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► **To cite this version:**

Jacques Demongeot, Nicolas Glade, Loïc Forest. Liénard systems and potential-Hamiltonian decomposition I - Methodology. Comptes rendus de l'Académie des sciences. Série I, Mathématique, 2007, 344, pp 121-126. hal-00211816

HAL Id: hal-00211816

<https://hal.science/hal-00211816>

Submitted on 23 Jan 2008

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Systèmes de Liénard et décomposition potentielle-Hamiltonienne I - Méthodologie

Liénard systems and potential-Hamiltonian decomposition I - Methodology

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Abstract

Following the Hodge decomposition of regular vector fields we can decompose the second member of any Liénard system into 2 (non unique) polynomials, first corresponding to potential and second to Hamiltonian dynamics. This polynomial Hodge decomposition is called potential-Hamiltonian, denoted PH-decomposition, and we give it for any polynomial differential system of dimension 2. We will give in a further Note an algorithm expliciting the PH-decomposition in the neighborhood of particular orbits like a limit-cycle for Liénard systems, the method being applicable for any polynomial differential system of dimension 2.

Résumé

Un système de Liénard est un système différentiel du second ordre, défini sur \mathbb{R}^2 , du type : $dx/dt = y$, $dy/dt = -g(x) + yf(x)$ où g et f sont des polynômes. Un tel système est susceptible d'être décomposé, de manière non unique, en 2 parties polynomiales, l'une potentielle et l'autre hamiltonienne, c'est-à-dire qu'il existe deux polynômes P et H , définis sur \mathbb{R}^2 à valeurs dans \mathbb{R} , vérifiant $dx/dt = -\partial P/\partial x + \partial H/\partial y$, $dy/dt = -\partial P/\partial y - \partial H/\partial x$.

On montre, en utilisant la décomposition de Hodge des champs de vecteurs réguliers, que le second membre d'un tel système est décomposable en 2 polynômes, l'un correspondant à une dynamique de gradient et l'autre à une dynamique hamiltonienne. Cette décomposition de Hodge polynomiale est appelée potentielle-hamiltonienne, notée PH-décomposition, et nous en donnons la formule pour tout système différentiel polynomial du plan. Nous donnons, dans une Note ultérieure, un algorithme permettant d'obtenir une formule explicite de la PH-décomposition au voisinage d'orbites particulières, telles qu'un cycle limite dans le cas des systèmes de Liénard, la méthode étant applicable à tout système différentiel polynomial du plan.

Version abrégée

Les systèmes de Liénard sont des Equations Différentielles Ordinaires de Dimension 2 (2D-EDO) du type : $dx/dt = y$, $dy/dt = -g(x) + yf(x)$, où f et g sont des polynômes.

Le système de van der Pol (cf. Figure 1) en est un bon exemple [2-4], utilisé pour modéliser de nombreux systèmes biologiques régulés, dont le système cardiaque. Les polynômes g et f d'un système de van der Pol

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sont définis par : $g(x) = x$ et $f(x) = \mu(1 - x^2)$

L'unique cycle-limite d'un système de van der Pol, lorsqu'il existe, est une courbe non algébrique difficile à approximer et il peut être très utile, dans les applications, d'en avoir une estimation polynomiale. Nous proposons donc, de manière générale, une décomposition des EDO régulières, dite potentielle-hamiltonienne, dont l'existence est fondée théoriquement sur la décomposition de Hodge [5-8].

Une ODE est dite potentiel-hamiltonienne décomposable, s'il existe un couple de polynômes (P, H) , tel que: $dx/dt = -\partial P/\partial x + \partial H/\partial y$, $dy/dt = -\partial P/\partial y - \partial H/\partial x$

La partie du flot de vecteurs vitesse définie par P correspond à une dynamique de descente de plus fort gradient sur la surface représentative de P et la partie définie par H correspond à une dynamique sur les courbes de niveau de la surface représentative de H .

