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SINGULAR BOUNDARY CONDITIONS AND REGULARITY FOR THE BIHARMONIC PROBLEM IN THE HALF-SPACE

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Abstract. In this paper, we are interested in some aspects of the biharmonic equation in the half-space $\mathbb{R}^N_+$, with $N \geq 2$. We study the regularity of generalized solutions in weighted Sobolev spaces, then we consider the question of singular boundary conditions. To finish, we envisage other sorts of boundary conditions.

1. **Introduction.** The purpose of this paper is the resolution of the biharmonic problem with nonhomogeneous boundary conditions

$$(\mathcal{P}) \quad \begin{cases} \Delta^2 u = f & \text{in } \mathbb{R}^N_+, \\ u = g_0 & \text{on } \Gamma = \mathbb{R}^{N-1}, \\ \partial_N u = g_1 & \text{on } \Gamma. \end{cases}$$

Since this problem is posed in the half-space, it is important to specify the behaviour at infinity for the data and solutions. We have chosen to impose such conditions by setting our problem in weighted Sobolev spaces, where the growth or decay of functions at infinity are expressed by means of weights. These weighted Sobolev spaces provide a correct functional setting for unbounded domains, in particular because the functions in these spaces satisfy an optimal weighted Poincaré-type inequality. Our analysis is based on the isomorphism properties of the biharmonic operator in the whole space and the resolution of the Dirichlet and Neumann problems for the Laplacian in the half-space. This last one is itself based on the isomorphism properties of the Laplace operator in the whole space and also on the reflection principle inherent in the half-space. Note here the double difficulty arising from the unboundedness of the domain in any direction and from the unboundedness of the boundary itself.

In a previous work (see [7]), we established the existence of generalized solutions to problem $(\mathcal{P})$, *i.e.* solutions which belong to weighted Sobolev spaces of type $W^{2,\,p}_{l}(\mathbb{R}^N_+)$. For the sake of convenience, we shall recall the necessary results of this paper in Section 2.4. Here, we are interested both in the existence of more regular solutions, as for instance strong solutions which belong to spaces of type $W^{4,\,p}_{l+2}(\mathbb{R}^N_+)$, and singular solutions which belong to $W^{0,\,p}_{l-2}(\mathbb{R}^N_+)$ in the case $f = 0$ with singular

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boundary conditions. We also establish the existence of solutions which belong to intermediate spaces as for example $W^{3,\,p}_{l+1}(\mathbb{R}^N_+)$.

As we saw (see [7]), it turns out that the use of classical Sobolev spaces is inadequate in this case, contrary to the study of elliptic problems of type:

$$(\mathcal{Q}) \qquad u + \Delta^2 u = f \ \text{ in } \mathbb{R}^N_+, \quad u = g_0 \ \text{ and } \ \partial_N u = g_1 \ \text{ on } \Gamma,$$

where it is more reasonable to consider data and solutions in standard Sobolev spaces. For example, if $f \in L^2(\mathbb{R}^N_+)$, $g_0 \in H^{7/2}(\mathbb{R}^{N-1})$ and $g_1 \in H^{5/2}(\mathbb{R}^{N-1})$, problem $(\mathcal{Q})$ admits an unique solution $u \in H^4(\mathbb{R}^N_+)$. In the case of problem $(\mathcal{P})$, if we assume that $f \in L^2(\mathbb{R}^N_+)$, the solution u can not be better than in $W^{4,\,2}_0(\mathbb{R}^N_+)$ and its traces $u|_\Gamma$ and $\partial_N u|_\Gamma$ respectively in $W^{7/2,\,2}_0(\mathbb{R}^{N-1})$ and $W^{5/2,\,2}_0(\mathbb{R}^{N-1})$. Moreover, we can observe that $H^4(\mathbb{R}^N_+)$ is a subspace of $W^{4,\,2}_0(\mathbb{R}^N_+)$ and the same remark holds for the traces spaces.

On the one hand, we can find in the literature an approach via homogeneous spaces. For instance, when $f \in L^2(\mathbb{R}^N_+)$, that consists in finding solutions to $(\mathcal{P})$ satisfying $\nabla^4 u \in L^2(\mathbb{R}^N_+)^{N^4}$, but that gives no information on the other derivatives, nor specifes the behavior at infinity for the data and solutions.

On the other hand, Boulmezaoud has established (see [9]) in a Hilbertian framework, the existence of solutions $u \in H^{3,\,2}_{l+1}(\mathbb{R}^N_+)$ for data $f \in H^{-1,\,2}_{l+1}(\mathbb{R}^N_+)$ and the corresponding regularity result. However, owing to some critical cases, this framework excludes in particular the dimensions 2 and 4 (see Section 3.3).

To reduce the set of critical values, we have used a special class of weighted Sobolev spaces with logarithmic factors (see Section 2.2 and Remark 2.1). We shall show the part of these logarithmic factors in the weights particularly in the question of singular boundary conditions.

This paper is organized as follows. Section 2 is devoted to the notations, functional setting and main results of our previous paper on the biharmonic operator, which is the source of the present work. In Section 3, we study the regularity of the solutions to problem $(\mathcal{P})$ according to the data. After that, we give a panorama of basic cases and we recall the Boulmezaoud theorem to throw light on our contribution in this study. In Section 4, we come back to the homogeneous problem ($f = 0$) with singular boundary conditions. Lastly, in Section 5, we shall consider the biharmonic equation, but with different types of boundary conditions. The main results of this paper are Theorem 3.1 for the regularity and Theorem 4.4 for the singular boundary conditions. In a forthcoming work, we shall use these results to solve the Stokes system.

2. **Notations, functional framework and useful results.**

2.1. **Notations.** For any real number $p > 1$, we always take p' to be the Hölder conjugate of p, *i.e.*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Let Ω be an open set of $\mathbb{R}^N$, $N \geq 2$. Writing a typical point $x \in \mathbb{R}^N$ as $x = (x', x_N)$, where $x' = (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$ and $x_N \in \mathbb{R}$, we will especially look on the upper half-space $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N;\ x_N > 0\}$. We let $\overline{\mathbb{R}^N_+}$ denote the closure of $\mathbb{R}^N_+$ in $\mathbb{R}^N$ and let $\Gamma = \{x \in \mathbb{R}^N;\ x_N = 0\} \equiv \mathbb{R}^{N-1}$ denote its boundary. Let $|x| = (x_1^2 + \cdots + x_N^2)^{1/2}$

denote the Euclidean norm of x, we will use two basic weights

$$\varrho = (1+|x|^2)^{1/2} \quad \text{and} \quad \lg \varrho = \ln(2+|x|^2).$$

We denote by ∂_i the partial derivative $\dfrac{\partial}{\partial x_i}$, similarly $\partial_i^2 = \partial_i \circ \partial_i = \dfrac{\partial^2}{\partial x_i^2}$, $\partial_{ij}^2 = \partial_i \circ \partial_j = \dfrac{\partial^2}{\partial x_i \partial x_j}$, ... More generally, if $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{N}^N$ is a multi-index, then

$$\partial^\lambda = \partial_1^{\lambda_1} \cdots \partial_N^{\lambda_N} = \frac{\partial^{|\lambda|}}{\partial x_1^{\lambda_1} \cdots \partial x_N^{\lambda_N}} \quad \text{, where } |\lambda| = \lambda_1 + \cdots + \lambda_N.$$

In the sequel, for any integer q, we shall use the following polynomial spaces:
— $\mathscr{P}_q$ is the space of polynomials of degree smaller than or equal to q;
— $\mathscr{P}_q^\Delta$ is the subspace of harmonic polynomials of $\mathscr{P}_q$;
— $\mathscr{P}_q^{\Delta^2}$ is the subspace of biharmonic polynomials of $\mathscr{P}_q$;
— $\mathscr{A}_q^\Delta$ is the subspace of polynomials of $\mathscr{P}_q^\Delta$, odd with respect to x_N, or equivalently, which satisfy the condition $\varphi(x', 0) = 0$;
— $\mathscr{N}_q^\Delta$ is the subspace of polynomials of $\mathscr{P}_q^\Delta$, even with respect to x_N, or equivalently, which satisfy the condition $\partial_N \varphi(x', 0) = 0$;
with the convention that these spaces are reduced to $\{0\}$ if $q < 0$.
For any real number s, we denote by $[s]$ the integer part of s.
Given a Banach space B, with dual space B' and a closed subspace X of B, we denote by $B' \perp X$ the subspace of B' orthogonal to X, *i.e.*

$$B' \perp X = \{f \in B';\ \forall v \in X,\ \langle f, v\rangle = 0\} = (B/X)'.$$

Lastly, if $k \in \mathbb{Z}$, we shall constantly use the notation $\{1, \dots, k\}$ for the set of the first k positive integers, with the convention that this set is empty if k is nonpositive.

2.2. **Weighted Sobolev spaces.** For any nonnegative integer m, real numbers $p > 1$, α and β, we define the following space (see [3], Section 7):

$$\begin{aligned} W^{m,p}_{\alpha,\beta}(\Omega) = \Big\{ & u \in \mathscr{D}'(\Omega);\ 0 \le |\lambda| \le k,\ \varrho^{\alpha-m+|\lambda|}\,(\lg \varrho)^{\beta-1}\,\partial^\lambda u \in L^p(\Omega); \\ & k+1 \le |\lambda| \le m,\ \varrho^{\alpha-m+|\lambda|}\,(\lg \varrho)^{\beta}\,\partial^\lambda u \in L^p(\Omega) \Big\}, \end{aligned} \tag{1}$$

where

$$k = \begin{cases} -1 & \text{if } \frac{N}{p} + \alpha \notin \{1, \dots, m\}, \\ m - \dfrac{N}{p} - \alpha & \text{if } \frac{N}{p} + \alpha \in \{1, \dots, m\}. \end{cases}$$

Note that $W^{m,p}_{\alpha,\beta}(\Omega)$ is a reflexive Banach space equipped with its natural norm:

$$\begin{aligned} \|u\|_{W^{m,p}_{\alpha,\beta}(\Omega)} = \Big(& \sum_{0 \le |\lambda| \le k} \|\varrho^{\alpha-m+|\lambda|}\,(\lg \varrho)^{\beta-1}\,\partial^\lambda u\|^p_{L^p(\Omega)} \\ & + \sum_{k+1 \le |\lambda| \le m} \|\varrho^{\alpha-m+|\lambda|}\,(\lg \varrho)^{\beta}\,\partial^\lambda u\|^p_{L^p(\Omega)} \Big)^{1/p}. \end{aligned}$$

We also define the semi-norm:

$$|u|_{W^{m,p}_{\alpha,\beta}(\Omega)} = \Big(\sum_{|\lambda| = m} \|\varrho^{\alpha}\,(\lg \varrho)^{\beta}\,\partial^\lambda u\|^p_{L^p(\Omega)} \Big)^{1/p}.$$

