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Two finiteness theorem for (a,b)-modules.

Daniel Barlet*

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Summary.

We prove the following two results

1. For a proper holomorphic function $f : X \rightarrow D$ of a complex manifold X on a disc such that $\{df = 0\} \subset f^{-1}(0)$, we construct, in a functorial way, for each integer p , a geometric (a,b)-module E^p associated to the (filtered) Gauss-Manin connexion of f .

This first theorem is an existence/finiteness result which shows that geometric (a,b)-modules may be used in global situations.

2. For any regular (a,b)-module E we give an integer $N(E)$, explicitly given from simple invariants of E , such that the isomorphism class of $E/b^{N(E)}.E$ determines the isomorphism class of E .

This second result allows to cut asymptotic expansions (in powers of b) of elements of E without losing any information.

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Key words : (a,b)-module or Brieskorn modules, Gauss-Manin connexion, vanishing cycles.

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1 Introduction.

The following situation is frequently met : we consider a vector space E of multi-valued holomorphic functions (possibly with values in a complex finite dimensional vector space V) with finite determination on a punctured disc around 0 in \mathbb{C} , stable by multiplication by the variable z and by "primitive". These functions are determined by there formal asymptotic expansions at 0 of the type

$$\sum_{(\alpha,j) \in A \times [0,n]} c_{\alpha,j}(z).z^\alpha. \frac{(\text{Log}z)^j}{j!}$$

where n is a fixed integer, where A is a finite set of complex numbers whose real parts are, for instance, in the interval $]-1, 0]$, and where the $c_{\alpha,j}$ are in $\mathbb{C}[[z]] \otimes V$. Define $e(\alpha, j) = z^\alpha. (\text{Log}z)^j / j!$ and $E(\alpha, n) := \bigoplus_{j=0}^n (\mathbb{C}[[z]] \otimes V).e(\alpha, j)$. Then for each α , $E(\alpha, n)$ is a free $\mathbb{C}[[z]]$ -module of rank $(n+1). \dim V$ which is stable by $b := \int_0^z$. To be precise, b is defined by induction on $j \geq 0$ by the "obvious" formulas :

$$b[e(\alpha, 0)] = \frac{e(\alpha + 1, 0)}{\alpha + 1} \quad \text{and for } j \geq 1$$

$$b[e(\alpha, j)] = \frac{e(\alpha + 1, j)}{\alpha + 1} - \frac{1}{\alpha + 1}.b[e(\alpha, j - 1)].$$

So we have an inclusion $E \subset \bigoplus_{\alpha \in A} E(\alpha)$ which is compatible with $a := \times z$ and by b . Note that each $E(\alpha)$ is also a free finite rank $\mathbb{C}[[b]]$ -module which is stable by a .

Let us assume now that E is also a $\mathbb{C}[[b]]$ -module which is stable by a . Then E has to be free and of finite rank over $\mathbb{C}[[b]]$. Our aim in this situation is to understand and to describe the relations between the coefficients $c_{\alpha,j}$ of the asymptotic expansions of elements in E . This leads to construct "invariants" associated to the given E .

As in the "geometric" situations we consider the complex numbers $\alpha \in A$ are rational numbers, a change of variable of type $t := z^{1/N}$, with $N \in \mathbb{N}^*$, allows to reduce the situation to the case where all α are 0 . Then one can use the "nilpotent" operator $\frac{d}{dt}$ and the nilpotent part of the monodromy $e_{\alpha,j} \rightarrow e_{\alpha,j-1}$ to construct some filtrations in order, in the good cases, to build a Mixte Hodge structure on a finite dimensional vector space associated to E . For instance, this is the case in A.N. Varchenko's description [V. 80] of the Mixte Hodge structure built by J. Steenbrink [St. 76] on the cohomology of the Milnor fiber of an holomorphic function with an isolated singularity at the origin of \mathbb{C}^{n+1} .

But it is clear that we loose some information in this procedure. The point of view which is to consider E itself as a left module on the (non commutative) algebra

$$\tilde{\mathcal{A}} := \left\{ \sum_{\nu \geq 0} P_\nu(a).b^\nu \right\}$$

where the P_ν are polynomials with complex coefficients is richer. This is evidenced by M. Saito result [Sa. 91].

The aim of the first part of this article is to build in a natural way, for any proper holomorphic function $f : X \rightarrow D$ of a complex manifold X , assumed to be smooth outside its 0-fiber $X_0 := f^{-1}(0)$, a regular (geometric) (a,b)-module for each degree $p \geq 0$, which represent a filtered version of the Gauss-Manin connexion of f at the origin.

This result is in fact a finiteness theorem which is a first step to refine the limite Mixte Hodge structure in this situation. It is interesting to remark that no Kähler assumption is used in this construction of these geometric (a,b)-modules.

This obviously shows that (a,b)-modules are basic objects and that they are important not only in the study of local singularities of holomorphic functions but more generally in complex geometry . So it is interesting to have some tools in order to compute them.

This is precisely the aim of the second part of this paper. We prove a finiteness result which gives, for a regular (a,b)-module E , an integer $N(E)$, bounded by simple numerical invariants of E , such that you may cut the asymptotic expansions (in powers of b) of elements of E without any lost of information on the structure of the (a,b)-module E . It is well known that the formal asymptotic expansions for solutions of a regular differential system always converge, and also that such an integer exists for any meromorphic connexion in one variable (see [M.91] proposition 1.12). But it is important to have an effective bound for such an integer easily computable from simple invariants of the (a,b)-module structure of E .

2 The existence theorem.

2.1 Preliminaries.

Here we shall complete and precise the results of the section 2 of [B.07]. The situation we shall consider is the following : let X be a connected complex manifold of dimension $n + 1$ and $f : X \rightarrow \mathbb{C}$ a non constant holomorphic function such that $\{x \in X / df = 0\} \subset f^{-1}(0)$. We introduce the following complexes of sheaves supported by $X_0 := f^{-1}(0)$

1. The formal completion "in f " $(\hat{\Omega}^\bullet, d^\bullet)$ of the usual holomorphic de Rham complex of X .
2. The sub-complexes $(\hat{K}^\bullet, d^\bullet)$ and $(\hat{I}^\bullet, d^\bullet)$ of $(\hat{\Omega}^\bullet, d^\bullet)$ where the subsheaves \hat{K}^p and \hat{I}^{p+1} are defined for each $p \in \mathbb{N}$ respectively as the kernel and the image of the map

$$\wedge df : \hat{\Omega}^p \rightarrow \hat{\Omega}^{p+1}$$

given par exterior multiplication by df . We have the exact sequence

$$0 \rightarrow (\hat{K}^\bullet, d^\bullet) \rightarrow (\hat{\Omega}^\bullet, d^\bullet) \rightarrow (\hat{I}^\bullet, d^\bullet)[+1] \rightarrow 0. \quad (1)$$

Note that \hat{K}^0 and \hat{I}^0 are zero by definition.

3. The natural inclusions $\hat{I}^p \subset \hat{K}^p$ for all $p \geq 0$ are compatible with the differential d . This leads to an exact sequence of complexes

$$0 \rightarrow (\hat{I}^\bullet, d^\bullet) \rightarrow (\hat{K}^\bullet, d^\bullet) \rightarrow ([\hat{K}/\hat{I}]^\bullet, d^\bullet) \rightarrow 0. \quad (2)$$

4. We have a natural inclusion $f^*(\hat{\Omega}_{\mathbb{C}}^1) \subset \hat{K}^1 \cap \text{Ker } d$, and this gives a sub-complex (with zero differential) of $(\hat{K}^\bullet, d^\bullet)$. As in [B.07], we shall consider also the complex $(\tilde{K}^\bullet, d^\bullet)$ quotient. So we have the exact sequence

$$0 \rightarrow f^*(\hat{\Omega}_{\mathbb{C}}^1) \rightarrow (\hat{K}^\bullet, d^\bullet) \rightarrow (\tilde{K}^\bullet, d^\bullet) \rightarrow 0. \quad (3)$$

We do not make the assumption here that $f = 0$ is a reduced equation of X_0 , and we do not assume that $n \geq 2$, so the cohomology sheaf in degree 1 of the complex $(\hat{K}^\bullet, d^\bullet)$, which is equal to $\hat{K}^1 \cap \text{Ker } d$ does not coincide, in general, with $f^*(\hat{\Omega}_{\mathbb{C}}^1)$. So the complex $(\tilde{K}^\bullet, d^\bullet)$ may have a non zero cohomology sheaf in degree 1.

Recall now that we have on the cohomology sheaves of the following complexes $(\hat{K}^\bullet, d^\bullet)$, $(\hat{I}^\bullet, d^\bullet)$, $([\hat{K}/\hat{I}]^\bullet, d^\bullet)$ and $f^*(\hat{\Omega}_{\mathbb{C}}^1)$, $(\tilde{K}^\bullet, d^\bullet)$ natural operations a and b with the relation $a.b - b.a = b^2$. They are defined in a naïve way by

$$a := \times f \quad \text{and} \quad b := \wedge df \circ d^{-1}.$$

The definition of a makes sens obviously. Let me precise the definition of b first in the case of $\mathcal{H}^p(\hat{K}^\bullet, d^\bullet)$ with $p \geq 2$: if $x \in \hat{K}^p \cap \text{Ker } d$ write $x = d\xi$ with $\xi \in \hat{\Omega}^{p-1}$ and let $b[x] := [df \wedge \xi]$. The reader will check easily that this makes sens. For $p = 1$ we shall choose $\xi \in \hat{\Omega}^0$ such that $\xi = 0$ on the smooth part of X_0 (set theoretically). This is possible because the condition $df \wedge d\xi = 0$ allows such a choice : near a smooth point of X_0 we can choose coordinnates such $f = x_0^k$ and the condition on ξ means independance of x_1, \dots, x_n . Then ξ has to be (set theoretically) locally constant on X_0 which is locally connected. So we may kill the value of such a ξ along X_0 .

The case of the complex $(\hat{I}^\bullet, d^\bullet)$ will be reduced to the previous one using the next lemma.

Lemme 2.1.1 *For each $p \geq 0$ there is a natural injective map*

$$\tilde{b} : \mathcal{H}^p(\hat{K}^\bullet, d^\bullet) \rightarrow \mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$$

which satisfies the relation $a.\tilde{b} = \tilde{b}.(a + b)$. For $p \neq 1$ this map is bijective.

PROOF. Let $x \in \hat{K}^p \cap \text{Ker } d$ and write $x = d\xi$ where $\xi \in \hat{\Omega}^{p-1}$ (with $\xi = 0$ on X_0 if $p = 1$), and set $\tilde{b}([x]) := [df \wedge \xi] \in \mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$. This is independant on the choice of ξ because, for $p \geq 2$, adding $d\eta$ to ξ does not modify the result as $[df \wedge d\eta] = 0$. For $p = 1$ remark that our choice of ξ is unique.

This is also independant of the the choice of x in $[x] \in \mathcal{H}^p(\hat{K}^\bullet, d^\bullet)$ because adding $\theta \in \hat{K}^{p-1}$ to ξ does not change $df \wedge \xi$.

Assume $\tilde{b}([x]) = 0$ in $\mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$; this means that we may find $\alpha \in \hat{\Omega}^{p-2}$ such $df \wedge \xi = df \wedge d\alpha$. But then, $\xi - d\alpha$ lies in \hat{K}^{p-1} and $x = d(\xi - d\alpha)$ shows that $[x] = 0$. So \tilde{b} is injective.

Assume now $p \geq 2$. If $df \wedge \eta$ is in $\hat{I}^p \cap \text{Ker } d$, then $df \wedge d\eta = 0$ and $y := d\eta$ lies in $\hat{K}^p \cap \text{Ker } d$ and defines a class $[y] \in \mathcal{H}^p(\hat{K}^\bullet, d^\bullet)$ whose image by \tilde{b} is $[df \wedge \eta]$. This shows the surjectivity of \tilde{b} for $p \geq 2$.

For $p = 1$ the map \tilde{b} is not surjective (see the remark below).

To finish the proof let us to compute $\tilde{b}(a[x] + b[x])$. Writing again $x = d\xi$, we get

$$a[x] + b[x] = [f.d\xi + df \wedge \xi] = [d(f.\xi)]$$

and so

$$\tilde{b}(a[x] + b[x]) = [df \wedge f.\xi] = a.\tilde{b}([x])$$

which concludes the proof. ■

Denote by $i : (\hat{I}^\bullet, d^\bullet) \rightarrow (\hat{K}^\bullet, d^\bullet)$ the natural inclusion and define the action of b on $\mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$ by $b := \tilde{b} \circ \mathcal{H}^p(i)$. As i is a -linear, we deduce the relation $a.b - b.a = b^2$ on $\mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$ from the relation of the previous lemma.

The action of a on the complex $([\hat{K}/\hat{I}]^\bullet, d^\bullet)$ is obvious and the action of b is zero.

The action of a and b on $f^*(\hat{\Omega}_{\mathbb{C}}^1) \simeq E_1 \otimes \mathbb{C}_{X_0}$ are the obvious one, where E_1 is the rank 1 (a,b)-module with generator e_1 satisfying $a.e_1 = b.e_1$ (or, if you prefer, $E_1 := \mathbb{C}[[z]]$ with $a := \times z$, $b := \int_0^z$ and $e_1 := 1$).

Remark that the natural inclusion $f^*(\hat{\Omega}_{\mathbb{C}}^1) \hookrightarrow (\hat{K}^\bullet, d^\bullet)$ is compatible with the actions of a and b . The actions of a and b on $\mathcal{H}^1(\hat{K}^\bullet, d^\bullet)$ are simply induced by the corresponding actions on $\mathcal{H}^1(\hat{K}^\bullet, d^\bullet)$.

REMARK. The exact sequence of complexes (1) induces for any $p \geq 2$ a bijection

$$\partial^p : \mathcal{H}^p(\hat{I}^\bullet, d^\bullet) \rightarrow \mathcal{H}^p(\hat{K}^\bullet, d^\bullet)$$

and a short exact sequence

$$0 \rightarrow \mathbb{C}_{X_0} \rightarrow \mathcal{H}^1(\hat{I}^\bullet, d^\bullet) \xrightarrow{\partial^1} \mathcal{H}^1(\hat{K}^\bullet, d^\bullet) \rightarrow 0 \quad (\textcircled{a})$$

because of the de Rham lemma. Let us check that for $p \geq 2$ we have $\partial^p = (\tilde{b})^{-1}$ and that for $p = 1$ we have $\partial^1 \circ \tilde{b} = Id$. If $x = d\xi \in \hat{K}^p \cap \text{Ker } d$ then $\tilde{b}([x]) = [df \wedge \xi]$ and $\partial^p[df \wedge \xi] = [d\xi]$. So $\partial^p \circ \tilde{b} = Id \quad \forall p \geq 0$. For $p \geq 2$ and

$df \wedge \alpha \in \hat{I}^p \cap \text{Ker } d$ we have $\partial^p[df \wedge \alpha] = [d\alpha]$ and $\tilde{b}[d\alpha] = [df \wedge \alpha]$, so $\tilde{b} \circ \partial^p = \text{Id}$. For $p = 1$ we have $\tilde{b}[d\alpha] = [df \wedge (\alpha - \alpha_0)]$ where $\alpha_0 \in \mathbb{C}$ is such that $\alpha|_{X_0} = \alpha_0$. This shows that in degree 1 \tilde{b} gives a canonical splitting of the exact sequence (4).