Un exemple de système hamiltonien pur classique est celui du système de Lotka-Volterra, utilisé pour modéliser les interactions proie/prédateur [9]. Des exemples de systèmes différentiels purs potentiels sont les systèmes de type n-switch, à second membre défini par une cinétique de Hill compétitive (équation 4), pour lesquels on peut obtenir explicitement la formule de définition de P et donc tracer la surface correspondante et en localiser les minima, qui sont les états stationnaires du n-switch (cf. Figure 3). Un exemple de système polynomial mixte potentiel-hamiltonien est le système chimique de Lotka, pour lequel il est facile d'exhiber P et H [22].

Pour généraliser la décomposition ci-dessus, on montre un théorème donnant une formule générale de décomposition potentielle-hamiltonienne (non unique), pour des ODE du type : $dx/dt = f(x, y)$, $dy/dt = g(x, y)$, où f et g sont des polynômes. A partir de cette formule générale, nous donnerons, dans deux prochaines notes, un algorithme permettant d'obtenir la décomposition potentielle-hamiltonienne dans le cas des systèmes de Liénard, puis de l'appliquer à des problèmes biologiques, dans lesquels il existe un grand intérêt à décomposer le flot en une partie potentielle (dont les paramètres sont responsables d'une modulation de fréquence d'un signal biologique, dans le cas d'une trajectoire du système de type cycle-limite) et une partie hamiltonienne (dont les paramètres sont responsables d'une modulation d'amplitude).

1. Introduction

Liénard systems [1] are 2-dimensional ordinary differential equations (2D ODEs) defined by $dx/dt = y$, $dy/dt = -g(x) + yf(x)$, where g and f are polynomials. The use of Liénard systems is universal in biological modeling, especially for physiological processes. A simple example of a Liénard system is given by the van der Pol system used to model the cardiac and respiratory rhythms [2-4], where :

$$g(x) = x, \quad f(x) = \mu(1 - x^2) \tag{1}$$

For $\mu > 0$, it presents a limit-cycle bifurcating from the stationary state located at the origin, which becomes progressively (as the anharmonic parameter μ increases) very far from a circle, and reciprocally becomes circular when μ tends to 0 (fig. 1).

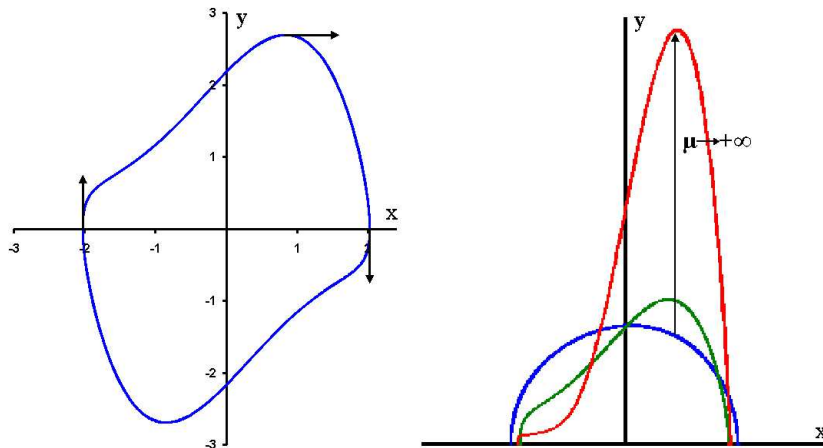


Figure 1. – van der Pol system ($\mu = 1$, left) and μ tending to infinity in the upper phase plane (right)

For many applications, it is essential to approximate the limit-cycle (which is a non-algebraic closed curve of the plane) by a polynomial closed curve. A manner to do it is to decompose the second member of the van der Pol differential system into 2 polynomial parts, a potential one and a Hamiltonian one, contour lines of the Hamiltonian part being good candidates for estimating the limit-cycle.

2. Potential-Hamiltonian decomposition and Hodge decomposition

A 2D ODE $dx/dt = f(x, y)$, $dy/dt = g(x, y)$ is called potential-Hamiltonian decomposable (PH-decomposable), if and only if there is at least one couple of polynomials (P, H) such as:

$$dx/dt = -\partial P/\partial x + \partial H/\partial y, \quad dy/dt = -\partial P/\partial y - \partial H/\partial x \quad (2)$$

