

# The stationary Navier-Stokes equations in 3D exterior domains. An approach in anisotropic weighted $L^q$ spaces

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## Abstract

We consider the three-dimensional exterior problem for stationary Navier-Stokes equations. We prove, under assumptions of smallness of the data, existence and uniqueness of solutions. By setting the problem in weighted spaces where the weights reflect the anisotropic decay properties of the fundamental solution of Oseen, we show the better decay of the solutions outside the wake region. Moreover, the solutions we obtained have a finite Dirichlet integral and under additional assumptions on the weights they are also  $\mathcal{PR}$ -solutions in the sense of Finn [10]. The study relies on a  $L^q$ -theory for  $1 < q < \infty$ .

**Keywords:** Fluid mechanics, Stationary Navier-Stokes equations,  $\mathcal{D}$ -solutions,  $\mathcal{PR}$ -solutions, Weighted spaces

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## 1 Introduction and Notation

Let  $\Omega'$  be a simply-connected bounded domain in  $\mathbb{R}^3$  assumed to have at least a Lipschitz-continuous boundary  $\partial\Omega$  and let  $\Omega$  denotes the complement of  $\overline{\Omega'}$ , in other words, the exterior of  $\Omega'$ . In this paper, we consider the stationary Navier-Stokes equations

$$\begin{aligned} -\nu\Delta\mathbf{u} + \rho\mathbf{u} \cdot \nabla\mathbf{u} + \nabla p &= \rho\mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{u}_* \quad \text{on } \partial\Omega, \\ \lim_{|\mathbf{x}| \rightarrow \infty} \mathbf{u}(\mathbf{x}) &= \mathbf{u}_\infty, \end{aligned} \tag{1.1}$$

describing the flow past the obstacle  $\overline{\Omega'}$  of a viscous incompressible fluid with viscosity  $\nu > 0$ , density  $\rho$ , velocity  $\mathbf{u}$  and pressure  $p$ . Furthermore, the prescribed quantities  $\mathbf{f}$ ,  $\mathbf{u}_*$  and  $\mathbf{u}_\infty$  denote the external force field acting on the fluid, the velocity of the fluid at the boundary  $\partial\Omega$  and its velocity at large distances from  $\partial\Omega$ , respectively. In a coordinate frame attached to  $\Omega'$ , we assume that  $\mathbf{u}_\infty = h\mathbf{e}_1$ , where  $\mathbf{e}_1 = (1, 0, 0)^t$  and  $h > 0$ . From a physical point of view, such flow is expected to have some properties which are closely connected with the behavior at large distances. The flow must indeed exhibit an infinite wake extending, in this case, in the direction  $x_1 > 0$ . Inside the wake region, the flow gets turbulent and the rate of convergence of the velocity  $\mathbf{u}$  to  $\mathbf{u}_\infty$  outside the wake is faster than inside it. To our knowledge, the first studies of (1.1) are due to Leray [19], who proved the existence of weak solutions with finite Dirichlet integral  $\int_\Omega |\nabla \mathbf{u}|^2 d\mathbf{x}$ , so-called *D-solutions* (see also [11], [14] and [18]). Then the investigation of the asymptotic behaviour of *D-solutions* is first due to Finn [9, 10], who introduced the class of *physically reasonable solutions*, so called *PR-solutions*, i.e.,

$$|\mathbf{u}(\mathbf{x}) - \mathbf{u}_\infty| = O(|\mathbf{x}|^{-1/2-\epsilon}), \quad \text{if } \mathbf{u}_\infty \neq \mathbf{0}, \quad (1.2)$$

where  $\epsilon > 0$  may be arbitrary small. Further investigations on the asymptotic behavior of solutions to (1.1) are due for instance to Clark [4], Babenko [3], Galdi [12, 13, 14], Farwig [5] (see also [7]), Farwig and Sohr [8] and references quoted there.

An important tool for the investigation of (1.1) is the study of linearized problems. We linearize  $\mathbf{u}$  at  $\mathbf{u} = \mathbf{u}_\infty$  and get the Oseen system ([20, 21]):

$$\begin{aligned} -\Delta \mathbf{v} + \lambda \frac{\partial \mathbf{v}}{\partial x_1} + \nabla p &= \mathbf{f}, \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{v} &= 0, \quad \text{in } \Omega, \\ \mathbf{v} &= \mathbf{u}_* - \mathbf{u}_\infty \quad \text{on } \partial\Omega, \\ \lim_{|\mathbf{x}| \rightarrow \infty} \mathbf{v}(\mathbf{x}) &= \mathbf{0}, \end{aligned} \quad (1.3)$$

for  $\mathbf{v} = \mathbf{u} - \mathbf{u}_\infty$  and  $\lambda = \rho h > 0$ . The fundamental solution  $(\mathcal{O}, \mathcal{P})$  of Oseen can be written in the form (see for instance [13]):

$$\mathcal{O}_{ij}(\mathbf{x}) = \left( \delta_{ij} \Delta - \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \right) \Phi(\mathbf{x}), \quad \mathcal{P}_i(\mathbf{x}) = \frac{1}{4\pi} \frac{x_i}{|\mathbf{x}|^3},$$

where

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\lambda} \int_0^{\lambda s(\mathbf{x})/2\nu} \frac{1 - e^{-t}}{t} dt, \quad s(\mathbf{x}) = |\mathbf{x}| - x_1.$$

It is well known that as  $|\mathbf{x}|$  tends to infinity, we have the following behaviors,

$$\begin{aligned}\mathcal{O}(\mathbf{x}) &= O(|\mathbf{x}|^{-1}(1 + s(\mathbf{x}))^{-1}), \\ \nabla\mathcal{O}(\mathbf{x}) &= O(|\mathbf{x}|^{-3/2}(1 + s(\mathbf{x}))^{-3/2}), \\ \frac{\partial^2\mathcal{O}(\mathbf{x})}{\partial x_i\partial x_j} &= O(|\mathbf{x}|^{-2}(1 + s(\mathbf{x}))^{-2}), \\ \frac{\partial\mathcal{O}(\mathbf{x})}{\partial x_1} &= O(|\mathbf{x}|^{-2}(1 + s(\mathbf{x}))^{-1}).\end{aligned}\tag{1.4}$$

The term  $s(\mathbf{x})$  shows the existence of the wake region behind the obstacle. Indeed in the set  $\mathbf{W} = \{\mathbf{x} \in \mathbb{R}^3, s(\mathbf{x}) \leq 1\}$  which is rotationally symmetric with respect to the  $x_1$ -axis and which has a paraboloidal shape opening for  $x_1 > 0$ , the fundamental solution  $\mathcal{O}$  decays as  $|\mathbf{x}|^{-1}$ . But in the set  $\mathbf{S} = \{\mathbf{x} \in \mathbb{R}^3, s(\mathbf{x}) \geq |\mathbf{x}|/2\}$ ,  $\mathcal{O}$  decays as  $|\mathbf{x}|^{-2}$  showing the faster decay of  $\mathcal{O}$  outside the wake.

The purpose of this paper is to prove, under suitable smallness assumptions, existence and uniqueness of solutions to (1.1) that have better decay properties outside the wake region. Since the flow domain is unbounded, the study is based on a  $L^q$ -theory for  $1 < q < \infty$ . Next, in our approach, we choose to include the wake behavior of the solutions in the definition of the function spaces. We therefore study the Navier-Stokes equations (1.1) in weighted Sobolev spaces where the weights reflect the behaviour of the fundamental solution of Oseen. In view of (1.4), weights of the type

$$\eta_\beta^\alpha(\mathbf{x}) = (1 + |\mathbf{x}|)^\alpha(1 + s(\mathbf{x}))^\beta, \quad \alpha, \beta \in \mathbb{R},$$

are used.

The study here is possible due to the investigation of the exterior Oseen problem in such anisotropic weighted  $L^q$  spaces (see [23]). Note that this paper extends the existence and uniqueness result proved by Farwig [5] where the study relies on a  $L^2$ -theory.

This paper is organized as follow. In the next section, we introduce the anisotropic weighted spaces and we recall some of their basic properties. In Section 3, we recall the main result on the Oseen equations proved in [23] that we will use for the investigation of the Navier-Stokes equations. In order to estimate the nonlinear term  $\mathbf{u} \cdot \nabla\mathbf{u}$ , in Section 4 we first prove imbedding theorems in weighted spaces. Here, we follow the ideas developed by Farwig [5] where integral representations involving the fundamental solution of a scalar model of the Oseen equations are used. We next prove pointwise decay estimates of functions that belong to the anisotropic weighted spaces considered. Finally Section 5 is devoted to the Navier-Stokes equations. After proving anisotropic weighted estimates for the nonlinear term, we prove our main existence and uniqueness result using the Banach's fixed point theorem for sufficiently small data  $\mathbf{f}$  and  $\mathbf{u}_* - \mathbf{u}_\infty$ . The solutions

we obtained are  $\mathcal{D}$ -solutions, *i.e.*  $\int_{\Omega} |\nabla \mathbf{u}|^q d\mathbf{x} < \infty$ . Furthermore, under additional assumptions on the weight  $\eta_{\beta}^{\alpha}(\mathbf{x})$ , they are also  $\mathcal{PR}$ -solutions in the sense of Finn [10] (see (1.2)).

