $\bar{E} := \sum_{i \geq 1} E_i$  properly. Then we have  $\mathcal{F}\alpha = \alpha'$ , where  $\alpha'$  is the proper transform of  $\alpha$  under  $f$ . If we denote by  $\tilde{\alpha}$  the proper transform of  $\alpha$  in  $A^*(\Gamma)$ , then we have

$$\phi'^* \alpha' = \tilde{\alpha} + \sum_C a_C F_C,$$

where  $F_C$  are some irreducible  $k$ -dimensional subvariety in  $\Gamma$  and  $a_C \in \mathbb{Z}$ .

For any  $C$ , note that  $\phi'(F_C)$  is contained in the support of  $\alpha' \cap E'_1$ . As  $\psi'(\alpha' \cap E'_1) = \psi(\alpha) \cap \bar{X}_{\text{sing}} = \psi(\alpha \cap E_1)$ ,  $F_C$  is contained in  $\phi^{-1}\psi^{-1}(B_C)$ , where  $B_C := \psi\phi(F_C) \subset \psi(\alpha \cap E_1)$ . Take the largest  $i$  such that there exists an irreducible component  $D$  of  $E_i$  with  $B_C \subset \psi(\alpha \cap D)$ . For a general point  $s \in B_C$ , we denote by  $F_{C,s}$  its fiber by the map  $\psi \circ \phi$ . Then we have

$$\dim F_{C,s} \geq \dim F_C - \dim B_C \geq \dim F_C - \dim(\alpha \cap D) = \text{codim } D.$$

By our assumption, we have  $\text{codim } D > \dim D - \dim \psi(D)$ , the latter being the dimension of a general fiber of  $\psi^{-1}(B_C) \rightarrow B_C$ . Thus the general fiber of the map  $\phi|_{F_C}$  has positive dimension, which gives that  $\phi_*(F_C) = 0$ . This gives that  $\mathcal{F}' \circ \mathcal{F} = \text{Id}$ , where  $\mathcal{F}' = \phi_*\phi'^*$ . A similar argument then shows that  $\mathcal{F} \circ \mathcal{F}' = \text{Id}$ , thus  $\mathcal{F}$  and  $\mathcal{F}'$  are isomorphisms.

Since  $F_C$  has positive fiber dimension in both  $\phi$  and  $\phi'$  directions, the statement on Poincaré pairing follows easily as in [LLW], Corollary 2.3.  $\square$

Now consider two (holomorphic) symplectic resolutions:  $X \xrightarrow{\psi} \bar{X} \xleftarrow{\psi'} X'$ . A conjecture in [FN] asserts that  $\psi$  and  $\psi'$  are deformation equivalent, i.e. there exist deformations of  $\psi$  and  $\psi'$  over  $\mathbb{C}$ :  $\mathfrak{X} \xrightarrow{\Psi} \bar{\mathfrak{X}} \xleftarrow{\Psi'} \mathfrak{X}'$ , such that for any  $t \neq 0$ , the morphisms  $\Psi_t, \Psi'_t$  are isomorphisms. This conjecture has been proved in various situations, such as nilpotent orbit closures of classical type [FN] [Na2], or when  $W$  is projective [Na1].

Assume this conjecture and consider the birational map  $\mathfrak{F} : \mathfrak{X} \dashrightarrow \mathfrak{X}'$ . Recall that every symplectic resolution is automatically semi-small by the work of Kaledin [Ka] and Namikawa [Na1]. We obtain that the deformed resolutions  $\Psi$  and  $\Psi'$  satisfy the condition of the precedent proposition. As a consequence, we obtain:

**Theorem 3.2.** *Consider two symplectic resolutions  $X \xrightarrow{\psi} \bar{X} \xleftarrow{\psi'} X'$ . Suppose that they are deformation equivalent (say given by  $\mathfrak{F} : \mathfrak{X} \dashrightarrow \mathfrak{X}'$ ). If we denote by  $\Gamma$  the graph of  $\mathfrak{F}$  and  $\Gamma_0$  its central fiber. Then the correspondence  $[\Gamma_0]$  induces an isomorphism of motives  $[X] \simeq [X']$  which preserves also the Poincaré pairing.*

#### 4. DEFORMATIONS OF LOCAL MODELS

From now on all the stratified Mukai flops are untwisted.

**4.1. Deformations of open local models.** Let  $S$  be a smooth variety and  $V \rightarrow S$  a vector bundle of rank  $n + 1$ . The relative Grassmanian bundle of  $k$ -planes in  $V$  is denoted by  $\psi : F := G_S(k, V) \rightarrow S$ . Let  $T$  be the universal sub-bundle of rank  $k$  on  $F$  and  $Q$  the universal quotient bundle of rank  $n + 1 - k$ . As is well-known, the relative cotangent bundle  $T_{F/S}^*$  is isomorphic to  $\text{Hom}_F(Q, T)$ . Thus it is natural to construct deformations of  $T_{F/S}^*$  inside the endomorphism bundle  $\text{End}_F \psi^*V = \psi^* \text{End}_S V$ .