2.2 $\tilde{\mathcal{A}}$ -structures.

Let us consider now the \mathbb{C} -algebra

$$\tilde{\mathcal{A}} := \left\{ \sum_{\nu \geq 0} P_\nu(a).b^\nu \right\}$$

where $P_\nu \in \mathbb{C}[z]$, and the commutation relation $a.b - b.a = b^2$, assuming that left and right multiplications by a are continuous for the b -adic topology of $\tilde{\mathcal{A}}$. Define the following complexes of sheaves of left $\tilde{\mathcal{A}}$ -modules on X :

$$(\Omega'^\bullet[[b]], D^\bullet) \quad \text{and} \quad (\Omega''^\bullet[[b]], D^\bullet) \quad \text{where} \quad (4)$$

$$\Omega'^\bullet[[b]] := \sum_{j=0}^{+\infty} b^j.\omega_j \quad \text{with} \quad \omega_0 \in \hat{K}^p$$

$$\Omega''^\bullet[[b]] := \sum_{j=0}^{+\infty} b^j.\omega_j \quad \text{with} \quad \omega_0 \in \hat{I}^p$$

$$D\left(\sum_{j=0}^{+\infty} b^j.\omega_j\right) = \sum_{j=0}^{+\infty} b^j.(d\omega_j - df \wedge \omega_{j+1})$$

$$a.\sum_{j=0}^{+\infty} b^j.\omega_j = \sum_{j=0}^{+\infty} b^j.(f.\omega_j + (j-1).\omega_{j-1}) \quad \text{with the convention} \quad \omega_{-1} = 0$$

$$b.\sum_{j=0}^{+\infty} b^j.\omega_j = \sum_{j=1}^{+\infty} b^j.\omega_{j-1}$$

It is easy to check that D is $\tilde{\mathcal{A}}$ -linear and that $D^2 = 0$. We have a natural inclusion of complexes of left $\tilde{\mathcal{A}}$ -modules

$$\tilde{i} : (\Omega''^\bullet[[b]], D^\bullet) \rightarrow (\Omega'^\bullet[[b]], D^\bullet).$$

Remark that we have natural morphisms of complexes

$$u : (\hat{I}^\bullet, d^\bullet) \rightarrow (\Omega''^\bullet[[b]], D^\bullet)$$

$$v : (\hat{K}^\bullet, d^\bullet) \rightarrow (\Omega'^\bullet[[b]], D^\bullet)$$

and that these morphisms are compatible with i . More precisely, this means that we have the commutative diagram of complexes

$$\begin{array}{ccc} (\hat{I}^\bullet, d^\bullet) & \xrightarrow{u} & (\Omega''^\bullet[[b]], D^\bullet) \\ \downarrow i & & \downarrow \tilde{i} \\ (\hat{K}^\bullet, d^\bullet) & \xrightarrow{v} & (\Omega'^\bullet[[b]], D^\bullet) \end{array}$$

The following theorem is a variant of theorem 2.2.1. of [B. 07].

Théorème 2.2.1 *Let X be a connected complex manifold of dimension $n + 1$ and $f : X \rightarrow \mathbb{C}$ a non constant holomorphic function with the following condition:*

$$\{x \in X / df = 0\} \subset f^{-1}(0).$$

Then the morphisms of complexes u and v introduced above are quasi-isomorphisms. Moreover, the isomorphisms that they induce on the cohomology sheaves of these complexes are compatible with the actions of a and b .

This theorem builds a natural structure of left $\tilde{\mathcal{A}}$ -modules on each of the complex $(\hat{K}^\bullet, d^\bullet)$, $(\hat{I}^\bullet, d^\bullet)$, $([\hat{K}/\hat{I}]^\bullet, d^\bullet)$ and $f^*(\hat{\Omega}_{\mathbb{C}}^1)$, $(\tilde{K}^\bullet, d^\bullet)$ in the derived category of bounded complexes of sheaves of \mathbb{C} -vector spaces on X .

Moreover the short exact sequences

$$0 \rightarrow (\hat{I}^\bullet, d^\bullet) \rightarrow (\hat{K}^\bullet, d^\bullet) \rightarrow ([\hat{K}/\hat{I}]^\bullet, d^\bullet) \rightarrow 0 \quad (2)$$

$$0 \rightarrow f^*(\hat{\Omega}_{\mathbb{C}}^1) \rightarrow (\hat{K}^\bullet, d^\bullet), (\hat{I}^\bullet, d^\bullet) \rightarrow (\tilde{K}^\bullet, d^\bullet) \rightarrow 0. \quad (3)$$

are equivalent to short exact sequences of complexes of left $\tilde{\mathcal{A}}$ -modules in the derived category.

PROOF. We have to prove that for any $p \geq 0$ the maps $\mathcal{H}^p(u)$ and $\mathcal{H}^p(v)$ are bijective and compatible with the actions of a and b . The case of $\mathcal{H}^p(v)$ is handled (at least for $n \geq 2$ and f reduced) in prop. 2.3.1. of [B.07]. To seek completeness and for the convenience of the reader we shall treat here the case of $\mathcal{H}^p(u)$.

First we shall prove the injectivity of $\mathcal{H}^p(u)$. Let $\alpha = df \wedge \beta \in \hat{I}^p \cap \text{Ker } d$ and assume that we can find $U = \sum_{j=0}^{+\infty} b^j \cdot u_j \in \Omega''^{p-1}[[b]]$ with $\alpha = DU$. Then we have the following relations

$$u_0 = df \wedge \zeta, \quad \alpha = du_0 - df \wedge u_1 \quad \text{and} \quad du_j = df \wedge u_{j+1} \quad \forall j \geq 1.$$

For $j \geq 1$ we have $[du_j] = b[du_{j+1}]$ in $\mathcal{H}^p(\hat{K}^\bullet, d^\bullet)$; using corollary 2.2. of [B.07] which gives the b -separation of $\mathcal{H}^p(\hat{K}^\bullet, d^\bullet)$, this implies $[du_j] = 0, \forall j \geq 1$ in $\mathcal{H}^p(\hat{K}^\bullet, d^\bullet)$. For instance we can find $\beta_1 \in \hat{K}^{p-1}$ such that $du_1 = d\beta_1$. Now, by de Rham, we can write $u_1 = \beta_1 + d\xi_1$ for $p \geq 2$, where $\xi_1 \in \hat{\Omega}^{p-2}$. Then we conclude that $\alpha = -df \wedge d(\xi_1 + \zeta)$ and $[\alpha] = 0$ in $\mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$.

For $p = 1$ we have $u_0 = 0, du_1 = 0$ so $[\alpha] = [-df \wedge d\xi_1] = 0$ in $\mathcal{H}^1(\hat{I}^\bullet, d^\bullet)$.

We shall show now that the image of $\mathcal{H}^p(u)$ is dense in $\mathcal{H}^p(\Omega''^\bullet[[b]], D^\bullet)$ for its b -adic topology. Let $\Omega := \sum_{j=0}^{+\infty} b^j \cdot \omega_j \in \Omega''^p[[b]]$ such that $D\Omega = 0$. The following relations holds $d\omega_j = df \wedge \omega_{j+1} \quad \forall j \geq 0$ and $\omega_0 \in \hat{I}^p$. The corollary 2.2. of [B.07] again allows to find $\beta_j \in \hat{K}^{p-1}$ for any $j \geq 0$ such that $d\omega_j = d\beta_j$. Fix $N \in \mathbb{N}^*$.

We have

$$D\left(\sum_{j=0}^N b^j \cdot \omega_j\right) = b^N \cdot d\omega_N = D(b^N \cdot \beta_N)$$

and $\Omega_N := \sum_{j=0}^N b^j \cdot \omega_j - b^N \cdot \beta_N$ is D -closed and in $\Omega''^p[[b]]$. As $\Omega - \Omega_N$ lies in $b^N \cdot \mathcal{H}^p(\Omega''^\bullet[[b]], D^\bullet)$, the sequence $(\Omega_N)_{N \geq 1}$ converges to Ω in $\mathcal{H}^p(\Omega''^\bullet[[b]], D^\bullet)$ for its b -adic topology. Let us show that each Ω_N is in the image of $\mathcal{H}^p(u)$.

Write $\Omega_N := \sum_{j=0}^N b^j \cdot w_j$. The condition $D\Omega_N = 0$ implies $dw_N = 0$ and $dw_{N-1} = df \wedge w_N = 0$. If we write $w_N = dv_N$ we obtain $d(w_{N-1} + df \wedge v_N) = 0$ and $\Omega_N - D(b^N \cdot v_N)$ is of degree $N - 1$ in b . For $N = 1$ we are left with $w_0 + b \cdot w_1 - (-df \wedge v_1 + b \cdot dv_1) = w_0 + df \wedge v_1$ which is in $\hat{I}^p \cap \text{Ker } d$ because $dw_0 = df \wedge dv_1$.

To conclude it is enough to know the following two facts

- i) The fact that $\mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$ is complete for its b -adic topology.
- ii) The fact that $\text{Im}(\mathcal{H}^p(u)) \cap b^N \cdot \mathcal{H}^p(\Omega''^\bullet[[b]], D^\bullet) \subset \text{Im}(\mathcal{H}^p(u) \circ b^N) \quad \forall N \geq 1$.

Let us first conclude the proof of the surjectivity of $\mathcal{H}^p(u)$ assuming i) and ii). For any $[\Omega] \in \mathcal{H}^p(\Omega''^\bullet[[b]], D^\bullet)$ we know that there exists a sequence $(\alpha_N)_{N \geq 1}$ in $\mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$ with $\Omega - \mathcal{H}^p(u)(\alpha_N) \in b^N \cdot \mathcal{H}^p(\Omega''^\bullet[[b]], D^\bullet)$. Now the property ii) implies that we may choose the sequence $(\alpha_N)_{N \geq 1}$ such that $[\alpha_{N+1}] - [\alpha_N]$ lies in $b^N \cdot \mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$. So the property i) implies that the Cauchy sequence $([\alpha_N])_{N \geq 1}$ converges to $[\alpha] \in \mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$. Then the continuity of $\mathcal{H}^p(u)$ for the b -adic topologies coming from its b -linearity, implies $\mathcal{H}^p(u)([\alpha]) = [\Omega]$.

The compatibility with a and b of the maps $\mathcal{H}^p(u)$ and $\mathcal{H}^p(v)$ is an easy exercise.

Let us now prove properties i) and ii).

The property i) is a direct consequence of the completion of $\mathcal{H}^p(\hat{K}^\bullet, d^\bullet)$ for its b -adic topology given by the corollary 2.2. of [B.07] and the b -linear isomorphism \tilde{b} between $\mathcal{H}^p(\hat{K}^\bullet, d^\bullet)$ and $\mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$ constructed in the lemma 2.1.1. above.

To prove ii) let $\alpha \in \hat{I}^p \cap \text{Ker } d$ and $N \geq 1$ such that

$$\alpha = b^N \cdot \Omega + DU$$

where $\Omega \in \Omega''^p[[b]]$ satisfies $D\Omega = 0$ and where $U \in \Omega''^{p-1}[[b]]$. With obvious notations we have

$$\begin{aligned} \alpha &= du_0 - df \wedge u_1 \\ &\dots \\ 0 &= du_j - df \wedge u_{j+1} \quad \forall j \in [1, N-1] \\ &\dots \\ 0 &= \omega_0 + du_N - df \wedge u_{N+1} \end{aligned}$$

which implies $D(u_0 + b \cdot u_1 + \dots + b^N \cdot u_N) = \alpha + b^N \cdot du_N$ and the fact that du_N lies in $\hat{I}^p \cap \text{Ker } d$. So we conclude that $[\alpha] + b^N \cdot [du_N]$ is in the kernel of $\mathcal{H}^p(u)$ which is 0. Then $[\alpha] \in b^N \cdot \mathcal{H}^p(\hat{I}^\bullet, d^\bullet)$. ■

REMARK. The map

$$\beta : (\Omega'[[b]]^\bullet, D^\bullet) \rightarrow (\Omega''[[b]]^\bullet, D^\bullet)$$

defined by $\beta(\Omega) = b.\Omega$ commutes to the differentials and with the action of b . It induces the isomorphism \tilde{b} of the lemma 2.1.1 on the cohomology sheaves. So it is a quasi-isomorphism of complexes of $\mathbb{C}[[b]]$ -modules.

To prove this fact, it is enough to verify that the diagram

$$\begin{array}{ccc} (\hat{K}^\bullet, d^\bullet) & \xrightarrow{v} & (\Omega'[[b]]^\bullet, D^\bullet) \\ \downarrow \tilde{b} & & \downarrow \beta \\ (\hat{I}^\bullet, d^\bullet) & \xrightarrow{u} & (\Omega''[[b]]^\bullet, D^\bullet) \end{array}$$

induces commutative diagrams on the cohomology sheaves.

But this is clear because if $\alpha = d\xi$ lies in $\hat{K}^p \cap \text{Ker } d$ we have $D(b.\xi) = b.d\xi - df \wedge \xi$ so $\mathcal{H}^p(\beta) \circ \mathcal{H}^p(v)([\alpha]) = \mathcal{H}^p(u) \circ \mathcal{H}^p(\tilde{b})([\alpha])$ in $\mathcal{H}^p(\Omega''[[b]]^\bullet, D^\bullet)$. ■

2.3 The existence theorem.

Let us recall some basic definitions on the left modules over the algebra $\tilde{\mathcal{A}}$.

Définition 2.3.1 An **(a,b)-module** is a left $\tilde{\mathcal{A}}$ -module which is free and of finite rank on the commutative sub-algebra $\mathbb{C}[[b]]$ of $\tilde{\mathcal{A}}$.

An (a,b) -module E is

1. **local** when $\exists N \in \mathbb{N}$ such that $a^N.E \subset b.E$;
2. **simple pole** when $a.E \subset b.E$;
3. **regular** when it is contained in a simple pole (a,b) -module;
4. **geometric** when it is contained in a simple pole (a,b) -module E^\sharp such that the minimal polynomial of the action of $b^{-1}.a$ on $E^\sharp/b.E^\sharp$ has its roots in \mathbb{Q}^{+*} .

We shall give more details and examples of (a,b) -modules in the section 3.

Now let E be any left $\tilde{\mathcal{A}}$ -module, and define $B(E)$ as the b -torsion of E , that is to say

$$B(E) := \{x \in E / \exists N \quad b^N.x = 0\}.$$

Define $A(E)$ as the a -torsion of E and

$$\hat{A}(E) := \{x \in E / \mathbb{C}[[b]].x \subset A(E)\}.$$

Remark that $B(E)$ and $\hat{A}(E)$ are sub- $\tilde{\mathcal{A}}$ -modules of E but that $A(E)$ is not stable by b .