We end this section with the Notations used throughout this paper. We denote by  $\mathbb{N}$  the set of all positive integer and  $q$  is a real number in the interval  $]1, +\infty[$ . The dual exponent of  $q$  denoted by  $q'$  is defined by the relation  $\frac{1}{q} + \frac{1}{q'} = 1$ . Vector or matrix fields will be denoted with boldface characters. A point in  $\mathbb{R}^3$  is denoted by  $\mathbf{x} = (x_1, x_2, x_3)$  and its norm by  $|\mathbf{x}| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$ . For any multi-index  $\mathbf{m} \in \mathbb{N}^3$ , we denote by  $\partial^{\mathbf{m}}$  the differential operator of order  $\mathbf{m}$ ,

$$\partial^{\mathbf{m}} = \frac{\partial^{|\mathbf{m}|}}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3}}, \quad |\mathbf{m}| = m_1 + m_2 + m_3.$$

By the same way, we denote by  $\partial_1$  the differential operator  $\partial_1 = \partial/\partial x_1$ . For any real number  $k$ , we denote by  $[k]$  the integer part of  $k$ . We denote by  $\mathcal{D}(\Omega)$  the space of  $C^{\infty}$  functions with compact support in  $\Omega$ . We recall that  $\mathcal{D}'(\Omega)$  is the well-known space of distributions defined on  $\Omega$  and  $L^q(\Omega)$  is the usual Lebesgue space on  $\Omega$ . For  $m \in \mathbb{N}$ , we recall the well-known Sobolev space  $W^{m,q}(\Omega)$ . We shall write  $u \in W_{loc}^{m,q}(\Omega)$  to mean  $u \in W^{m,q}(\tilde{\Omega})$ , for any bounded domain  $\tilde{\Omega}$ , with  $\bar{\tilde{\Omega}} \subset \Omega$ . We introduce the trace space  $W^{1+1/q',q}(\partial\Omega)$  of functions which belong to  $W^{2,q}(\Omega)$ . For  $R > 0$ , we denote by  $B_R(\mathbf{x})$  the open ball of radius  $R$  centered at  $\mathbf{x}$  and  $B'_R(\mathbf{x})$  the complement of  $\bar{B}_R(\mathbf{x})$ . To simplify the notations, we denote by  $B_R$  the open ball centered at the origin. Finally,  $C, C_0, C_1$  denote generic constants the values of which may change from line to line. Sometimes we indicate some parameters such as  $\delta$  these constants will depend on.

## 2 Anisotropic weighted spaces

Let us recall that  $s(\mathbf{x}) = |\mathbf{x}| - x_1$  and

$$\eta_{\beta}^{\alpha}(\mathbf{x}) = (1 + |\mathbf{x}|)^{\alpha} (1 + s(\mathbf{x}))^{\beta}, \quad \alpha, \beta \in \mathbb{R}.$$

Observe that we have

$$\nabla s(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|} - \mathbf{e}_1, \quad \partial_1 s(\mathbf{x}) = -\frac{s(\mathbf{x})}{|\mathbf{x}|}, \quad |\nabla s(\mathbf{x})|^2 = \frac{2s(\mathbf{x})}{|\mathbf{x}|} \quad (2.5)$$

and as  $|\mathbf{x}|$  tends to infinity,

$$|\nabla \eta_{\beta}^{\alpha}(\mathbf{x})| \leq C \eta_{\beta-1/2}^{\alpha-1/2}(\mathbf{x}), \quad |\partial_1 \eta_{\beta}^{\alpha}(\mathbf{x})| \leq C \eta_{\beta}^{\alpha-1}(\mathbf{x}) \quad \text{and} \quad |\partial^2 \eta_{\beta}^{\alpha}(\mathbf{x})| \leq C \eta_{\beta-1}^{\alpha-1}(\mathbf{x}). \quad (2.6)$$

Let  $\Omega$  be an exterior domain of  $\mathbb{R}^3$ . Given  $\alpha, \beta \in \mathbb{R}$ , we consider the following anisotropic weighted spaces

$$\begin{aligned}
L_{\alpha,\beta}^q(\Omega) &= \{v \in \mathcal{D}'(\Omega), \eta_\beta^\alpha v \in L^q(\Omega)\}, \\
X_{\alpha,\beta}^{1,q}(\Omega) &= \{v \in L_{\alpha-\frac{1}{2},\beta-\frac{1}{2}}^q(\Omega), \nabla v \in L_{\alpha,\beta}^q(\Omega)\}, \\
W_{\alpha,\beta}^{2,q}(\Omega) &= \{v \in L_{\alpha-1,\beta}^q(\Omega), \nabla v \in L_{\alpha-\frac{1}{2},\beta}^q(\Omega), \partial^2 v \in L_{\alpha,\beta}^q(\Omega)\}, \\
\widetilde{W}_{\alpha,\beta}^{2,q}(\Omega) &= \{v \in W_{\alpha,\beta}^{2,q}(\Omega), \partial_1 v \in L_{\alpha,\beta}^q(\Omega)\},
\end{aligned}$$

These are Banach spaces when endowed with their respective norms

$$\begin{aligned}
\|v\|_{L_{\alpha,\beta}^q(\Omega)} &= \|\eta_\beta^\alpha v\|_{L^q(\Omega)}, \\
\|v\|_{X_{\alpha,\beta}^{1,q}(\Omega)} &= \|v\|_{L_{\alpha-\frac{1}{2},\beta-\frac{1}{2}}^q(\Omega)} + \|\nabla v\|_{L_{\alpha,\beta}^q(\Omega)}, \\
\|v\|_{W_{\alpha,\beta}^{2,q}(\Omega)} &= \|v\|_{L_{\alpha-1,\beta}^q(\Omega)} + \|\nabla v\|_{L_{\alpha-\frac{1}{2},\beta}^q(\Omega)} + \|\partial^2 v\|_{L_{\alpha,\beta}^q(\Omega)}. \\
\|v\|_{\widetilde{W}_{\alpha,\beta}^{2,q}(\Omega)} &= \|v\|_{W_{\alpha,\beta}^{2,q}(\Omega)} + \|\partial_1 v\|_{L_{\alpha,\beta}^q(\Omega)}.
\end{aligned}$$

The space  $\mathcal{D}(\Omega)$  is dense in  $L_{\alpha,\beta}^q(\Omega)$  and, proceeding as in [16], we can prove that the space  $\mathcal{D}(\overline{\Omega})$  is dense in  $X_{\alpha,\beta}^{1,q}(\Omega)$ ,  $W_{\alpha,\beta}^{2,q}(\Omega)$  and  $\widetilde{W}_{\alpha,\beta}^{2,q}(\Omega)$ . We have  $L_{\alpha,\beta}^q(\Omega) \subset L_{\text{loc}}^q(\Omega)$  and, by the same way, all the local properties of the spaces  $X_{\alpha,\beta}^{1,q}(\Omega)$ ,  $W_{\alpha,\beta}^{2,q}(\Omega)$  and  $\widetilde{W}_{\alpha,\beta}^{2,q}(\Omega)$  coincide with those of the Sobolev spaces  $W^{1,q}(\Omega)$  and  $W^{2,q}(\Omega)$  respectively. Hence, they also satisfy the usual trace theorems on  $\partial\Omega$ . Observe that the operator defined by

$$u \in L_{\alpha,\beta}^q(\Omega) \mapsto \eta_b^a u \in L_{\alpha-a,\beta-b}^q(\Omega) \quad (2.7)$$

is an isomorphism. For further properties of the anisotropic weighted Sobolev spaces considered, the reader can refer to [6], [2] and [1].

### 3 The linear problem: the stationary Oseen equations

Now that the functional framework is introduced and before going any further, we recall the existence and uniqueness result for the linearized problem proved in [23] that we will use in the sequel. We consider the stationary Oseen equations in an exterior domain  $\Omega$

$$\begin{aligned}
-\nu \Delta \mathbf{u} + \lambda \partial_1 \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\
\operatorname{div} \mathbf{u} &= 0 \quad \text{in } \Omega, \\
\mathbf{u} &= \mathbf{u}_* \quad \text{on } \partial\Omega.
\end{aligned} \quad (3.8)$$

**Theorem 3.1.** *Let  $\Omega$  be an exterior domain of  $\mathbb{R}^3$  and suppose that  $\partial\Omega$  is of class  $\mathcal{C}^2$ . Let  $\alpha, \beta \in \mathbb{R}$  satisfy*

$$\max\left(0, \frac{1}{2} - \frac{1}{q}\right) < \beta \leq 1 - \frac{1}{q} \quad \text{and} \quad \frac{1}{2} - \frac{1}{q} \leq \alpha < \frac{1}{2} - \frac{1}{q} - \beta. \quad (3.9)$$

Then for any  $\mathbf{f} \in L^q_{\alpha+\frac{1}{2},\beta}(\Omega)$  and  $\mathbf{u}_* \in W^{1+1/q',q}(\partial\Omega)$ , problem (3.8) has a unique solution  $(\mathbf{u}, p) \in \widetilde{W}^{2,q}_{\alpha+\frac{1}{2},\beta}(\Omega) \times X^{1,q}_{\alpha+\frac{1}{2},\beta}(\Omega)$ . Moreover, we have the estimate

$$\|\mathbf{u}\|_{\widetilde{W}^{2,q}_{\alpha+\frac{1}{2},\beta}(\Omega)} + \|p\|_{X^{1,q}_{\alpha+\frac{1}{2},\beta}(\Omega)} \leq C \left( \|\mathbf{f}\|_{L^q_{\alpha+\frac{1}{2},\beta}(\Omega)} + \|\mathbf{u}_*\|_{W^{1+1/q',q}(\partial\Omega)} \right).$$