Consider the vector bundle  $\mathcal{E}$  over  $F$  defined as follows: For  $x \in F$ ,

$$\mathcal{E}_x := \{(p, t) \in \text{End } V_{\psi(x)} \times \mathbb{C} \mid \text{Im } p \subset T_x, p|_{T_x} = t \text{Id}_{T_x}\}.$$



We have an inclusion  $T_{F/S,x}^* = \text{Hom}(Q_x, T_x) \rightarrow \mathcal{E}_x$  which sends  $q \in \text{Hom}(Q_x, T_x)$  to  $(\tilde{q}, 0) \in \mathcal{E}_x$ , where  $\tilde{q}$  is the composition

$$\tilde{q} : V_{\psi(x)} \rightarrow Q_x \xrightarrow{q} T_x \hookrightarrow V_{\psi(x)}.$$

The projection to the second factor  $\pi : \mathcal{E} \rightarrow \mathbb{C}$  is then an one-dimensional deformation of  $\pi^{-1}(0) = T_{F/S}^*$ .

The projection to the first factor, followed by  $\psi_*$

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\quad} & \psi^* \text{End}_S V & \xrightarrow{\quad \psi_* \quad} & \text{End}_S V \\ & \searrow & \downarrow & & \downarrow \\ & & F & \xrightarrow{\quad \psi \quad} & S \end{array}$$

gives rise to a map  $\mathcal{E} \rightarrow \text{End}_S V$ , which is a birational morphism onto its image  $\bar{\mathcal{E}}$ . Indeed,  $\Psi : \mathcal{E} \rightarrow \bar{\mathcal{E}}$  is isomorphic over the loci with rank  $p = k$ . In particular, it is isomorphic outside  $\pi^{-1}(0)$ . For any  $s \in S$ ,

$$\bar{\mathcal{E}}_s := \overline{\{p \in \text{End } V_s \mid \text{rank } p = k \text{ and } p^2 = tp \text{ for some } t \in \mathbb{C}\}}$$

is the *cone of scaled projectors* with rank at most  $k$ . Thus  $\pi = \bar{\pi} \circ \Psi$ , where

$$\bar{\pi} : \bar{\mathcal{E}} \rightarrow \mathbb{C} \quad \text{via} \quad \phi \mapsto \frac{1}{k} \text{Tr } \phi.$$

For  $t \neq 0$ ,  $\Psi_t : \mathcal{E}_t \xrightarrow{\sim} \bar{\mathcal{E}}_t$ . For  $t = 0$ ,  $\psi := \Psi_0 : T_{F/S}^* = \mathcal{E}_0 \rightarrow \bar{\mathcal{E}}_0$  is the open local model of an  $A_{n,k}$  contraction.

We do a similar construction for the dual bundle  $V^* \rightarrow S$ . Under the canonical isomorphism  $\text{End}_S V \simeq \text{End}_S V^*$ , we see that  $\bar{\mathcal{E}}$  is identified with  $\bar{\mathcal{E}}' = \bar{\mathcal{E}}(V^*)$ . Thus we get a birational map  $\mathfrak{F} : \mathcal{E} \dashrightarrow \mathcal{E}'$  over  $\bar{\mathcal{E}}$ . This proves

**Proposition 4.1.** *The birational map  $\mathfrak{F}$  over  $\mathbb{C}$ :*

$$\begin{array}{ccc} \mathcal{E} & \overset{\mathfrak{F}}{\dashrightarrow} & \mathcal{E}' \\ \Psi \searrow & & \swarrow \Psi' \\ & \bar{\mathcal{E}} & \\ \pi \searrow & \downarrow & \swarrow \pi' \\ & \mathbb{C} & \end{array}$$

*deforms the birational map ( $A_{n,k}$  flop)  $f : T_{F/S}^* \dashrightarrow T_{F'/S}^*$  into isomorphisms.*

Let  $\Gamma$  be the graph closure of  $\mathfrak{F} : \mathcal{E} \dashrightarrow \mathcal{E}'$  and  $\Gamma_0$  be its central fiber. By Proposition 3.1, the map  $\Gamma_* : A^*(\mathcal{E}) \rightarrow A^*(\mathcal{E}')$  is an isomorphism. Since  $\Gamma \rightarrow \mathcal{E} \times_{\bar{\mathcal{E}}} \mathcal{E}'$  is birational,  $(\mathcal{E} \times_{\bar{\mathcal{E}}} \mathcal{E}')_* : A^*(\mathcal{E}) \rightarrow A^*(\mathcal{E}')$  is again an isomorphism. It follows that its central fiber  $\mathcal{F}_{\text{open}} := T_{F/S}^* \times_{\bar{\mathcal{E}}_0} T_{F'/S}^*$  induces an isomorphism  $A^*(T_{F/S}^*) \rightarrow A^*(T_{F'/S}^*)$ .