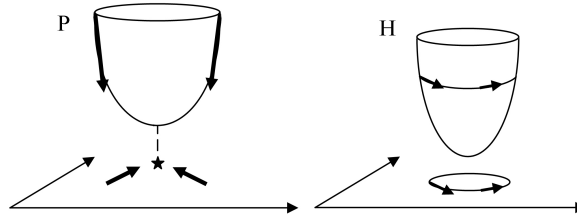


Figure 2. – Schematic representation of the gradient (potential P) flow (left) and of the conservative (Hamiltonian H) flow (right)

The potential flow is the projection on the 2D phase plane of a steepest descent velocity vector field on the surface defined by the potential P, and the Hamiltonian flow the vector field tangential to contour lines of the surface generated by H (fig. 2). The Hodge decomposition [5–8] ensures the existence of regular P and H functions (e.g. L^2 or C^1) for sufficiently smooth differential systems, e.g. ODEs having a second member of the same level of regularity L^2 or C^1 . We try here to obtain explicit expressions (if possible polynomials) for P and H. As example, we have the Lotka-Volterra system, a pure Hamiltonian system, introduced by V. Volterra to account for the fluctuations observed in the struggle for life between a prey population of size x and a predator population of size y [9]:

$$\begin{aligned} dx/dt &= x(a - by), & dX/dt &= \partial H/\partial Y \\ dy/dt &= y(cx - d), & dY/dt &= -\partial H/\partial X, \end{aligned} \quad \text{with } H(X, Y) = -ce^X + dX - be^Y + aY \quad (3)$$

by changing variables $X = \log(x)$ and $Y = \log(y)$ (population 'affinities'). The trajectories are just contour lines of H.

3. A pure potential system, the n-switch model

A n-switch model is used to formalize fully connected interaction networks with only inhibitory interactions. Such a system has been proposed in [10] to model genetic and metabolic regulations and is encountered in plant growth [11–13], embryogenesis [14, 15] and neural networks [16, 18], in which hormones (like auxine), proteins (like transduction peptide) or neuro-transmitters (like Gaba) secreted by a part of the system (e.g. apex in plants) inhibit the growth or activity of the other parts (e.g. cotyledonary buds in plants) (fig. 3). We can represent each inhibition by a Hill competition with a decay rate ν , constant entry coefficients α_i 's, a cooperativity coefficient c, a maximal reaction rate σ and affinity coefficients a_i 's as in the following system of equations, where x_i denotes the concentration (or the size, or the amount) of the component X_i in inhibitory interaction with all other components of the n-switch system [10]:

$$\forall i = 1..n, \quad \frac{dx_i}{dt} = -\nu x_i + \frac{\sigma a_i x_i^c}{1 + \sum_{j=1}^n a_j x_j^c} + \alpha_i \quad (4)$$

If we consider a n-switch with $c = 1$ and $\forall i = 1..n, \alpha_i = 0, a_i = 1$: $dx_i/dt = -\nu x_i + \sigma x_i/(1 + \sum_{j=1}^n x_j)$, with $\sigma > \nu$, then the non-zero stationary states have $m \geq 1$ components x_i^* equal to $(\sigma - \nu)/m\nu > 0$ and the others equal to 0. The Jacobian matrix in these stationary states has only one non-zero eigenvalue $\lambda = \nu(\nu - \sigma)/\sigma < 0$ and we need a complete stability analysis done in [10] in the general case. Studying the

stability basins topography is easier if we decompose the flow. By changing variables $y_i = (x_i)^{1/2}$ we see that the system is purely gradient with : $P(y) = \nu \sum_i y_i^2/4 - \sigma \log(1 + \sum_i a_i y_i^{2c})/4c - \sum_i \alpha_i \log y_i/2$ (cf. in fig. 3 a 2-switch with 3 stable stationary states). Replacing Hill competition by an allosteric inhibition, the system remains pure potential (cf. [19, 20] for a first proof of this result):

$$dx_i/dt = -\nu x_i + \sigma x_i ((1+x_i)^{c-1} \prod_{j \neq i} (1+x_j)^c + a_i (1+a_i x_i)^{c-1} \prod_{j \neq i} (1+a_j x_j)^c) / (\prod_{k=1}^n (1+x_k)^c + \prod_{k=1}^n (1+a_k x_k)^c) + \alpha_i.$$