The weights in the definition (1) are chosen so that the corresponding space satisfies two fundamental properties. On the one hand, $\mathscr{D}\big(\overline{\mathbb{R}^N_+}\big)$ is dense in $W^{m,\,p}_{\alpha,\,\beta}(\mathbb{R}^N_+)$. On the other hand, the following Poincaré-type inequality holds in $W^{m,\,p}_{\alpha,\,\beta}(\mathbb{R}^N_+)$ (see [5], Theorem 1.1): if

$$\frac{N}{p}+\alpha \notin \{1,\dots,m\} \quad \text{or} \quad (\beta-1)p \neq -1, \tag{2}$$

then the semi-norm $|\cdot|_{W^{m,\,p}_{\alpha,\,\beta}(\mathbb{R}^N_+)}$ defines on $W^{m,\,p}_{\alpha,\,\beta}(\mathbb{R}^N_+)/\mathscr{P}_{q'}$ a norm which is equivalent to the quotient norm,

$$\forall u \in W^{m,\,p}_{\alpha,\,\beta}(\mathbb{R}^N_+), \quad \|u\|_{W^{m,\,p}_{\alpha,\,\beta}(\mathbb{R}^N_+)/\mathscr{P}_{q'}} \leq C\,|u|_{W^{m,\,p}_{\alpha,\,\beta}(\mathbb{R}^N_+)}, \tag{3}$$

with $q'=\inf(q,m-1)$, where q is the highest degree of the polynomials contained in $W^{m,\,p}_{\alpha,\,\beta}(\mathbb{R}^N_+)$. Now, we define the space

$$\overset{\circ}{W}{}^{m,\,p}_{\alpha,\,\beta}(\mathbb{R}^N_+)=\overline{\mathscr{D}(\mathbb{R}^N_+)}^{\|\cdot\|_{W^{m,\,p}_{\alpha,\,\beta}(\mathbb{R}^N_+)}};$$

which will be characterized in Lemma 2.2 as the subspace of functions with null traces in $W^{m,\,p}_{\alpha,\,\beta}(\mathbb{R}^N_+)$. From that, we can introduce the space $W^{-m,\,p'}_{-\alpha,\,-\beta}(\mathbb{R}^N_+)$ as the dual space of $\overset{\circ}{W}{}^{m,\,p}_{\alpha,\,\beta}(\mathbb{R}^N_+)$. In addition, under the assumption (2), $|\cdot|_{W^{m,\,p}_{\alpha,\,\beta}(\mathbb{R}^N_+)}$ is a norm on $\overset{\circ}{W}{}^{m,\,p}_{\alpha,\,\beta}(\mathbb{R}^N_+)$ which is equivalent to the full norm $\|\cdot\|_{W^{m,\,p}_{\alpha,\,\beta}(\mathbb{R}^N_+)}$.
We will still recall some properties of the weighted Sobolev spaces $W^{m,\,p}_{\alpha,\,\beta}(\mathbb{R}^N_+)$. We have the algebraic and topological imbeddings:

$$W^{m,\,p}_{\alpha,\,\beta}(\mathbb{R}^N_+) \hookrightarrow W^{m-1,\,p}_{\alpha-1,\,\beta}(\mathbb{R}^N_+) \hookrightarrow \cdots \hookrightarrow W^{0,\,p}_{\alpha-m,\,\beta}(\mathbb{R}^N_+) \ \text{ if } \frac{N}{p}+\alpha \notin \{1,\dots,m\}.$$

When $\dfrac{N}{p}+\alpha=j\in\{1,\dots,m\}$, then we have:

$$W^{m,\,p}_{\alpha,\,\beta} \hookrightarrow \cdots \hookrightarrow W^{m-j+1,\,p}_{\alpha-j+1,\,\beta} \hookrightarrow W^{m-j,\,p}_{\alpha-j,\,\beta-1} \hookrightarrow \cdots \hookrightarrow W^{0,\,p}_{\alpha-m,\,\beta-1}.$$

Note that in the first case, for any $\gamma\in\mathbb{R}$ such that $\dfrac{N}{p}+\alpha-\gamma \notin \{1,\dots,m\}$ and $m\in\mathbb{N}$, the mapping

$$u \in W^{m,\,p}_{\alpha,\,\beta}(\mathbb{R}^N_+) \longmapsto \varrho^{\gamma}u \in W^{m,\,p}_{\alpha-\gamma,\,\beta}(\mathbb{R}^N_+)$$

is an isomorphism. In both cases and for any multi-index $\lambda\in\mathbb{N}^N$, the mapping

$$u \in W^{m,\,p}_{\alpha,\,\beta}(\mathbb{R}^N_+) \longmapsto \partial^{\lambda}u \in W^{m-|\lambda|,\,p}_{\alpha,\,\beta}(\mathbb{R}^N_+)$$

is continuous. Finally, it can be readily checked that the highest degree q of the polynomials contained in $W^{m,\,p}_{\alpha,\,\beta}(\mathbb{R}^N_+)$ is given by

$$q=\begin{cases} m-\left(\dfrac{N}{p}+\alpha\right)-1, & \text{if } \begin{cases} \frac{N}{p}+\alpha\in\{1,\dots,m\} \ \text{ and } \ (\beta-1)p\geq -1, \\ \text{or} \\ \frac{N}{p}+\alpha\in\{j\in\mathbb{Z};\ j\leq 0\} \ \text{ and } \ \beta p\geq -1, \end{cases} \\ \left[m-\left(\dfrac{N}{p}+\alpha\right)\right], & \text{otherwise.} \end{cases} \tag{4}$$

Remark 2.1. In the case $\beta = 0$, we simply denote the space $W^{m,\,p}_{\alpha,\,0}(\Omega)$ by $W^{m,\,p}_{\alpha}(\Omega)$. In [11], Hanouzet introduced a class of weighted Sobolev spaces without logarithmic factors, with the same notation $W^{m,\,p}_{\alpha}(\Omega)$. We recall his definition under the notation $H^{m,\,p}_{\alpha}(\Omega)$:

$$H^{m,\,p}_{\alpha}(\Omega) = \Big\{ u \in \mathscr{D}'(\Omega);\ 0 \le |\lambda| \le m,\ \varrho^{\alpha-m+|\lambda|}\,\partial^{\lambda} u \in L^p(\Omega) \Big\}.$$

It is clear that if $\dfrac{N}{p} + \alpha \notin \{1, \dots, m\}$, we have $W^{m,\,p}_{\alpha}(\Omega) = H^{m,\,p}_{\alpha}(\Omega)$. The fundamental difference between these two families of spaces is that the assumption (2) and thus the Poincaré-type inequality (3), hold for any value of $(N,\,p,\,\alpha)$ in $W^{m,\,p}_{\alpha}(\Omega)$, but not in $H^{m,\,p}_{\alpha}(\Omega)$ if $\dfrac{N}{p} + \alpha \in \{1, \dots, m\}$. □

2.3. **The spaces of traces.** In order to define the traces of functions of $W^{m,\,p}_{\alpha}(\mathbb{R}^N_+)$ (here we don't consider the case $\beta \neq 0$), for any $\sigma \in \,]0,1[$, we introduce the space:

$$\begin{aligned} W^{\sigma,\,p}_{0}(\mathbb{R}^N) = \Big\{ & u \in \mathscr{D}'(\mathbb{R}^N);\ w^{-\sigma} u \in L^p(\mathbb{R}^N) \ \text{ and } \ \forall i = 1, \dots, N, \\ & \int_0^{+\infty} t^{-1-\sigma p}\,\mathrm{d}t \int_{\mathbb{R}^N} |u(x + te_i) - u(x)|^p\,\mathrm{d}x < \infty \Big\}, \end{aligned} \tag{5}$$

where $w = \varrho$ if $N/p \neq \sigma$ and $w = \varrho\,(\lg \varrho)^{1/\sigma}$ if $N/p = \sigma$, and $e_1, \dots, e_N$ is the canonical basis of $\mathbb{R}^N$. It is a reflexive Banach space equipped with its natural norm:

$$\|u\|_{W^{\sigma,\,p}_0(\mathbb{R}^N)} = \left(\Big\| \frac{u}{w^{\sigma}} \Big\|^p_{L^p(\mathbb{R}^N)} + \sum_{i=1}^{N} \int_0^{+\infty} t^{-1-\sigma p}\,\mathrm{d}t \int_{\mathbb{R}^N} |u(x + te_i) - u(x)|^p\,\mathrm{d}x \right)^{1/p}$$

which is equivalent to the norm

$$\left(\Big\| \frac{u}{w^{\sigma}} \Big\|^p_{L^p(\mathbb{R}^N)} + \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+\sigma p}}\,\mathrm{d}x\,\mathrm{d}y \right)^{1/p}.$$

Similarly, for any real number $\alpha \in \mathbb{R}$, we define the space:

$$\begin{aligned} W^{\sigma,\,p}_{\alpha}(\mathbb{R}^N) = \Big\{ & u \in \mathscr{D}'(\mathbb{R}^N);\ w^{\alpha-\sigma} u \in L^p(\mathbb{R}^N), \\ & \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\varrho^{\alpha}(x)\,u(x) - \varrho^{\alpha}(y)\,u(y)|^p}{|x - y|^{N+\sigma p}}\,\mathrm{d}x\,\mathrm{d}y < \infty \Big\}, \end{aligned}$$

where $w = \varrho$ if $N/p + \alpha \neq \sigma$ and $w = \varrho\,(\lg \varrho)^{1/(\sigma-\alpha)}$ if $N/p + \alpha = \sigma$. For any $s \in \mathbb{R}^+$, we set

$$\begin{aligned} & W^{s,\,p}_{\alpha}(\mathbb{R}^N) = \Big\{ u \in \mathscr{D}'(\mathbb{R}^N);\ 0 \le |\lambda| \le k,\ \varrho^{\alpha-s+|\lambda|}\,(\lg \varrho)^{-1}\,\partial^{\lambda} u \in L^p(\mathbb{R}^N); \\ & k+1 \le |\lambda| \le [s] - 1,\ \varrho^{\alpha-s+|\lambda|}\,\partial^{\lambda} u \in L^p(\mathbb{R}^N);\ |\lambda| = [s],\ \partial^{\lambda} u \in W^{\sigma,\,p}_{\alpha}(\mathbb{R}^N) \Big\}, \end{aligned}$$

where $k = s - N/p - \alpha$ if $N/p + \alpha \in \{\sigma, \dots, \sigma + [s]\}$, with $\sigma = s - [s]$ and $k = -1$ otherwise. It is a reflexive Banach space equipped with the norm:

$$\|u\|_{W^{s,\,p}_{\alpha}(\mathbb{R}^N)} = \Big(\sum_{0\le|\lambda|\le k} \|\varrho^{\alpha-s+|\lambda|}\,(\lg\varrho)^{-1}\,\partial^\lambda u\|^p_{L^p(\mathbb{R}^N)} + \sum_{k+1\le|\lambda|\le[s]-1} \|\varrho^{\alpha-s+|\lambda|}\,\partial^\lambda u\|^p_{L^p(\mathbb{R}^N)}\Big)^{1/p} + \sum_{|\lambda|=[s]} \|\partial^\lambda u\|_{W^{\sigma,\,p}_{\alpha}(\mathbb{R}^N)}.$$

We can similarly define, for any real number β, the space:

$$W^{s,\,p}_{\alpha,\,\beta}(\mathbb{R}^N) = \Big\{ v \in \mathscr{D}'(\mathbb{R}^N);\ (\lg\varrho)^\beta\, v \in W^{s,\,p}_{\alpha}(\mathbb{R}^N) \Big\}.$$