Consider the fiber product

$$\mathcal{F}_{\text{loc}} := \mathbb{P}_F(T_{F/S}^* \oplus \mathcal{O}) \times_{\mathbb{P}_S(\bar{\mathcal{E}}_0 \times \mathbb{C})} \mathbb{P}_{F'}(T_{F'/S}^* \oplus \mathcal{O})$$

and  $\mathcal{F}_\infty := \mathbb{P}_F(T_{F/S}^*) \times_{\mathbb{P}_S(\bar{\mathcal{E}}_0)} \mathbb{P}_{F'}(T_{F'/S}^*)$ . Note that the push-forward map  $A_*(\mathbb{P}_F(T_{F/S}^*)) \rightarrow A_*(\mathbb{P}_F(T_{F/S}^* \oplus \mathcal{O}))$  is injective.

**Proposition 4.2.** *We have the following commutative diagrams with exact horizontal rows (induced by the localization formula in Chow groups):*

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_*(\mathbb{P}(T_{F/S}^*)) & \longrightarrow & A_*(\mathbb{P}(T_{F/S}^* \oplus \mathcal{O})) & \longrightarrow & A_*(T_{F/S}^*) \longrightarrow 0 \\ & & \mathcal{F}_\infty \downarrow & & \mathcal{F}_{loc} \downarrow & & \mathcal{F}_{open} \downarrow \\ 0 & \longrightarrow & A_*(\mathbb{P}(T_{F'/S}^*)) & \longrightarrow & A_*(\mathbb{P}(T_{F'/S}^* \oplus \mathcal{O})) & \longrightarrow & A_*(T_{F'/S}^*) \longrightarrow 0 \end{array}$$

Thus if  $\mathcal{F}_\infty$  is an isomorphism, so is  $\mathcal{F}_{loc}$ .

Note that  $\mathbb{P}_F(T_{F/S}^*) \dashrightarrow \mathbb{P}_{F'}(T_{F'/S}^*)$  is a stratified Mukai flop of type  $A_{n-2,k-1}$ . This allows us to perform inductive argument later.

**4.2. Deformations of projective local models.** Consider the rational map  $\pi : \mathbb{P}(\mathcal{E} \oplus \mathcal{O}) \dashrightarrow \mathbb{P}^1$  which extends the map  $\pi : \mathcal{E} \rightarrow \mathbb{C}$  and maps  $\mathbb{P}(\mathcal{E})$  to  $\infty$ . The map  $\pi$  is undefined exactly along  $E := \mathbb{P}(\mathcal{E}_0)$ . Blow up  $\mathbb{P}(\mathcal{E} \oplus \mathcal{O})$  along  $E$  resolves the map  $\pi$ , thus we obtain  $\hat{\pi} : \mathfrak{X} := \text{Bl}_E \mathbb{P}(\mathcal{E} \oplus \mathcal{O}) \rightarrow \mathbb{P}^1$ .

Since  $E$  is a divisor of the central fiber  $\mathbb{P}(\mathcal{E} \oplus \mathcal{O})_0 = \mathbb{P}(T_{F/S}^* \oplus \mathcal{O})$ , we have  $\hat{\pi}^{-1}(0) \simeq \mathbb{P}(T_{F/S}^* \oplus \mathcal{O})$ . When  $t \neq 0$ ,  $\mathfrak{X}_t \simeq \mathbb{P}(\mathcal{E})$  which is the compactification of  $\mathcal{E}_t$  by  $\mathbb{P}(T_{F/S}^*)$ . This gives a deformation of  $\mathbb{P}(T_{F/S}^* \oplus \mathcal{O})$  over  $\mathbb{P}^1$  with other fibers isomorphic to  $\mathbb{P}(\mathcal{E})$ .

We do a similar construction on the dual side, which gives a deformation of  $\mathbb{P}(T_{F'/S}^* \oplus \mathcal{O})$  by  $\hat{\pi}' : \mathfrak{X}' \rightarrow \mathbb{P}^1$ . We get also an induced birational map  $\mathfrak{F} : \mathfrak{X} \dashrightarrow \mathfrak{X}'$  over  $\mathbb{P}^1$  extending  $\mathfrak{F} : \mathcal{E} \dashrightarrow \mathcal{E}'$  over  $\bar{\mathcal{E}}$ . This proves

**Proposition 4.3.** *The birational map  $\mathfrak{F}$  over  $\mathbb{P}^1$ :*

$$\begin{array}{ccc} \mathfrak{X} & \overset{\mathfrak{F}}{\dashrightarrow} & \mathfrak{X}' \\ \Psi \searrow & & \swarrow \Psi' \\ & \mathfrak{X} & \\ \hat{\pi} \searrow & \downarrow & \swarrow \hat{\pi}' \\ & \mathbb{P}^1 & \end{array}$$

deforms the  $A_{n,k}$  flop  $f = \mathfrak{F}_0 : \mathbb{P}(T_{F/S}^* \oplus \mathcal{O}) \dashrightarrow \mathbb{P}(T_{F'/S}^* \oplus \mathcal{O})$  into  $A_{n-2,k-1}$  flops  $\mathfrak{F}_t : \mathbb{P}(\mathcal{E}) \dashrightarrow \mathbb{P}(\mathcal{E}')$  for  $t \neq 0$ .