Définition 2.3.2 A left $\tilde{\mathcal{A}}$ -module E is **small** when the following conditions hold

1. E is a finite type $\mathbb{C}[[b]]$ -module ;
2. $B(E) \subset \hat{A}(E)$;
3. $\exists N / a^N \cdot \hat{A}(E) = 0$;

Recall that for E small we have always the equality $B(E) = \hat{A}(E)$ and that this complex vector space is finite dimensional. The quotient $E/B(E)$ is an (a,b) -module called **the associate (a,b) -module** to E .

Conversely, any left $\tilde{\mathcal{A}}$ -module E such that $B(E)$ is a finite dimensional \mathbb{C} -vector space and such that $E/B(E)$ is an (a,b) -module is small.

The following easy criterium to be small will be used later :

Lemme 2.3.3 *A left $\tilde{\mathcal{A}}$ -module E is small if and only if the following conditions hold :*

1. $\exists N / a^N \cdot \hat{A}(E) = 0$;
2. $B(E) \subset \hat{A}(E)$;
3. $\cap_{m \geq 0} b^m \cdot E \subset \hat{A}(E)$;
4. $\text{Ker } b$ and $\text{Coker } b$ are finite dimensional complex vector spaces.

As the condition 3) in the previous lemma has been omitted in [B.07] (but this does not affect the results of this article because this lemma was used only in a case where this condition 3) was satisfied, thanks to proposition 2.2.1. of *loc. cit.*), we shall give the (easy) proof.

PROOF. First the conditions 1) to 4) are obviously necessary. Conversely, assume that E satisfies these four conditions. Then condition 2) implies that the action of b on $\hat{A}(E)/B(E)$ is injective. But the condition 1) implies that $b^{2N} = 0$ on $\hat{A}(E)$ (see [B.06]). So we conclude that $\hat{A}(E) = B(E) \subset \text{Ker } b^{2N}$ which is a finite dimensional complex vector space using condition 4) and an easy induction. Now $E/B(E)$ is a $\mathbb{C}[[b]]$ -module which is separated for its b -adic topology. The finiteness of $\text{Coker } b$ now shows that it is a free finite type $\mathbb{C}[[b]]$ -module concluding the proof. ■

Définition 2.3.4 *We shall say that a left $\tilde{\mathcal{A}}$ -module E is **geometric** when E is small and when its associated (a,b) -module $E/B(E)$ is geometric.*

The main result of this section is the following theorem, which shows that the Gauss-Manin connexion of a proper holomorphic function produces geometric $\tilde{\mathcal{A}}$ -modules associated to vanishing cycles and nearby cycles.

Théorème 2.3.5 *Let X be a connected complex manifold of dimension $n + 1$ where $n \in \mathbb{N}$, and let $f : X \rightarrow D$ be an non constant proper holomorphic function to an open disc D in \mathbb{C} with center 0. Let us assume that df is nowhere vanishing outside of $X_0 := f^{-1}(0)$.*

Then the $\tilde{\mathcal{A}}$ -modules

$$\mathbb{H}^j(X, (\hat{K}^\bullet, d^\bullet)) \quad \text{and} \quad \mathbb{H}^j(X, (\hat{I}^\bullet, d^\bullet))$$

are geometric for any $j \geq 0$.

In the proof we shall use the \mathcal{C}^∞ version of the complex $(\hat{K}^\bullet, d^\bullet)$. We define K_∞^p as the kernel of $\wedge df : \mathcal{C}^{\infty,p} \rightarrow \mathcal{C}^{\infty,p+1}$ where $\mathcal{C}^{\infty,j}$ denote the sheaf of \mathcal{C}^∞ -forms on X of degree j , let \hat{K}_∞^p be its formal f -completion and $(\hat{K}_\infty^\bullet, d^\bullet)$ the corresponding de Rham complex.

The next lemma is proved in [B.07] (lemma 6.1.1.)

Lemme 2.3.6 *The natural inclusion*

$$(\hat{K}^\bullet, d^\bullet) \hookrightarrow (\hat{K}_\infty^\bullet, d^\bullet)$$

is a quasi-isomorphism.

REMARK. As the sheaves \hat{K}_∞^\bullet are fine, so we have a natural isomorphism

$$\mathbb{H}^p(X, (\hat{K}^\bullet, d^\bullet)) \simeq H^p(\Gamma(X, \hat{K}_\infty^\bullet), d^\bullet).$$

Let us denote by X_1 the generic fiber of f . Then X_1 is a smooth compact complex manifold of dimension n and the restriction of f to $f^{-1}(D^*)$ is a locally trivial \mathcal{C}^∞ bundle with typical fiber X_1 on $D^* = D \setminus \{0\}$, if the disc D is small enough around 0. Fix now $\gamma \in H_p(X_1, \mathbb{C})$ and let $(\gamma_s)_{s \in D^*}$ the corresponding multivalued horizontal family of p -cycles $\gamma_s \in H_p(X_s, \mathbb{C})$. Then for $\omega \in \Gamma(X, \hat{K}_\infty^p \cap \text{Ker } d)$ define the multivalued holomorphic function

$$F_\omega(s) := \int_{\gamma_s} \frac{\omega}{df}.$$

Let now

$$\Xi := \sum_{\alpha \in \mathbb{Q} \cap]-1, 0], j \in [0, n]} \mathbb{C}[[s]].s^\alpha. \frac{(\text{Log } s)^j}{j!}.$$

This is an $\tilde{\mathcal{A}}$ -modules with a acting as multiplication by s and b as the primitive in s without constant. Now if \hat{F}_ω is the asymptotic expansion at 0 of F_ω , it is an element in Ξ , and we obtain in this way an $\tilde{\mathcal{A}}$ -linear map

$$\text{Int} : \mathbb{H}^p(X, (\hat{K}^\bullet, d^\bullet)) \rightarrow H^p(X_1, \mathbb{C}) \otimes_{\mathbb{C}} \Xi.$$

To simplify notations, let $E := \mathbb{H}^p(X, (\hat{K}^\bullet, d^\bullet))$. Now using Grothendieck theorem [G.66], there exists $N \in \mathbb{N}$ such that $\text{Int}(\omega) \equiv 0$, implies $a^N.[\omega] = 0$ in E .

As the converse is clear we conclude that $\hat{A}(E) = Ker(Int)$. It is also clear that $B(E) \subset Ker(Int)$ because Ξ has no b -torsion. So we conclude that E satisfies properties 1) and 2) of the lemma 2.3.3.

The property 3) is also true because of the regularity of the Gauss-Manin connexion of f .

END OF THE PROOF OF THEOREM 2.3.5. To show that $E := \mathbb{H}^p(X, (\hat{K}^\bullet, d^\bullet))$ is small, it is enough to prove that E satisfies the condition 4) of the lemma 2.3.3. Consider now the long exact sequence of hypercohomology of the exact sequence of complexes

$$0 \rightarrow (\hat{I}^\bullet, d^\bullet) \rightarrow (\hat{K}^\bullet, d^\bullet) \rightarrow ([\hat{K}/\hat{I}]^\bullet, d^\bullet) \rightarrow 0.$$

It contains the exact sequence

$$\mathbb{H}^{p-1}(X, ([\hat{K}/\hat{I}]^\bullet, d^\bullet)) \rightarrow \mathbb{H}^p(X, (\hat{I}^\bullet, d^\bullet)) \xrightarrow{\mathbb{H}^p(i)} \mathbb{H}^p(X, (\hat{K}^\bullet, d^\bullet)) \rightarrow \mathbb{H}^p(X, ([\hat{K}/\hat{I}]^\bullet, d^\bullet))$$

and we know that b is induced on the complex of $\tilde{\mathcal{A}}$ -modules quasi-isomorphic to $(\hat{K}^\bullet, d^\bullet)$ by the composition $i \circ \tilde{b}$ where \tilde{b} is a quasi-isomorphism of complexes of $\mathbb{C}[[b]]$ -modules. This implies that the kernel and the cokernel of $\mathbb{H}^p(i)$ are isomorphic (as \mathbb{C} -vector spaces) to $Ker b$ and $Coker b$ respectively. Now to prove that E satisfies condition 4) of the lemma 2.3.3 it is enough to prove finite dimensionality for the vector spaces $\mathbb{H}^j(X, ([\hat{K}/\hat{I}]^\bullet, d^\bullet))$ for all $j \geq 0$.

But the sheaves $[\hat{K}/\hat{I}]^j \simeq [Ker df / Im df]^j$ are coherent on X and supported in X_0 . The spectral sequence

$$E_2^{p,q} := H^q(H^p(X, [\hat{K}/\hat{I}]^\bullet, d^\bullet))$$

which converges to $\mathbb{H}^j(X, ([\hat{K}/\hat{I}]^\bullet, d^\bullet))$, is a bounded complex of finite dimensional vector spaces by Cartan-Serre. This gives the desired finite dimensionality.

To conclude the proof, we want to show that $E/B(E)$ is geometric. But this is an easy consequence of the regularity of the Gauss-Manin connexion of f and of the Monodromy theorem, which are already incoded in the definition of Ξ : the injectivity on $E/B(E)$ of the $\tilde{\mathcal{A}}$ -linear map Int implies that $E/B(E)$ is geometric. Remark now that the piece of exact sequence above gives also the fact that $\mathbb{H}^p(X, (\hat{I}^\bullet, d^\bullet))$ is geometric, because it is an exact sequence of $\tilde{\mathcal{A}}$ -modules. \blacksquare

3 Basic properties.

3.1 Definition and examples.

First recall in a more naïve way the definition of an (a,b) -module.

Définition 3.1.1 *An (a,b) -module E is a free finite type $\mathbb{C}[[b]]$ -module with a \mathbb{C} -linear endomorphism $a : E \rightarrow E$ which is continuous for the b -adic topology of E and satisfies $a.b - b.a = b^2$.*

The rank of E , denote by $rank(E)$, will be the rank of E as a $\mathbb{C}[[b]]$ -module.

REMARKS.

1. Let (e_1, \dots, e_k) a $\mathbb{C}[[b]]$ -basis of a free finite type $\mathbb{C}[[b]]$ -module. Then choosing arbitrarily elements $(\varepsilon_1, \dots, \varepsilon_k)$ and defining $a.e_j = \varepsilon_j \quad \forall j \in [1, k]$ gives an (a, b) -module: the commutation relation implies that $\forall n \in \mathbb{N}$ we have $a.b^n = b^n.a + n.b^{n+1}$ so a is defined on $\sum_{j=1}^k \mathbb{C}[b].e_j$. The continuity assumption gives its (unique) extension.
2. There is a natural (a, b) -module associated to every algebraic linear differential system (see [B.95] p.42)

$$Q(z). \frac{dF}{dz} = M(z).F(z), \quad Q \in \mathbb{C}[z], \quad M \in \text{End}(\mathbb{C}^n) \otimes_{\mathbb{C}} \mathbb{C}[z].$$

In the sequel of this article we shall mainly consider regular (a, b) -modules (see definition recalled below). To try to convince the reader that the "general" (a, b) -module structure is interesting, let me quote the following result, which is quite elementary in the regular case, but which is not so easy in general.

Théorème 3.1.2 ([B.95] th.1bis p.31) *Let E be an (a, b) -module. Then the kernel and cokernel of "a" are finite dimensional.*

This result implies a general finiteness theorem for extensions of (a, b) -modules (see [B.95] and also section 1.3).

Définition 3.1.3 *We shall say that an (a, b) -module E has a **simple pole** when the inclusion $a.E \subset b.E$ is satisfied.*

This terminology comes from the terminology of meromorphic connexions (see for instance [D.70]).

EXAMPLE. For any $\lambda \in \mathbb{C}$ define the simple pole rank 1 (a, b) -module E_λ as $E := \mathbb{C}[[b]].e_\lambda$ where "a" is defined by the relation $a.e_\lambda = \lambda.b.e_\lambda$. \square

As an introduction to our second theorem, the reader may solve the following exercise by direct computation.

EXERCICE. For any $S \in \mathbb{C}[[b]]$ show that the simple pole (a, b) -module defined by $E := \mathbb{C}[[b]].e_S$ and $a.e_S = b.S(b).e_S$ is isomorphic to E_λ with $\lambda = S(0)$ (hint: begin by looking for $\alpha_1 \in \mathbb{C}$ such that $(a - S(0).b)(e + \alpha_1.b.e) \in b^3.E$). \square

For a simple pole (a, b) -module, the linear map $b^{-1}.a : E \rightarrow E$ is well defined and induces an endomorphism $f := b^{-1}.a : E/b.E \rightarrow E/b.E$. For any $\lambda \in \mathbb{C}$ we shall denote by λ_{min} the smallest eigenvalue of f which is in $\lambda + \mathbb{Z}$. Then for

$\lambda = \lambda_{min} - k$ with $k \in \mathbb{N}^*$ the bijectivity of the map $f - \lambda$ on $E/b.E$ implies easily its bijectivity on E (see the exercise above). It gives then the equality

$$(a - \lambda.b).E = b.E.$$

Using this remark, it is not difficult to prove the following result from [B.93] (prop.1.3. p.11) that we shall use later on.

Proposition 3.1.4 *Let E be a simple pole (a,b) -module, and let $\lambda \in \mathbb{C}$ and $\kappa \in \mathbb{N}$ such that $\lambda - \kappa \leq \lambda_{min}$. If $y \in E$ satisfies $(a - \lambda.b).y \in b^{\kappa+2}.E$ then there exists a unique $\tilde{y} \in E$ such that $(a - \lambda.b).\tilde{y} = 0$ and $\tilde{y} - y \in b^{\kappa+1}.E$.*

An easy consequence of this proposition is that for an eigenvalue λ of f such that $\lambda = \lambda_{min}$ there always exists a non zero $x \in E$ such that $(a - \lambda.b).x = 0$. This gives an embedding of E_λ in E . Remark also that if E is a non zero simple pole (a,b) -module, such a λ always exists. This leads to a rather precise description a of "general" simple pole (a,b) -module (see [B.93] th. 1.1 p.15).

Définition 3.1.5 *An (a,b) -module E is **regular** when its saturation by $b^{-1}.a$ in $E[b^{-1}]$ is finitely generated on $\mathbb{C}[[b]]$.*

We shall denote E^\sharp this saturation. It is a simple pole (a,b) -module and it is the smallest simple pole (a,b) -module containing E in the sense that for any (a,b) -linear morphism $j : E \rightarrow F$ where F is a simple pole (a,b) -module, there exists a unique (a,b) -linear extension $j^\sharp : E^\sharp \rightarrow F$ of j .