We can prove some properties of the weighted Sobolev spaces $W^{s,\,p}_{\alpha,\,\beta}(\mathbb{R}^N)$. We have the algebraic and topological imbeddings if $N/p + \alpha \notin \{\sigma, \dots, \sigma + [s] - 1\}$:

$$W^{s,\,p}_{\alpha,\,\beta}(\mathbb{R}^N) \hookrightarrow W^{s-1,\,p}_{\alpha-1,\,\beta}(\mathbb{R}^N) \hookrightarrow \cdots \hookrightarrow W^{\sigma,\,p}_{\alpha-[s],\,\beta}(\mathbb{R}^N),$$
$$W^{s,\,p}_{\alpha,\,\beta}(\mathbb{R}^N) \hookrightarrow W^{[s],\,p}_{\alpha+[s]-s,\,\beta}(\mathbb{R}^N) \hookrightarrow \cdots \hookrightarrow W^{0,\,p}_{\alpha-s,\,\beta}(\mathbb{R}^N).$$

When $N/p + \alpha = j \in \{\sigma, \dots, \sigma + [s] - 1\}$, then we have:

$$W^{s,\,p}_{\alpha,\,\beta} \hookrightarrow \cdots \hookrightarrow W^{s-j+1,\,p}_{\alpha-j+1,\,\beta} \hookrightarrow W^{s-j,\,p}_{\alpha-j,\,\beta-1} \hookrightarrow \cdots \hookrightarrow W^{\sigma,\,p}_{\alpha-[s],\,\beta-1},$$
$$W^{s,\,p}_{\alpha,\,\beta} \hookrightarrow W^{[s],\,p}_{\alpha+[s]-s,\,\beta} \hookrightarrow \cdots \hookrightarrow W^{[s]-j+1,\,p}_{\alpha-\sigma-j+1,\,\beta} \hookrightarrow W^{[s]-j,\,p}_{\alpha-\sigma-j,\,\beta-1} \hookrightarrow \cdots \hookrightarrow W^{0,\,p}_{\alpha-s,\,\beta-1}.$$

If u is a function on $\mathbb{R}^N_+$, we denote its trace of order j on the hyperplane Γ by:

$$\forall j \in \mathbb{N}, \quad \gamma_j u : x' \in \mathbb{R}^{N-1} \longmapsto \partial^j_N u(x', 0).$$

Let's recall the following trace lemma due to Hanouzet (see [11]) for the functions of $H^{m,\,p}_{\alpha}(\mathbb{R}^N_+)$ and extended by Amrouche-Nečasová (see [5]) to the class of weighted Sobolev spaces $W^{m,\,p}_{\alpha}(\mathbb{R}^N_+)$:

Lemma 2.2. *For any integer $m \ge 1$ and real number α, the mapping*

$$\gamma = (\gamma_0, \gamma_1, \dots, \gamma_{m-1}) :\ \mathscr{D}\big(\overline{\mathbb{R}^N_+}\big) \longrightarrow \prod_{j=0}^{m-1} \mathscr{D}(\mathbb{R}^{N-1}),$$

can be extended to a linear continuous mapping, still denoted by γ,

$$\gamma :\ W^{m,\,p}_{\alpha}(\mathbb{R}^N_+) \longrightarrow \prod_{j=0}^{m-1} W^{m-j-1/p,\,p}_{\alpha}(\mathbb{R}^{N-1}).$$

Moreover γ is surjective and $\mathrm{Ker}\gamma = \overset{\circ}{W}{}^{m,\,p}_{\alpha}(\mathbb{R}^N_+)$.

2.4. **Previous results on Δ^2.** First note that if we replace $\mathbb{R}^N_+$ by a bounded open set Ω of $\mathbb{R}^N$ in problem $(\mathcal{P})$, the natural functional framework in which we find the solution u is the one of classical Sobolev spaces. But as the things are well made, in this case the weighted spaces coincide with the classical spaces. Concerning the regularity results on bounded domains, we can refer to Luckhaus (see [12]) for Dirichlet boundary conditions and to the Appendix B in the paper by R. van der Vorst (see [15]) for boundary conditions on $(u,\, \Delta u)$ that we shall see in Section 5.

In unbounded domains, we must recall the mains results of our previous paper on the biharmonic equation (see [7]), which are the start point of this study. First, let's give the global result for the biharmonic operator in the whole space:

Theorem 2.3 (Amrouche-Raudin [7]). *Let $l \in \mathbb{Z}$ and $m \in \mathbb{N}$ and assume that*

$$\frac{N}{p'} \notin \{1, \dots, l + \min\{m, 2\}\} \quad \text{and} \quad \frac{N}{p} \notin \{1, \dots, -l - m\}, \tag{6}$$

then the biharmonic operator

$$\Delta^2 : W^{m+2,\,p}_{m+l}(\mathbb{R}^N)/\mathscr{P}^{\Delta^2}_{[2-l-N/p]} \longrightarrow W^{m-2,\,p}_{m+l}(\mathbb{R}^N) \perp \mathscr{P}^{\Delta^2}_{[2+l-N/p']}$$

is an isomorphism.

About the half-space, the first point concerns the kernel $\mathscr{K}^m$ of the operator $(\Delta^2, \gamma_0, \gamma_1)$ in $W^{m+2,\,p}_{m+l}(\mathbb{R}^N_+)$. For any $q \in \mathbb{Z}$, we introduce the space $\mathscr{B}_q$ as a subspace of $\mathscr{P}^{\Delta^2}_q$:

$$\mathscr{B}_q = \left\{ u \in \mathscr{P}^{\Delta^2}_q;\ u = \partial_N u = 0 \text{ on } \Gamma \right\}.$$

Then we define the two operators Π_D and Π_N by:

$$\forall r \in \mathscr{A}^{\Delta}_k, \quad \Pi_D r = \frac{1}{2} \int_0^{x_N} t\, r(x', t)\, \mathrm{d}t,$$
$$\forall s \in \mathscr{N}^{\Delta}_k, \quad \Pi_N s = \frac{1}{2}\, x_N \int_0^{x_N} s(x', t)\, \mathrm{d}t,$$

satisfying the following properties:

$$\forall r \in \mathscr{A}^{\Delta}_k, \qquad \Delta \Pi_D r = r \ \text{ in } \mathbb{R}^N_+, \qquad \Pi_D r = \partial_N \Pi_D r = 0 \ \text{ on } \Gamma,$$
$$\forall s \in \mathscr{N}^{\Delta}_k, \qquad \Delta \Pi_N s = s \ \text{ in } \mathbb{R}^N_+, \qquad \Pi_N s = \partial_N \Pi_N s = 0 \ \text{ on } \Gamma.$$

So we have the following characterizations for this kernel (see [7], Lemma 4.4): let $l \in \mathbb{Z}$ and $m \in \mathbb{N}$ and assume that

$$\frac{N}{p} \notin \{1, \dots, -l - m\}, \tag{7}$$

then

$$\mathscr{K}^m = \mathscr{B}_{[2-l-N/p]} = \Pi_D \mathscr{A}^{\Delta}_{[-l-N/p]} \oplus \Pi_N \mathscr{N}^{\Delta}_{[-l-N/p]}. \tag{8}$$

Everywhere in the sequel, we shall denote this kernel by $\mathscr{B}_{[2-l-N/p]}$. Now, the fundamental result that we showed and which is the pivot of the present work, concerns the generalized solutions to problem $(\mathcal{P})$:

Theorem 2.4 (Amrouche-Raudin [7]). *Let $l \in \mathbb{Z}$ and assume that*

$$\frac{N}{p'} \notin \{1, \dots, l\} \quad \text{and} \quad \frac{N}{p} \notin \{1, \dots, -l\}. \tag{9}$$

For any $f \in W^{-2,\,p}_l(\mathbb{R}^N_+)$, $g_0 \in W^{2-1/p,\,p}_l(\Gamma)$ and $g_1 \in W^{1-1/p,\,p}_l(\Gamma)$ satisfying the compatibility condition

$$\forall \varphi \in \mathscr{B}_{[2+l-N/p']},$$
$$\langle f, \varphi \rangle_{W^{-2,\,p}_l(\mathbb{R}^N_+) \times \mathring{W}^{2,\,p'}_{-l}(\mathbb{R}^N_+)} + \langle g_1, \Delta\varphi \rangle_\Gamma - \langle g_0, \partial_N \Delta \varphi \rangle_\Gamma = 0, \tag{10}$$

problem $(\mathcal{P})$ has a solution $u \in W^{2,\,p}_l(\mathbb{R}^N_+)$, unique up to an element of $\mathscr{B}_{[2-l-N/p]}$, with the estimate

$$\inf_{q \in \mathscr{B}_{[2-l-N/p]}} \|u + q\|_{W^{2,\,p}_l(\mathbb{R}^N_+)} \ \leq$$
$$C \left(\|f\|_{W^{-2,\,p}_l(\mathbb{R}^N_+)} + \|g_0\|_{W^{2-1/p,\,p}_l(\Gamma)} + \|g_1\|_{W^{1-1/p,\,p}_l(\Gamma)} \right).$$

We also established a global result for the homogeneous problem in the half-space:

$$(\mathcal{P}^0)\quad \begin{cases} \Delta^2 u = 0 & \text{in } \mathbb{R}^N_+, \\ u = g_0 & \text{on } \Gamma, \\ \partial_N u = g_1 & \text{on } \Gamma. \end{cases}$$

Proposition 2.5 (Amrouche-Raudin [7])**.** *Let $l \in \mathbb{Z}$ and $m \in \mathbb{N}$ and assume that*

$$\frac{N}{p'} \notin \{1, \dots, l\} \quad \text{and} \quad \frac{N}{p} \notin \{1, \dots, -l-m\}. \tag{11}$$

For any $g_0 \in W^{m+2-1/p,\,p}_{m+l}(\Gamma)$ and $g_1 \in W^{m+1-1/p,\,p}_{m+l}(\Gamma)$, satisfying the compatibility condition

$$\forall \varphi \in \mathscr{B}_{[2+l-N/p']}\ , \quad \langle g_1, \Delta\varphi\rangle_\Gamma - \langle g_0, \partial_N \Delta\varphi\rangle_\Gamma = 0, \tag{12}$$

problem $(\mathcal{P}^0)$ admits a solution $u \in W^{m+2,\,p}_{m+l}(\mathbb{R}^N_+)$, unique up to an element of $\mathscr{B}_{[2-l-N/p]}$, with the estimate

$$\inf_{q\in\mathscr{B}_{[2-l-N/p]}} \|u+q\|_{W^{m+2,\,p}_{m+l}(\mathbb{R}^N_+)} \ \le\ C\left(\|g_0\|_{W^{m+2-1/p,\,p}_{m+l}(\Gamma)} + \|g_1\|_{W^{m+1-1/p,\,p}_{m+l}(\Gamma)}\right).$$

3. **Weak solutions, strong solutions and regularity.** The first part of the present work consists in the study of solutions to problem $(\mathcal{P})$ for more regular data. We shall now establish a global result which extends our previous result to different types of data.