Clearly for  $t \neq 0$ , the map  $\mathfrak{F}_t : \mathbb{P}(\mathcal{E}) \dashrightarrow \mathbb{P}(\mathcal{E}')$  is an isomorphism only for the case of ordinary Mukai flop, i.e.  $k = 1$ . This case has been treated in [LLW], Theorem 6.2. The blow-up argument here is motivated by it.

**Corollary 4.4.** *For projective local model of  $A_{n,k}$  flops  $X \xrightarrow{\psi} \bar{X} \xleftarrow{\psi'} X'$ , the correspondence defined by fiber product  $\mathcal{F} = X \times_{\bar{X}} X'$  induces isomorphism of Chow motives  $[X] \cong [X']$  which preserves also the Poincaré pairing.*

*Proof.* By definition, the  $A_{n,k}$  contraction satisfies

$$2 \operatorname{codim} D = \operatorname{codim} \psi(D)$$

for each irreducible component  $D$  of  $E_i$ . The deformation

$$\mathfrak{X} \xrightarrow{\Psi} \bar{\mathfrak{X}} \xleftarrow{\Psi'} \mathfrak{X}'$$

constructed by Proposition 4.3 is not isomorphic on general fibers, instead it gives  $A_{n-2,k-1}$  flops. Thus the additional deformation dimension makes it satisfying the assumption of Proposition 3.1. The result follows by noticing that the graph closure restricts to  $\mathcal{F}$  on the central fiber.  $\square$

*Remark 4.5.* Proposition 4.3 suggests an inductive structure of  $A_{n,k}$  flops. It will become more useful (e.g. for the discussion of global  $A_{n,k}$  flops or Gromov-Witten theory) after we develop detailed analysis on correspondences.

## 5. DEGENERATION OF CORRESPONDENCES

Let  $f : X \dashrightarrow X'$  be a stratified Mukai flop, say of type  $A_{n,k}$  with  $2k < n + 1$ . The aim of the following theorem is to show that the degeneration to normal cone for  $(X, F_k)$  and  $(X', F'_k)$  splits the correspondence  $\mathcal{F}^f$  defined by  $X \times_{\bar{X}} X'$  into the one  $\mathcal{F}^g$  defined by  $Y \times_{\bar{Y}} Y'$  of type  $A_{n-2,k-1}$  and its version  $\mathcal{F}_{\text{loc}}^f$  on projectivized local models relative to  $F_k$  and  $F'_k$ . Conversely we may define  $\mathcal{F}$  inductively by gluing these two parts. Here is the blow-up diagram

$$\begin{array}{ccccc}
 Y = \operatorname{Bl}_{F_k} X & \xrightarrow{\quad g \quad} & Y' = \operatorname{Bl}_{F'_k} X' & & \\
 \phi \downarrow & \searrow & \swarrow & \downarrow \phi' & \\
 X & \xrightarrow{\quad \quad} & \bar{Y} = \operatorname{Bl}_{S_k} \bar{X} & \xrightarrow{\quad \quad} & X' \\
 & \searrow \psi & \downarrow & \swarrow \psi' & \\
 & & \bar{X} & & 
 \end{array}$$

with  $g : Y \dashrightarrow Y'$  being the induced  $A_{n-2,k-1}$  flop. To save the notations we use the same symbol  $\mathcal{F}$  for  $\mathcal{F}^f$  and its local models as well if no confusion is likely to arise.

We consider degenerations to the normal cone  $W \rightarrow \mathbb{A}^1$  of  $X$ , where  $W$  is the blow-up of  $X \times \mathbb{A}^1$  along  $F_k \times \{0\}$ . Similarly we get  $W' \rightarrow \mathbb{A}^1$  for  $X'$ .

Note that the central fiber

$$W_0 := Y_1 \cup Y_2 = Y \cup X_{\text{loc}}, \quad W'_0 := Y'_1 \cup Y'_2 = Y' \cup X'_{\text{loc}},$$

where  $X_{\text{loc}} = \mathbb{P}(T_{F_k/S}^* \oplus \mathcal{O})$  and  $X'_{\text{loc}} = \mathbb{P}(T_{F'_k/S}^* \oplus \mathcal{O})$ . The intersections  $E := Y \cap X_{\text{loc}}$  and  $E' := Y' \cap X'_{\text{loc}}$  are isomorphic respectively to  $\mathbb{P}(T_{F_k/S}^*)$  and  $\mathbb{P}(T_{F'_k/S}^*)$ . The map  $f : X \dashrightarrow X'$  induces the Mukai flop of the same

type for local models:  $f : X_{loc} \dashrightarrow X'_{loc}$  and Mukai flop  $g : Y \dashrightarrow Y'$  of type  $A_{n-2,k-1}$ . Let  $p : X_{loc} \rightarrow F_k$  be the projection and similarly we get  $p'$ .