## 4 Imbedding and asymptotic behavior results

In this section, we first prove two imbedding theorems which will allow to estimate the nonlinear term  $\mathbf{u} \cdot \nabla \mathbf{u}$  in anisotropic weighted spaces. We introduce

$$\mathcal{O}(\mathbf{x}) = \frac{1}{4\pi} \frac{e^{-s(\mathbf{x})/2}}{|\mathbf{x}|},$$

the fundamental solution of the following model equation

$$-\Delta u + \partial_1 u = f \quad \text{in } \mathbb{R}^3.$$

Observe that we have

$$|\mathcal{O}(\mathbf{x})| \leq \frac{C}{|\mathbf{x}|(1+s(\mathbf{x}))^d}, \quad \forall d \in \mathbb{R} \quad (4.10)$$

and, using (2.5) ,

$$|\nabla \mathcal{O}(\mathbf{x})| \leq \frac{C}{(1+s(\mathbf{x}))^d} \left( \frac{1}{|\mathbf{x}|^{3/2}} + \frac{1}{|\mathbf{x}|^2} \right), \quad \forall d \in \mathbb{R}. \quad (4.11)$$

Next, let  $u \in \mathcal{D}(\mathbb{R}^3)$ . Then we have the following integral representation

$$u(\mathbf{x}) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \mathcal{O}(\mathbf{x} - \mathbf{y}) (-\Delta u(\mathbf{y}) + \partial_1 u(\mathbf{y})) d\mathbf{y}. \quad (4.12)$$

Besides, we need a technical lemma which is a version of a more general result proved by Farwig (see [5], Section 3):

**Lemma 4.1.** *Let  $a, b, c, d$  be real numbers satisfying:*

$$\begin{aligned} a < b + 1, \quad 0 \leq b < 1, \quad 0 < c < 2, \quad d > 1, \quad a + b + c - 2 > 0, \\ a + b \neq 1, \quad c + d \neq 1, \quad 2b + c + d \neq 3. \end{aligned} \quad (4.13)$$

Then we have

$$\int_{\mathbb{R}^3} \eta_{-b}^{-a}(\mathbf{y}) |\mathbf{x} - \mathbf{y}|^{-c} (1 + s(\mathbf{x} - \mathbf{y}))^{-d} d\mathbf{y} \leq C \eta_{-b}^{-(a+c-2)}(\mathbf{x}). \quad (4.14)$$

Inequality (4.14) still holds if  $|\mathbf{x} - \mathbf{y}|^{-c}$  is replaced by  $|\mathbf{x} - \mathbf{y}|^{-c-\gamma}$  for any  $\gamma$  such that  $c + \gamma < 3$ .

**Remark 4.2.** (i) Let us mention that in [5], Farwig has given conditions on  $a, b, c, d, e, f$  such that,

$$\int_{\mathbb{R}^3} \eta_{-b}^{-a}(\mathbf{y}) \eta_{-d}^{-c}(\mathbf{x} - \mathbf{y}) d\mathbf{y} \leq C \eta_{-f}^{-e}(\mathbf{x}). \quad (4.15)$$

The whole space  $\mathbb{R}^3$  is divided into 16 subdomains  $\Omega_i$ ,  $i = 0, \dots, 15$  and in each  $\Omega_i$ ,  $i = 0, \dots, 15$ , the inequalities

$$\int_{\Omega_i} \eta_{-b}^{-a}(\mathbf{y}) \eta_{-d}^{-c}(\mathbf{x} - \mathbf{y}) d\mathbf{y} \leq C \eta_{-f_i}^{-e_i}(\mathbf{x}), \quad (4.16)$$

are proved where  $e_i = e_i(a, b, c, d)$  and  $f_i = f_i(a, b, c, d)$ . From these results and under the assumptions (4.14), it is straightforward to verify that in each  $\Omega_i$ ,  $i = 0, \dots, 15$ , we have  $\eta_{-f_i}^{-e_i}(\mathbf{x}) \leq C \eta_{-b}^{-(a+c+2)}(\mathbf{x})$ .

(ii) A generalization of (4.15) to  $\mathbb{R}^n$ , with  $n \geq 2$  can be found in [17] (see also [22]).

(iii) Assume now that inequality (4.14) holds. First we write

$$\int_{\mathbb{R}^3} \eta_{-b}^{-a}(\mathbf{y}) |\mathbf{x} - \mathbf{y}|^{-c-\gamma} (1 + s(\mathbf{x} - \mathbf{y}))^{-d} d\mathbf{y} = I_1(\mathbf{x}) + I_2(\mathbf{x}),$$

where

$$I_1(\mathbf{x}) = \int_{B_1(\mathbf{x})} \eta_{-b}^{-a}(\mathbf{y}) |\mathbf{x} - \mathbf{y}|^{-c-\gamma} (1 + s(\mathbf{x} - \mathbf{y}))^{-d} d\mathbf{y}$$

and

$$I_2(\mathbf{x}) = \int_{B'_1(\mathbf{x})} \eta_{-b}^{-a}(\mathbf{y}) |\mathbf{x} - \mathbf{y}|^{-c-\gamma} (1 + s(\mathbf{x} - \mathbf{y}))^{-d} d\mathbf{y}.$$

Clearly we have

$$I_2(\mathbf{x}) \leq \int_{\mathbb{R}^3} \eta_{-b}^{-a}(\mathbf{y}) |\mathbf{x} - \mathbf{y}|^{-c} (1 + s(\mathbf{x} - \mathbf{y}))^{-d} d\mathbf{y}.$$

Next, we have

$$\begin{aligned} I_1(\mathbf{x}) &\leq \max_{\mathbf{y} \in B_1(\mathbf{x})} \eta_{-b}^{-(a+c-2)}(\mathbf{y}) \int_{B_1(\mathbf{x})} \eta_0^{c-2}(\mathbf{y}) |\mathbf{x} - \mathbf{y}|^{-c-\gamma} (1 + s(\mathbf{x} - \mathbf{y}))^{-d} d\mathbf{y} \\ &\leq C \eta_{-b}^{-(a+c-2)}(\mathbf{x}) \int_{B_1(\mathbf{x})} \eta_0^{c-2}(\mathbf{y}) |\mathbf{x} - \mathbf{y}|^{-c-\gamma} (1 + s(\mathbf{x} - \mathbf{y}))^{-d} d\mathbf{y}. \end{aligned}$$

Since  $c + \gamma < 3$ , we easily get

$$\int_{B_1(\mathbf{x})} \eta_0^{c-2}(\mathbf{y}) |\mathbf{x} - \mathbf{y}|^{-c-\gamma} (1 + s(\mathbf{x} - \mathbf{y}))^{-d} d\mathbf{y} \leq C.$$

This proves that inequality (4.14) still holds if  $|\mathbf{x} - \mathbf{y}|^{-c}$  is replaced by  $|\mathbf{x} - \mathbf{y}|^{-c-\gamma}$  for any  $\gamma$  such that  $c + \gamma < 3$ .

The next results prove imbedding theorems in anisotropic weighted spaces.

**Theorem 4.3.** *Let  $\alpha, \beta, q, r$  be real numbers such that  $\frac{1}{q} - \frac{1}{2} < \frac{1}{r} < \frac{1}{q}$ . Then for any  $u \in \widetilde{W}_{\alpha+\frac{1}{2}, \beta}^{2,q}(\mathbb{R}^3)$ , we have*

$$\|u\|_{L^r_{\alpha-\frac{1}{2}+\frac{2}{q}-\frac{2}{r}, \beta}(\mathbb{R}^3)} \leq C \|u\|_{\widetilde{W}_{\alpha+\frac{1}{2}, \beta}^{2,q}(\mathbb{R}^3)}.$$

**Proof.** It is enough to consider a function  $u \in \mathcal{D}(\mathbb{R}^3)$ . We split the proof into two parts.

1) In this first part, we assume

$$0 < \beta < 1 - \frac{1}{q} \quad \text{and} \quad \frac{1}{2} - \frac{2}{q} + \frac{1}{r} - \beta < \alpha < \beta + \frac{1}{2} - \frac{1}{q}. \quad (4.17)$$

Now, for any  $\mathbf{y} \in \mathbb{R}^3$ , let  $g(\mathbf{y}) = -\Delta u(\mathbf{y}) + \partial_1 u(\mathbf{y})$ . Then from (4.12), we have

$$u(\mathbf{x}) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{-s(\mathbf{x}-\mathbf{y})/2}}{|\mathbf{x}-\mathbf{y}|} g(\mathbf{y}) d\mathbf{y}.$$

Introducing two positive real numbers  $t$  and  $\delta$ , with  $0 < t < 1$ , we can write

$$\begin{aligned} u(\mathbf{x}) &= \frac{1}{4\pi} \int_{\mathbb{R}^3} \left( |g(\mathbf{y})|^q \eta_{\beta q}^{(\alpha+\frac{1}{2})q}(\mathbf{y}) \right)^{\frac{r-q}{qr}} \times \\ &\times |g(\mathbf{y})|^{q/r} \frac{e^{-s(\mathbf{x}-\mathbf{y})t/2}}{|\mathbf{x}-\mathbf{y}|^t} \eta_{\beta q/r}^{-(\alpha+\frac{1}{2})\frac{r-q}{r}+\delta}(\mathbf{y}) \frac{e^{-s(\mathbf{x}-\mathbf{y})(1-t)/2}}{|\mathbf{x}-\mathbf{y}|^{1-t}} \eta_{-\beta}^{-\delta}(\mathbf{y}) d\mathbf{y}. \end{aligned}$$