3.1. **Extension of Theorem 2.4.**

Theorem 3.1. *Let $l \in \mathbb{Z}$ and $m \in \mathbb{N}$. Under hypothesis (6), for any $f \in W^{m-2,\,p}_{m+l}(\mathbb{R}^N_+)$, $g_0 \in W^{m+2-1/p,\,p}_{m+l}(\Gamma)$ and $g_1 \in W^{m+1-1/p,\,p}_{m+l}(\Gamma)$ satisfying the compatibility condition (10), problem $(\mathcal{P})$ has a solution $u \in W^{m+2,\,p}_{m+l}(\mathbb{R}^N_+)$, unique up to an element of $\mathscr{B}_{[2-l-N/p]}$, with the estimate*

$$\inf_{q\in\mathscr{B}_{[2-l-N/p]}} \|u+q\|_{W^{m+2,\,p}_{m+l}(\mathbb{R}^N_+)}$$
$$\le\ C\left(\|f\|_{W^{m-2,\,p}_{m+l}(\mathbb{R}^N_+)} + \|g_0\|_{W^{m+2-1/p,\,p}_{m+l}(\Gamma)} + \|g_1\|_{W^{m+1-1/p,\,p}_{m+l}(\Gamma)}\right).$$

Proof. Note at first that if $m = 0$, we find Theorem 2.4. The kernel has been globally characterized by (8) (see [7], Lemma 4.4). Let's recall that this kernel is reduced to $\{0\}$ if $l \ge 0$ and symmetrically the compatibility condition (10) vanishes if $l \le 0$. Moreover under hypothesis (6), the imbeddings $W^{m-2,\,p}_{m+l}(\mathbb{R}^N_+) \hookrightarrow W^{-2,\,p}_{l}(\mathbb{R}^N_+)$, $W^{m+2-1/p,\,p}_{m+l}(\Gamma) \hookrightarrow W^{2-1/p,\,p}_{l}(\Gamma)$ and $W^{m+1-1/p,\,p}_{m+l}(\Gamma) \hookrightarrow W^{1-1/p,\,p}_{l}(\Gamma)$ hold for all $l \ge 1$, hence the necessity of (10) for any $m \in \mathbb{N}$. So it suffices to show the existence of a solution. By Lemma 2.2, we can consider the problem with homogeneous boundary conditions

$$(\mathcal{P}^\star)\quad \begin{cases} \Delta^2 u = f & \text{in } \mathbb{R}^N_+, \\ u = 0 & \text{on } \Gamma, \\ \partial_N u = 0 & \text{on } \Gamma, \end{cases}$$

with $f \perp \mathscr{B}_{[2+l-N/p']}$. This orthogonality condition naturally corresponds to the compatibility condition (10). For more details on these questions, see [7].

Let's now give the plan of the proof of the existence for $m \ge 1$:

i) If $l \le -2$, we establish globally the existence of a solution.
ii) If $l \ge -1$ and $m = 1$, we show that by a direct construction.

iii) If $l \geq -1$ and $m \geq 1$, we show that by induction on m from the previous case ($m = 1$), thanks to a regularity argument.

i) Assume that $l \leq -2$. Then hypothesis (6) is reduced to (7). Let $f \in W^{m-2,\,p}_{m+l}(\mathbb{R}^N_+)$. Let's first suppose $m \geq 2$. We know that there exists a continuous linear extension operator from $W^{m-2,\,p}_{m+l}(\mathbb{R}^N_+)$ to $W^{m-2,\,p}_{m+l}(\mathbb{R}^N)$ and thus $\tilde{f} \in W^{m-2,\,p}_{m+l}(\mathbb{R}^N)$ which extends f to $\mathbb{R}^N$. Then we use Theorem 2.3 to obtain $\tilde{z} \in W^{m+2,\,p}_{m+l}(\mathbb{R}^N)$ such that $\tilde{f} = \Delta^2\tilde{z}$ in $\mathbb{R}^N$ and thus $f = \Delta^2 z$ in $\mathbb{R}^N_+$, with $z = \tilde{z}|_{\mathbb{R}^N_+} \in W^{m+2,\,p}_{m+l}(\mathbb{R}^N_+)$. Then Proposition 2.5 asserts the existence of a solution $v \in W^{m+2,\,p}_{m+l}(\mathbb{R}^N_+)$ to the homogeneous problem

$$\Delta^2 v = 0 \ \text{ in } \mathbb{R}^N_+, \qquad v = z \ \text{ and } \ \partial_N v = \partial_N z \ \text{ on } \Gamma,$$

with $z|_\Gamma \in W^{m+2-1/p,\,p}_{m+l}(\Gamma)$ and $\partial_N z|_\Gamma \in W^{m+1-1/p,\,p}_{m+l}(\Gamma)$. We remark again that $\mathscr{B}_{[2+l-N/p']} = \{0\}$ because $l \leq 0$, thus there is no compatibility condition. Then the function $u = z - v$ answer to problem $(\mathcal{P}^\star)$ in this case.

Let's now consider the case $m = 1$, *i.e.* $f \in W^{-1,\,p}_{l+1}(\mathbb{R}^N_+)$. As we did for the distributions of $W^{-2,\,p}_{l}(\mathbb{R}^N_+)$ in [7], we can show that there exists $F = (F_i)_{1\leq i\leq N} \in W^{0,\,p}_{l+1}(\mathbb{R}^N_+)^N$ such that $f = \operatorname{div} F = \displaystyle\sum_{i=1}^N \partial_i F_i$, with the estimate $\displaystyle\sum_{i=1}^N \|F_i\|_{W^{0,\,p}_{l+1}(\mathbb{R}^N_+)} \leq C\,\|f\|_{W^{-1,\,p}_{l+1}(\mathbb{R}^N_+)}$. Let's denote by $\tilde{F} \in W^{0,\,p}_{l+1}(\mathbb{R}^N)^N$ the extension by 0 of F to $\mathbb{R}^N$. Since $\dfrac{N}{p} \notin \{1, \dots, -l-1\}$, by Theorem 2.3, there exists $\tilde{\Psi} \in W^{4,\,p}_{l+1}(\mathbb{R}^N)^N$ such that $\tilde{F} = \Delta^2\tilde{\Psi}$ in $\mathbb{R}^N$. Setting $\tilde{\psi} = \operatorname{div}\tilde{\Psi}$ and $\psi = \tilde{\psi}|_{\mathbb{R}^N_+}$, so we have $\psi \in W^{3,\,p}_{l+1}(\mathbb{R}^N_+)$ and by Proposition 2.5, there exists $v \in W^{3,\,p}_{l+1}(\mathbb{R}^N_+)$ such that

$$\Delta^2 v = 0 \ \text{ in } \mathbb{R}^N_+, \qquad v = \psi \ \text{ and } \ \partial_N v = \partial_N \psi \ \text{ on } \Gamma.$$

The function $u = \psi - v \in W^{3,\,p}_{l+1}(\mathbb{R}^N_+)$ is a solution to Problem $(\mathcal{P}^\star)$ in this case.

ii) Assume that $l \geq -1$ and $m = 1$. Note that the distribution $f \in W^{-1,\,p}_{l+1}(\mathbb{R}^N_+)$ defines the linear functional L on $\mathscr{A}^\Delta_{[l+2-N/p']}$ by

$$L:\ r \longmapsto \langle f, r\rangle_{W^{-1,\,p}_{l+1}(\mathbb{R}^N_+)\times \mathring{W}^{1,\,p'}_{-l-1}(\mathbb{R}^N_+)},$$

and introduce the inner product Φ on $\mathscr{A}^\Delta_{[l+2-N/p']} \times \mathscr{A}^\Delta_{[l+2-N/p']}$ defined by

$$\Phi:\ (\mu, r) \longmapsto \int_\Gamma \varrho'^{-2l-1-N/p+N/p'}\, \partial_N\mu\, \partial_N r\, \mathrm{d}x'.$$

Note that

$$r \in \mathscr{A}^\Delta_{[l+2-N/p']} \implies \varrho'^{-l-1+1/p'}\, \partial_N r \in L^{p'}(\Gamma),$$

and

$$\mu \in \mathscr{A}^\Delta_{[l+2-N/p']} = \mathscr{A}^\Delta_{[l+2-N/p'+N/p-N/p]} \implies \varrho'^{-l-1-N/p+N/p'+1/p}\, \partial_N\mu \in L^p(\Gamma).$$

Thus, thanks to Hölder inequality, Φ is well-defined. Then, there exists an unique $\mu \in \mathscr{A}^\Delta_{[l+2-N/p']}$ such that

$$\forall r \in \mathscr{A}^\Delta_{[l+2-N/p']},\ L(r) = \Phi(\mu, r),$$

i.e.

$$\forall r \in \mathscr{A}^{\Delta}_{[l+2-N/p']},$$
$$\langle f, r\rangle_{W^{-1,\,p}_{l+1}(\mathbb{R}^N_+)\times \mathring{W}^{1,\,p'}_{-l-1}(\mathbb{R}^N_+)} = \int_\Gamma \varrho'^{-2l-1-N/p+N/p'}\,\partial_N\mu\,\partial_N r\,\mathrm{d}x'. \tag{13}$$

Let's set $\xi_0 = \varrho'^{-2l-1-N/p+N/p'}\,\partial_N\mu$, then we have $\xi_0 \in W^{1-1/p,\,p}_{l+1}(\Gamma)$ and (13) becomes

$$\begin{aligned}&\forall r \in \mathscr{A}^{\Delta}_{[l+2-N/p']},\\ &\langle f, r\rangle_{W^{-1,\,p}_{l+1}(\mathbb{R}^N_+)\times \mathring{W}^{1,\,p'}_{-l-1}(\mathbb{R}^N_+)} = \langle \xi_0, \partial_N r\rangle_{W^{1-1/p,\,p}_{l+1}(\Gamma)\times W^{-1/p',\,p'}_{-l-1}(\Gamma)}.\end{aligned} \tag{14}$$

That is precisely the compatibility condition of the Dirichlet problem

$$(\mathcal{Q})\quad \begin{cases}\Delta\xi = f & \text{in } \mathbb{R}^N_+,\\ \xi = \xi_0 & \text{on } \Gamma,\end{cases}$$

Thus (see [5], Theorem 3.1), $(\mathcal{Q})$ admits a solution $\xi \in W^{1,\,p}_{l+1}(\mathbb{R}^N_+)$ under hypothesis (6). Here we shall use the characterization (8):

$$\mathscr{B}_{[2+l-N/p']} = \Pi_D\mathscr{A}^{\Delta}_{[l-N/p']} \oplus \Pi_N\mathscr{N}^{\Delta}_{[l-N/p']}.$$

Since $f \perp \mathscr{B}_{[2+l-N/p']}$, we have

$$\forall r \in \mathscr{A}^{\Delta}_{[l-N/p']},\quad \langle \Delta\xi, \Pi_D r\rangle_{W^{-1,\,p}_{l+1}(\mathbb{R}^N_+)\times \mathring{W}^{1,\,p'}_{-l-1}(\mathbb{R}^N_+)} = \langle f, \Pi_D r\rangle = 0.$$