A *lifting* of an element  $a \in A^*(X)$  is a couple  $(a_1, a_2)$  with  $a_1 \in A^*(Y)$  and  $a_2 \in A^*(X_{loc})$  such that  $\phi_* a_1 + p_* a_2 = a$  and  $a_1|_E = a_2|_E$ . Similarly one defines the lifting of an element in  $A^*(X')$ .

**Theorem 5.1.** *Let  $a \in A^*(X)$  with  $(a_1, a_2)$  and  $(a'_1, a'_2)$  being liftings of  $a$  and  $\mathcal{F}a$  respectively. Then*

$$\mathcal{F}a_1 = a'_1 \iff \mathcal{F}a_2 = a'_2.$$

Moreover it is always possible to pick such liftings.

It is instructive to re-examine the Mukai case ( $k = 1$ ) first, which is stated in [LLW], Section 6. In this case  $Y = Y'$  and  $f$  is an isomorphism outside the blow-up loci  $Z = F_1$  and  $Z' = F'_1$ . Let us denote  $\mathcal{F} = \mathcal{F}_0 + \mathcal{F}_1$  with  $\mathcal{F}_0 = [\bar{\Gamma}_f] = \phi'_* \phi^*$  and  $\mathcal{F}_1$  the degenerate correspondence  $[Z \times_S Z']$ .

By Lemma 4.2 in [LLW], it is enough to prove the result for any single choice of  $a_1 = a'_1$ . Consider the standard liftings

$$\begin{aligned} a(0) &= (a_1, a_2) = (\phi^* a, p^*(a|_Z)), \\ (\mathcal{F}a)(0) &= (\phi'^* \mathcal{F}a, p'^*(\mathcal{F}a|_{Z'})). \end{aligned}$$

Since  $\phi'^* \mathcal{F}a = \phi^* a + \lambda$  with  $\lambda$  supported on  $E' = E$ , we may select lifting  $(a'_1, a'_2)$  with  $a'_1 = a_1$ . In that case,

$$(\mathcal{F}a_2 - a'_2)|_{E'} = a_2|_E - a'_2|_{E'} = a_1|_E - a'_1|_E = 0$$

by the compatibility constraint on  $E$  and the fact that  $\mathcal{F}$  restricts to an isomorphism on  $E$ . Hence  $\mathcal{F}a_2 - a'_2 = z' \in A^*(Z')$ .

To prove that  $z' = 0$ , consider

$$z' = p'_* z' = p'_* \mathcal{F}a_2 - p'_* a'_2.$$

By substituting  $\phi'_* a'_1 + p'_* a'_2 = \mathcal{F}a$ ,  $a'_1 = \phi^* a$  and  $a_2 = p^*(a|_Z)$ , we get

$$p'_* a'_2 = \mathcal{F}a - \phi'_* \phi^* a = \mathcal{F}_1 a.$$

Let  $q, q'$  be the projections of  $Z \times_S Z'$  to the two factors and

$$j : X_{loc} \times_{\bar{X}_{loc}} X'_{loc} \rightarrow Z \times_S Z'$$

the natural morphism. Then

$$z' = p'_* \mathcal{F}_{loc} p^*(a|_Z) - \mathcal{F}_1 a = q'_* j_* j^* q^*(a|_Z) - q'_* q^*(a|_Z).$$

Note that there exists a unique irreducible component in  $X_{loc} \times_{\bar{X}_{loc}} X'_{loc}$  birational to  $Z \times_S Z'$  via  $j$ , so  $j_* j^* = \text{Id}$ , which gives  $z' = 0$ .

Now we proceed for general  $A_{n,k}$  flops. It is enough to prove the result for any single choice of  $a_1$  and  $a'_1$ . To make  $a'_1 = \mathcal{F}a_1$ , notice that  $g$  is an isomorphism outside  $\tilde{F}_1 = \phi_*^{-1}(F_1)$  and  $\tilde{F}'_1 = \phi'^*{}^{-1}(F'_1)$  but we may adjust the standard lifting  $\phi'^* \mathcal{F}a$  only by elements lying over  $F'_k$ , namely classes in  $E' = \mathbb{P}(T_{F'_k/S_k}^*)$ .

The following simple observation resolves this as well as later difficulties. Recall that  $\mathcal{F} = \sum_j \mathcal{F}_j$  with  $\mathcal{F}_j = [F_j \times_{S_j} F'_j]$ .

**Lemma 5.2.** *We have decomposition of correspondences:*

$$\mathcal{F}^f = \phi'_* \mathcal{F}^g \phi^* + \mathcal{F}_k^f.$$

In particular,  $\phi'^* \mathcal{F}^f = \mathcal{F}^g \phi^*$  modulo  $A^*(E')$ .