It follows from the Hölder's inequality with the exponents  $\frac{qr}{r-q}$ ,  $r$  and  $q'$ , that

$$\begin{aligned} |u(\mathbf{x})| &\leq C \|g\|_{L^q_{\alpha+\frac{1}{2},\beta}(\mathbb{R}^3)}^{(r-q)/r} \left( \int_{\mathbb{R}^3} |g(\mathbf{y})|^q \frac{e^{-s(\mathbf{x}-\mathbf{y})rt/2}}{|\mathbf{x}-\mathbf{y}|^{rt}} \eta_{\beta q}^{-(\alpha+\frac{1}{2})(r-q)+\delta r}(\mathbf{y}) d\mathbf{y} \right)^{1/r} \times \\ &\times \left( \int_{\mathbb{R}^3} \frac{e^{-s(\mathbf{x}-\mathbf{y})(1-t)q'/2}}{|\mathbf{x}-\mathbf{y}|^{(1-t)q'}} \eta_{-\beta q'}^{-\delta q'}(\mathbf{y}) d\mathbf{y} \right)^{1/q'}. \end{aligned} \quad (4.18)$$

We set

$$J_1(\mathbf{x}) = \int_{\mathbb{R}^3} \frac{e^{-s(\mathbf{x}-\mathbf{y})(1-t)q'/2}}{|\mathbf{x}-\mathbf{y}|^{(1-t)q'}} \eta_{-\beta q'}^{-\delta q'}(\mathbf{y}) d\mathbf{y}.$$

From (4.10), we have

$$J_1(\mathbf{x}) \leq C \int_{\mathbb{R}^3} \eta_{-\beta q'}^{-\delta q'}(\mathbf{y}) |\mathbf{x}-\mathbf{y}|^{-(1-t)q'} (1+s(\mathbf{x}-\mathbf{y}))^{-d} d\mathbf{y}, \quad \forall d > 1.$$

We set  $a = \delta q'$ ,  $b = \beta q'$  and  $c = (1-t)q'$ . Choosing

$$\frac{2}{q'} - 1 + t - \beta < \delta < \beta + \frac{1}{q'}, \quad \delta \neq \frac{1}{q'} - \beta \quad \text{and} \quad t > 1 - \frac{2}{q'}, \quad (4.19)$$

then all the assumptions (4.13) of Lemma 4.1 are satisfied and we get

$$J_1(\mathbf{x}) \leq C \eta_{-\beta q'}^{-\delta q' - (1-t)q' + 2}(\mathbf{x}).$$

From (4.18) we deduce that

$$\begin{aligned} \int_{\mathbb{R}^3} |u(\mathbf{x})|^r \eta_{\beta r}^{(\alpha-\frac{1}{2}+\frac{2}{q})r-2}(\mathbf{x}) d\mathbf{x} &\leq C \|g\|_{L^q_{\alpha+\frac{1}{2},\beta}(\mathbb{R}^3)}^{r-q} \int_{\mathbb{R}^3} |g(\mathbf{y})|^q \eta_{\beta q}^{-(\alpha+\frac{1}{2})(r-q)+\delta r}(\mathbf{y}) \times \\ &\times \left( \int_{\mathbb{R}^3} \frac{e^{-s(\mathbf{x}-\mathbf{y})rt/2}}{|\mathbf{x}-\mathbf{y}|^{rt}} \eta_0^{-\delta r + (\alpha+\frac{1}{2})r+rt-2}(\mathbf{x}) d\mathbf{x} \right) d\mathbf{y}. \end{aligned} \quad (4.20)$$



We set now

$$J_2(\mathbf{y}) = \int_{\mathbb{R}^3} \frac{e^{-s(\mathbf{x}-\mathbf{y})rt/2}}{|\mathbf{x}-\mathbf{y}|^{rt}} \eta_0^{-\delta r + (\alpha + \frac{1}{2})r + rt - 2}(\mathbf{x}) d\mathbf{x}.$$

Using again (4.10), we have

$$J_2(\mathbf{y}) \leq C \int_{\mathbb{R}^3} \eta_0^{-\delta r + (\alpha + \frac{1}{2})r + rt - 2}(\mathbf{x}) |\mathbf{x}-\mathbf{y}|^{-rt} (1 + s(\mathbf{x}-\mathbf{y}))^{-d} d\mathbf{x}, \quad \forall d > 1.$$

Setting  $a = \delta r - (\alpha + \frac{1}{2})r - rt + 2$ ,  $b = 0$ ,  $c = rt$  and choosing

$$\alpha + \frac{1}{2} < \delta < -\frac{1}{r} + t + \alpha + \frac{1}{2} \quad \text{and} \quad t < \frac{2}{r}, \quad (4.21)$$

then the assumptions (4.13) are satisfied and we get

$$J_2(\mathbf{y}) \leq C \eta_0^{-\delta r + (\alpha + \frac{1}{2})r}(\mathbf{y}).$$

Inserting this last inequality in (4.20), we easily obtain

$$\|u\|_{L^r_{\alpha - \frac{1}{2} + \frac{2}{q} - \frac{2}{r}, \beta}(\mathbb{R}^3)} \leq C \|g\|_{L^q_{\alpha + \frac{1}{2}, \beta}(\mathbb{R}^3)} \quad (4.22)$$

which proves the statement of the theorem. Observe now that to get (4.22), we need to choose  $\delta$  and  $t$  satisfying both (4.19) and (4.21). In other words, we need to choose  $\delta$  and  $t$  such that

$$\max\left(-1 + \frac{2}{q}, \frac{1}{r}\right) < t < \min\left(1, 2\beta + \frac{1}{q}, \frac{2}{r}\right), \quad \delta \neq 1 - \frac{1}{q} - \beta$$

and

$$\max\left(1 - \frac{2}{q} + t - \beta, \alpha + \frac{1}{2}\right) < \delta < \min\left(-\frac{1}{r} + t + \alpha + \frac{1}{2}, \beta + 1 - \frac{1}{q}\right).$$

The assumptions  $\frac{1}{q} - \frac{1}{2} < \frac{1}{r} < \frac{1}{q}$  and (4.17) ensure that such  $t$  and  $\delta$  exist.

2) In this second part, let  $\alpha$  and  $\beta$  be two arbitrary real numbers and consider  $v = \eta_{-\epsilon}^{-\tau} u$ , where  $\epsilon, \tau \in \mathbb{R}$ . We can write

$$\int_{\mathbb{R}^3} |u(\mathbf{x})|^r \eta_{\beta r}^{(\alpha - \frac{1}{2} + \frac{2}{q})r - 2}(\mathbf{x}) d\mathbf{x} = \int_{\mathbb{R}^3} |v(\mathbf{x})|^r \eta_{(\beta + \epsilon)r}^{(\alpha + \tau - \frac{1}{2} + \frac{2}{q})r - 2}(\mathbf{x}) d\mathbf{x}.$$

Setting now  $\alpha' = \alpha + \tau$ ,  $\beta' = \beta + \epsilon$ , choosing  $\epsilon$  and  $\tau$  such that  $0 < \beta' < 1 - \frac{1}{q}$  and  $\frac{1}{2} - \frac{2}{q} + \frac{1}{r} - \beta' < \alpha' < \beta' + \frac{1}{2} - \frac{1}{q}$  and using the first part, we easily get

$$\|v\|_{L^r_{\alpha' - \frac{1}{2} + \frac{2}{q} - \frac{2}{r}, \beta'}(\mathbb{R}^3)} \leq C \|\Delta v + \partial_1 v\|_{L^q_{\alpha' + \frac{1}{2}, \beta'}(\mathbb{R}^3)}.$$

Thanks to (2.6) we arrive at

$$\|v\|_{L^r_{\alpha' - \frac{1}{2} + \frac{2}{q} - \frac{2}{r}, \beta'}(\mathbb{R}^3)} \leq C \|u\|_{\widetilde{W}^{2,q}_{\alpha + \frac{1}{2}, \beta}(\mathbb{R}^3)}$$

which ends the proof.  $\square$

**Theorem 4.4.** Let  $\alpha, \beta, q$  and  $r$  be real numbers such that  $\frac{1}{q} - \frac{1}{4} < \frac{1}{r} < \frac{1}{q}$ . Then for any  $u \in \widetilde{W}_{\alpha+\frac{1}{2}, \beta}^{2,q}(\mathbb{R}^3)$ , we have

$$\|\nabla u\|_{L^r_{\alpha+\frac{2}{q}-\frac{2}{r}, \beta}(\mathbb{R}^3)} \leq C \|u\|_{\widetilde{W}_{\alpha+\frac{1}{2}, \beta}^{2,q}(\mathbb{R}^3)}.$$

**Proof.** As for the proof of Theorem 4.3, it is enough to consider a function  $u \in \mathcal{D}(\mathbb{R}^3)$  and we split the proof in two parts.