It is easy to show that a regular (a,b) -module of rank 1 is isomorphic to some E_λ for some $\lambda \in \mathbb{C}$. The classification of rank 2 regular (a,b) -module is not so obvious. We recall it here for a later use

Proposition 3.1.6 *(see [B.93] prop.2.4 p. 34) The list of rank 2 regular (a,b) -modules is, up to isomorphism, the following :*

1. $E_\lambda \oplus E_\mu$ for $(\lambda, \mu) \in \mathbb{C}^2/\mathfrak{S}_2$.
2. For any $\lambda \in \mathbb{C}$ and any $n \in \mathbb{N}$ let $E_\lambda(n)$ be the simple pole (a,b) -module with basis (x, y) such that

$$a.x = (\lambda + n).b.x + b^{n+1}.y \quad \text{and} \quad a.y = \lambda.b.y.$$

3. For any $(\lambda, \mu) \in \mathbb{C}^2/\mathfrak{S}_2$ let $E_{\lambda, \mu}$ the rank 2 regular (a,b) -module with basis (y, t) such that

$$a.y = \mu.b.y \quad \text{and} \quad a.t = y + (\lambda - 1).b.t.$$

4. For any $\lambda \in \mathbb{C}$, any $n \in \mathbb{N}^*$ and any $\alpha \in \mathbb{C}^*$ let $E_{\lambda, \lambda-n}(\alpha)$ be the rank 2 regular (a,b) -module with basis (y, t) such that

$$a.y = (\lambda - n).b.y \quad \text{and} \quad a.t = y + (\lambda - 1)b.t + \alpha.b^n.y$$

Note that the first two cases are simple pole (a,b) -modules.

The saturation by $b^{-1}.a$ in case 3. is generated by $b^{-1}.y$ and t as a $\mathbb{C}[[b]]$ -module. It is isomorphic to $E_{\lambda-1} \oplus E_{\mu-1}$ for $\lambda \neq \mu$ and to $E_{\lambda-1}(0)$ for $\lambda = \mu$.

The saturation by $b^{-1}.a$ in case 4. is generated by $b^{-1}.y$ and t as a $\mathbb{C}[[b]]$ -module. It is isomorphic to $E_{\lambda-n-1}(n)$ for any non zero value of α .

To conclude this first section, let me recall also the theorem of existence of Jordan-Hölder sequences for regular (a,b) -module, which will be useful in the induction in the proof of our result .

Théorème 3.1.7 (see [B.93] th. 2.1 p.30) For any regular rank k (a,b) -module E there exists a sequence of sub- (a,b) -modules

$$0 = E^0 \subset E^1 \subset \dots \subset E^{k-1} \subset E^k = E$$

such that for any $j \in [1, k]$ the quotient E^j/E^{j-1} is isomorphic to E_{λ_j} . Moreover we may choose for E^1 any normal¹ rank 1 sub- (a,b) -module of E .

The number $\alpha(E) := \sum_{j=1}^k \lambda_j$ is independant of the choice of the Jordan-Hölder sequence. It is given by the following formula

$$\alpha(E) = \text{trace}(b^{-1}.a : E^\sharp/b.E^\sharp \rightarrow E^\sharp/b.E^\sharp) + \dim_{\mathbb{C}}(E^\sharp/E).$$

3.2 The regularity order.

Définition 3.2.1 Let E be a regular (a,b) -module. We define the **regularity order** of E as the smallest integer $k \in \mathbb{N}$ such that the inclusion

$$a^{k+1}.E \subset \sum_{j=0}^k a^j.b^{k-j+1}.E \tag{reg.}$$

is valid. We shall note this integer $or(E)$.

We define also the **index** $\delta(E)$ of E as the smallest integer $m \in \mathbb{N}$ such that $E^\sharp \subset b^{-m}.E$.

REMARKS.

i) The (a,b) -module E has a simple pole if and only if $or(E) = 0$.

¹normal means $E^1 \cap b.E = b.E^1$, so that E/E^1 is again free on $\mathbb{C}[[b]]$.

ii) The inclusion (reg.) implies that $(b^{-1}.a)^{k+1}.E \subset \Phi_k(E) := \sum_{j=0}^k (b^{-1}.a)^j.E$ and this implies that $\Phi_k(E)$ is stable by $b^{-1}.a$. So $\Phi_k(E)$ is a simple pole (a,b)-module contained in $b^{-k}.E \subset E[b^{-1}]$. This implies clearly the regularity of E .

For $k = \text{or}(E)$ we have $E^\sharp = \Phi_k(E) \subset b^{-k}.E$. So we have $\delta(E) \leq \text{or}(E)$.

iii) As the quotient $b^{-k}.E/E$ is a finite dimensional \mathbb{C} -vector space, the quotient E^\sharp/E is always a finite dimensional \mathbb{C} -vector space. \square

The remark iii) shows that for a regular (a,b)-module E there always exists a simple pole sub-(a,b)-module of E which is a finite codimensional vector space in E . This comes from the fact that for $k = \delta(E)$ we have $b^k.E^\sharp \subset E$ and that $b^k.E^\sharp$ has a simple pole.

EXAMPLE. The inequality $\delta(E) \leq \text{or}(E)$ may be strict for $\text{or}(E) \geq 2$. For instance the (a,b)-module of rank 3 with $\mathbb{C}[[b]]$ -basis e_1, e_2, e_3 with $a.e_1 = e_2, a.e_2 = b.e_3, a.e_3 = 0$ has index 1 and regularity order 2 : an easy computation gives that a $\mathbb{C}[[b]]$ -basis for E^\sharp is given by $e_1, b^{-1}.e_2, b^{-1}.e_3$, and that a $\mathbb{C}[[b]]$ -basis for $E + b^{-1}.a.E$ is given by $e_1, b^{-1}.e_2, e_3$. \square

Définition 3.2.2 *Let E be a regular (a,b)-module. The biggest simple pole sub-(a,b)-module of E exists and has finite \mathbb{C} -codimension in E . We shall note it E^b .*

In general, for $k = \delta(E)$ the inclusion $b^k.E^\sharp \subset E^b$ is strict. For instance this is the case for $E_{\lambda,\mu} \oplus E_\nu$.

Lemme 3.2.3 *Let E be a regular (a,b)-module. The smallest integer m such we have $b^m.E \subset E^b$ is equal to $\delta(E)$.*

PROOF. Let $k := \delta(E)$. Then $b^k.E^\sharp$ is a simple pole sub-(a,b)-module of E . So we have $b^k.E \subset b^k.E^\sharp \subset E^b$. Conversely, an inclusion $b^m.E \subset E^b$ gives $E \subset b^{-m}.E^b$. As $b^{-m}.E^b$ has a simple pole this implies $E^\sharp \subset b^{-m}.E^b \subset b^{-m}.E$. So $\delta(E) \leq m$. \blacksquare

EXAMPLES. In the case 3 of the proposition 3.1.6 E^b is generated as a $\mathbb{C}[[b]]$ -module by y and $b.t$, so $E^b = b.E^\sharp$.

In case 4 we have also $E^b = b.E^\sharp$.

Lemme 3.2.4 *Let E be a regular (a,b)-module. For any exact sequence of (a,b)-modules*

$$0 \rightarrow E' \rightarrow E \xrightarrow{\pi} E'' \rightarrow 0 \quad (*)$$

we have $\text{or}(E'') \leq \text{or}(E) \leq \text{rank}(E') + \text{or}(E'')$.

As a consequence, the order of regularity of E is at most $\text{rank}(E) - 1$ for any regular non zero (a,b)-module.

PROOF. The inequality $or(E'') \leq or(E)$ is trivial because an inequality

$$a^{k+1}.E \subset \sum_{j=0}^k a^j.b^{k-j+1}.E$$

implies the same for E'' and, by definition, the best such integer k is the order of regularity.

The crucial case is when E' is of rank 1. So we may assume that $E' \simeq E_\lambda$ for some $\lambda \in \mathbb{C}$ (see 3.1.7 or [B.93] prop.2.2 p.23). Let $k = or(E'')$. Then the inclusion

$$a^{k+1}.E'' \subset \sum_{j=0}^k a^j.b^{k-j+1}.E'' \quad (1)$$

implies that

$$a^{k+1}.E \subset \sum_{j=0}^k a^j.b^{k-j+1}.E + b^l.E_\lambda \quad (2)$$

for some $l \in \mathbb{N}$. In fact we can take for l the smallest integer such that the generator e_λ of E_λ (defined up to \mathbb{C}^* by the relation $a.e_\lambda = \lambda.b.e_\lambda$) satisfies $b^l.e_\lambda \in \Psi_k = \sum_{j=0}^k a^j.b^{k-j+1}.E$.

Remark that this integer $l \geq 0$ is well defined because $b^{k+1}.e_\lambda \in \Psi_k$. Moreover, as Ψ_k is a $\mathbb{C}[[b]]$ -submodule of E , $b^l.e_\lambda \in \Psi_k$ implies $b^l.E_\lambda \subset \Psi_k$.

Now, thanks to (2) we have

$$a^{k+2}.E \subset \sum_{j=0}^k a^{j+1}.b^{k+1-j}.E + a.b^l.E_\lambda \quad (3)$$

which gives

$$a^{k+2}.E \subset \sum_{j=0}^{k+1} a^j.b^{k-j+2}.E \quad (4)$$

because $a.b^l.E_\lambda = b.b^l.E_\lambda \subset b.\Psi_k$.

This proves that $or(E)$ is at most $k+1 = or(E'') + rank(E')$.

Assume now that our inequality is proved for E' of rank $p-1$ and consider an exact sequence (*) with $rank(E')$ equal $p \geq 2$. Let $E_\lambda \subset E'$ be a normal rank 1 sub-(a,b)-module of E' (see 3.1.7 or [B.93] prop.2.2 p.23 for a proof of the existence of such sub-(a,b)-module) and consider the exact sequence of (a,b)-modules (using the fact that E_λ is also normal in E ; see lemma 2.5 of [B.93])

$$0 \rightarrow E'/E_\lambda \rightarrow E/E_\lambda \rightarrow E'' \rightarrow 0$$

Using the induction hypothesis and the rank 1 case we get

$$or(E) \leq or(E/E_\lambda) + 1 \leq p-1 + or(E'') + 1 = p + or(E'').$$

Now using an easy induction (or a Jordan-Hölder sequence for E) we obtain $or(E) \leq rank(E) - 1$ for any regular E . ■

REMARK. In the situation of the previous lemma we have $\delta(E') \leq \delta(E)$. This is a consequence of the obvious inclusion $(E')^\# \subset E'[b^{-1}] \cap E^\#$: assume that $x \in E'[b^{-1}] \cap E^\#$; then, for $k := \delta(E)$ we have $b^k.x \in E'[b^{-1}] \cap E$ so that $b^{N+k}.x \in E'$ for N large enough. As E/E' has no b -torsion, we conclude that $b^k.x \in E'$. So our initial inclusion implies $\delta(E') \leq k$. \square

3.3 Duality.

In this section we consider more carefully the associative and unitary \mathbb{C} -algebra

$$\tilde{\mathcal{A}} := \left\{ \sum_0^\infty P_n(a).b^n \quad \text{with} \quad P_n \in \mathbb{C}[z] \right\}$$

with the commutation relation $a.b - b.a = b^2$, and such that the left and right multiplications by a are continuous for the b -adic topology² of $\tilde{\mathcal{A}}$.

THE RIGHT STRUCTURE AS A COMMUTING LEFT-STRUCTURE ON $\tilde{\mathcal{A}}$.

There exists a unique \mathbb{C} -linear (bijective) map $\theta : \tilde{\mathcal{A}} \rightarrow \tilde{\mathcal{A}}$ with the following properties

- i) $\theta(1) = 1, \quad \theta(a) = a, \quad \theta(b) = -b;$
- ii) $\theta(x.y) = \theta(y).\theta(x) \quad \forall x, y \in \tilde{\mathcal{A}}.$
- iii) θ is continuous for the b -adic topology of $\tilde{\mathcal{A}}$

The uniqueness is an easy consequence of iii) and the fact that the conditions i) and ii) implies $\theta(b^p.a^q) = (-1)^p.a^q.b^p \quad \forall p, q \in \mathbb{N}$. Existence is then clear from the explicit formula deduced from this remark.

We define a new structure of left $\tilde{\mathcal{A}}$ -module on $\tilde{\mathcal{A}}$, **called the θ -structure** and denote by $x_*\square$, by the formula

$$x_*y = y.\theta(x).$$

It is easy to see that this new left-structure on $\tilde{\mathcal{A}}$ commutes with the ordinary one and that with this θ -structure $\tilde{\mathcal{A}}$ is still free of rank one as a left $\tilde{\mathcal{A}}$ -module.

Définition 3.3.1 *Let E be a (left) $\tilde{\mathcal{A}}$ -module. On the \mathbb{C} -vector space $\text{Hom}_{\tilde{\mathcal{A}}}(E, \tilde{\mathcal{A}})$ we define a left $\tilde{\mathcal{A}}$ -module structure using the θ -structure on $\tilde{\mathcal{A}}$. Explicitly this means that for $\varphi \in \text{Hom}_{\tilde{\mathcal{A}}}(E, \tilde{\mathcal{A}})$ and $x \in \tilde{\mathcal{A}}$ we let*

$$\forall e \in E \quad (x.\varphi)(e) := x_*\varphi(e) = \varphi(e).\theta(x).$$

We obtain in this way a left $\tilde{\mathcal{A}}$ -module that we shall still denote $\text{Hom}_{\tilde{\mathcal{A}}}(E, \tilde{\mathcal{A}})$.

²remark that for each $k \in \mathbb{N}$ $b^k.\tilde{\mathcal{A}} = \tilde{\mathcal{A}}.b^k$.

It is clear that $E \rightarrow \text{Hom}_{\tilde{\mathcal{A}}}(E, \tilde{\mathcal{A}})$ is a contravariant functor which is left exact in the category of left $\tilde{\mathcal{A}}$ -modules. As every finite type left $\tilde{\mathcal{A}}$ -module has a resolution of length ≤ 2 by free finite type modules (see [B.95] cor.2 p.29), we shall denote by $\text{Ext}_{\tilde{\mathcal{A}}}^i(E, \tilde{\mathcal{A}}), i \in [0, 2]$ the right derived functors of this functor. They are finite type left $\tilde{\mathcal{A}}$ -modules when E is finitely generated because $\tilde{\mathcal{A}}$ is left noetherian (see [B.95] prop.2 p.26).

Any (a,b)-module is a left $\tilde{\mathcal{A}}$ -module. They are characterized by the existence of special simple resolutions.

Lemma 3.3.2 *Let M be a (p, p) matrix with entries in the ring $\mathbb{C}[[b]]$. Then the left $\tilde{\mathcal{A}}$ -linear map $\text{Id}_p.a - M : \tilde{\mathcal{A}}^p \rightarrow \tilde{\mathcal{A}}^p$ given by*

$${}^tX := (x_1, \dots, x_p) \rightarrow {}^tX.(Id_p.a - M)$$

is injective. Its cokernel is the (a,b)-module E given as follows :

E has a $\mathbb{C}[[b]]$ base $e := (e_1, \dots, e_p)$ and a is defined by the two conditions

1. $a.e := M(b).e$;
2. *the left action of a is continuous for the b -adic topology of E .*

Any (a,b)-module is obtained in this way and so, as a $\tilde{\mathcal{A}}$ -left-module, has a resolution of the form

$$0 \rightarrow \tilde{\mathcal{A}}^p \xrightarrow{{}^t\Box.(Id_p.a - M)} \tilde{\mathcal{A}}^p \rightarrow E \rightarrow 0. \quad (@)$$

PROOF. First remark that for $x \in \tilde{\mathcal{A}}$ the condition $x.a \in b.\tilde{\mathcal{A}}$ implies $x \in b.\tilde{\mathcal{A}}$. Now let us prove, by induction on $n \geq 1$, that, for any (p, p) matrix M with entries in $\mathbb{C}[[b]]$ the condition ${}^tX.(Id_p.a - M) = 0$ implies ${}^tX \in b^n.\tilde{\mathcal{A}}^p$.