*Proof.* This follows from the definition and the base change property of fiber product.  $\square$

Thus we may pick

$$a_1 = \phi^* a, \quad a'_1 = \mathcal{F}^g(\phi^* a) = \mathcal{F}a_1.$$

Then

$$\begin{aligned} (\mathcal{F}a_2 - a'_2)|_{E'} &= \mathcal{F}(a_2|_E) - a'_2|_{E'} \\ &= \mathcal{F}(a_1|_E) - a'_1|_{E'} = (\mathcal{F}a_1 - a'_1)|_{E'} = 0 \end{aligned}$$

and so  $\mathcal{F}a_2 - a'_2 = z' \in A^*(F'_k)$ .

To prove that  $z' = 0$ , consider

$$z' = p'_* z' = p'_* \mathcal{F}a_2 - p'_* a'_2.$$

By substituting  $\phi'_* a'_1 + p'_* a'_2 = \mathcal{F}a$ ,  $a'_1 = \mathcal{F}\phi^* a$  and  $a_2 = p^*(a|_{F_k})$ , we get

$$p'_* a'_2 = \mathcal{F}a - \phi'_* \mathcal{F}\phi^* a = \mathcal{F}_k a$$

by the above lemma.

Let  $q, q'$  be the projections of  $F_k \times_{S_k} F'_k$  to the two factors and

$$j : X_{loc} \times_{\bar{X}_{loc}} X'_{loc} \rightarrow F_k \times_{S_k} F'_k$$

the natural morphism. Then

$$z' = p'_* \mathcal{F}_{loc} p^*(a|_{F_k}) - \mathcal{F}_k a = q'_* j_* j^* q^*(a|_{F_k}) - q'_* q^*(a|_{F_k}).$$

Note that there exists a unique irreducible component in  $X_{loc} \times_{\bar{X}_{loc}} X'_{loc}$  birational to  $F_k \times_{S_k} F'_k$  via  $j$ , so  $j_* j^* = \text{Id}$ , which gives  $z' = 0$ . The proof is complete.

Besides correspondences, we also need to understand the effect on the Poincaré pairing under degeneration. We will in fact degenerate classical cup product and this works for any degenerations to normal cones  $W \rightarrow X \times \mathbb{A}^1$  with respect to  $Z \subset X$ . Let  $W_0 = Y_1 \cup Y_2$ , where  $\phi : Y_1 = Y \rightarrow X$  is the blow up along  $Z$ ,  $p : Y_2 = \tilde{E} = \mathbb{P}_Z(N_{Z/X} \oplus \mathcal{O}) \rightarrow Z$  is the local model and  $Y_1 \cap Y_2 = E$  is the  $\phi$  exceptional divisor. Let  $i_1 : E \hookrightarrow Y_1$ ,  $i_2 : E \hookrightarrow Y_2$ .

**Lemma 5.3.** *Let  $a, b \in H^*(X)$ . Then for any lifting  $(a_1, a_2)$  of  $a$  and any lifting  $(b_1, b_2)$  of  $b$ , the pair  $(a_1 b_1, a_2 b_2)$  is a lifting of  $ab$ .*

*In particular, if  $a, b$  are of complementary degree, then we have an orthogonal splitting of the Poincaré pairing:  $(a \cdot b)_X = (a_1 \cdot b_1)_{Y_1} + (a_2 \cdot b_2)_{Y_2}$ .*

*Proof.* We compute

$$a.b = \phi_* a_1.b + p_* a_2.b = \phi_*(a_1.\phi^*b) + p_*(a_2.p^*(b|_Z)).$$

Since  $a_1.\phi^*b|_E = a_2.p^*(b|_Z)|_E$ ,  $(a_1b_1, a_2b_2)$  is a lifting of  $ab$  for the special lifting  $(b_1, b_2) = (\phi^*b, p^*(b|_Z))$  of  $b$ . By [LLW], Lemma 4.2, any other lifting of  $b$  is of the form  $(b_1 + i_{1*}e, b_2 - i_{2*}e)$  for some class  $e$  in  $E$ .

Since  $i_{1*}a_1.e = i_{2*}a_2.e$  is a class  $e' \in H^*(E)$ . The correction terms are

$$a_1.i_{1*}e = i_{1*}(i_{1*}^*a_1.e) = i_{1*}e', \quad -a_2.i_{2*}e = -i_{2*}(i_{2*}^*a_2.e) = -i_{2*}e'.$$

The lemma then follows from

$$i_{1*}^*(i_{1*}e') = e'.c_1(N_{E/Y_1}) = -e'.c_1(N_{E/Y_2}) = i_{2*}^*(-i_{2*}e')$$

and  $\phi_*i_{1*}e' - p_*i_{2*}e' = (\phi|_E)_*e' - (p|_E)_*e' = 0.$   $\square$

**Theorem 5.4.** *For  $A_{n,k}$  flops,  $\mathcal{F}$  induces an isomorphism on Chow motives and the Poincaré pairing. Moreover, the cohomology rings are also isomorphic.*

*Proof.* If  $f : X \dashrightarrow X'$  is an  $A_{n,k}$  flop,  $f \times \text{id} : X \times T \dashrightarrow X' \times T$  is also an  $A_{n,k}$  flop. Thus by the identity principle, to prove  $[X] \cong [X']$  we only need to prove the equivalence of Chow groups under  $\mathcal{F}$  for any  $A_{n,k}$  flop.