1) In this first part, we assume

$$0 < \beta < 1 - \frac{1}{q} \quad \text{and} \quad -\frac{2}{q} + \frac{1}{r} - \beta < \alpha < \beta + \frac{1}{2} - \frac{1}{q}. \quad (4.23)$$

For any  $\mathbf{y} \in \mathbb{R}$ , we set  $g(\mathbf{y}) = -\Delta u(\mathbf{y}) + \partial_1 u(\mathbf{y})$  and we can write

$$\nabla u(\mathbf{x}) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \nabla \left( \frac{e^{-s(\mathbf{x}-\mathbf{y})/2}}{|\mathbf{x}-\mathbf{y}|} \right) g(\mathbf{y}) d\mathbf{y}.$$

Using (4.11), for any  $d > 1$ , we have

$$\begin{aligned} |\nabla u(\mathbf{x})| &\leq C \left( \int_{\mathbb{R}^3} (1+s(\mathbf{x}-\mathbf{y}))^{-d} |\mathbf{x}-\mathbf{y}|^{-3/2} g(\mathbf{y}) d\mathbf{y} + \int_{\mathbb{R}^3} (1+s(\mathbf{x}-\mathbf{y}))^{-d} |\mathbf{x}-\mathbf{y}|^{-2} g(\mathbf{y}) d\mathbf{y} \right) \\ &\equiv C(I_1(\mathbf{x}) + I_2(\mathbf{x})). \end{aligned}$$

Introducing two positive real numbers  $t$  and  $\delta$  with  $0 < t < 1$  and since  $d$  is any real number greater than 1, we can write

$$\begin{aligned} I_1(\mathbf{x}) &= \int_{\mathbb{R}^3} \left( |g(\mathbf{y})|^q \eta_{\beta q}^{(\alpha+\frac{1}{2})q}(\mathbf{y}) \right)^{\frac{r-q}{qr}} |g(\mathbf{y})|^{q/r} (1+s(\mathbf{x}-\mathbf{y}))^{-d/r} |\mathbf{x}-\mathbf{y}|^{-\frac{3t}{2}} \eta_{\beta q/r}^{-(\alpha+\frac{1}{2})(\frac{r-q}{r})+\delta}(\mathbf{y}) \times \\ &\times (1+s(\mathbf{x}-\mathbf{y}))^{-d/q'} |\mathbf{x}-\mathbf{y}|^{-\frac{3}{2}(1-t)} \eta_{-\beta}^{-\delta}(\mathbf{y}) d\mathbf{y}. \end{aligned}$$

From the Hölder's inequality with the exponents  $\frac{qr}{r-q}$ ,  $r$  and  $q'$ , we have

$$\begin{aligned} I_1(\mathbf{x}) &\leq C \left( \int_{\mathbb{R}^3} |g(\mathbf{y})|^q \eta_{\beta q}^{(\alpha+\frac{1}{2})q}(\mathbf{y}) d\mathbf{y} \right)^{\frac{r-q}{qr}} \times \\ &\times \left( \int_{\mathbb{R}^3} |g(\mathbf{y})|^q (1+s(\mathbf{x}-\mathbf{y}))^{-d} |\mathbf{x}-\mathbf{y}|^{-\frac{3rt}{2}} \eta_{\beta q}^{-(\alpha+\frac{1}{2})(r-q)+\delta r}(\mathbf{y}) d\mathbf{y} \right)^{1/r} \times \\ &\times \left( \int_{\mathbb{R}^3} (1+s(\mathbf{x}-\mathbf{y}))^{-d} |\mathbf{x}-\mathbf{y}|^{-\frac{3}{2}(1-t)q'} \eta_{-\beta q'}^{-\delta q'}(\mathbf{y}) d\mathbf{y} \right)^{1/q'}. \end{aligned}$$

Let

$$I_{11}(\mathbf{x}) = \int_{\mathbb{R}^3} \eta_{-\beta q'}^{-\delta q'}(\mathbf{y}) |\mathbf{x}-\mathbf{y}|^{-\frac{3}{2}(1-t)q'} (1+s(\mathbf{x}-\mathbf{y}))^{-d} d\mathbf{y}.$$

We set  $a = \delta q'$ ,  $b = \beta q'$ ,  $c = \frac{3}{2}(1-t)q'$  and we choose

$$\frac{2}{q'} - \frac{3}{2}(1-t) - \beta < \delta < \beta + \frac{1}{q'}, \quad \delta \neq \frac{1}{q'} - \beta, \quad t > 1 - \frac{4}{3q'}. \quad (4.24)$$

Then all the assumptions (4.13) are satisfied and Lemma 4.1 yields

$$I_{11}(\mathbf{x}) \leq C\eta_{-\beta q'}^{-\delta q' - \frac{3}{2}(1-t)q'+2}(\mathbf{x}). \quad (4.25)$$

Proceeding by the same way for  $I_2(\mathbf{x})$ , we will have the term

$$\begin{aligned} I_{21}(\mathbf{x}) &= \int_{\mathbb{R}^3} \eta_{-\beta q'}^{-\delta q'}(\mathbf{y}) |\mathbf{x} - \mathbf{y}|^{-2(1-t)q'} (1 + s(\mathbf{x} - \mathbf{y}))^{-d} d\mathbf{y} \\ &= \int_{\mathbb{R}^3} \eta_{-\beta q'}^{-\delta q'}(\mathbf{y}) |\mathbf{x} - \mathbf{y}|^{-\frac{3}{2}(1-t)q' - \frac{1}{2}(1-t)q'} (1 + s(\mathbf{x} - \mathbf{y}))^{-d} d\mathbf{y}. \end{aligned}$$

Setting  $\gamma = \frac{1}{2}(1-t)q'$ , then from Lemma 4.1, the condition  $c + \gamma < 3$  yields the additional assumption  $t > 1 - \frac{3}{2q'}$  which is already satisfied by (4.24). Hence inequality (4.25) holds if we replace  $I_{11}(\mathbf{x})$  by  $I_{21}(\mathbf{x})$ . Thus we deduce that

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla u(\mathbf{x})|^r \eta_{\beta r}^{(\alpha + \frac{2}{q})r-2}(\mathbf{x}) d\mathbf{x} &\leq C \|g\|_{L_{\alpha + \frac{1}{2}, \beta}^q(\mathbb{R}^3)}^{r-q} \int_{\mathbb{R}^3} |g(\mathbf{y})|^q \eta_{\beta q}^{-(\alpha + \frac{1}{2})(r-q) + \delta r}(\mathbf{y}) \times \\ &\times \left( \int_{\mathbb{R}^3} (1 + s(\mathbf{x} - \mathbf{y}))^{-d} |\mathbf{x} - \mathbf{y}|^{-\frac{3rt}{2}} (1 + |\mathbf{x} - \mathbf{y}|^{-\frac{rt}{2}}) \eta_0^{-\delta r + \frac{3rt}{2} + (\alpha + \frac{1}{2})r-2}(\mathbf{x}) d\mathbf{x} \right) d\mathbf{y}. \end{aligned} \quad (4.26)$$

Let

$$I_{12}(\mathbf{y}) = \int_{\mathbb{R}^3} (1 + s(\mathbf{x} - \mathbf{y}))^{-d} |\mathbf{x} - \mathbf{y}|^{-\frac{3rt}{2}} \eta_0^{-\delta r + \frac{3rt}{2} + (\alpha + \frac{1}{2})r-2}(\mathbf{x}) d\mathbf{x}.$$

Setting  $a = \delta r - \frac{3rt}{2} - (\alpha + \frac{1}{2})r + 2$ ,  $b = 0$ ,  $c = \frac{3rt}{2}$  and choosing

$$\alpha + \frac{1}{2} < \delta < -\frac{1}{r} + \frac{3t}{2} + \alpha + \frac{1}{2}, \quad t < \frac{4}{3r}, \quad (4.27)$$

then all the assumptions (4.13) are satisfied and we get

$$I_{12}(\mathbf{y}) \leq C\eta_0^{-\delta r + (\alpha + \frac{1}{2})r}(\mathbf{y}). \quad (4.28)$$

Let

$$I_{22}(\mathbf{y}) = \int_{\mathbb{R}^3} (1 + s(\mathbf{x} - \mathbf{y}))^{-d} |\mathbf{x} - \mathbf{y}|^{-2rt} \eta_0^{-\delta r + \frac{3rt}{2} + (\alpha + \frac{1}{2})r-2}(\mathbf{x}) d\mathbf{x}.$$

Then, again from Lemma 4.1, we have

$$I_{22}(\mathbf{y}) \leq C\eta_0^{-\delta r + (\alpha + \frac{1}{2})r}(\mathbf{y}) \quad (4.29)$$

if  $2rt < 3$  which implies that  $t < \frac{3}{2r}$ . But this condition is already satisfied by (4.27).

From (4.26), (4.28) and (4.29), it follows that

$$\int_{\mathbb{R}^3} |\nabla u(\mathbf{x})|^r \eta_{\beta r}^{(\alpha + \frac{2}{q})r-2}(\mathbf{x}) d\mathbf{x} \leq C \|g\|_{L_{\alpha + \frac{1}{2}, \beta}^q(\mathbb{R}^3)}^r.$$

To get this last inequality,  $\delta$  and  $t$  must be chosen such that

$$\max\left(-\frac{1}{3} + \frac{4}{3q}, \frac{2}{3r}\right) < t < \min\left(1, \frac{4}{3r}\right)$$

and

$$\max\left(\frac{1}{2} - \frac{2}{q} + \frac{3t}{2} - \beta, \alpha + \frac{1}{2}\right) < \delta < \min\left(\beta + 1 - \frac{1}{q}, -\frac{1}{r} + \frac{3t}{2} + \alpha + \frac{1}{2}\right),$$

which summarize the assumptions (4.24) and (4.27). Since  $\frac{1}{q} - \frac{1}{4} < \frac{1}{r} < \frac{1}{q}$  and under the assumptions (4.23), such  $\delta$  and  $t$  exist.