For $n = 1$ this comes from the previous remark. Let assume that the assertion is proved for $n \geq 1$ and consider an $X \in \tilde{\mathcal{A}}^p$ such that ${}^tX.(Id_p.a - M) = 0$. Using the induction hypothesis we can find $Y \in \tilde{\mathcal{A}}^p$ such that $X = b^n.Y$. Now we obtain, using $a.b^n = b^n.a + n.b^{n+1}$ and the fact that $\tilde{\mathcal{A}}$ has no zero divisor, the relation

$${}^tY(Id_p.a - (M + n.Id_p.b)) = 0$$

and using again our initial remark we conclude that $Y \in b.\tilde{\mathcal{A}}^p$ so $X \in b^{n+1}.\tilde{\mathcal{A}}^p$.

So such an X is in $\bigcap_{n \geq 1} b^n.\tilde{\mathcal{A}}^p = (0)$.

The other assertions of the lemma are obvious. ■

We recall now a construction given in [B.95] which allows to compute more easily the vector spaces $\text{Ext}_{\tilde{\mathcal{A}}}^i(E, F)$ when E, F are (a,b)-modules

Définition 3.3.3 *Let E, F two (a,b)-modules. Then the $\mathbb{C}[[b]]$ -module $\text{Hom}_b(E, F)$ is again a free and finitely generated $\mathbb{C}[[b]]$ -module. Define on it an (a,b)-module structure in the following way.*

1. First change the sign of the action of b . So $S(b) \in \mathbb{C}[[b]]$ will act as $\check{S}(b) = S(-b)$.
2. Define a using the linear map $\Lambda : Hom_b(E, F) \rightarrow Hom_b(E, F)$ given by $\Lambda(\varphi)(e) = \varphi(a.e) - a.\varphi(e)$.

We shall denote $Hom_{a,b}(E, F)$ the corresponding (a,b) -module.

The verification that $\Lambda(\varphi)$ is $\mathbb{C}[[b]]$ -linear and that $\Lambda.\check{b} - \check{b}.\Lambda = \check{b}^2$ are easy (and may be found in [B.95] p.31).

REMARK. In *loc. cit.* we defined the (a,b) -module structure on $Hom_{a,b}(E, F)$ with opposite signs for a and b . The present convention is better because it fits with the usual definition of the formal adjoint of a differential operator : $z^* = z$ and $(\partial/\partial z)^* = -\partial/\partial z$. \square

The following lemma is also proved in *loc.cit.*

Lemme 3.3.4 *Let E, F two (a,b) -modules. Then there is a functorial isomorphism of \mathbb{C} -vector spaces*

$$H^i\left(Hom_{a,b}(E, F) \xrightarrow{a} Hom_{a,b}(E, F)\right) \rightarrow Ext_{\check{\mathcal{A}}}^i(E, F) \quad \forall i \geq 0.$$

Here the map a of the complex $Hom_{a,b}(E, F) \xrightarrow{a} Hom_{a,b}(E, F)$ is equal to the Λ defined above which is, by definition, the operator " a " of the (a,b) -module $Hom_{a,b}(E, F)$.

Now the following corollary of the lemma 3.3.2 gives that the two natural ways of defining the dual of an (a,b) -module give the same answer.

Corollaire 3.3.5 *Let E an (a,b) -module. There is a functorial isomorphism of (a,b) -modules between the following two (a,b) -modules constructed as follows :*

1. $Ext_{\check{\mathcal{A}}}^1(E, \check{\mathcal{A}})$ with the $\check{\mathcal{A}}$ -structure defined by the θ -structure of $\check{\mathcal{A}}$.
2. $Hom_{a,b}(E, E_0)$ where $E_0 := \check{\mathcal{A}}/\check{\mathcal{A}}.a$.

PROOF. Using a free resolution (\textcircled{a}) of E deduced from a $\mathbb{C}[[b]]$ -basis $e := (e_1, \dots, e_p)$ we obtain, by the previous lemma, an exact sequence

$$0 \rightarrow \check{\mathcal{A}}^p \xrightarrow{(Id_p, a-tM).\square} \check{\mathcal{A}}^p \rightarrow Ext_{\check{\mathcal{A}}}^1(E, \check{\mathcal{A}}) \rightarrow 0. \quad (\textcircled{a}\textcircled{a})$$

of left $\check{\mathcal{A}}$ -modules where $\check{\mathcal{A}}^p$ is endowed with its θ -structure. Writing the same exact sequence with the ordinary left-module structure of $\check{\mathcal{A}}^p$ gives

$$0 \rightarrow \check{\mathcal{A}}^p \xrightarrow{t\square.(Id_p, a-t\check{M})} \check{\mathcal{A}}^p \rightarrow Ext_{\check{\mathcal{A}}}^1(E, \check{\mathcal{A}}) \rightarrow 0. \quad (\textcircled{a}\textcircled{a} \text{ bis})$$

where ${}^t\check{M}(b) := {}^tM(-b)$.

Denote by $e^* := (e_1^*, \dots, e_p^*)$ the dual basis of $\text{Hom}_{\mathbb{C}[[b]]}(E, E_0)$. By definition of the action of a on $\text{Hom}_{a,b}(E, E_0)$ we get, if ω is the class of 1 in E_0 :

$$(a.e_i^*)(e_j) = e_i^*(a.e_j) - a.e_i^*(e_j) = e_i^*\left(\sum_{h=1}^p m_{j,h}.e_h\right) - a.\delta_{i,j}.\omega = \check{m}_{j,i}.\omega$$

because $a.\omega = 0$ in E_0 , and the definition of the action of b on $\text{Hom}_{a,b}(E, E_0)$. So we have $a.e^* = {}^t\check{M}.e^*$ concluding the proof. \blacksquare

Définition 3.3.6 For any (a,b) -module E the dual of E , denoted by E^* , is the (a,b) -module $\text{Ext}_{\tilde{\mathcal{A}}}^1(E, \tilde{\mathcal{A}}) \simeq \text{Hom}_{a,b}(E, E_0)$.

Of course, for any $\tilde{\mathcal{A}}$ -linear map $f : E \rightarrow F$ between two (a,b) -modules we have an $\tilde{\mathcal{A}}$ -linear "dual" map $f^* : F^* \rightarrow E^*$.

It is an easy consequence of our previous description of $\text{Ext}_{\tilde{\mathcal{A}}}^1(E, \tilde{\mathcal{A}})$ that we have a functorial isomorphism $(E^*)^* \rightarrow E$.

EXAMPLES.

1. For each $\lambda \in \mathbb{C}$ we have $(E_\lambda)^* \simeq E_{-\lambda}$.
2. For $(\lambda, \mu) \in \mathbb{C}^2$ we have $E_{\lambda, \mu}^* \simeq E_{-\mu+1, -\lambda+1}$.
3. Let E be the rank two simple pole (a,b) -module $E_1(0)$ defined by $a.e_1 = b.e_1 + b.e_2$ and $a.e_2 = b.e_2$. Then its dual is isomorphic to $E_{-1}(0)$.
It is also an elementary exercise to show the following isomorphisms :

$$E_1(0) \simeq \mathbb{C}[[z]] \oplus \mathbb{C}[[z]].\text{Log}z \quad \text{and} \quad E_{-1}(0) \simeq \mathbb{C}[[z]]\frac{1}{z^2} \oplus \mathbb{C}[[z]].\frac{\text{Log}z}{z^2}$$

with $a := \times z$ and $b := \int_0^z$.

Proposition 3.3.7 For any exact sequence of (a,b) -modules

$$0 \rightarrow E' \xrightarrow{u} E \xrightarrow{v} E'' \rightarrow 0$$

we have an exact sequence of (a,b) -modules

$$0 \rightarrow (E'')^* \xrightarrow{v^*} E^* \xrightarrow{u^*} (E')^* \rightarrow 0.$$

If E is a simple pôle (a,b) -module, E^* has a simple pole.

For any regular (a,b) -module E its dual E^* is regular. Moreover, if E^b and E^\sharp are respectively the biggest simple pole submodule of E and the saturation of E by $b^{-1}.a$ in $E[b^{-1}]$, we have

$$(E^\sharp)^* \simeq (E^*)^b \quad \text{and} \quad (E^b)^* \simeq (E^*)^\sharp.$$

PROOF. The first assertion is a direct consequence of the vanishing of $Ext_{\tilde{\mathcal{A}}}^i(E, \tilde{\mathcal{A}})$ for $i = 0, 2$, for any (a,b)-module and the long exact sequence for the "Ext".

The condition that E has a simple pole is equivalent to the fact that for any choosen basis e of E the matrix M has its coefficients in $b.\tilde{\mathcal{A}} = \tilde{\mathcal{A}}.b$. Then this remains true for ${}^t\tilde{M}$.

To prove the regularity of E^* when E is regular, we shall use induction on the rank of E . The rank 1 case is obvious because we have a simple pole in this case. Assume that the assertion is true for $rank < p$ and consider a $rank = p$ regular (a,b)-module E . Using the theorem 3.1.7 we have an exact sequence of (a,b)-modules

$$0 \rightarrow E_\lambda \rightarrow E \rightarrow F \rightarrow 0$$

where F is regular of rank $p - 1$. This gives a short exact sequence

$$0 \rightarrow F^* \rightarrow E^* \rightarrow E_{-\lambda} \rightarrow 0$$

and the regularity of F^* and of $E_{-\lambda}$ implies the regularity of E^* .

Now the inclusions $E^b \subset E \subset E^\sharp$ gives exact sequences

$$\begin{aligned} 0 \rightarrow Ext_{\tilde{\mathcal{A}}}^1(E/E^b, \tilde{\mathcal{A}}) \rightarrow E^* \rightarrow (E^b)^* \rightarrow Ext_{\tilde{\mathcal{A}}}^2(E/E^b, \tilde{\mathcal{A}}) \rightarrow 0 \\ 0 \rightarrow Ext_{\tilde{\mathcal{A}}}^1(E^\sharp/E, \tilde{\mathcal{A}}) \rightarrow (E^\sharp)^* \rightarrow E^* \rightarrow Ext_{\tilde{\mathcal{A}}}^2(E^\sharp/E, \tilde{\mathcal{A}}) \rightarrow 0 \end{aligned}$$

and the next lemma will show that the $Ext_{\tilde{\mathcal{A}}}^1(V, \tilde{\mathcal{A}}) = 0$ for any $\tilde{\mathcal{A}}$ -module which is a finite dimensional vector space, and also the finiteness (as a vector space) of $Ext_{\tilde{\mathcal{A}}}^2(V, \tilde{\mathcal{A}})$. This implies that we have, for any regular (a,b)-module, the inclusions

$$E^* \subset (E^b)^* \quad \text{and} \quad (E^\sharp)^* \subset E^*.$$

They imply, thanks to the fact that $(E^b)^*$ and $(E^\sharp)^*$ have simple poles,

$$(E^*)^\sharp \subset (E^b)^* \quad \text{and} \quad (E^\sharp)^* \subset (E^*)^b.$$

But the inclusion $(E^*)^b \subset E^*$ gives

$$E = (E^*)^* \subset ((E^*)^b)^* \subset ((E^\sharp)^*)^* = E^\sharp$$

and the minimality of E^\sharp gives $((E^*)^b)^* = E^\sharp$ because $((E^*)^b)^*$ has a simple pole and contains E . Dualizing again gives $(E^\sharp)^* \simeq (E^*)^b$. The last equality is obtained in a similar way from $E^* \subset (E^*)^\sharp$. ■

Lemme 3.3.8 *Let V be an $\tilde{\mathcal{A}}$ -module of finite dimension over \mathbb{C} . Then we have $Ext_{\tilde{\mathcal{A}}}^i(V, \tilde{\mathcal{A}}) = 0$ for $i = 0, 1$ and $Ext_{\tilde{\mathcal{A}}}^2(V, \tilde{\mathcal{A}})$ is again a $\tilde{\mathcal{A}}$ -module (via the θ -structure of $\tilde{\mathcal{A}}$) which is a finite dimensional vector space. Moreover it has the same dimension than V and there is a canonical $\tilde{\mathcal{A}}$ -module isomorphism*

$$Ext_{\tilde{\mathcal{A}}}^2(Ext_{\tilde{\mathcal{A}}}^2(V, \tilde{\mathcal{A}}), \tilde{\mathcal{A}}) \simeq V.$$

PROOF. We begin by proving the first assertion of the lemma for the special case $V_\lambda := \tilde{\mathcal{A}}/\tilde{\mathcal{A}}.(a - \lambda) + \tilde{\mathcal{A}}.b$ for any $\lambda \in \mathbb{C}$. Let us show that we have the free resolution

$$0 \rightarrow \tilde{\mathcal{A}} \xrightarrow{\alpha} \tilde{\mathcal{A}}^2 \xrightarrow{\beta} \tilde{\mathcal{A}} \rightarrow V_\lambda \rightarrow 0$$

where $\alpha(x) := (x.b, -x.(a - b - \lambda))$, $\beta(u, v) := u.(a - \lambda) + v.b$. The map α is clearly injective and $\beta(\alpha(x)) = x.(b.a - \lambda.b - (a - b - \lambda).b) = 0$. If we have $\beta(u, v) = 0$ then $u \in \tilde{\mathcal{A}}.b$; let $u = x.b$. Then we get

$$x.(a - b - \lambda).b + v.b = 0 \quad \text{and so} \quad v = -x.(a - b - \lambda).$$

This gives the exactness of our resolution.

Now the $Ext_{\tilde{\mathcal{A}}}^i(V_\lambda, \tilde{\mathcal{A}})$ are given by the cohomology of the complex

$$0 \rightarrow \tilde{\mathcal{A}} \xrightarrow{\beta^*} \tilde{\mathcal{A}}^2 \xrightarrow{\alpha^*} \tilde{\mathcal{A}} \rightarrow 0.$$

The map $\beta^*(x) = ((a - \lambda).x, b.x)$ and $\alpha^*(u, v) = b.u - (a - b - \lambda).v$ are $\tilde{\mathcal{A}}$ -linear for the θ -structure of $\tilde{\mathcal{A}}$. Clearly β^* is injective and $\alpha^*(\beta^*(x)) \equiv 0$. If $\alpha^*(u, v) = 0$ set $v = b.y$ and conclude that $u = (a - \lambda).y$. This gives the vanishing of the Ext^i for $i = 0, 1$. The Ext^2 is the cokernel of β^* which is easily seen to be isomorphic to V_λ .

Consider now any finite dimensional $\tilde{\mathcal{A}}$ -module V over \mathbb{C} . We make an induction on $\dim_{\mathbb{C}}(V)$ to prove the vanishing of the Ext^i for $i = 0, 1$ and the assertion on the dimension of the Ext^2 .