By a Green formula, we can deduce that

$$\forall r \in \mathscr{A}^{\Delta}_{[l-N/p']},\quad \langle \xi, \Delta\Pi_D r\rangle_{W^{-1,\,p}_{l-1}(\mathbb{R}^N_+)\times \mathring{W}^{1,\,p'}_{-l+1}(\mathbb{R}^N_+)} = 0,$$

because $W^{1,\,p}_{l+1}(\mathbb{R}^N_+) \hookrightarrow W^{-1,\,p}_{l-1}(\mathbb{R}^N_+)$ unless $\dfrac{N}{p} = -l$ or $\dfrac{N}{p'} = l$. The second possibility is excluded by (6), and since $l \geq -1$, the only problematic case is $l = -1$. But then $[l - N/p'] < 0$ and the condition vanishes. Thus, we have

$$\forall r \in \mathscr{A}^{\Delta}_{[l-N/p']},\quad \langle \xi, r\rangle_{W^{-1,\,p}_{l-1}(\mathbb{R}^N_+)\times \mathring{W}^{1,\,p'}_{-l+1}(\mathbb{R}^N_+)} = 0,$$

which is the compatibility condition for the Dirichlet problem

$$(\mathcal{R}^\star)\quad \begin{cases}\Delta\vartheta = \xi & \text{in } \mathbb{R}^N_+,\\ \vartheta = 0 & \text{on } \Gamma.\end{cases}$$

Thus (see [5], Corollary 3.4), $(\mathcal{R}^\star)$ admits a solution $\vartheta \in W^{3,\,p}_{l+1}(\mathbb{R}^N_+)$ under hypothesis (6). Similarly we have

$$\forall s \in \mathscr{N}^{\Delta}_{[l-N/p']},\quad \langle \Delta\xi, \Pi_N s\rangle_{W^{-1,\,p}_{l+1}(\mathbb{R}^N_+)\times \mathring{W}^{1,\,p'}_{-l-1}(\mathbb{R}^N_+)} = \langle f, \Pi_N s\rangle = 0,$$

therefore as previously, we have

$$\forall s \in \mathscr{N}^{\Delta}_{[l-N/p']},\quad \langle \xi, s\rangle_{W^{-1,\,p}_{l-1}(\mathbb{R}^N_+)\times \mathring{W}^{1,\,p'}_{-l+1}(\mathbb{R}^N_+)} = 0,$$

which is the compatibility condition for Neumann problem

$$(\mathcal{S}^\star)\quad \begin{cases}\Delta\zeta = \xi & \text{in } \mathbb{R}^N_+,\\ \partial_N\zeta = 0 & \text{on } \Gamma.\end{cases}$$

As for problem $(\mathcal{R}^\star)$, we can show that $(\mathcal{S}^\star)$ admits a solution $\zeta \in W^{3,\,p}_{l+1}(\mathbb{R}^N_+)$ under hypothesis (6) (see [7], Theorem 2.8). Then the function defined by

$$u = x_N\,\partial_N(\zeta - \vartheta) + \vartheta \tag{15}$$

is a solution to $(\mathcal{P}^\star)$. It remains to show that $u \in W^{3,\,p}_{l+1}(\mathbb{R}^N_+)$.

If $\dfrac{N}{p} \neq -l$, then we have the imbedding $W^{3,\,p}_{l+1}(\mathbb{R}^N_+) \hookrightarrow W^{2,\,p}_{l}(\mathbb{R}^N_+)$ therefore $u \in W^{2,\,p}_{l}(\mathbb{R}^N_+)$. Moreover u satisfies the system

$$(\mathcal{T}^\star)\ \begin{cases} \Delta u = 2\,\partial_N^2(\zeta - \vartheta) + \xi & \text{in } \mathbb{R}^N_+, \\ u = 0 & \text{on } \Gamma, \end{cases}$$

with $2\,\partial_N^2(\zeta - \vartheta) + \xi \in W^{1,\,p}_{l+1}(\mathbb{R}^N_+)$. As for $(\mathcal{R}^\star)$, we know that problem $(\mathcal{T}^\star)$ has a solution $y \in W^{3,\,p}_{l+1}(\mathbb{R}^N_+)$. We can deduce that $u - y \in \mathscr{A}^{\Delta}_{[2-l-N/p]} \subset W^{3,\,p}_{l+1}(\mathbb{R}^N_+)$, *i.e.* $u \in W^{3,\,p}_{l+1}(\mathbb{R}^N_+)$.

If $\dfrac{N}{p} = -l$, then we have necessary $l = -1$ and moreover the imbedding $W^{3,\,p}_{0}(\mathbb{R}^N_+) \hookrightarrow W^{2,\,p}_{-1,\,-1}(\mathbb{R}^N_+)$ with $l - \dfrac{N}{p'} = l + \dfrac{N}{p} - N = -N < 0$, therefore no compatibility condition for $(\mathcal{T}^\star)$. So we can still deduce that $u \in W^{3,\,p}_{l+1}(\mathbb{R}^N_+)$.

iii) Assume that $l \geq -1$ and $m \geq 1$. Consider $f \in W^{m-2,\,p}_{m+l}(\mathbb{R}^N_+) \perp \mathscr{B}_{[2+l-N/p']}$. Remark at first that we have the imbedding

$$W^{m-2,\,p}_{m+l}(\mathbb{R}^N_+) \hookrightarrow W^{-1,\,p}_{l+1}(\mathbb{R}^N_+) \quad \text{if} \quad \frac{N}{p'} \neq l+2 \ \text{ or } \ m = 1.$$

Then, thanks to the previous step, there exists a solution $u \in W^{3,\,p}_{l+1}(\mathbb{R}^N_+)$ to problem $(\mathcal{P}^\star)$. Let's prove by induction that, under hypothesis (6),

$$f \in W^{m-2,\,p}_{m+l}(\mathbb{R}^N_+) \implies u \in W^{m+2,\,p}_{m+l}(\mathbb{R}^N_+). \tag{16}$$

For $m = 1$, (16) is true. Assume that (16) is true for $1, 2, \ldots, m$ and suppose that $f \in W^{m-1,\,p}_{m+1+l}(\mathbb{R}^N_+)$. Let's prove that $u \in W^{m+3,\,p}_{m+1+l}(\mathbb{R}^N_+)$. Let's first observe that $W^{m-1,\,p}_{m+1+l}(\mathbb{R}^N_+) \hookrightarrow W^{m-2,\,p}_{m+l}(\mathbb{R}^N_+)$, hence u belongs to $W^{m+2,\,p}_{m+l}(\mathbb{R}^N_+)$ thanks to the induction hypothesis. Now, for any $i \in \{1, \ldots, N-1\}$,

$$\Delta(\varrho\,\partial_i u) = \varrho\,\Delta\partial_i u + \frac{2}{\varrho}\,x.\nabla\partial_i u + \left(\frac{N-1}{\varrho} + \frac{1}{\varrho^3}\right)\partial_i u.$$

Then let's set $v_i = \dfrac{2}{\varrho}\,x.\nabla\partial_i u + \left(\dfrac{N-1}{\varrho} + \dfrac{1}{\varrho^3}\right)\partial_i u$. We can remark that $v_i \in W^{m,\,p}_{m+l}(\mathbb{R}^N_+)$ and moreover we can write

$$\Delta^2(\varrho\,\partial_i u) = \Delta(\varrho\,\Delta\partial_i u) + \Delta v_i,$$

with $\Delta v_i \in W^{m-2,\,p}_{m+l}(\mathbb{R}^N_+)$. It remains to see the first term, *i.e.*

$$\Delta(\varrho\,\partial_i\Delta u) = \varrho\,\partial_i f + \frac{2}{\varrho}\,x.\nabla\partial_i\Delta u + \left(\frac{N-1}{\varrho} + \frac{1}{\varrho^3}\right)\partial_i\Delta u.$$

We can see that $\Delta(\varrho\,\partial_i\Delta u) \in W^{m-2,\,p}_{m+l}(\mathbb{R}^N_+)$, hence $\Delta^2(\varrho\,\partial_i u) \in W^{m-2,\,p}_{m+l}(\mathbb{R}^N_+)$. Let's set $z_i = \varrho\,\partial_i u$ and $f_i = \Delta^2 z_i \in W^{m-2,\,p}_{m+l}(\mathbb{R}^N_+)$. A priori, we only have $z_i \in W^{m+1,\,p}_{m-1+l}(\mathbb{R}^N_+)$. However, we know that $\gamma_0 u = \gamma_1 u = 0$, then we can deduce that

$$\gamma_0 z_i = (\varrho\,\partial_i u)|_\Gamma = 0 \ \text{ and } \ \gamma_1 z_i = (\partial_N \varrho\,\partial_i u + \varrho\,\partial^2_{iN} u)|_\Gamma = 0 \ \text{ since } i \neq N.$$

Therefore

$$\Delta^2 z_i = f_i \ \text{ in } \mathbb{R}^N_+, \qquad z_i = \partial_N z_i = 0 \ \text{ on } \Gamma,$$

with $f_i \in W^{m-2,\,p}_{m+l}(\mathbb{R}^N_+) \hookrightarrow W^{-2,\,p}_{l}(\mathbb{R}^N_+)$ under hypothesis (6). Moreover, thanks to the Green formula, we have for any $\varphi \in \mathscr{B}_{[2+l-N/p']}$:

$$\left\langle \Delta^2 z_i, \varphi \right\rangle_{W^{-2,\,p}_{l}(\mathbb{R}^N_+)\times \mathring{W}^{2,\,p'}_{-l}(\mathbb{R}^N_+)} = \left\langle z_i, \Delta^2 \varphi \right\rangle_{\mathring{W}^{2,\,p}_{l}(\mathbb{R}^N_+)\times W^{-2,\,p'}_{-l}(\mathbb{R}^N_+)} = 0.$$

So the orthogonality condition $f_i \perp \mathscr{B}_{[2+l-N/p']}$ is satisfied for the problem

$$(\mathcal{Q}_i) \quad \begin{cases} \Delta^2 \zeta_i = f_i & \text{in } \mathbb{R}^N_+, \\ \zeta_i = 0 & \text{on } \Gamma, \\ \partial_N \zeta_i = 0 & \text{on } \Gamma, \end{cases}$$

which admits, by the induction hypothesis, a solution $\zeta_i \in W^{m+2,\,p}_{m+l}(\mathbb{R}^N_+)$, unique up to an element of $\mathscr{B}_{[2-l-N/p]}$. Thus $z_i - \zeta_i \in \mathscr{B}_{[2-l-N/p]}$, hence we can deduce that $z_i \in W^{m+2,\,p}_{m+l}(\mathbb{R}^N_+)$. Since $z_i = \varrho\, \partial_i u$, that implies

$$\forall i \in \{1, \dots, N-1\}, \quad \partial_i u \in W^{m+2,\,p}_{m+1+l}(\mathbb{R}^N_+) \tag{17}$$

and consequently, for any $(i,j,k) \in \{1,\dots,N\}^2 \times \{1,\dots,N-1\}$,

$$\partial^3_{ijk}(\partial_N u) = \partial^3_{ijN}(\partial_k u) \ \in W^{m-1,\,p}_{m+1+l}(\mathbb{R}^N_+). \tag{18}$$

Furthermore, (17) gives us

$$\forall (i,j) \in \{1,\dots,N\} \times \{1,\dots,N-1\}, \quad \partial_i^2 \partial_j^2 u \in W^{m-1,\,p}_{m+1+l}(\mathbb{R}^N_+),$$

which implies

$$\partial^4_N u = f - \sum_{\substack{i,j=1 \\ (i,j)\neq(N,N)}}^{N} \partial_i^2 \partial_j^2 u \ \in W^{m-1,\,p}_{m+1+l}(\mathbb{R}^N_+). \tag{19}$$