2) For arbitrary  $\alpha, \beta \in \mathbb{R}$ , we proceed as in the proof of Theorem 4.3.  $\square$

**Remark 4.5.** The imbedding results proved previously are still valid if  $\mathbb{R}^3$  is replaced by  $\Omega$ . Indeed let  $u \in W_{\alpha+\frac{1}{2},\beta}^{1,q}(\Omega)$ . Then the trace  $\gamma_0 u \in W^{1+1/q',q}(\partial\Omega)$ . Hence, recalling that  $\Omega'$  is the bounded domain complement of  $\Omega$ , there exists a unique  $\tilde{u} \in W^{2,q}(\Omega')$  such that (see for instance [15])

$$\Delta \tilde{u} = 0 \quad \text{in } \Omega', \quad \tilde{u} = \gamma_0 u \quad \text{on } \partial\Omega.$$

Moreover, we have the estimates

$$\|\tilde{u}\|_{W^{2,q}(\Omega')} \leq C \|\gamma_0 u\|_{W^{1+1/q',q}(\partial\Omega)} \leq C \|u\|_{\widetilde{W}_{\alpha+\frac{1}{2},\beta}^{2,q}(\Omega)}. \quad (4.30)$$

Consider the extension of  $u$ :

$$\bar{u} = u \quad \text{in } \Omega \quad \text{and} \quad \bar{u} = \tilde{u} \quad \text{in } \Omega'.$$

Then,  $\bar{u} \in \widetilde{W}_{\alpha+\frac{1}{2},\beta}^{2,q}(\mathbb{R}^3)$  and, thanks to (4.30),

$$\|\bar{u}\|_{\widetilde{W}_{\alpha+\frac{1}{2},\beta}^{2,q}(\mathbb{R}^3)} \leq C \|u\|_{\widetilde{W}_{\alpha+\frac{1}{2},\beta}^{2,q}(\Omega)}.$$

From this last inequality, it is straightforward to prove that Theorems 4.3 and 4.4 are still valid if  $\mathbb{R}^3$  is replaced by  $\Omega$ .

For the last result of this section, we study the asymptotic behavior of functions that belong to anisotropic weighted Sobolev spaces.

**Theorem 4.6.** *Assume  $\frac{3}{2} < q < \infty$  and let  $\theta = 2 - \frac{3}{q} - [2 - \frac{3}{q}]$ . If  $u \in W_{\alpha,\beta}^{2,q}(\mathbb{R}^3)$ , then there exists  $C > 0$  such that*

$$\forall \mathbf{x} \in \mathbb{R}^3, \quad |u(\mathbf{x})| \leq C \eta_{-\beta}^{\theta-\alpha}(\mathbf{x}) \|u\|_{W_{\alpha,\beta}^{2,q}(\mathbb{R}^3)}.$$

**Proof.** 1) We assume first  $\alpha = \beta = 0$ . Let  $R > 0$  be a real number. Using the following partition of unity

$$\begin{aligned} \varphi_1, \varphi_2 &\in C^\infty(\mathbb{R}^3), \quad 0 \leq \varphi_1, \varphi_2 \leq 1, \quad \varphi_1 + \varphi_2 = 1 \quad \text{in } \mathbb{R}^3 \\ \varphi_1 &= 1 \quad \text{in } B_R, \quad \text{supp } \varphi_1 \subset B_{R+1}. \end{aligned}$$

we split  $u$  into  $u = u\varphi_1 + u\varphi_2 = u_1 + u_2$ . Since  $\frac{3}{2} < q < \infty$ , the Sobolev's imbedding theorems yield

$$\|u_1\|_{L^\infty(B_{R+1})} \leq C\|\partial^2 u_1\|_{L^q(B_{R+1})}$$

and

$$\forall \mathbf{x} \in \mathbb{R}^3, \quad |u_2(\mathbf{x}) - u_2(\mathbf{0})| \leq C\|\partial^2 u_2\|_{L^q(\mathbb{R}^3)}|\mathbf{x}|^\theta.$$

By the definitions of  $\varphi_1, \varphi_2$  and since  $0 < \theta < 1$ , we have

$$\forall \mathbf{x} \in \mathbb{R}^3, \quad |u(\mathbf{x})| \leq C\eta_0^\theta(\mathbf{x})\|u\|_{W_{0,0}^{2,q}(\mathbb{R}^3)}.$$

2) Let now  $\alpha, \beta$  be two arbitrary real numbers and  $u \in W_{\alpha,\beta}^{2,q}(\mathbb{R}^3)$ . Then thanks to (2.6) and (2.7),  $\eta_\beta^\alpha u \in W_{0,0}^{2,q}(\mathbb{R}^3)$  and we have

$$\forall \mathbf{x} \in \mathbb{R}^3, \quad \eta_\beta^\alpha |u(\mathbf{x})| \leq C\eta_0^\theta(\mathbf{x})\|\eta_\beta^\alpha u\|_{W_{0,0}^{2,q}(\mathbb{R}^3)},$$

which implies that

$$\forall \mathbf{x} \in \mathbb{R}^3, \quad |u(\mathbf{x})| \leq C\eta_{-\beta}^{\theta-\alpha}(\mathbf{x})\|u\|_{W_{\alpha,\beta}^{2,q}(\mathbb{R}^3)}. \quad \square$$

## 5 The stationary Navier-Stokes equations

Let  $\Omega$  be an exterior domain with boundary  $\partial\Omega$  of class  $\mathcal{C}^2$ . For any  $\mathbf{w} \in \widetilde{W}_{\alpha+\frac{1}{2},\beta}^{2,q}(\Omega)$ , we introduce the nonlinear operator  $T$  such that  $\mathbf{v} = T\mathbf{w}$  satisfies the problem

$$\begin{aligned} -\nu\Delta\mathbf{v} + k\partial_1\mathbf{v} + \nabla p &= \rho\mathbf{f} - \rho\mathbf{w} \cdot \nabla\mathbf{w} \quad \text{in } \Omega, \\ \operatorname{div}\mathbf{v} &= 0 \quad \text{in } \Omega, \\ \mathbf{v} &= \mathbf{u}_* - \mathbf{u}_\infty \quad \text{on } \partial\Omega, \\ \lim_{|\mathbf{x}| \rightarrow \infty} \mathbf{v}(\mathbf{x}) &= \mathbf{0}. \end{aligned} \tag{5.31}$$

Our aim is to prove that the operator  $T$  has a fixed point. Using Section 4, we estimate the nonlinear term in anisotropic weighted spaces.

**Lemma 5.1.** *We assume  $\frac{4}{3} < q \leq 2$ .*

(i) *Let  $\mathbf{v} \in \widetilde{W}_{\alpha+\frac{1}{2},\beta}^{2,q}(\Omega)$ , then  $\mathbf{v} \cdot \nabla\mathbf{v} \in L_{2\alpha+\frac{1}{2},2\beta}^q(\Omega)$  and satisfies*

$$\|\mathbf{v} \cdot \nabla\mathbf{v}\|_{L_{2\alpha+\frac{1}{2},2\beta}^q(\Omega)} \leq C\|\mathbf{v}\|_{\widetilde{W}_{\alpha+\frac{1}{2},\beta}^{2,q}(\Omega)}^2. \tag{5.32}$$

(ii) *Let  $\mathbf{v}_1, \mathbf{v}_2 \in \widetilde{W}_{\alpha+\frac{1}{2},\beta}^{2,q}(\Omega)$ , then we have the estimate*

$$\begin{aligned} &\|\mathbf{v}_1 \cdot \nabla\mathbf{v}_1 - \mathbf{v}_2 \cdot \nabla\mathbf{v}_2\|_{L_{2\alpha+\frac{1}{2},2\beta}^q(\Omega)} \\ &\leq C\|\mathbf{v}_1 - \mathbf{v}_2\|_{\widetilde{W}_{\alpha+\frac{1}{2},\beta}^{2,q}(\Omega)} \left( \|\mathbf{v}_1\|_{\widetilde{W}_{\alpha+\frac{1}{2},\beta}^{2,q}(\Omega)} + \|\mathbf{v}_2\|_{\widetilde{W}_{\alpha+\frac{1}{2},\beta}^{2,q}(\Omega)} \right). \end{aligned} \tag{5.33}$$

**Proof.** (i) We have

$$\int_{\Omega} |\mathbf{v} \cdot \nabla \mathbf{v}|^q \eta_{2\beta q}^{(2\alpha + \frac{1}{2})q} d\mathbf{x} \leq C \int_{\Omega} |\mathbf{v}|^q |\nabla \mathbf{v}|^q \eta_{2\beta q}^{(2\alpha + \frac{1}{2})q} d\mathbf{x}.$$

Introducing a real number  $t > 1$ , we can write

$$\int_{\Omega} |\mathbf{v}|^q |\nabla \mathbf{v}|^q \eta_{2\beta q}^{(2\alpha + \frac{1}{2})q} d\mathbf{x} \leq C \int_{\Omega} |\mathbf{v}|^q \eta_{\beta q}^{(\alpha - \frac{1}{2} + \frac{2}{q})q - \frac{2}{t}} |\nabla \mathbf{v}|^q \eta_{\beta q}^{(\alpha + 1)q - 2 + \frac{2}{t}} d\mathbf{x}.$$

From the Höder's inequality and since  $q \leq 2$ , we have

$$\begin{aligned} & \int_{\Omega} |\mathbf{v}|^q |\nabla \mathbf{v}|^q \eta_{2\beta q}^{(2\alpha + \frac{1}{2})q} d\mathbf{x} \\ & \leq C \left( \int_{\Omega} |\mathbf{v}|^{qt} \eta_{\beta qt}^{(\alpha - \frac{1}{2} + \frac{2}{q})qt - 2} d\mathbf{x} \right)^{1/t} \left( \int_{\Omega} |\nabla \mathbf{v}|^{qt'} \eta_{\beta qt'}^{(\alpha + 1)qt' - 2t' + \frac{2t'}{t}} d\mathbf{x} \right)^{1/t'} \\ & \leq C \left( \int_{\Omega} |\mathbf{v}|^{qt} \eta_{\beta qt}^{(\alpha - \frac{1}{2} + \frac{2}{q})qt - 2} d\mathbf{x} \right)^{1/t} \left( \int_{\Omega} |\nabla \mathbf{v}|^{qt'} \eta_{\beta qt'}^{(\alpha + \frac{2}{q})qt' - 2} d\mathbf{x} \right)^{1/t'}. \end{aligned}$$

Since  $q > \frac{4}{3}$ , we can choose  $t$  such that  $1 - \frac{q}{2} < \frac{1}{t} < \frac{q}{4}$ . Thus, the assumptions of Theorem 4.3 and Theorem 4.4 are both satisfied and we obtain

$$\int_{\Omega} |\mathbf{v}|^q |\nabla \mathbf{v}|^q \eta_{2\beta q}^{(2\alpha + \frac{1}{2})q} d\mathbf{x} \leq C \|\mathbf{v}\|_{\widetilde{W}_{\alpha + \frac{1}{2}, \beta}^{2, q}(\Omega)}^2$$

which ends the proof of (i).