The $\dim V = 1$ case is clear because reduced to the case $V = V_\lambda$ for some $\lambda \in \mathbb{C}$. Assume that the case $\dim V = p$ is proved, for $p \geq 1$ and consider some V with $\dim V = p + 1$. Then $Ker b$ is not $\{0\}$ and is stable by a . Let $\lambda \in \mathbb{C}$ an eigenvalue of a acting on $Ker b$. Then a eigenvector generates in V a sub- $\tilde{\mathcal{A}}$ -module isomorphic to V_λ .

The exact sequence of $\tilde{\mathcal{A}}$ -modules

$$0 \rightarrow V_\lambda \rightarrow V \rightarrow W \rightarrow 0$$

where $W := V/V_\lambda$ has dimension p allows us to conclude, looking at the long exact sequence of Ext.

The last assertion follows from the remark that we produce a free resolution of $Ext_{\tilde{\mathcal{A}}}^2(V, \tilde{\mathcal{A}})$ by taking $Hom_{\tilde{\mathcal{A}}}(-, \tilde{\mathcal{A}})$ of a free (length two, see [B.97]) resolution of V because of the already proved vanishing of the Ext^i for $i = 0, 1$. Doing this again gives back the initial resolution (remark that we use here that the $\theta \circ \theta$ -structure on $Hom_{\tilde{\mathcal{A}}}(Hom_{\tilde{\mathcal{A}}}(\tilde{\mathcal{A}}, \tilde{\mathcal{A}}), \tilde{\mathcal{A}})$ is the usual left structure on $\tilde{\mathcal{A}}$). \blacksquare

Corollaire 3.3.9 *For a simple pole (a, b) module E denote by $S(E)$ the spectrum of $b^{-1}.a$ acting on $E/b.E$. Then we have*

$$S(E^*) = -S(E).$$

PROOF. We make an induction on the rank of E . In rank 1 the result is clear because we have $E \simeq E_\lambda$ for some $\lambda \in \mathbb{C}$, and $S(E_\lambda) = \{\lambda\}$. But we know that $E_\lambda^* = E_{-\lambda}$.

Assume the assertion proved for any rank $p \geq 1$ simple pole (a,b)-module, and consider E with rank $p + 1$. Using theorem 3.1.7, there exists $\lambda \in \mathbb{C}$ and an exact sequence (a,b)-modules

$$0 \rightarrow E_\lambda \rightarrow E \rightarrow F \rightarrow 0$$

where $\text{rank}(F) = p$ and where F has a simple pole (because a quotient of a simple pole (a,b)-module has a simple pole !). The exact sequence of vector spaces

$$0 \rightarrow E_\lambda/b.E_\lambda \rightarrow E/b.E \rightarrow F/b.F \rightarrow 0$$

shows that $S(E) = S(F) \cup \{\lambda\}$. Now proposition 3.3.7 gives the exact sequence

$$0 \rightarrow F^* \rightarrow E^* \rightarrow E_{-\lambda} \rightarrow 0$$

which implies, as before, $S(E^*) = S(F^*) \cup \{-\lambda\}$. The induction hypothesis $S(F^*) = -S(F)$ allows to conclude. \blacksquare

Lemme 3.3.10 *For any pair of (a,b)-modules E and F there is a canonical isomorphism of vector spaces*

$$D : \text{Ext}_{\mathcal{A}}^1(E, F) \rightarrow \text{Ext}_{\mathcal{A}}^1(F^*, E^*)$$

associated to the correspondance between 1-extensions (i.e. short exact sequences)

$$(0 \rightarrow F \rightarrow G \rightarrow E \rightarrow 0) \xrightarrow{D} (0 \rightarrow E^* \rightarrow G^* \rightarrow F^* \rightarrow 0).$$

PROOF. We have a obvious isomorphism of $\mathbb{C}[[b]]$ -modules³

$$I : \text{Hom}_b(E, F) \rightarrow \text{Hom}_b(\text{Hom}_b(F, E_0), \text{Hom}_b(E, E_0)) \simeq \text{Hom}_b(F^*, E^*)$$

because $E_0 \simeq \mathbb{C}[[b]]$ as a $\mathbb{C}[[b]]$ -module. But recall that $\text{Ext}_{\mathcal{A}}^1(E, F)$ (resp. $\text{Ext}_{\mathcal{A}}^1(F^*, E^*)$) is the cokernel of the \mathbb{C} -linear map "a" defined on $\text{Hom}_b(E, F)$ by the formula

$$(a.\varphi)(x) = \varphi(a.x) - a.\varphi(x)$$

So it is enough to check that the isomorphism I commutes with "a" in order to get an isomorphism between the cokernels of "a" in these two spaces.

Let $\varphi \in \text{Hom}_b(E, F)$ and $\xi \in F^*$. Then $I(\varphi)(\xi) = \varphi \circ \xi$. So, for $x \in E$ we have (using Λ to avoid too many "a")

$$\begin{aligned} \Lambda(I(\varphi)(\xi) &= I(\varphi)(a.\xi) - a.(I(\varphi)(\xi)) \\ \Lambda(I(\varphi)(\xi)(x) &= (\varphi \circ \xi)(a.x) - a.\xi(\varphi(x)) - (\xi(\varphi(a.x)) - a.\xi(\varphi(x))) \\ &= [(\Lambda(\varphi)) \circ \xi](x) = I(\Lambda(\varphi))(x). \end{aligned}$$

³but be carefull with the $b \rightarrow \check{b}$!

So $\Lambda \circ I = I \circ \Lambda$. The map I gives an isomorphism of complexes

$$\begin{array}{ccc} \text{Hom}_{a,b}(E, F) & \xrightarrow{\Lambda} & \text{Hom}_{a,b}(E, F) \\ \downarrow I & & \downarrow I \\ \text{Hom}_{a,b}(F^*, E^*) & \xrightarrow{\Lambda} & \text{Hom}_{a,b}(F^*, E^*) \end{array}$$

and this conclude the proof, using lemma 3.3.4. ■

For an (a,b) -module E and an integer $m \in \mathbb{N}$ it is clear that $b^m.E$ is again an (a,b) -module. This can be generalize for any $m \in \mathbb{C}$.

Définition 3.3.11 For any (a,b) -module E and any complex number $m \in \mathbb{C}$ define the (a,b) -module $b^m.E$ as follows : as an $\mathbb{C}[[b]]$ -module we let $b^m.E \simeq E \simeq \mathbb{C}[[b]]^{\text{rank}(E)}$; the operator a is defined as $a + m.b$.

Precisely, this means that if (e_1, \dots, e_k) is a $\mathbb{C}[[b]]$ -basis of E such that we have $a.e = M(b).e$ where $M \in \text{End}(\mathbb{C}^p) \otimes_{\mathbb{C}} \mathbb{C}[[b]]$, the (a,b) -module $b^m.E$ admit a basis, denote by $(b^m.e_1, \dots, b^m.e_k)$, such that the operator a is defined by the relation $a.(b^m.e) := (M(b) + m.b.Id_k).(b^m.e)$.

Remark that for $m \in \mathbb{N}$ this notation is compatible with the preexisting one, because of the relation $a.b^m = b^m.(a + m.b)$.

For any $m \in \mathbb{N}$ there exists a canonical (a,b) -morphism

$$b^m.E \rightarrow E$$

which is an isomorphism of $b^m.E$ on $\text{Im}(b^m : E \rightarrow E)$. But remark that the map $b^m : E \rightarrow E$ is not a -linear (but the image is stable by a).

For any $m \in \mathbb{N}$ there is also a canonical (a,b) -morphism

$$E \rightarrow b^{-m}.E$$

which induces an isomorphism of E on $\text{Im}(b^m : b^{-m}.E \rightarrow b^{-m}.E)$. So we may write, via this canonical identification, $b^m.(b^{-m}.E) = E$.

It is easy to see that for any $m, m' \in \mathbb{C}$ we have a natural isomorphism

$$b^{m'}.(b^m.E) \simeq b^{m+m'}.E \quad \text{and also} \quad b^0.E \simeq E.$$

REMARK. It is easy to show that for any $m \in \mathbb{C}$ there exists an unique \mathbb{C} -algebra automorphism

$$\eta_m : \tilde{\mathcal{A}} \rightarrow \tilde{\mathcal{A}} \quad \text{such that} \quad \eta(1) = 1, \eta(b) = b \quad \text{and} \quad \eta(a) = a + m.b.$$

Using this automorphism, one can define a left $\tilde{\mathcal{A}}$ -module $b^m.F$ for any left $\tilde{\mathcal{A}}$ -module F and any $m \in \mathbb{C}$. This is, of course compatible with our definition in the context of (a,b) -modules. □

The behaviour of the correspondance $E \rightarrow b^m.E$ by duality is given by the following easy lemma; the proof is left as an exercice.

Lemme 3.3.12 For any (a,b) -module E and any $m \in \mathbb{C}$ there is natural (a,b) -isomorphism

$$(b^m.E)^* \rightarrow b^{-m}.E^*.$$

The following corollary of the lemma 3.2.3 and the proposition 3.3.7 allows to show that duality preserves the index.

Lemme 3.3.13 Let E be a regular (a,b) -module. Then we have $\delta(E^*) = \delta(E)$.

PROOF. By definition $\delta(E)$ is the smallest integer $k \in \mathbb{N}$ such that $E^\sharp \subset b^{-k}.E$. Now $E^\sharp \subset b^{-m}.E$ implies by duality that $b^m.E^* \subset (E^*)^\flat$. So, by lemma 3.2.3, we have $m \geq \delta(E^*)$. This proves that $\delta(E) \leq \delta(E^*)$ and we obtain the equality by symetry. ■

REMARK. Duality does not preserve the order of regularity : in the example given before the definition 3.2.2 we have $or(E) = 2$ and $or(E^*) = 1$. □

Let us conclude this section by an easy exercice.

EXERCICE. For any (a,b) -modules E, F and any $\lambda \in \mathbb{C}$ there are natural (a,b) -isomorphisms

1. $b^\lambda.E_\mu \simeq E_{\lambda+\mu}$.
2. $b^\lambda.Hom_{a,b}(E, F) \simeq Hom_{a,b}(b^{-\lambda}.E, F) \simeq Hom_{a,b}(E, b^\lambda.F)$.
3. Then deduce from the previous isomorphisms that $Hom_{a,b}(E, E_\lambda) \simeq b^{-\lambda}.E^*$, and $Ext_{\mathcal{A}}^1(E, E_\lambda) \simeq E^*/(a + \lambda.b).E^*$.

3.4 Width of a regular (a,b) -module.

NOTATION. For a complex number λ we shall note by $\tilde{\lambda}$ is class in \mathbb{C}/\mathbb{Z} . We shall order elements in each class *modulo* \mathbb{Z} by its natural order on real parts. □

Définition 3.4.1 Let E be a regular (a,b) -module and let $\tilde{\lambda} \in \mathbb{C}/\mathbb{Z}$. We define the following complex numbers :

$$\begin{aligned} \tilde{\lambda}_{min}(E) &:= \inf\{\lambda \in \tilde{\lambda} / \exists \text{ a non zero morphism } E_\lambda \rightarrow E\} \\ \tilde{\lambda}_{max}(E) &= \sup\{\lambda \in \tilde{\lambda} / \exists \text{ a non zero morphism } E \rightarrow E_\lambda\} \\ L_{\tilde{\lambda}}(E) &= \tilde{\lambda}_{max}(E) - \tilde{\lambda}_{min}(E) \in \mathbb{Z} \\ L(E) &= \sup\{\tilde{\lambda} \in \mathbb{C}/\mathbb{Z} / L_{\tilde{\lambda}}(E)\} \end{aligned}$$

with the following conventions :

$$\begin{aligned} \inf\{\emptyset\} &= +\infty, \quad \sup\{\emptyset\} = -\infty \quad \text{and} \\ -\infty - \lambda &= -\infty \quad \forall \lambda \in]-\infty, +\infty] \\ +\infty - \lambda &= +\infty \quad \forall \lambda \in [-\infty, +\infty[. \end{aligned}$$

We shall call $L(E)$ **the width of E** .

REMARKS.

1. A non zero morphism $E_\lambda \rightarrow E$ is necessarily injective. Either its image is a normal submodule in E or there exists an integer $k \geq 1$ and a morphism $E_{\lambda-k} \rightarrow E$ whose image is normal and contains the image of the previous one.
2. In a dual way, a non zero morphism $E \rightarrow E_\lambda$ has an image equal to $b^k.E_\lambda \simeq E_{\lambda+k}$, where $k \in \mathbb{N}$.
3. A non zero morphism $E_\lambda \rightarrow E_\mu$ implies that λ lies in $\mu + \mathbb{N}$. It is possible that for some E we have $\tilde{\lambda}_{max}(E) < \tilde{\lambda}_{min}(E)$. For instance this is the case for the rank 2 regular (a,b)-module $E_{\lambda,\mu}$ from 3.1.6. So the width of a regular but not simple pole (a,b)-module is not necessarily a non negative integer.
4. Let E and F be regular (a,b)-modules. If there is a surjective morphism $E \rightarrow F$ then for all $\tilde{\lambda} \in \mathbb{C}/\mathbb{Z}$ we have $\tilde{\lambda}_{max}(E) \geq \tilde{\lambda}_{max}(F)$.
If there is an injective morphism $E' \rightarrow E$ then for all $\tilde{\lambda} \in \mathbb{C}/\mathbb{Z}$ we have $\tilde{\lambda}_{min}(E) \leq \tilde{\lambda}_{min}(E')$.
5. Every submodule of E isomorphic to E_λ is contained in E^b . So we have $\tilde{\lambda}_{min}(E) = \tilde{\lambda}_{min}(E^b)$, for every regular (a,b)-module E and every $\tilde{\lambda} \in \mathbb{C}/\mathbb{Z}$.
6. In a dual way, every morphism $E \rightarrow E_\lambda$ extends uniquely to a morphism $E^\# \rightarrow E_\lambda$ with the same image. So for every regular (a,b)-module E and every $\tilde{\lambda} \in \mathbb{C}/\mathbb{Z}$, we get $\tilde{\lambda}_{max}(E) = \tilde{\lambda}_{max}(E^\#)$. \square

Lemme 3.4.2 1. Let E a simple pole (a,b)-module and let $S(E)$ denotes the spectrum of the linear map $b^{-1}.a : E/b.E \rightarrow E/b.E$, we have

$$\tilde{\lambda}_{min}(E) = \inf\{\lambda \in S(E) \cap \tilde{\lambda}\} \quad \text{and} \quad \tilde{\lambda}_{max}(E) = \sup\{\lambda \in S(E) \cap \tilde{\lambda}\} \quad (@)$$

2. For any regular (a,b)-module E we have

$$\widetilde{(-\lambda)}_{max}(E^*) = -\tilde{\lambda}_{min}(E) \quad \widetilde{(-\lambda)}_{min}(E^*) = -\tilde{\lambda}_{max}(E).$$

This implies $L_{-\tilde{\lambda}}(E^*) = L_{\tilde{\lambda}}(E) \quad \forall \tilde{\lambda} \in \mathbb{C}/\mathbb{Z}$, and so $L(E^*) = L(E)$.