Then combining (18) and (19), we obtain that $\nabla^3(\partial_N u) \in W^{m-1,\,p}_{m+1+l}(\mathbb{R}^N_+)^{N^3}$ and knowing that $\partial_N u \in W^{m+1,\,p}_{m+l}(\mathbb{R}^N_+)$, we can deduce that $\partial_N u \in W^{m+2,\,p}_{m+1+l}(\mathbb{R}^N_+)$, because $m \geq 1$. Adding this last point to (17), we have $\nabla u \in W^{m+2,\,p}_{m+1+l}(\mathbb{R}^N_+)^N$, hence we can conclude that $u \in W^{m+3,\,p}_{m+1+l}(\mathbb{R}^N_+)$. □

3.2. **Panorama of basic cases.** The purpose of this part is to extract the basic cases included in Theorem 3.1. We give them for the lifted problem $(\mathcal{P}^\star)$. There is no orthogonality condition in these cases because $l \in \{-2,-1,0\}$, hence $\mathscr{B}_{[2+l-N/p']} = \{0\}$. For $m \geq 3$, we introduce the notation

$$\overset{\star}{W}{}^{m,\,p}_{l}(\mathbb{R}^N_+) = \left\{ u \in W^{m,\,p}_{l}(\mathbb{R}^N_+);\ u = \partial_N u = 0 \text{ on } \Gamma \right\}.$$

Corollary 3.2. *The following biharmonic operators are isomorphisms:*

i) For $l = 0$

$$\begin{array}{lrcll} \Delta^2: & \mathring{W}^{2,\,p}_{0}(\mathbb{R}^N_+) & \longrightarrow & W^{-2,\,p}_{0}(\mathbb{R}^N_+). & \\ \Delta^2: & \overset{\star}{W}{}^{3,\,p}_{1}(\mathbb{R}^N_+) & \longrightarrow & W^{-1,\,p}_{1}(\mathbb{R}^N_+), & \text{if } N/p' \neq 1. \\ \Delta^2: & \overset{\star}{W}{}^{4,\,p}_{2}(\mathbb{R}^N_+) & \longrightarrow & W^{0,\,p}_{2}(\mathbb{R}^N_+), & \text{if } N/p' \notin \{1,2\}. \end{array}$$

ii) For $l=-1$

$$\begin{aligned}
\Delta^2:\ & \overset{\circ}{W}{}^{2,\,p}_{-1}(\mathbb{R}^N_+)/\mathscr{B}_{[3-N/p]} \longrightarrow W^{-2,\,p}_{-1}(\mathbb{R}^N_+), \quad \textit{if } N/p\neq 1.\\
\Delta^2:\ & \overset{\star}{W}{}^{3,\,p}_{0}(\mathbb{R}^N_+)/\mathscr{B}_{[3-N/p]} \longrightarrow W^{-1,\,p}_{0}(\mathbb{R}^N_+).\\
\Delta^2:\ & \overset{\star}{W}{}^{4,\,p}_{1}(\mathbb{R}^N_+)/\mathscr{B}_{[3-N/p]} \longrightarrow W^{0,\,p}_{1}(\mathbb{R}^N_+), \quad \textit{if } N/p'\neq 1.
\end{aligned}$$

iii) For $l=-2$

$$\begin{aligned}
\Delta^2:\ & \overset{\circ}{W}{}^{2,\,p}_{-2}(\mathbb{R}^N_+)/\mathscr{B}_{[4-N/p]} \longrightarrow W^{-2,\,p}_{-2}(\mathbb{R}^N_+), \quad \textit{if } N/p\notin\{1,2\}.\\
\Delta^2:\ & \overset{\star}{W}{}^{3,\,p}_{-1}(\mathbb{R}^N_+)/\mathscr{B}_{[4-N/p]} \longrightarrow W^{-1,\,p}_{-1}(\mathbb{R}^N_+), \quad \textit{if } N/p\neq 1.\\
\Delta^2:\ & \overset{\star}{W}{}^{4,\,p}_{0}(\mathbb{R}^N_+)/\mathscr{B}_{[4-N/p]} \longrightarrow L^p(\mathbb{R}^N_+).
\end{aligned}$$

Remark 3.3. Note that we have without any critical value, the isomorphism $\Delta^2:\overset{\star}{W}{}^{3,\,p}_{0}(\mathbb{R}^N_+)/\mathscr{B}_{[3-N/p]}\longrightarrow W^{-1,\,p}_{0}(\mathbb{R}^N_+)$. On the other hand, we have the isomorphism $\Delta^2:\ W^{3,\,p}_{0}(\mathbb{R}^N_+)\cap\overset{\circ}{W}{}^{2,\,p}_{-1}(\mathbb{R}^N_+)/\mathscr{B}_{[3-N/p]}\longrightarrow W^{-1,\,p}_{0}(\mathbb{R}^N_+)$ only if $N/p\neq 1$, which is necessary for the imbedding $W^{3,\,p}_{0}(\mathbb{R}^N_+)\hookrightarrow W^{2,\,p}_{-1}(\mathbb{R}^N_+)$. Hence the specificity of spaces $\overset{\star}{W}{}^{m,\,p}_{l}(\mathbb{R}^N_+)$. □

3.3. **What is new?** Recall the Boulmezaoud theorem on the biharmonic problem (see [9]), using the spaces $H^{m,\,p}_{\alpha}(\mathbb{R}^N_+)$ instead of $W^{m,\,p}_{\alpha}(\mathbb{R}^N_+)$ (see Remark 2.1):

Theorem (Boulmezaoud)**.** *Let $l\in\mathbb{Z}$ and $m\in\mathbb{N}$ and assume that*

$$\frac{N}{2}\notin\{1,\dots,|l|+2\}.$$

For any $f\in H^{m-1,\,2}_{m+l+1}(\mathbb{R}^N_+)$, $g_0\in H^{m+5/2,\,2}_{m+l+1}(\Gamma)$ and $g_1\in H^{m+3/2,\,2}_{m+l+1}(\Gamma)$ satisfying the compatibility condition (10), problem $(\mathcal{P})$ has a solution $u\in H^{m+3,\,2}_{m+l+1}(\mathbb{R}^N_+)$, unique up to an element of $\mathscr{B}_{[2-l-N/2]}$ and this solution continuously depends on the data with respect to the quotient norm.

The most important point is about the regularity of data. In Theorem 3.1, we can take $f\in W^{-2,\,2}_{l}(\mathbb{R}^N_+)$, $g_0\in W^{3/2,\,2}_{l}(\Gamma)$ and $g_1\in W^{1/2,\,2}_{l}(\Gamma)$, whereas the lower level in Boulmezaoud theorem is for $f\in H^{-1,\,2}_{l+1}(\mathbb{R}^N_+)$, $g_0\in H^{5/2,\,2}_{l+1}(\Gamma)$ and $g_1\in H^{3/2,\,2}_{l+1}(\Gamma)$. The second point is about critical values which appear for all the even dimensions. Particularly for the dimensions $N=2$ or $N=4$, the Boulmezaoud theorem unfortunately does not give any answer to problem $(\mathcal{P})$, whereas we can see in Corollary 3.2 that Theorem 3.1 gives solutions with f in $W^{-2,\,2}_{0}(\mathbb{R}^N_+)$, $W^{-1,\,2}_{0}(\mathbb{R}^N_+)$ or $L^2(\mathbb{R}^N_+)$ … The last point concerns the underlying functional setting of our work, which is that of Lebesgue spaces $L^p(\Omega)$, with $1<p<\infty$.

4. **Singular boundary conditions.** The second part of this work consists now to find some solutions to the homogeneous problem $(\mathcal{P}^0)$ for singular boundary conditions. Theorem 4.4 and Theorem 4.5 answer to this question.

4.1. **Extension of traces.** In this section, we establish the existence of traces in special cases we shall use for the study of singular boundary conditions.

For any $l\in\mathbb{Z}$, we introduce the spaces

$$\begin{aligned}
Y^p_l(\mathbb{R}^N_+)&=\left\{v\in W^{0,\,p}_{l-2}(\mathbb{R}^N_+);\ \Delta^2 v\in W^{0,\,p}_{l+2}(\mathbb{R}^N_+)\right\},\\
Y^p_{l,\,1}(\mathbb{R}^N_+)&=\left\{v\in W^{0,\,p}_{l-2}(\mathbb{R}^N_+);\ \Delta^2 v\in W^{0,\,p}_{l+2,\,1}(\mathbb{R}^N_+)\right\}.
\end{aligned}$$

They are reflexive Banach spaces equipped with their natural norms:

$$\|v\|_{Y^p_l(\mathbb{R}^N_+)} = \|v\|_{W^{0,\,p}_{l-2}(\mathbb{R}^N_+)} + \|\Delta^2 v\|_{W^{0,\,p}_{l+2}(\mathbb{R}^N_+)},$$
$$\|v\|_{Y^p_{l,\,1}(\mathbb{R}^N_+)} = \|v\|_{W^{0,\,p}_{l-2}(\mathbb{R}^N_+)} + \|\Delta^2 v\|_{W^{0,\,p}_{l+2,\,1}(\mathbb{R}^N_+)}.$$

Lemma 4.1. *Let $l \in \mathbb{Z}$ such that*

$$\frac{N}{p'} \notin \{1, \ldots, l-2\} \quad \text{and} \quad \frac{N}{p} \notin \{1, \ldots, -l+2\}, \tag{20}$$

then the space $\mathscr{D}\big(\overline{\mathbb{R}^N_+}\big)$ is dense in $Y^p_l(\mathbb{R}^N_+)$ and in $Y^p_{l,\,1}(\mathbb{R}^N_+)$.