(ii) Using the following identity,

$$\mathbf{v}_1 \cdot \nabla \mathbf{v}_1 - \mathbf{v}_2 \cdot \nabla \mathbf{v}_2 = (\mathbf{v}_1 - \mathbf{v}_2) \cdot \nabla (\mathbf{v}_1 - \mathbf{v}_2) + \mathbf{v}_2 \cdot \nabla (\mathbf{v}_1 - \mathbf{v}_2) + (\mathbf{v}_1 - \mathbf{v}_2) \cdot \nabla \mathbf{v}_2$$

and the inequality (5.32), the proof of (5.33) is straightforward.  $\square$

We are now in a position to prove our main result.

**Theorem 5.2.** *Let  $\Omega$  be an exterior domain of  $\mathbb{R}^3$  with boundary  $\partial\Omega$  of class  $C^2$  and  $\frac{4}{3} < q \leq 2$ . Let  $\alpha$  and  $\beta$  be two real numbers satisfying*

$$\max\left(0, \frac{1}{2} - \frac{1}{q}\right) < \beta \leq 1 - \frac{1}{q}, \quad \text{and} \quad 0 \leq \alpha < \frac{1}{2} - \frac{1}{q} + \beta. \quad (5.34)$$

*Then there are  $\delta, \varepsilon > 0$  such that for any external forces  $\mathbf{f} \in L^q_{\alpha + \frac{1}{2}, \beta}(\Omega)$  and boundary velocity  $\mathbf{u}_* \in W^{1+1/q', q}(\partial\Omega)$  satisfying*

$$\|\mathbf{f}\|_{L^q_{\alpha + \frac{1}{2}, \beta}(\Omega)} \leq \varepsilon \quad \text{and} \quad \|\mathbf{u}_\infty - \mathbf{u}_*\|_{W^{1+1/q', q}(\partial\Omega)} \leq \varepsilon,$$

*the stationary Navier-Stokes equations (1.1) have a unique solution  $(\mathbf{u}, p) \in W_{loc}^{2, q}(\overline{\Omega}) \times W_{loc}^{1, q}(\overline{\Omega})$  such that  $(\mathbf{u} - \mathbf{u}_\infty, p) \in \widetilde{W}_{\alpha + \frac{1}{2}, \beta}^{2, q}(\Omega) \times X_{\alpha + \frac{1}{2}, \beta}^{1, q}(\Omega)$  and*

$$\|\mathbf{u} - \mathbf{u}_\infty\|_{\widetilde{W}_{\alpha + \frac{1}{2}, \beta}^{2, q}(\Omega)} \leq \delta.$$

Moreover, there exists  $C(\delta) > 0$  such that

$$\|\mathbf{u} - \mathbf{u}_\infty\|_{\widetilde{W}_{\alpha+\frac{1}{2},\beta}^{2,q}(\Omega)} + \|p\|_{X_{\alpha+\frac{1}{2},\beta}^{1,q}(\Omega)} \leq C(\delta) \left( \|\mathbf{f}\|_{L^q_{\alpha+\frac{1}{2},\beta}(\Omega)} + \|\mathbf{u}_* - \mathbf{u}_\infty\|_{W^{1+1/q',q}(\partial\Omega)} \right).$$

Assume in addition that  $\frac{3}{2} < q < 2$ . Then we have the following pointwise decay estimate

$$|\mathbf{u}(\mathbf{x}) - \mathbf{u}_\infty| = O\left(\eta_{-\beta}^{\frac{3}{2}-\frac{3}{q}-\alpha}(\mathbf{x})\right). \quad (5.35)$$

**Proof.** We apply the Banach's fixed point theorem. Let  $\delta > 0$  be a real number and

$$K(\delta) = \left\{ \mathbf{w} \in \widetilde{W}_{\alpha+\frac{1}{2},\beta}^{2,q}(\Omega), \|\mathbf{w}\|_{\widetilde{W}_{\alpha+\frac{1}{2},\beta}^{2,q}(\Omega)} \leq \delta \right\}.$$

1) The set  $K(\delta)$  is clearly a closed subset of  $\widetilde{W}_{\alpha+\frac{1}{2},\beta}^{2,q}(\Omega)$ .

2) Let us prove that  $T(K(\delta)) \subset K(\delta)$ . Let  $\mathbf{w} \in K(\delta)$ ,  $\mathbf{f} \in L^q_{\alpha+\frac{1}{2},\beta}(\Omega)$ ,  $\mathbf{u}_* \in W^{1+1/q',q}(\partial\Omega)$  satisfy

$$\|\mathbf{f}\|_{L^q_{\alpha+\frac{1}{2},\beta}(\Omega)} \leq \varepsilon \quad \text{and} \quad \|\mathbf{u}_\infty - \mathbf{u}_*\|_{W^{1+1/q',q}(\partial\Omega)} \leq \varepsilon.$$

Since  $\alpha \geq 0$  and  $\beta > 0$ , from Lemma 5.1,  $\mathbf{w} \cdot \nabla \mathbf{w} \in L^q_{\alpha+\frac{1}{2},\beta}(\Omega)$  and we have

$$\|\mathbf{w} \cdot \nabla \mathbf{w}\|_{L^q_{\alpha+\frac{1}{2},\beta}(\Omega)} \leq C \|\mathbf{w}\|_{\widetilde{W}_{\alpha+\frac{1}{2},\beta}^{2,q}(\Omega)}^2.$$

Hence thanks to Theorem 3.1, problem (5.31) has a unique solution  $(\mathbf{v} = T\mathbf{w}, p) \in \widetilde{W}_{\alpha+\frac{1}{2},\beta}^{2,q}(\Omega) \times X_{\alpha+\frac{1}{2},\beta}^{1,q}(\Omega)$ . Moreover the following estimate holds

$$\begin{aligned} & \|T\mathbf{w}\|_{\widetilde{W}_{\alpha+\frac{1}{2},\beta}^{2,q}(\Omega)} + \|p\|_{X_{\alpha+\frac{1}{2},\beta}^{1,q}(\Omega)} \\ & \leq C_0 \left( \|\mathbf{f}\|_{L^q_{\alpha+\frac{1}{2},\beta}(\Omega)} + \|\mathbf{u}_* - \mathbf{u}_\infty\|_{W^{1+1/q',q}(\partial\Omega)} + \|\mathbf{w}\|_{\widetilde{W}_{\alpha+\frac{1}{2},\beta}^{2,q}(\Omega)}^2 \right) \\ & \leq C_0 (2\varepsilon + \delta^2). \end{aligned}$$

Choosing now  $\varepsilon \leq \frac{\delta}{4C_0}$  and  $\delta \leq \frac{1}{2C_0}$ , we have  $C_0(2\varepsilon + \delta^2) \leq \delta$  which proves that  $T(K(\delta)) \subset K(\delta)$ .

3) Let us prove that  $T$  is a contraction. Let  $\mathbf{w}_1, \mathbf{w}_2 \in K(\delta)$  and  $\mathbf{v}_1 = T\mathbf{w}_1$ ,  $\mathbf{v}_2 = T\mathbf{w}_2$ . We can write

$$\begin{aligned} \|\mathbf{v}_1 - \mathbf{v}_2\|_{\widetilde{W}_{\alpha+\frac{1}{2},\beta}^{2,q}(\Omega)} & = \|T\mathbf{w}_1 - T\mathbf{w}_2\|_{\widetilde{W}_{\alpha+\frac{1}{2},\beta}^{2,q}(\Omega)} \\ & \leq C_0 \|\rho(\mathbf{w}_1 \cdot \nabla \mathbf{w}_1 - \mathbf{w}_2 \cdot \nabla \mathbf{w}_2)\|_{L^q_{\alpha+\frac{1}{2},\beta}(\Omega)} \\ & \leq C_1 \|\mathbf{w}_1 - \mathbf{w}_2\|_{\widetilde{W}_{\alpha+\frac{1}{2},\beta}^{2,q}(\Omega)} (\|\mathbf{w}_1\|_{\widetilde{W}_{\alpha+\frac{1}{2},\beta}^{2,q}(\Omega)} + \|\mathbf{w}_2\|_{\widetilde{W}_{\alpha+\frac{1}{2},\beta}^{2,q}(\Omega)}) \\ & \leq 2\delta C_1 \|\mathbf{w}_1 - \mathbf{w}_2\|_{\widetilde{W}_{\alpha+\frac{1}{2},\beta}^{2,q}(\Omega)}, \end{aligned}$$

where the second inequality is obtained thanks to Lemma 5.1 (ii). Choosing additionally  $\delta < \frac{1}{2C_1}$ , then the operator  $T : K(\delta) \mapsto K(\delta)$  is a contraction. Hence from the Banach's fixed point theorem, the statement of the theorem is proved. If  $\frac{3}{2} < q < 2$ , then Theorem 4.6 yields the pointwise decay (5.35).  $\square$