Hence, because $dy_i/dt = 1/(2y_i) dx_i/dt$, we have $dy_i/dt = -\partial P/\partial y_i$, with:

$$P(y) = \nu \sum_{i=1}^n y_i^2/4 - \sigma \log(\prod_{i=1}^n (1+y_i^2)^c + \prod_{i=1}^n (1+a_i y_i^2)^c)/4c - \sum_{i=1}^n \alpha_i \log y_i/2, \text{ where } y_i = (x_i)^{1/2}.$$

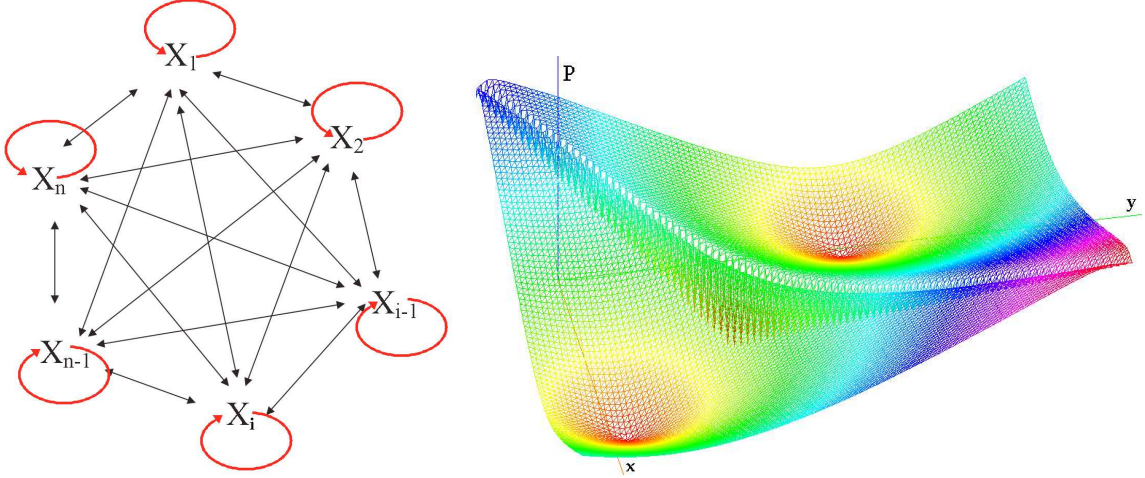


Figure 3. – Fully connected inhibitory interactions (except eventually auto-loops) of a n-switch (left) and potential P, if n=2 (right).

We represent on Figure 3 (right) the surface of the potential P for a 2-switch ($\nu = 1, c=2, \sigma = 2, a_i = 0.1, \alpha = 1$) over the square $[-5,35] \times [-5,35]$. The local minima are located in $(1.24, 24)$, $(24, 1.24)$, and $(12.35, 12.35)$ on the bisectrix.

4. Application to mixed potential-Hamiltonian systems, the Lotka and the Liénard equations

Chemical dynamical systems frequently offer equations susceptible to be PH-decomposed [21]. A historical example is the Lotka system [22] built to model a bi-reactant oscillatory reaction:

$$dx/dt = -xy - Kx - Ly, \quad dy/dt = xy + Ly \quad (5)$$

The decomposition is obtained by choosing

$$P(x, y) = (K + y)x^2/2 - Ly^2/2 - x^3/6 \text{ and } H(x, y) = -yx^2/2 - Ly^2/2 - x^3/6.$$

An other simple non-linear system is the Lorenz pendulum [23]:

$$dx/dt = y, \quad dy/dt = -ax - x^3 \quad (6)$$

This system is pure Hamiltonian with $H(x, y) = y^2/2 + ax^2/2 + x^4/4$. More generally, Liénard systems are PH-decomposable, if h and l are polynomials of order respectively m and p , and if $h^{(k)}$ denotes the k^{th} derivative of h , we have:

$$P(x, y) = \sum_{k=1}^n (-1)^k f^{(2k-2)} y^{2k} / (2k)! + \sum_{k=1}^m (-1)^k h^{(2k-2)} y^{2k-1} / (2k-1)! + \sum_{k=1}^p (-1)^k l^{(2k-2)} x^{2k-2} / (2k-2)!$$

$$H(x, y) = \int (g + h) + y^2/2 + \sum_{k=1}^n (-1)^k f^{(2k-1)} y^{2k+1} / (2k+1)! \\ + \sum_{k=1}^m (-1)^k h^{(2k-1)} y^{2k} / (2k)! + \sum_{k=1}^p (-1)^{k+1} l^{(2k-1)} x^{2k-1} / (2k-1)!$$

Proof is just the consequence of the following Proposition, Lemmas and Theorem, available for any 2D

ODE with sufficiently regular second member.