Proof. i) We use an extension of the Riesz representation theorem to weighted Sobolev spaces: Given $T \in \big(Y^p_l(\mathbb{R}^N_+)\big)'$, there exists a unique pair $(u_1,\, u_2) \in \Big(L^{p'}(\mathbb{R}^N_+)\Big)^2$ such that

$$\forall \varphi \in Y^p_l(\mathbb{R}^N_+), \quad \langle T, \varphi\rangle = \int_{\mathbb{R}^N_+} u_1\, \varrho^{l-2}\, \varphi \,\mathrm{d}x + \int_{\mathbb{R}^N_+} u_2\, \varrho^{l+2}\, \Delta^2\varphi \,\mathrm{d}x. \tag{21}$$

Let's suppose that $T = 0$ on $\mathscr{D}\big(\overline{\mathbb{R}^N_+}\big)$, thus on $\mathscr{D}(\mathbb{R}^N_+)$. Then we can deduce from (21) that

$$\varrho^{l-2}\, u_1 + \Delta^2\left(\varrho^{l+2}\, u_2\right) = 0 \quad \text{in } \mathbb{R}^N_+. \tag{22}$$

We set $v_1 = \varrho^{l-2}\, u_1$ and $v_2 = \varrho^{l+2}\, u_2$, and we respectively denote by $\tilde{v}_1$ and $\tilde{v}_2$ the extensions by 0 of v_1 and v_2 to $\mathbb{R}^N$. We have for any $\varphi \in \mathscr{D}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \tilde{v}_1\, \varphi \,\mathrm{d}x + \int_{\mathbb{R}^N} \tilde{v}_2\, \Delta^2\varphi \,\mathrm{d}x = \int_{\mathbb{R}^N_+} v_1\, \varphi \,\mathrm{d}x + \int_{\mathbb{R}^N_+} v_2\, \Delta^2\varphi \,\mathrm{d}x = 0, \tag{23}$$

according to the assumption on T, since $\varphi|_{\mathbb{R}^N_+} \in \mathscr{D}\big(\overline{\mathbb{R}^N_+}\big)$. Therefore we can deduce that $\tilde{v}_1 + \Delta^2\tilde{v}_2 = 0$ in $\mathbb{R}^N$. We know that $\tilde{v}_1 \in W^{0,\,p'}_{-l+2}(\mathbb{R}^N)$, then we also have $\Delta^2\tilde{v}_2 \in W^{0,\,p'}_{-l+2}(\mathbb{R}^N)$. Moreover, we have the following Green formula: for any $\varphi \in \mathscr{D}(\mathbb{R}^N)$,

$$\left\langle \Delta^2\tilde{v}_2, \varphi\right\rangle_{W^{0,\,p'}_{-l+2}(\mathbb{R}^N)\times W^{0,\,p}_{l-2}(\mathbb{R}^N)} = \left\langle \tilde{v}_2, \Delta^2\varphi\right\rangle_{W^{0,\,p'}_{-l-2}(\mathbb{R}^N)\times W^{0,\,p}_{l+2}(\mathbb{R}^N)}, \tag{24}$$

and we know that $\mathscr{P}^{\Delta^2}_{[2-l-N/p]} \subset W^{4,\,p}_{l+2}(\mathbb{R}^N) \hookrightarrow W^{0,\,p}_{l-2}(\mathbb{R}^N)$ under the hypothesis $\dfrac{N}{p} \notin \{1, \ldots, -l+2\}$. Since $\mathscr{D}(\mathbb{R}^N)$ is dense in $W^{4,\,p}_{l+2}(\mathbb{R}^N)$, we can deduce that (24) holds for any $\varphi \in \mathscr{P}^{\Delta^2}_{[2-l-N/p]}$ and consequently that $\Delta^2\tilde{v}_2 \perp \mathscr{P}^{\Delta^2}_{[2-l-N/p]}$. Thanks to Theorem 2.3, with $m = 2$, $-l$ instead of l and exchanging p and p', we can deduce that under hypothesis (20), we have $\tilde{v}_2 \in W^{4,\,p'}_{-l+2}(\mathbb{R}^N)$. Since $\tilde{v}_2$ is an extension by 0, it follows that $v_2 \in \overset{\circ}{W}{}^{4,\,p'}_{-l+2}(\mathbb{R}^N_+)$. Now, thanks to the density of $\mathscr{D}(\mathbb{R}^N_+)$ in $\overset{\circ}{W}{}^{4,\,p'}_{-l+2}(\mathbb{R}^N_+)$, we have the following Green formula:

$$\begin{aligned}\forall \varphi \in Y^p_l(\mathbb{R}^N_+),\ \forall w \in \overset{\circ}{W}{}^{4,\,p'}_{-l+2}(\mathbb{R}^N_+) \cap W^{0,\,p'}_{-l-2}(\mathbb{R}^N_+),&\\ \int_{\mathbb{R}^N_+} \varphi\, \Delta^2 w \,\mathrm{d}x = \int_{\mathbb{R}^N_+} w\, \Delta^2\varphi \,\mathrm{d}x.&\end{aligned} \tag{25}$$

Then it suffices to come back to (21) and to use (25) with $w = v_2$ which belongs to $\mathring{W}^{4,\,p'}_{-l+2}(\mathbb{R}^N_+) \cap W^{0,\,p'}_{-l-2}(\mathbb{R}^N_+)$, to obtain for any $\varphi \in Y^p_l(\mathbb{R}^N_+)$:

$$\langle T, \varphi\rangle = \int_{\mathbb{R}^N_+} v_1\,\varphi\,\mathrm{d}x + \int_{\mathbb{R}^N_+} v_2\,\Delta^2\varphi\,\mathrm{d}x = \int_{\mathbb{R}^N_+} \left(v_1 + \Delta^2 v_2\right)\,\varphi\,\mathrm{d}x = 0,$$

according to (22). Then we have proved that $T = 0$ on $Y^p_l(\mathbb{R}^N_+)$, and the Hahn-Banach theorem assures us that $\mathscr{D}\big(\overline{\mathbb{R}^N_+}\big)$ is dense in $Y^p_l(\mathbb{R}^N_+)$.

ii) Likewise, we can prove the density of $\mathscr{D}\big(\overline{\mathbb{R}^N_+}\big)$ in $Y^p_{l,\,1}(\mathbb{R}^N_+)$. The differences only concern the logarithmic factors in the weights.

Given $T \in \big(Y^p_{l,\,1}(\mathbb{R}^N_+)\big)'$, there exists a unique pair $(u_1,\,u_2) \in \Big(L^{p'}(\mathbb{R}^N_+)\Big)^2$ such that

$$\forall \varphi \in Y^p_{l,\,1}(\mathbb{R}^N_+), \quad \langle T, \varphi\rangle = \int_{\mathbb{R}^N_+} u_1\,\varrho^{l-2}\,\varphi\,\mathrm{d}x + \int_{\mathbb{R}^N_+} u_2\,\varrho^{l+2}\,\lg\varrho\,\Delta^2\varphi\,\mathrm{d}x. \tag{26}$$

Let's suppose that $T = 0$ on $\mathscr{D}\big(\overline{\mathbb{R}^N_+}\big)$, then we have

$$\varrho^{l-2}\,u_1 + \Delta^2\left(\varrho^{l+2}\,\lg\varrho\,u_2\right) = 0 \quad \text{ in } \mathbb{R}^N_+. \tag{27}$$

We set $v_1 = \varrho^{l-2}\,u_1$ and $v_2 = \varrho^{l+2}\,\lg\varrho\,u_2$, and we respectively denote by $\tilde{v}_1$ and $\tilde{v}_2$ the extensions by 0 of v_1 and v_2 to $\mathbb{R}^N$. We have the analog of identity (23) for any $\varphi \in \mathscr{D}(\mathbb{R}^N)$. Therefore we can deduce that $\tilde{v}_1 + \Delta^2\tilde{v}_2 = 0$ in $\mathbb{R}^N$. We know that $\tilde{v}_1 \in W^{0,\,p'}_{-l+2}(\mathbb{R}^N)$, then we also have $\Delta^2\tilde{v}_2 \in W^{0,\,p'}_{-l+2}(\mathbb{R}^N)$, whence the analog of Green formula (24) where the duality of the right side is replaced by $W^{0,\,p'}_{-l-2,\,-1}(\mathbb{R}^N) \times W^{0,\,p}_{l+2,\,1}(\mathbb{R}^N)$. Since $\mathscr{P}^{\Delta^2}_{[2-l-N/p]} \subset W^{4,\,p}_{l+2,\,1}(\mathbb{R}^N) \hookrightarrow W^{0,\,p}_{l-2}(\mathbb{R}^N)$ if $\dfrac{N}{p} \notin \{1, \ldots, -l+2\}$, we can deduce by density that this formula holds for any $\varphi \in \mathscr{P}^{\Delta^2}_{[2-l-N/p]}$ and consequently that $\Delta^2\tilde{v}_2 \perp \mathscr{P}^{\Delta^2}_{[2-l-N/p]}$. Thanks to Theorem 2.3, we can deduce that under hypothesis (20), we have $\tilde{v}_2 \in W^{4,\,p'}_{-l+2}(\mathbb{R}^N)$. It follows that $v_2 \in \mathring{W}^{4,\,p'}_{-l+2}(\mathbb{R}^N_+)$. We also have the analog of Green formula (25) for any $w \in \mathring{W}^{4,\,p'}_{-l+2}(\mathbb{R}^N_+) \cap W^{0,\,p'}_{-l-2,\,-1}(\mathbb{R}^N_+)$. The end of the proof is quite similar to the previous case. □

Thanks to this density lemma, we can prove the following result of traces:

Lemma 4.2. *Let $l \in \mathbb{Z}$. Under hypothesis (20), the mapping*

$$(\gamma_0,\,\gamma_1) : \mathscr{D}\big(\overline{\mathbb{R}^N_+}\big) \longrightarrow \mathscr{D}(\mathbb{R}^{N-1})^2,$$

can be extended to a linear continuous mapping

$$(\gamma_0,\,\gamma_1) : Y^p_{l,\,1}(\mathbb{R}^N_+) \longrightarrow W^{-1/p,\,p}_{l-2}(\Gamma) \times W^{-1-1/p,\,p}_{l-2}(\Gamma),$$

and we have the following Green formula:

$$\begin{aligned}
&\forall v \in Y^p_{l,\,1}(\mathbb{R}^N_+),\ \forall \varphi \in \mathring{W}^{\star\,4,\,p'}_{-l+2}(\mathbb{R}^N_+),\\
&\left\langle \Delta^2 v, \varphi\right\rangle_{W^{0,\,p}_{l+2,\,1}(\mathbb{R}^N_+) \times W^{0,\,p'}_{-l-2,\,-1}(\mathbb{R}^N_+)} - \left\langle v, \Delta^2\varphi\right\rangle_{W^{0,\,p}_{l-2}(\mathbb{R}^N_+) \times W^{0,\,p'}_{-l+2}(\mathbb{R}^N_+)}\\
&= \left\langle v, \partial_N\Delta\varphi\right\rangle_{W^{-1/p,\,p}_{l-2}(\Gamma) \times W^{1/p,\,p'}_{-l+2}(\Gamma)} - \left\langle \partial_N v, \Delta\varphi\right\rangle_{W^{-1-1/p,\,p}_{l-2}(\Gamma) \times W^{1+1/p,\,p'}_{-l+2}(\Gamma)}.
\end{aligned} \tag{28}$$

Proof. Let's first remark that for any $\varphi \in \overset{\star}{W}{}^{4,\,p'}_{-l+2}(\mathbb{R}^N_+)$, we have

$$\Delta\varphi = \partial_N^2\varphi \ \text{ and } \ \partial_N\Delta\varphi = \partial_N^3\varphi \ \text{ on } \Gamma.$$

Moreover, we always have the imbedding $W^{4,\,p'}_{-l+2}(\mathbb{R}^N_+) \hookrightarrow W^{0,\,p'}_{-l-2,\,-1}(\mathbb{R}^N_+)$. So we can write the following Green formula:

$$\begin{aligned}&\forall v \in \mathscr{D}\big(\overline{\mathbb{R}^N_+}\big),\ \forall \varphi \in \overset{\star}{W}{}^{4,\,p'}_{-l+2}(\mathbb{R}^N_+),\\ &\qquad \int_{\mathbb{R}^N_+} \varphi\,\Delta^2 v\,\mathrm{d}x - \int_{\mathbb{R}^N_+} v\,\Delta^2\varphi\,\mathrm{d}x = \int_\Gamma v\,\partial_N\Delta\varphi\,\mathrm{d}x' - \int_\Gamma \partial_N v\,\Delta\varphi\,\mathrm{d}x'.\end{aligned} \tag{29}$$