We prove this for all  $n$  with  $n+1 > 2k$  by induction on  $k$ . We start with  $k=0$ , which is trivial. (We may also start with  $k=1$  which is proved in [LLW].)

Given  $k \geq 1$ , by Theorem 5.1, the equivalence of Chow groups is reduced to the  $A_{n-2,k-1}$  case and the local  $A_{n,k}$  case. The former is true by induction. The later again follows by induction and Proposition 4.2 or by Corollary 4.4 directly.

The same induction procedure proves the isomorphism of Poincaré pairings by using Theorem 5.1, Lemma 5.3 and Corollary 4.4.

To prove isomorphism on cohomology rings we need to proceed more carefully. By induction on  $k$  and using Theorem 5.1, Lemma 5.3 the problem is reduced to the local  $A_{n,k}$  case.

By Proposition 4.3, the local  $A_{n,k}$  flop  $f$  can be deformed into  $A_{n-2,k-1}$  flops  $\mathfrak{F}_t$  for  $t \neq 0$ . The key point is that the correspondences  $\mathfrak{F}_t$ 's are compatible for all  $t$  since they all come from the graph closure of  $\mathfrak{F}$ . The classical cohomology ring is invariant under (continuous) deformations, so the problem is reduced to  $A_{n-2,k-1}$  flops which holds by induction.  $\square$

## 6. QUANTUM INVARIANCE

**6.1. An algebro-geometrical proof.** The aim of this section is to complete the proof of Theorem 1.1, namely to prove the equivalence of Gromov-Witten theory under stratified Mukai flops.

As before, we will give the proof only for  $A_{n,k}$  flops. The proof proceeds by induction on  $k \in \mathbb{N}$  (for all  $n$  with  $n+1 > 2k$ ).

As in the precedent section, we consider degenerations to the normal cone  $W \rightarrow \mathbb{A}^1$  of  $X$  and  $W' \rightarrow \mathbb{A}^1$  of  $X'$ . Note that the map  $f : X \dashrightarrow X'$

induces the Mukai flop of the same type for local models:  $f : X_{loc} \dashrightarrow X'_{loc}$  and Mukai flop  $g : Y \dashrightarrow Y'$  of type  $A_{n-2,k-1}$ .

By the degeneration formula (for the algebraic version used here, c.f. [Li]), any Gromov-Witten invariant on  $X$  splits into products of relative invariants of  $(Y, E)$  and  $(X_{loc}, E)$ :

$$\langle a \rangle_{g,n,\beta}^X = \sum_{\eta \in \Omega(g,\beta)} C_\eta \left( \langle a_1 \rangle_{\Gamma_1}^{\bullet(Y,E)} \cdot \langle a_2 \rangle_{\Gamma_2}^{\bullet(X_{loc},E)} \right)^{E^\rho}.$$

Here  $\rho$  is the number of gluing points (in  $E$ ). The relative invariants take values in  $H^*(E^\rho)$  and the formula is in terms of the Poincaré pairing of  $E^\rho$ .

We apply it to  $X'$  as well and get:

$$\langle \mathcal{F}a \rangle_{g,n,\mathcal{F}\beta}^{X'} = \sum_{\eta' \in \Omega(g,\mathcal{F}\beta)} C_{\eta'} \left( \langle a'_1 \rangle_{\Gamma'_1}^{\bullet(Y',E')} \cdot \langle a'_2 \rangle_{\Gamma'_2}^{\bullet(X'_{loc},E')} \right)^{E'^\rho}.$$

There is an one to one correspondence between admissible triples  $\eta = (\Gamma_1, \Gamma_2, I_\rho)$  and  $\eta' = (\Gamma'_1, \Gamma'_2, I_{\rho'})$  via  $\eta' := \mathcal{F}\eta$ . The combinatorial structure is kept the same, while the curve classes are related by  $\mathcal{F}$ . We do still need the cohomology class splitting on  $X$  and  $X'$  be to compatible.