Proposition 1 *Let's consider $V = f(x, y)\partial./\partial x + g(x, y)\partial./\partial y$ being a polynomial vector field, then there is a (non unique) PH-decomposition of V into two parts, a potential part, which is the gradient of a polynomial P and a conservative part, which is the Hamiltonian of a polynomial H such that: $V = -\text{grad}P + \text{ham}H$*

Proof 1 *Calculate $\text{div}V = D(x, y)$ and use complex coordinates $z = x + iy$, $\bar{z} = x - iy$. Then there is a (non unique) polynomial P such that $\partial^2 P/\partial z\partial\bar{z} = \partial^2 P/\partial x^2 + \partial^2 P/\partial y^2 = -D(x, y)$. The divergence $V + \text{grad}P$ equals 0. Hence this divergence is a Hamiltonian polynomial vector field deriving from a Hamiltonian function H : $V + \text{grad}P = (\partial H/\partial y, -\partial H/\partial x) = \text{ham}H$ ■*

Lemma 1 *Let's consider the 2D ODE : $dx/dt = 0$, $dy/dt = 0$ then its PH-decomposition can be any kernel of the type $P(x, y) = (f + ig)(x + iy) + (f - ig)(x - iy)$, $H(x, y) = -i(f + ig)(x + iy) + i(f - ig)(x - iy)$, f and g being differentiable functions.*

Proof 2 *It's sufficient to check that: $-\partial P/\partial x + \partial H/\partial y = 0$ and $-\partial P/\partial y - \partial H/\partial x = 0$ ■*

Lemma 2 *Let's consider the 2D ODE:*

$$dx/dt = f_1(x, y) + f_2(x, y), \quad dy/dt = g_1(x, y) + g_2(x, y) \quad (7)$$

where the functions f_1, f_2, g_1, g_2 are infinitely differentiable and integrable. If $f^{[n]^{(m)}}(x, y)$ represents the function obtained from f by integrating n times in x and derivating m times in y then, for any integer $k \geq 4$, the potential P and the Hamiltonian H defined by :

$$P = \sum_{j \leq k, j=4p+1} -f_1^{[j](j-1)} - g_1^{(j-1)[j]} + \sum_{j \leq k, j=4p+2} -f_2^{(j-1)[j]} - g_2^{[j](j-1)} +$$

$$\sum_{j \leq k, j=4p+3} f_1^{[j](j-1)} \pm g_1^{(j-1)[j]} + \sum_{j \leq k, j=4p+4} f_2^{(j-1)[j]} + g_2^{[j](j-1)}$$

$$H = \sum_{j \leq k, j=4p+1} f_2^{(j-1)[j]} - g_2^{[j](j-1)} + \sum_{j \leq k, j=4p+2} f_1^{[j](j-1)} - g_1^{(j-1)[j]} +$$

$$\sum_{j \leq k, j=4p+3} -f_2^{(j-1)[j]} + g_2^{[j](j-1)} + \sum_{j \leq k, j=4p+4} -f_1^{[j](j-1)} + g_1^{(j-1)[j]}$$

constitute the PH-decomposition of the following ODE's:

$$\text{for } k = 4p + 1, \quad dx/dt = f_1 + f_2 + g_1^{(k)[k]} - g_2^{k}, \quad dy/dt = g_1 + g_2 + f_1^{k} - f_2^{(k)[k]}$$

$$\text{for } k = 4p + 2, \quad dx/dt = f_1 + f_2 + f_1^{k} + f_2^{(k)[k]}, \quad dy/dt = g_1 + g_2 + g_1^{(k)[k]} + g_2^{k}$$

$$\text{for } k = 4p + 3, \quad dx/dt = f_1 + f_2 - g_1^{(k)[k]} + g_2^{k}, \quad dy/dt = g_1 + g_2 - f_1^{k} + f_2^{(k)[k]}$$

$$\text{for } k = 4p + 4, \quad dx/dt = f_1 + f_2 - f_1^{k} - f_2^{(k)[k]}, \quad dy/dt = g_1 + g_2 - g_1^{(k)[k]} - g_2^{k}$$

If f_1, f_2, g_1, g_2 are polynomials of order less than n , then all these ODE's are equal to (7), since $k > n$.

