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Amel Bentata

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A note about conditional Ornstein-Uhlenbeck processes

Amel Bentata*

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Abstract

In this short note, the identity in law, which was obtained by P. Salminen [6], between on one hand, the Ornstein-Uhlenbeck process with parameter γ , killed when it reaches 0, and on the other hand, the 3-dimensional radial Ornstein-Uhlenbeck process killed exponentially at rate γ and conditioned to hit 0, is derived from a simple absolute continuity relationship.

Keywords: Ornstein-Uhlenbeck process, Doob's h-transform, absolute continuity relationship.

Mathematical Subject Classification: 60J60; 60G15.

e-mail address: bentata@clipper.ens.fr

- All probability distributions considered in this note are defined on $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$, where $(X_t, t \geq 0)$ denotes the coordinate process, and $\mathcal{F}_t = \sigma\{X_s, s \leq t\}$, its natural filtration.
- For $\gamma > 0$, and $a > 0$, we denote by \mathbb{P}_a^γ the law of the Ornstein-Uhlenbeck process with parameter γ , starting from a , i.e : under \mathbb{P}_a^γ , one has :

$$X_t = a + B_t - \gamma \int_0^t X_s ds \quad (1)$$

for a Brownian motion $(B_t, t \geq 0)$, starting from 0.

*Université Paris 6, Laboratoire de Probabilités et Modèles Aléatoires, CNRS-UMR 7599, 16, rue Clisson, 75013 Paris Cedex, France

Note that $(X_t \exp(\gamma t), t \geq 0)$ is a \mathbb{P}_a^γ martingale which is indeed equal to :

$$a + \int_0^t \exp(\gamma s) dB_s, t \geq 0, \quad (2)$$

thus leading us to Doob's well-known representation of the Ornstein-Uhlenbeck process (see [2]) :

$$X_t = e^{-\gamma t} \left(a + \beta \left(\frac{e^{2\gamma t} - 1}{2\gamma} \right) \right), t \geq 0, \quad (3)$$

for another Brownian motion $(\beta(u), u \geq 0)$, starting from 0.

- In this note, the 3-dimensional Ornstein-Uhlenbeck process (\vec{X}_t) with parameter γ and starting from \vec{a} , i.e : a solution of (1) where (B_t) is replaced by a 3-dimensional Brownian motion (\vec{B}_t) , and its radial part $R_t = |\vec{X}_t|$ play an important role; (R_t) solves the *SDE* :

$$R_t = a + \tilde{B}_t + \int_0^t \frac{ds}{R_s} - \gamma \int_0^t R_s ds \quad (4)$$

where $a = |\vec{a}|$ and $(\tilde{B}_t, t \geq 0)$ is a 1-dimensional Brownian motion, starting from 0.

- The main result of this note is the following :

Proposition 1. *Define the probability \mathbb{Q}_a^γ via :*

$$\mathbb{Q}_a^\gamma |_{\mathcal{F}_t} = \frac{X_{t \wedge T_0}}{a} e^{\gamma t} \cdot \mathbb{P}_a^\gamma |_{\mathcal{F}_t} \quad (5)$$

Then, \mathbb{Q}_a^γ is the law of the 3-dimensional radial Ornstein-Uhlenbeck process starting from a .

Proof. Due to Girsanov's theorem, and (2), (B_t) considered under \mathbb{Q}_a^γ solves :

$$B_t = \tilde{B}_t + \int_0^{t \wedge T_0} \frac{\exp(\gamma s)}{X_s e^{\gamma s}} ds \quad (6)$$

Now, noting that $T_0 = \infty$, \mathbb{Q}_a^γ a.s, we obtain that X_t under \mathbb{Q}_a^γ satisfies (4), as a consequence of (2) and (6). \square

- We now make a few comments :

1. Note that, for $\gamma = 0$, (5) is nothing else but Doob's h-transform relationship between Brownian motion killed when it reaches 0 and the 3-dimensional Bessel process.
2. We now write (5) in the equivalent form :

$$1_{\{t < T_0\}} \cdot \mathbb{P}_a^\gamma |_{\mathcal{F}_t} = \frac{a}{X_t} e^{-\gamma t} \cdot \mathbb{Q}_a^\gamma |_{\mathcal{F}_t} \quad (7)$$

which is nothing else but the identity in law obtained by P. Salminen ([6], Theorem 1,(i)) between, on the left hand-side, the Ornstein-Uhlenbeck process with parameter γ , killed when it reaches 0, and on the right hand-side the 3-dimensional radial Ornstein-Uhlenbeck process killed exponentially at rate γ , and conditioned to hit 0, that is the density $(\frac{1}{X_t})$ corresponding to such a conditioning may be seen a posteriori from the formula :

$$\mathbb{Q}_a^\gamma(F_t \frac{1}{X_t}) = \mathbb{Q}_a^\gamma(\frac{1}{X_t}) \mathbb{P}_a^\gamma(F_t | t < T_0), \quad \forall F_t \in \mathcal{F}_t, \quad (8)$$

which is a consequence of(7)

It also follows from (7) that $(\frac{1}{X_t} \exp(-\gamma t), t \geq 0)$ is a strictly local martingale with respect to \mathbb{Q}_a^γ , thus extending very simply the well-known result for the case $\gamma = 0$, when (X_t) is a 3-dimensional Bessel process.

3. Note that (7) leads us to the computation of the semi-group of the killed Ornstein-Uhlenbeck, via :

$$\mathbb{E}_a^\gamma[(\frac{e^{\gamma t}}{a}) f(X_t) 1_{\{t \wedge T_0\}}] = \mathbb{Q}_a^\gamma[f(X_t) \frac{1}{X_t}], \quad (9)$$

allowing us to recover the result in [7], p122, formula (53).

4. What about $\gamma < 0$? Formula (5) is still valid thanks to the same arguments; it is closely related to Theorem 1,(ii) of [6].
5. Related informations about (radial) Ornstein-Uhlenbeck processes, and their hitting times, may be found in [1], [3], [4], [5].

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