By Theorem 5.1 we may split the cohomology classes  $a \in H^*(X)^{\oplus m}$  into  $(a_1, a_2)$  with  $a_i \in H^*(Y_i)^{\oplus m}$  and  $\mathcal{F}a \in H^*(X')^{\oplus m}$  into  $(a'_1, a'_2)$  with  $a'_i \in H^*(Y'_i)^{\oplus m}$ , such that

$$\mathcal{F}a_1 = a'_1, \quad \mathcal{F}a_2 = a'_2.$$

By Theorem 5.4, the Poincaré pairing is preserved by  $\mathcal{F}$  under stratified Mukai flops  $E \dashrightarrow E'$ , the same holds true for  $E^\rho \dashrightarrow E'^\rho$  by  $\mathcal{F}^\rho$ , which for simplicity is still denoted by  $\mathcal{F}$ . Thus by the degeneration formula the problem is reduced to showing that  $\mathcal{F}$  maps the relative invariants of  $(Y, E)$  and  $(X_{loc}, E)$  to the corresponding ones of  $(Y', E')$  and  $(X'_{loc}, E')$ .

**Lemma 6.1.**  $\mathcal{F}$  maps isomorphically the relative GW invariants of  $(Y, E)$  to those of  $(Y', E')$ .

*Proof.* By construction the birational map  $g : Y \dashrightarrow Y'$  is a Mukai flop of type  $A_{n-2,k-1}$ . The birational map  $E \dashrightarrow E'$  is also a Mukai flop of type  $A_{n-2,k-1}$ , which is compatible with  $g$ .

By the induction hypothesis, the absolute GW invariants of  $Y$  (resp.  $E$ ) are mapped isomorphically to those of  $Y'$  (resp.  $E'$ ) under  $\mathcal{F}$ . The lemma then follows from theorem 2 in [MP].  $\square$

**Lemma 6.2.**  $\mathcal{F}$  maps isomorphically the relative GW invariants of  $(X_{loc}, E)$  to those of  $(X'_{loc}, E')$ .

*Proof.* By Proposition 4.3, the  $A_{n,k}$  flop  $f : X_{loc} \dashrightarrow X'_{loc}$  can be projectively deformed into  $A_{n-2,k-1}$  flops  $\mathfrak{F}_t : \mathfrak{X}_t \dashrightarrow \mathfrak{X}'_t$  for  $t \neq 0$ . This deformation keeps the  $A_{n-2,k-1}$  flop on the infinity divisors  $E \dashrightarrow E'$  unchanged.

Since the correspondences  $\mathcal{F}_t$ 's are all compatible, by the *deformation invariance of relative Gromov-Witten theory*, the lemma is reduced to the relative theory of  $A_{n-2,k-1}$  case. But this is exactly the content of Lemma 6.1.  $\square$

*Remark 6.3.* The degeneration formula is in terms of the Poincaré pairing of relative GW invariants. Thus *invariance of the Poincaré pairing* is crucial in our study. Indeed, as is explained in [LLW], the Poincaré pairing is the 2-point function which together with 3-point functions determine the (small) quantum product. So far this is *the only constraint* we have found for the expected correspondence  $\mathcal{F}$  under  $K$  equivalence to be *canonical*.

**6.2. A differential-geometrical heuristic.** A closer look at our construction indicates that any  $A_{n,k}$  flop  $f_0 : X \dashrightarrow X'$  can be deformed into isomorphisms  $f_t : \mathfrak{X}_t \cong \mathfrak{X}'_t$  in the almost complex category. The isomorphism can be constructed inductively via deformation to the normal cone.

Indeed, if both  $X_{loc} \dashrightarrow X'_{loc}$  and  $Y \dashrightarrow Y'$  admit small deformations into isomorphisms in the almost complex category and satisfy certain compatibility along  $E \dashrightarrow E'$ , then it should be possible to glue both deformations into deformations  $f_t$  of  $\mathfrak{X}_t \dashrightarrow \mathfrak{X}'_t$ . The deformation of  $Y \dashrightarrow Y'$  is achieved by induction. The deformation of  $X_{loc} \dashrightarrow X'_{loc}$  is achieved by Proposition 4.3 and then by induction. In particular,  $X$  and  $X'$  are diffeomorphic.

The morphisms  $f_t$ 's are not just isomorphisms of almost complex manifolds. Indeed, given real symplectic forms  $\omega$  (resp.  $\omega'$ ) on  $X$  (resp.  $X'$ ), we should be able to construct  $X_t$  (resp.  $X'_t$ ) very close to  $X$  (resp.  $X'$ ) so that it (the almost complex structure) is still tamed by  $\omega$  (resp.  $\omega'$ ). Thus through  $f_t$  we may conclude that  $(X, \omega)$  is symplectic deformation equivalent to  $(X', \omega')$  when  $\omega$  and  $\omega'$  are *chosen suitably*. In particular they have isomorphic Gromov-Witten theory.

Instead of the above heuristic argument, we have chosen to work within algebraic geometry via the degeneration formula of Gromov-Witten theory in this section. Besides the reason of rigor, the more important issue is that we believe the inductive structure developed in this paper can be applied to much more general situations than the stratified Mukai flops.

Nevertheless the viewpoint of  $C^\infty$  symplectic deformation fits naturally into the deformation/decomposition conjecture about  $K$  equivalent varieties [W] and deserves further study. We plan to write it up in a later work.

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