Proof 3 *It's sufficient to check that: $dx/dt = -\partial P/\partial x + \partial H/\partial y$, $dy/dt = -\partial P/\partial y - \partial H/\partial x$ and $f_1^{k} = f_2^{(k)[k]} = g_1^{(k)[k]} = g_2^{k} = 0$, $k > n$ ■*

Theorem 1 *Let's consider the 2D ODE: $dx/dt = f(x, y)$, $dy/dt = g(x, y)$ where f and g are polynomials: $f(x, y) = \sum_{i \geq j} a_{ij} x^i y^j + \sum_{i < j} c_{ij} x^i y^j$ and $g(x, y) = \sum_{i \geq j} b_{ij} x^i y^j + \sum_{i < j} d_{ij} x^i y^j$*

Then the most general PH-decomposition is obtained by choosing

$$P = \sum_{j \leq k, j=4p+1} -f_1^{[j](j-1)} - g_1^{(j-1)[j]} + \sum_{j \leq k, j=4p+2} -f_2^{(j-1)[j]} - g_2^{[j](j-1)} +$$

$$\sum_{j \leq k, j=4p+3} f_1^{[j](j-1)} + g_1^{(j-1)[j]} + \sum_{j \leq k, j=4p+4} f_2^{(j-1)[j]} + g_2^{[j](j-1)} + (f + ig)(x + iy) + (f - ig)(x - iy)$$

$$H = \sum_{j \leq k, j=4p+1} f_2^{(j-1)[j]} - g_2^{[j](j-1)} + \sum_{j \leq k, j=4p+2} f_1^{[j](j-1)} - g_1^{(j-1)[j]} + \sum_{j \leq k, j=4p+3} -f_2^{(j-1)[j]} + g_2^{[j](j-1)} + \sum_{j \leq k, j=4p+4} -f_1^{[j](j-1)} + g_1^{(j-1)[j]} - i(f + ig)(x + iy) + i(f - ig)(x - iy)$$

where f, g are any differentiable functions and where:

$$f_1 = \sum_{i > j} a_{ij} x^i y^j + \frac{1}{2} \sum_{i=j} a_{ij} x^i y^j, \quad f_2 = \sum_{i < j} c_{ij} x^i y^j + \frac{1}{2} \sum_{i=j} a_{ij} x^i y^j$$

$$g_1 = \sum_{i > j} b_{ij} x^i y^j + \frac{1}{2} \sum_{i=j} b_{ij} x^i y^j, \quad g_2 = \sum_{i < j} d_{ij} x^i y^j + \frac{1}{2} \sum_{i=j} b_{ij} x^i y^j$$

Proof 4 The existence of a PH-decomposition is ensured by the above Proposition. Then to obtain P and H , it's sufficient to apply the above Lemmas 1 and 2 ■

5. Conclusion

In this article, we have shown the possibility to decompose any sufficiently regular 2D-differential system into two parts : a potential one having a dissipative gradient-like behaviour responsible, in general, for the amplitude of a periodic signal generated by the system, and another one, Hamiltonian, expressing the conservative part of the flow responsible for the frequency of this periodic signal. This decomposition, called PH-decomposition if the potential P and the Hamiltonian H are polynomials, is non unique, but we easily conceive that we can estimate the coefficients of P and H by obliging the dynamical flow to pass through certain known points on orbits of the studied system. Then we have obtained a way to algebraically estimate these orbits. The algorithm for doing this approximation (in particular in the case of limit-cycles) will be given in a further Note.

Acknowledgements

We are very indebted to R. Thom (died in October 2002, formerly President of the French Society for Theoretical Biology) for opening the new field of modeling biological systems with a qualitative approach using singularity theory, and to J.P. Françoise for stimulating discussions and helpful suggestions.

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