3. For any regular (a,b)-module E and any $\tilde{\lambda} \in \mathbb{C}/\mathbb{Z}$ we have equivalence between

$$\tilde{\lambda}_{min}(E) \neq +\infty \quad \text{and} \quad \tilde{\lambda}_{max}(E) \neq -\infty.$$

PROOF. Let E be a simple pole (a,b)-module. We have already seen (in proposition 3.1.4) that if $\lambda \in S(E)$ is minimal in its class modulo 1, there exists a non zero $x \in E$ such that $a.x = \lambda.b.x$. This implies that $\tilde{\lambda}_{min} \leq \inf\{\lambda \in S(E) \cap \tilde{\lambda}\}$. But the opposite inequality is obvious, so the first part of (ⓐ) is proved. Using corollary 3.3.9 and the result already obtained for E^* gives

$$\widetilde{(-\lambda)}_{min}(E^*) = \inf\{-\lambda \in S(E^*) \cap \widetilde{(-\lambda)}\} = -\sup\{\lambda \in S(E) \cap \tilde{\lambda}\}.$$

So for $\mu = \sup\{\lambda \in S(E) \cap \tilde{\lambda}\}$ we have an exact sequence of (a,b)-modules

$$0 \rightarrow E_{-\mu} \rightarrow E^* \rightarrow F \rightarrow 0$$

and by duality, a surjective map $E \rightarrow E_{\mu}$. This implies $\tilde{\lambda}_{max} \geq \mu$. As, again, the opposite inequality is obvious, the second part of (ⓐ) is proved.

Let us prove now the relations in 2.

Remark first that these equalities are true for a simple pole (a,b)-module because of (ⓐ) and corollary 3.3.9.

For any regular (a,b)-module E we know that

$$\tilde{\lambda}_{min}(E) = \tilde{\lambda}_{min}(E^b) = \inf\{\lambda \in S(E^b) \cap \tilde{\lambda}\} \quad \text{and} \quad \widetilde{(-\lambda)}_{max}(E^*) = \widetilde{(-\lambda)}_{max}((E^*)^{\sharp}).$$

But we have

$$\widetilde{(-\lambda)}_{max}((E^*)^{\sharp}) = \sup\{-\lambda \in S((E^*)^{\sharp}) \cap \widetilde{(-\lambda)}\} = -\inf\{\lambda \in S((E^*)^{\sharp})^* \cap \tilde{\lambda}\}$$

because $(E^*)^{\sharp}$ has a simple pole, using corollary 3.3.9. So we obtain

$$\widetilde{(-\lambda)}_{max}(E^*) = -\tilde{\lambda}_{min}(E^b) = -\tilde{\lambda}_{min}(E)$$

because $(E^*)^{\sharp})^* = E^b$ (see proposition 3.3.7).

The second relation is analogous.

The equivalence in 3 is obvious in the simple pole case using (ⓐ).

The general case is an easy consequence using E^b, E^{\sharp} : if $\tilde{\lambda}_{min}(E) \neq +\infty$ so is $\tilde{\lambda}_{min}(E^{\sharp})$ because $E \subset E^{\sharp}$. Then $\tilde{\lambda}_{max}(E^{\sharp}) \neq -\infty$ and so is $\tilde{\lambda}_{max}(E)$. The converse is analogous using E^b . ■

REMARKS.

1. If E has a simple pole, we have $L_{\tilde{\lambda}}(E) \geq 0$ or $L_{\tilde{\lambda}}(E) = -\infty$ for any $\tilde{\lambda}$ in \mathbb{C}/\mathbb{Z} . So $L(E)$ is always ≥ 0 .
2. In cases 1 and 2 of the proposition 3.1.6 the formula (ⓐ) gives the values of $\tilde{\lambda}_{min}$ and $\tilde{\lambda}_{max}$ for any $\tilde{\lambda} \in \mathbb{C}/\mathbb{Z}$.
For the remaining cases we can compute these numbers using the fact that we already know the corresponding E^b and E^{\sharp} and the remark 5 and 6 before the preceding lemma. □

Proposition 3.4.3 *Let E be a regular (a,b) -module and let $\tilde{\lambda} \in \mathbb{C}/\mathbb{Z}$. Assume that $\lambda = \tilde{\lambda}_{min}(E) < +\infty$. Consider an exact sequence of (a,b) -modules*

$$0 \rightarrow E_\lambda \rightarrow E \xrightarrow{\pi} F \rightarrow 0.$$

Then we have for all $\tilde{\mu} \in \mathbb{C}/\mathbb{Z}$ the inequality

$$L_{\tilde{\mu}}(F) \leq L_{\tilde{\mu}}(E) + 1. \quad (\text{i})$$

PROOF. As $\tilde{\mu}_{max}(F) \leq \tilde{\mu}_{max}(E)$ for any $\mu \in \mathbb{C}$ it is enough to prove that we have $\tilde{\mu}_{min}(E) \leq \tilde{\mu}_{min}(F) + 1$ for all $\tilde{\mu} \in \mathbb{C}/\mathbb{Z}$.

Let begin by the case of $\tilde{\mu} = \tilde{\lambda}$. We want to show the inequality

$$\tilde{\lambda}_{min}(F) \geq \lambda - 1 \quad (\text{ii})$$

Let $E_{\lambda-d} \hookrightarrow F$ with $d \geq 0$. The rank 2 (a,b) -module $G := \pi^{-1}(E_{\lambda-d})$ is contained in E , so $\lambda = \tilde{\lambda}_{min}(G)$. We have the exact sequence of (a,b) -modules

$$0 \rightarrow E_\lambda \rightarrow \pi^{-1}(E_{\lambda-d}) \xrightarrow{\pi} E_{\lambda-d} \rightarrow 0.$$

Now let us compare G with the list in proposition 3.1.6.

If G is in case 1, we have $E_{\lambda-d} \subset G$ so $d = 0$ because $\lambda = \tilde{\lambda}_{min}(G)$.

If G is in case 2, we have $\lambda - d = \lambda + n$ with $n \in \mathbb{N}$, so $d = 0$.

If G is in case 3, we have $G \simeq E_{\lambda, \lambda+k}$ with $k \in \mathbb{N}$. Then the theorem 3.1.7 gives $2\lambda - d = 2\lambda + k - 1$ and so $d = 1 - k \leq 1$.

If G is in case 4, we have $G \simeq E_{\lambda, \lambda+n}(\alpha)$. Again theorem 3.1.7 gives $2\lambda - d = 2\lambda + n - 1$ so $d = 1 - n \leq 0$ because $n \in \mathbb{N}^*$. So $d = 0$.

We conclude that we always have $d \leq 1$ and this proves (ii).

For $\tilde{\mu} \neq \tilde{\lambda}$ let us prove now the following inequality :

$$\tilde{\mu}_{min}(F) \leq \tilde{\mu}_{min}(E) \leq \tilde{\mu}_{min}(F) + 1. \quad (\text{iii})$$

Consider an injective morphism $E_\mu \rightarrow E$ with $\mu = \tilde{\mu}_{min}(E)$. The restriction of π to E_μ is injective and so it gives $\tilde{\mu}_{min}(E) \geq \tilde{\mu}_{min}(F)$. Assume now that we have an injective morphism $E_{\mu'} \hookrightarrow F$ with $\mu' = \tilde{\mu}_{min}(F)$, and consider the rank 2 (a,b) -module $\pi^{-1}(E_{\mu'})$. Using the proposition 3.1.6 where only cases 1 or 3 are possible now, it can be easily check that (iii) is satisfied. \blacksquare

REMARKS.

1. In the situation of the previous proposition we have either $\tilde{\lambda}_{min}(E) \geq \tilde{\lambda}_{max}(E)$ or $\tilde{\lambda}_{max}(E) = \tilde{\lambda}_{max}(F)$: Assume that we have $\lambda < \lambda' := \tilde{\lambda}_{max}(E)$. Then there exists a surjective morphism $q : E \rightarrow E_{\lambda'}$, and, as the restriction of q to E_λ is zero, the map q can be factorized and gives a surjective morphism $\tilde{q} : F \rightarrow E_{\lambda'}$. So we get $\tilde{\lambda}_{max}(E) \leq \tilde{\lambda}_{max}(F)$, and the desired equality thanks to the preceding lemma.

2. We shall use later that in the situation of the previous proposition we have the inequality $\tilde{\lambda}_{max}(F) \leq \lambda + L(E)$. \square

Corollaire 3.4.4 *In the situation of the previous proposition we have the inequality $L(E) + rank(E) \geq L(F) + rank(F)$. So this integer is always positive for any non zero regular (a,b) -module.*

PROOF. As the rank 1 case is obvious, an easy induction on the rank of E using the propositions 3.1.4 and 3.4.3 gives the proof. \blacksquare

EXAMPLES.

1. The (a,b) -module

$$J_k(\lambda) := \tilde{\mathcal{A}}/\tilde{\mathcal{A}}.(a - (\lambda + k - 1).b)(a - (\lambda + k - 2).b) \cdots (a - \lambda.b)$$

which has rank k , satisfies $\lambda_{max} = \lambda$ and $\lambda_{min} = \lambda + k - 1$. So its width is $L(J_k(\lambda)) = -k + 1$.

To understand easily the (a,b) -module $J_k(\lambda)$ the reader may use the following alternative definition of it : there is a $\mathbb{C}[[b]]$ -basis (e_1, \dots, e_k) in which the action of "a" is given by

$$a.e_1 = e_2 + \lambda.b.e_1, \quad a.e_2 = e_3 + (\lambda + 1).b.e_2, \dots, a.e_k = (\lambda + k - 1).b.e_k.$$

2. The rank 2 (a,b) -module $E_\lambda \oplus E_{\lambda+n}$ has width n . This shows that, despite the fact that the width is always bigger than $-rank(E) + 1$, the width may be arbitrarily big, even for a rank 2 regular (a,b) -module. \square

4 Finite determination of regular (a,b) -modules.

4.1 Some more preliminaries.

Lemme 4.1.1 *Let E be a regular (a,b) -module of index $\delta(E) = k$. For $N \geq k+1$ the quotient map $q_N : E \rightarrow E/b^N.E$ induces a bijection between simple pole sub- (a,b) -modules F containing $b^k.E^\sharp$ and sub $\tilde{\mathcal{A}}$ -modules $\mathcal{F} \subset E/b^N.E$ satisfying the following two conditions*

- i) $a.\mathcal{F} \subset b.\mathcal{F}$;
- ii) $b^k.E^\sharp/b^N.E \subset \mathcal{F}$.

PROOF. It is clear that if F is a simple pole sub- (a,b) -module of E containing $b^k.E^\sharp$ the image $\mathcal{F} := q_N(F)$ is a $\tilde{\mathcal{A}}$ -submodule of $E/b^N.E$ such that i) and ii) are fulfilled. Conversely, if a $\tilde{\mathcal{A}}$ -submodule \mathcal{F} satisfies i) and ii), let $F := q_N^{-1}(\mathcal{F})$. Of course, F is a sub- (a,b) -module of E and contains $b^k.E^\sharp$. The only point to see is that F has a simple pole. If $x \in F$ then $a.q_N(x) \in b.\mathcal{F}$ so $a.x \in b.F + b^N.E$. As $N \geq k+1$ we may write $a.x = b.y + b.z$ with $y \in F$ and $z \in b^{N-1}.E \subset b^k.E^\sharp \subset F$. This completes the proof. \blacksquare

REMARKS.

1. we may replace $b^k.E^\sharp$ by $b^k.E$ in the second condition imposed on F and \mathcal{F} : if a simple pole (a,b) -submodule F contains $b^k.E$ it contains $b^k.E^\sharp$ by definition of E^\sharp . This allows to avoid the use of E^\sharp in the previous lemma.
2. The biggest \mathcal{F} satisfying i) and ii) corresponds to E^b . So we may recover E^b from the quotient $E/b^N.E$ for $N \geq \delta(E) + 1$. \square

Corollaire 4.1.2 *Let E be a regular (a,b) -module of order of regularity k . Fix $N \geq k + 1$ and assume that we has an isomorphism of $\tilde{\mathcal{A}}$ -modules*

$$\varphi : E/b^N.E \rightarrow E'/b^N.E'$$

where E' is an (a,b) -module. Then E' is regular and has order of regularity k . Moreover we have the equality $\varphi(E^b/b^N.E) = (E')^b/b^N.E'$.

PROOF. As k is the order of regularity of E we have $a^{k+1}.E \subset \sum_{j=0}^k a^j.b^{k-j+1}.E$. The inequality $N \geq k + 1$ gives $a^{k+1}.E/b^N.E \subset \sum_{j=0}^k a^j.b^{k-j+1}.E/b^N.E$, and this is also true for $E'/b^N.E'$, and implies $a^{k+1}.E' \subset \sum_{j=0}^k a^j.b^{k-j+1}.E'$. So the order of regularity of E' is at most k . We conclude that it is exactly k by symetry. The last stament comes from the second remark above, as $or(E) \geq \delta(E)$. \blacksquare

4.2 Finite determination for a rank one extension.

Lemme 4.2.1 *Let E be an (a,b) -module et fix a complex number λ . There exists $N(E, \lambda) \in \mathbb{N}$ such that for any $N \geq N(E, \lambda)$ we have the following inclusion :*

$$b^N.E \subset (a - \lambda.b).E.$$

PROOF. With the b -adic topology, E is a Frechet space. The \mathbb{C} -linear map $a - \lambda.b : E \rightarrow E$ is continuous. The finiteness theorem of [B.95], theorem 1.bis p.31 gives that the kernel and cokernel of this map are finite dimensional vector spaces. So the subspace $(a - \lambda.b).E$ is closed in E . This statement corresponds to the equality

$$\bigcap_{N \geq 0} [(a - \lambda.b).E + b^N.E] = (a - \lambda.b).E \quad (\textcircled{a})$$

But the images of the subspaces $b^N.E$ in the finite dimensional vector space $E/(a - \lambda.b).E$ is a decreasing sequence. So it is stationnary, and, as the intersection is $\{0\}$ thanks to (\textcircled{a}) , the result follows. \blacksquare

Proposition 4.2.2 *Let F be an (a,b) -module and λ a complex number. Consider a short exact sequence of (a,b) -modules*

$$0 \rightarrow E_\lambda \xrightarrow{\alpha} E \xrightarrow{\beta} F \rightarrow 0 \quad (\textcircled{a}\textcircled{a})$$

where $E_\lambda := \tilde{\mathcal{A}}/\tilde{\mathcal{A}}.(a - \lambda.b)$. Then, for any $N \geq N(F^*, -\lambda)$, the extension $(@@)$ is uniquely determined by the following extension of $\tilde{\mathcal{A}}$ -modules which are finite dimensional vectors spaces

$$0 \rightarrow E_\lambda/b^N.E_\lambda \xrightarrow{\alpha} E/b^N.E \xrightarrow{\beta} F/b^N.F \rightarrow 0 \quad (@@_N)$$

obtained from $(@@)$ by "quotient by b^N ".