**Remark 5.3.** (i) We recall the wake  $\mathbf{W} = \{\mathbf{x} \in \mathbb{R}^3, s(\mathbf{x}) \leq 1\}$  and the set  $\mathbf{S} = \{\mathbf{x} \in \mathbb{R}^3, s(\mathbf{x}) \geq |\mathbf{x}|/2\}$ . For  $|\mathbf{x}| \rightarrow \infty$ , the velocity  $\mathbf{u}$  given by Theorem 5.2 satisfies  $|\mathbf{u}(\mathbf{x}) - \mathbf{u}_\infty| = O(|\mathbf{x}|^{\frac{3}{2} - \frac{3}{q} - \alpha})$  in  $\mathbf{W}$  and  $|\mathbf{u}(\mathbf{x}) - \mathbf{u}_\infty| = O(|\mathbf{x}|^{\frac{3}{2} - \frac{3}{q} - (\alpha + \beta)})$  in  $\mathbf{S}$ . Since  $\beta > 0$ , this shows the better decay of the velocity outside the wake.

(ii) The decay properties of the velocity  $\mathbf{u}$  for the case  $q = 2$  is proved in [5].

(iii) Observe that the velocity  $\mathbf{u}$  given by Theorem 5.2 satisfies  $\nabla \mathbf{u} \in L^q_{\alpha, \beta}(\Omega)$ , with  $\alpha \geq 0$  and  $\beta > 0$ . Then  $\mathbf{u}$  is a  $\mathcal{D}$ -solution, *i.e.*  $\int_{\mathbb{R}^3} |\nabla \mathbf{u}|^q dx < \infty$ . Additionally, if we assume  $\beta > \frac{3}{2} - \frac{2}{q}$  and  $\alpha > 2 - \frac{3}{q}$ , then  $\mathbf{u}$  is also a *physically reasonable* solution in the sense of Finn [10] (see (1.2)).

Combining Lemma 5.1 and Theorem 5.2, we easily get the following stability result.

**Corollary 5.4.** *We assume the assumptions of Theorem 5.2 to be satisfied. Let  $\mathbf{f}_1, \mathbf{f}_2 \in L^q_{\alpha + \frac{1}{2}, \beta}(\Omega)$  and  $\mathbf{u}_*^1, \mathbf{u}_*^2 \in W^{1+1/q', q}(\partial\Omega)$ ,  $(\mathbf{u}_1, p_1), (\mathbf{u}_2, p_2) \in W^{2, q}_{loc}(\overline{\Omega}) \times W^{1, q}_{loc}(\overline{\Omega})$  be solutions of (1.1), such that  $(\mathbf{u}_1 - \mathbf{u}_\infty, p_1) \in \widetilde{W}^{2, q}_{\alpha + \frac{1}{2}, \beta}(\Omega) \times X^{1, q}_{\alpha + \frac{1}{2}, \beta}(\Omega)$  and  $(\mathbf{u}_2 - \mathbf{u}_\infty, p_2) \in \widetilde{W}^{2, q}_{\alpha + \frac{1}{2}, \beta}(\Omega) \times X^{1, q}_{\alpha + \frac{1}{2}, \beta}(\Omega)$ , with*

$$\|\mathbf{u}_1 - \mathbf{u}_\infty\|_{\widetilde{W}^{2, q}_{\alpha + \frac{1}{2}, \beta}(\Omega)} \leq \delta \quad \text{and} \quad \|\mathbf{u}_2 - \mathbf{u}_\infty\|_{\widetilde{W}^{2, q}_{\alpha + \frac{1}{2}, \beta}(\Omega)} \leq \delta.$$

*Then we have the estimate*

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_{\widetilde{W}^{2, q}_{\alpha + \frac{1}{2}, \beta}(\Omega)} + \|p_1 - p_2\|_{X^{1, q}_{\alpha + \frac{1}{2}, \beta}(\Omega)} \leq C(\delta) \left( \|\mathbf{f}_1 - \mathbf{f}_2\|_{L^q_{\alpha + \frac{1}{2}, \beta}(\Omega)} + \|\mathbf{u}_*^1 - \mathbf{u}_*^2\|_{W^{1+1/q', q}(\partial\Omega)} \right).$$

## References

- [1] C. Amrouche and U. Razafison. Anisotropically weighted Poincaré-type inequalities; application to the Oseen problem. *Math. Nachr.*, **279**(9-10):931–947, 2006.
- [2] C. Amrouche and U. Razafison. Weighted Sobolev spaces for a scalar model of the stationary Oseen equations in  $\mathbb{R}^3$ . *J. Math. Fluid Mech.*, **9**:181–210, 2007.
- [3] K.I. Babenko. On stationary solutions of the problem of flow past a body of a viscous incompressible fluid. *Math. USSR, Sb.*, **20**:1–25, 1973.
- [4] D. C. Clark. The vorticity at infinity for solutions of the stationary Navier-Stokes equations in exterior domains. *Indiana Univ. Math. J.*, **20**:633–654, 1970/1971.



- [5] R. Farwig. The stationary exterior 3D-problem of Oseen and Navier-Stokes equations in anisotropically weighted Sobolev spaces. *Math. Z.*, **211**(3):409–447, 1992.
- [6] R. Farwig. A variational approach in weighted Sobolev spaces to the operator  $-\Delta + \partial/\partial x_1$  in exterior domains of  $\mathbb{R}^3$ . *Math. Z.*, **210**(3):449–464, 1992.
- [7] R. Farwig. The stationary Navier-Stokes equations in a 3D-exterior domain. In *Recent topics on mathematical theory of viscous incompressible fluid (Tsukuba, 1996)*, volume 16 of *Lecture Notes Numer. Appl. Anal.*, pages 53–115. Kinokuniya, Tokyo, 1998.
- [8] R. Farwig and H. Sohr. Weighted estimates for the Oseen equations and the Navier-Stokes equations in exterior domains. In *Theory of the Navier-Stokes equations*, volume 47 of *Ser. Adv. Math. Appl. Sci.*, pages 11–30. World Sci. Publishing, River Edge, NJ, 1998.
- [9] R. Finn. Estimates at infinity for stationary solutions of the Navier-Stokes equations. *Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.)*, 3 (**51**):387–418, 1959.
- [10] R. Finn. On the exterior stationary problem for the Navier-Stokes equations, and associated perturbation problems. *Arch. Rational Mech. Anal.*, **19**:363–406, 1965.
- [11] H. Fujita. On the existence and regularity of the steady-state solutions of the Navier-Stokes equations. *J. Fac. Sci. Univ. Tokyo*, **9**:59–102, 1961.
- [12] G. P. Galdi. On the asymptotic structure of  $D$ -solutions to steady Navier-Stokes equations in exterior domains. In *Mathematical problems relating to the Navier-Stokes equation*, volume 11 of *Ser. Adv. Math. Appl. Sci.*, pages 81–104. World Sci. Publishing, River Edge, NJ, 1992.
- [13] G. P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I*, volume 38 of *Springer Tracts in Natural Philosophy*. Springer-Verlag, New York, 1994.
- [14] G. P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. II*, volume 39 of *Springer Tracts in Natural Philosophy*. Springer-Verlag, New York, 1994.
- [15] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1985.
- [16] B. Hanouzet. Espaces de Sobolev avec poids application au problème de Dirichlet dans un demi espace. *Rend. Sem. Mat. Univ. Padova*, **46**:227–272, 1971.

- [17] S. Kračmar, A. Novotný, and M. Pokorný. Estimates of Oseen kernels in weighted  $L^p$  spaces. *J. Math. Soc. Japan*, **53**(1):59–111, 2001.
- [18] O. A. Ladyzhenskaya. *The mathematical theory of viscous incompressible flow*. Second English edition, revised and enlarged. Translated from the Russian by Richard A. Silverman and John Chu. Mathematics and its Applications, Vol. 2. Gordon and Breach Science Publishers, New York, 1969.
- [19] J. Leray. Étude de diverses équations intégrales non linéaires et de quelques problèmes que pose l'hydrodynamique. *J. Math. Pures Appl.*, **12**:1–82, 1933.
- [20] C. W. Oseen. Über die Stokesesche Formel und Über eine Verwandte aufgabe in der Hydrodynamik. *Ark. Mat. Astron. Fys.(29)*, **6**:1–20, 1910.
- [21] C. W. Oseen. *Neuere Methoden und Ergebnisse in der Hydrodynamik*. XXIV + 337 S. mit 7 Fig. Leipzig, Akadem. Verlagsgesellschaft (Mathematik und ihre Anwendungen in Monographien und Lehrbchern Bd. 1). , 1927.
- [22] M. Pokorný. *Asymptotic behaviour of some equations describing the flow of fluids in unbounded domains*. PhD thesis, Charles University Prague and University of Toulon, 1999.
- [23] U. Razafison. Anisotropic weighted  $L^p$  spaces for the stationary exterior 3D-problem of Oseen. *J. Math. Anal. Appl.*, **323**(1):275–292, 2006.