COMMENTS. This statement needs some more explanations. Denote by K_N the kernel of the obvious map (forget "a")

$$ob_N : Ext_{\tilde{\mathcal{A}}}^1(F/b^N.F, E_\lambda/b^N.E_\lambda) \rightarrow Ext_{\mathbb{C}[[b]]}^1(F/b^N.F, E_\lambda/b^N.E_\lambda)$$

where $Ext_b^1(-, -)$ is a short notation for $Ext_{\mathbb{C}[[b]]}^1(-, -)$. The short exact sequence correspondance $(@@) \rightarrow (@@_N)$ gives a map

$$\delta_N : Ext_{\tilde{\mathcal{A}}}^1(F, E_\lambda) \rightarrow Ext_{\tilde{\mathcal{A}}}^1(F/b^N.F, E_\lambda/b^N.E_\lambda)$$

whose range lies in K_N , because the $\mathbb{C}[[b]]$ -exact sequence $(@@)$ is split as F is $\mathbb{C}[[b]]$ -free, and so is the exact sequence $(@@_N)$. The precise signification of the previous proposition is that for $N \geq N(F^*, -\lambda)$ the map δ_N is a \mathbb{C} -linear isomorphism between the vector spaces $Ext_{\tilde{\mathcal{A}}}^1(F, E_\lambda)$ and K_N . \square

PROOF. As a first step to realize the map δ_N let us consider the following commutative diagram of complex vector spaces, deduced from the exact sequences of $\tilde{\mathcal{A}}$ -modules:

$$\begin{array}{ccccc} 0 \rightarrow E_{\lambda+N} \rightarrow E_\lambda \rightarrow E_\lambda/b^N.E_\lambda \rightarrow 0 \\ 0 \rightarrow b^N.F \rightarrow F \rightarrow F/b^N.F \rightarrow 0 \\ \\ Ext^1(F/b^N.F, E_{\lambda+N}) \longrightarrow Ext^1(F, E_{\lambda+N}) \longrightarrow Ext^1(b^N.F, E_{\lambda+N}) \\ \downarrow \qquad \qquad \qquad \downarrow \alpha \qquad \qquad \qquad \downarrow \\ Ext^1(F/b^N.F, E_\lambda) \longrightarrow Ext^1(F, E_\lambda) \xrightarrow{u} Ext^1(b^N.F, E_\lambda) \\ \downarrow \qquad \qquad \qquad \downarrow \beta \qquad \qquad \qquad \downarrow v \\ Ext^1(F/b^N.F, E_\lambda/b^N.E_\lambda) \xrightarrow{\gamma} Ext^1(F, E_\lambda/b^N.E_\lambda) \xrightarrow{w} Ext^1(b^N.F, E_\lambda/b^N.E_\lambda) \end{array}$$

We have the following properties :

1. The surjectivity of the map β is consequence of the vanishing of the vector space $Ext_{\tilde{\mathcal{A}}}^2(F, E_{\lambda+N})$ thanks to the proposition 3.3.7.

The surjectivity of β implies that the map $i \circ \gamma$ is surjective (we know that the extensions in the image of δ_N comes from K_N , so δ_N factors in $\hat{\delta}_N \circ i$). We have $i(K_N) \cap \text{Im}(\partial_N) = (0)$ because ob_N is injective on $\text{Im}(\partial_N)$. So i induces an isomorphism of vector spaces from K_N to

$$\text{Ext}_{\tilde{\mathcal{A}}}^1(F/b^N.F, E_\lambda/b^N.E_\lambda)/\text{Im}(\partial_N) \stackrel{\sim}{\simeq} \text{Ext}_{\tilde{\mathcal{A}}}^1(F, E_\lambda/b^N.E_\lambda) \stackrel{\beta^{-1}}{\simeq} \text{Ext}_{\tilde{\mathcal{A}}}^1(F, E_\lambda).$$

This completes the proof . ■

We shall need some bound for the integer $N(F^*, -\lambda)$ which appears in the previous proposition for the proof of our theorem.

Lemme 4.2.3 *Let G be a regular (a,b) -module and let $\mu \in \mathbb{C}$. A sufficient condition on $N \in \mathbb{N}$ in order to have the inclusion $b^N.G \subset (a - \mu.b).G$ is*

$$N \geq \mu - \tilde{\mu}_{\min}(G) + \delta(G) + 2.$$

PROOF. As we know that $\tilde{\mu}_{\min}(G^b) = \tilde{\mu}_{\min}(G)$, for $M \in \mathbb{N}$, the assumption $M > \mu - \tilde{\mu}_{\min}(G)$ implies that $(a - (\mu - M).b).G^b = b.G^b$ (see the remark before proposition 3.1.4). By definition of the index of G we have $b^{\delta(G)}.G \subset G^b$. Combining both gives

$$b^{M+\delta(G)+1}.G \subset b^M.(a - (\mu - M).b).G = (a - \mu.b).b^M.G \subset (a - \mu.b).G.$$

Now let $N = M + \delta(G) + 1$; a sufficient condition on the integer N is now $N \geq \mu - \tilde{\mu}_{\min}(G) + \delta(G) + 2$. ■

Corollaire 4.2.4 *A sufficient condition for $N \geq N(F^*, -\lambda)$ in the situation of prop. 4.2.2 in the regular case is that $N \geq \text{or}(E) + L(E) + \text{rank}(E) + 1$.*

Remark that the inequality $L(E) + \text{rank}(E) \geq 1$ for any non zero regular E implies that we have $\text{or}(E) + L(E) + \text{rank}(E) + 1 \geq \text{or}(E) + 2$.

PROOF. We apply the previous lemma with $F^* = G$ and $\mu = -\lambda = -\tilde{\lambda}_{\min}(E)$. The conclusion comes now from the following facts :

1. $-\widetilde{(-\lambda)}_{\min}(F^*) = \tilde{\lambda}_{\max}(F) \leq \lambda + L(E)$ this last inequality is proved in 3.4.3.
2. $\delta(F^*) = \delta(F) \leq \text{or}(F) \leq \text{or}(E)$ proved in 3.3.13 and 3.2.4 ■

4.3 The theorem.

Théorème 4.3.1 *Let E be a regular (a,b) -module. There exists an integer $N(E)$ such that for any (a,b) -module E' , any integer $N \geq N(E)$ and any $\tilde{\mathcal{A}}$ -isomorphism*

$$\varphi : E/b^N.E \rightarrow E'/b^N.E' \tag{1}$$

there exists an unique $\tilde{\mathcal{A}}$ -isomorphism $\Phi : E \rightarrow E'$ inducing the given φ . Moreover the choice $N(E) = N_0(E) := \text{or}(E) + L(E) + \text{rank}(E) + 1$ is possible.

REMARKS.

1. It is easy to see that for a rank 1 regular (a,b)-module the integer 2 is the best possible.
2. In our final lemma 4.3.2 we show that the integer given in the theorem is optimal for the rank k (a,b)-module $J_k(\lambda)$, (defined in the lemma), for any $k \in \mathbb{N}^*$.
3. For the rank 2 (a,b)-modules $E_{\lambda,\mu}$ the integer given by the theorem is $or(E) + L(E) + 2 + 1 = 3$ is again optimal, as it can be shown in the same maner that in our final lemma.
4. For the rank 2 simple pole (a,b)-module $E_\lambda(0)$ the integer given by the theorem is $3 = L(E) + rank(E) + 1$ and the best possible is 2 : the action of $b^{-1}.a$ on $E/b.E$ which is determined by $E/b^2.E$ characterizes this rank 2 regular (a,b)-module in the classification given in proposition 3.1.6.
5. For the (a,b)-module E associated to an holomorphic germ at the origine of \mathbb{C}^{n+1} with an isolated singularity we have the uniform bounds $or(E) \leq n + 1$ and $L(E) \leq n$ so the choice $N(E) = 2n + \mu + 2$ is always possible, where μ is the Milnor number (equal to the rank). \square

PROOF. We shall make an induction on the rank of E . So we shall assume that the result is proved for a rank $p - 1$ (a,b)-module and we shall consider a regular (a,b)-module E of rank $p \geq 1$, an (a,b)-module E' , an integer $N \geq N_0(E)$ and an $\tilde{\mathcal{A}}$ -isomorphism φ as in (1). From 4.1.2 we know that E' is then regular and has order of regularity $or(E') = or(E)$.

Choose now a complex number λ which is minimal *modulo* \mathbb{Z} such there exists an exact sequence of (a,b)-module (so $\lambda = \tilde{\lambda}_{min}(E)$ with the terminology of §1.3)

$$0 \rightarrow E_\lambda \xrightarrow{\alpha} E \xrightarrow{\beta} F \rightarrow 0. \quad (2)$$

This exists from theorem 3.1.7. The (a,b)-module F has rank $p - 1$ and from 4.2.4 we have $N_0(E) \geq N(F^*, -\lambda)$. So we know from 4.2.2 that the extension (2) is determined by the extension

$$0 \rightarrow E_\lambda/b^N.E_\lambda \xrightarrow{\alpha_N} E/b^N.E \xrightarrow{\beta_N} F/b^N.F \rightarrow 0. \quad (2_N)$$

Now, using the $\tilde{\mathcal{A}}$ -isomorphism φ we obtain an injective $\tilde{\mathcal{A}}$ -linear map

$$j_N : E_\lambda/b^N.E_\lambda \hookrightarrow E'/b^N.E'.$$

Using the proposition 3.1.4 with the fact that $N \geq or(E') + 2$ there exists a unique normal inclusion $j : E_\lambda \hookrightarrow E'$ inducing j_N .

Define $F' := E'/j(E_\lambda)$. Then F' is a rank $p - 1$ (a,b) -module and the exact sequence

$$0 \rightarrow E_\lambda \xrightarrow{j} E' \rightarrow F' \rightarrow 0 \quad (2')$$

induced the extension (2_N) . Using the induction hypothesis, because the inequalities $or(E) \geq or(F)$ from 4.1.2 and $L(E) + rank(E) \geq L(F) + rank(F)$ from 3.4.4 implies $N_0(E) \geq N_0(F)$, we have a unique isomorphism $\Psi : F \rightarrow F'$ compatible with the one induced by φ between $F/b^N.F$ and $F'/b^N.F'$. Using 4.2.2, 4.2.4 and the inequality $N_0(E) \geq N(F^*, -\lambda)$ we have an unique isomorphism of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_\lambda & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & F & \longrightarrow & 0 \\ & & \downarrow = & & \downarrow \Phi & & \downarrow \Psi & & \\ 0 & \longrightarrow & E_\lambda & \xrightarrow{j} & E' & \longrightarrow & F' & \longrightarrow & 0 \end{array}$$

concluding the proof. ■

Lemma 4.3.2 *Let $E := J_k(\lambda)$ the rank k (a,b) -module defined by the $\mathbb{C}[[b]]$ -basis e_1, \dots, e_k and by the following relations*

$$a.e_j = (\lambda + j - 1).b.e_j + e_{j+1} \quad \forall j \in [1, k]$$

with the convention $e_{k+1} = 0$. We have $\delta(E) = or(E) = k - 1, L(E) = -k + 1$. The integer $or(E) + L(E) + rank(E) + 1 = k + 1$ is the best possible for the theorem.

PROOF. It is easy to see that the saturation $E^\#$ is generated by $e_1, b^{-1}.e_2, \dots, b^{-k+1}.e_k$.

This gives the equality $\delta(E) = or(E) = k - 1$.

Assume that we have an inclusion $E_\mu \hookrightarrow E$ such that $e_\mu \notin b.E$. Then there exists $(\alpha_1, \dots, \alpha_k) \in \mathbb{C}^k \setminus \{0\}$ such that

$$a. \left(\sum_{h=1}^k \alpha_h . e_h \right) = \mu . b \left(\sum_{h=1}^k \alpha_h . e_h \right) + b^2 . E.$$

Then we obtain

$$\sum_{h=1}^k \alpha_h . ((\lambda + j - 1).b.e_h + e_{h+1}) = \sum_{h=1}^k \alpha_h . \mu . b . e_h + b^2 . E$$

and so $\alpha_1 = \dots = \alpha_{k-1} = 0$ and we conclude that $\mu = \lambda + k - 1$.

An easy computation shows that $J_k(\lambda)^* = J_k(-\lambda - 2k + 2)$ and so we have $\lambda_{max} = \lambda$. So $L(E) = -k + 1$.

Now we shall prove that the integer $k + 1$ is the best possible in the theorem 4.3.1 for $E = J_k(\lambda)$ by giving a regular (a,b) -module F such that $F/b^k.F \simeq E/b^k.E$ and not isomorphic to E .

Let consider the rank k (a,b)-module F defined by $\sum_{j=1}^k \mathbb{C}[[b]].e_j$ with the following relations

$$\begin{aligned} a.e_j &= (\lambda + j - 1).b.e_j + e_{j+1} \quad \forall j \in [1, k] \\ a.e_k &= (\lambda + k - 1).b.e_k + \sum_{h=1}^{k-1} \alpha_h.b^{k-h+1}.e_h \end{aligned}$$

Then define, for $\beta_1, \dots, \beta_{k-1} \in \mathbb{C}$,

$$\varepsilon := e_k + \sum_{j=1}^{k-1} \beta_j.b^{k-j}.e_j.$$

We have

$$\begin{aligned} a.\varepsilon &:= (\lambda + k - 1).b.e_k + \sum_{h=1}^{k-1} \alpha_h.b^{k-h+1}.e_h + \\ &\quad \sum_{j=1}^{k-1} \beta_j.[b^{k-j}.((\lambda + j - 1).b.e_j + e_{j+1}) + (k - j).b^{k-j+1}.e_j] \\ a.\varepsilon &:= (\lambda + k - 1).b.e_k + \sum_{h=1}^{k-1} (\alpha_h + \beta_h.(\lambda + k - 1) + \beta_{h-1}).b^{k-h+1}.e_h \end{aligned}$$

Let now choose $\beta_1, \dots, \beta_{k-1}$ such that we have

$$\alpha_h + \beta_h.(\lambda + k - 1) + \beta_{h-1} = (\lambda + k - 1 + \beta_{k-1}).\beta_h \quad \forall h \in [1, k - 1]$$

with the convention $\beta_0 = 0$. We obtain the system of equations

$$\alpha_h + \beta_{h-1} = \beta_{k-1}.\beta_h \quad \forall h \in [1, k - 1].$$

This implies, assuming $\beta_{k-1} \neq 0$, that β_{k-1} is solution of the equation

$$x^k = \alpha_{k-1}.x^{k-2} + \dots + \alpha_2.x + \alpha_1.$$

Now choose $\alpha_2 = \dots = \alpha_{k-1} = 0$ and $\alpha_1 := \rho^k$ with $\rho \in]0, 1[$. Then choose $\beta_j = \rho^{k-j} \quad \forall j \in [1, k - 1]$. It is clear that the corresponding F_ρ satisfies $F/b^k.F \simeq E/b^k.E$ as $a.e_k = e_k + \rho.b^k.e_1$ in F_ρ . But the relation $a.\varepsilon = (\lambda + k - 1 + \rho^k).b.\varepsilon$ with $\varepsilon \neq 0$ shows that F_ρ cannot be isomorphic to $J_k(\lambda)$. \blacksquare

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