In particular, if $\varphi \in W^{4,\,p'}_{-l+2}(\mathbb{R}^N_+)$ and such that $\varphi = \partial_N\varphi = \partial_N^2\varphi = 0$ on Γ, we have

$$\left|\int_\Gamma v\,\partial_N\Delta\varphi\,\mathrm{d}x'\right| \le \ \|v\|_{Y^p_{l,\,1}(\mathbb{R}^N_+)}\ \|\varphi\|_{W^{4,\,p'}_{-l+2}(\mathbb{R}^N_+)}.$$

For all $g \in W^{1-1/p',\,p'}_{-l+2}(\Gamma)$, thanks to Lemma 2.2, there exists a lifting function $\varphi_0 \in W^{4,\,p'}_{-l+2}(\mathbb{R}^N_+)$ such that $\varphi_0 = \partial_N\varphi_0 = \partial_N^2\varphi_0 = 0$ on Γ and $\partial_N^3\varphi_0 = g$ on Γ, satisfying moreover

$$\|\varphi_0\|_{W^{4,\,p'}_{-l+2}(\mathbb{R}^N_+)} \le C\ \|g\|_{W^{1-1/p',\,p'}_{-l+2}(\Gamma)},$$

where C is a constant not depending on φ_0 and g. Then we have

$$\left|\int_\Gamma v\,g\,\mathrm{d}x'\right| \ \le\ \|v\|_{Y^p_{l,\,1}(\mathbb{R}^N_+)}\ \|\varphi_0\|_{W^{4,\,p'}_{-l+2}(\mathbb{R}^N_+)} \ \le\ C\ \|v\|_{Y^p_{l,\,1}(\mathbb{R}^N_+)}\ \|g\|_{W^{1-1/p',\,p'}_{-l+2}(\Gamma)}.$$

Thus

$$\|\gamma_0 v\|_{W^{-1/p,\,p}_{l-2}(\Gamma)} \ \le\ C\ \|v\|_{Y^p_{l,\,1}(\mathbb{R}^N_+)}.$$

Therefore, the linear mapping $\gamma_0 : v \longmapsto v|_\Gamma$ defined on $\mathscr{D}\big(\overline{\mathbb{R}^N_+}\big)$ is continuous for the norm of $Y^p_{l,\,1}(\mathbb{R}^N_+)$. Since $\mathscr{D}\big(\overline{\mathbb{R}^N_+}\big)$ is dense in $Y^p_{l,\,1}(\mathbb{R}^N_+)$, γ_0 can be extended by continuity to a mapping $\gamma_0 \in \mathscr{L}\big(Y^p_{l,\,1}(\mathbb{R}^N_+);\ W^{-1/p,\,p}_{l-2}(\Gamma)\big)$.

To define the trace γ_1 on $Y^p_{l,\,1}(\mathbb{R}^N_+)$, we consider $\varphi \in W^{4,\,p'}_{-l+2}(\mathbb{R}^N_+)$ such that $\varphi = \partial_N\varphi = \partial_N^3\varphi = 0$ on Γ. In this case, we have

$$\left|\int_\Gamma \partial_N v\,\Delta\varphi\,\mathrm{d}x'\right| \le \ \|v\|_{Y^p_{l,\,1}(\mathbb{R}^N_+)}\ \|\varphi\|_{W^{4,\,p'}_{-l+2}(\mathbb{R}^N_+)}.$$

For all $g \in W^{2-1/p',\,p'}_{-l+2}(\Gamma)$, thanks to Lemma 2.2, there exists a lifting function $\varphi_0 \in W^{4,\,p'}_{-l+2}(\mathbb{R}^N_+)$ such that $\varphi_0 = \partial_N\varphi_0 = \partial_N^3\varphi_0 = 0$ on Γ and $\partial_N^2\varphi_0 = g$ on Γ, satisfying moreover

$$\|\varphi_0\|_{W^{4,\,p'}_{-l+2}(\mathbb{R}^N_+)} \le C\ \|g\|_{W^{2-1/p',\,p'}_{-l+2}(\Gamma)},$$

where C is a constant independent of φ_0 and g. Once again, the linear mapping $\gamma_1 : v \longmapsto \partial_N v|_\Gamma$, defined on $\mathscr{D}\big(\overline{\mathbb{R}^N_+}\big)$ is continuous for the norm of $Y^p_{l,\,1}(\mathbb{R}^N_+)$, and it can be extended by continuity to a mapping $\gamma_1 \in \mathscr{L}\big(Y^p_{l,\,1}(\mathbb{R}^N_+);\ W^{-1-1/p,\,p}_{l-2}(\Gamma)\big)$.

To conclude this proof, we can deduce the formula (28) from (29) by density of $\mathscr{D}\big(\overline{\mathbb{R}^N_+}\big)$ in $Y^p_{l,\,1}(\mathbb{R}^N_+)$. □

Remark 4.3. Note that the logarithmic factors are unnecessary in the case where $\frac{N}{p'} \notin \{l-1, l, l+1, l+2\}$, because the imbedding $W^{4,p'}_{-l+2}(\mathbb{R}^N_+) \hookrightarrow W^{0,p'}_{-l-2}(\mathbb{R}^N_+)$ holds. So we can replace the space $Y^p_{l,1}(\mathbb{R}^N_+)$ by $Y^p_l(\mathbb{R}^N_+)$ in the lemma, with a Green formula without logarithmic factors, *i.e.* where the first term of the left side is replaced by $\left\langle \Delta^2 v, \varphi \right\rangle_{W^{0,p}_{l+2}(\mathbb{R}^N_+) \times W^{0,p'}_{-l-2}(\mathbb{R}^N_+)}$. □

4.2. **Singular boundary conditions.** We now come back to the homogeneous problem, and we consider here singular boundary conditions. Let $g_0 \in W^{-1/p,\,p}_{l-2}(\Gamma)$ and $g_1 \in W^{-1-1/p,\,p}_{l-2}(\Gamma)$, we search $u \in W^{0,p}_{l-2}(\mathbb{R}^N_+)$ solution to the problem

$$(\mathcal{P}^0) \quad \begin{cases} \Delta^2 u = 0 & \text{in } \mathbb{R}^N_+, \\ u = g_0 & \text{on } \Gamma, \\ \partial_N u = g_1 & \text{on } \Gamma. \end{cases}$$

Let's first remark that if $u \in W^{0,p}_{l-2}(\mathbb{R}^N_+)$ verifies $(\mathcal{P}^0)$ under hypothesis (20), then it belongs to $Y^p_{l,1}(\mathbb{R}^N_+)$ and thanks to Lemma 4.2, $\gamma_0 u \in W^{-1/p,\,p}_{l-2}(\Gamma)$ and $\gamma_1 u \in W^{-1-1/p,\,p}_{l-2}(\Gamma)$, which gives a sense to $(\mathcal{P}^0)$.

Theorem 4.4. *Let $l \in \mathbb{Z}$. Under hypothesis (20), for any $g_0 \in W^{-1/p,\,p}_{l-2}(\Gamma)$ and $g_1 \in W^{-1-1/p,\,p}_{l-2}(\Gamma)$ satisfying the compatibility condition*

$$\forall \varphi \in \mathscr{B}_{[2+l-N/p']}\ , \quad \langle g_1, \Delta\varphi \rangle_\Gamma - \langle g_0, \partial_N \Delta\varphi \rangle_\Gamma = 0, \tag{30}$$

problem $(\mathcal{P}^0)$ has a solution $u \in W^{0,p}_{l-2}(\mathbb{R}^N_+)$, unique up to an element of $\mathscr{B}_{[2-l-N/p]}$, with the estimate

$$\inf_{q \in \mathscr{B}_{[2-l-N/p]}} \|u+q\|_{W^{0,p}_{l-2}(\mathbb{R}^N_+)} \ \le\ C \left(\|g_0\|_{W^{-1/p,\,p}_{l-2}(\Gamma)} + \|g_1\|_{W^{-1-1/p,\,p}_{l-2}(\Gamma)} \right).$$

Proof. Let $\mathscr{K}^{-2}$ denote the kernel of the operator associated to this problem. We can observe that problem $(\mathcal{P}^0)$ is equivalent to the formulation:

$$(\mathcal{Q}) \quad \begin{cases} \text{Find } u \in Y^p_{l,1}(\mathbb{R}^N_+)/\mathscr{K}^{-2} \text{ such that for any } v \in \overset{\star}{W}{}^{4,p'}_{-l+2}(\mathbb{R}^N_+), \\ \quad \left\langle u, \Delta^2 v \right\rangle_{W^{0,p}_{l-2}(\mathbb{R}^N_+) \times W^{0,p'}_{-l+2}(\mathbb{R}^N_+)} \ =\ \langle g_1, \Delta v \rangle_\Gamma - \langle g_0, \partial_N \Delta v \rangle_\Gamma , \end{cases}$$

where we have used the Green formula (28) of Lemma 4.2.

Now, let's solve problem $(\mathcal{Q})$. For any $f \in W^{0,p'}_{-l+2}(\mathbb{R}^N_+) \perp \mathscr{B}_{[2-l-N/p]}$, according to Theorem 3.1, with $m=2$, $-l$ instead of l and exchanging p and p', the problem

$$(\mathcal{P}^\star) \quad \begin{cases} \Delta^2 v = f & \text{in } \mathbb{R}^N_+, \\ v = 0 & \text{on } \Gamma, \\ \partial_N v = 0 & \text{on } \Gamma, \end{cases}$$

admits an unique solution $v \in W^{4,p'}_{-l+2}(\mathbb{R}^N_+)/\mathscr{B}_{[2+l-N/p']}$, under hypothesis (20). Moreover, v satisfies the estimate

$$\|v\|_{W^{4,p'}_{-l+2}(\mathbb{R}^N_+)/\mathscr{B}_{[2+l-N/p']}} \ \le\ C\ \|f\|_{W^{0,p'}_{-l+2}(\mathbb{R}^N_+)},$$

where C denotes as usual a generic constant not depending on v and f. Consider the linear form $T:\ f \longmapsto \langle g_1, \Delta v \rangle_\Gamma - \langle g_0, \partial_N \Delta v \rangle_\Gamma$ defined on $W^{0,p'}_{-l+2}(\mathbb{R}^N_+) \perp \mathscr{B}_{[2-l-N/p]}$.

We have for any $q \in \mathscr{B}_{[2+l-N/p']}$,

$$|T(f)| = \Big| \langle g_1, \Delta(v+q)\rangle_\Gamma - \langle g_0, \partial_N \Delta(v+q)\rangle_\Gamma \Big| \\ \leq C \, \|v+q\|_{W^{4,\,p'}_{-l+2}(\mathbb{R}^N_+)} \Big(\|g_0\|_{W^{-1/p,\,p}_{l-2}(\Gamma)} + \|g_1\|_{W^{-1-1/p,\,p}_{l-2}(\Gamma)} \Big).$$

Thus

$$|T(f)| \leq C \, \|v\|_{W^{4,\,p'}_{-l+2}(\mathbb{R}^N_+)/\mathscr{B}_{[2+l-N/p']}} \Big(\|g_0\|_{W^{-1/p,\,p}_{l-2}(\Gamma)} + \|g_1\|_{W^{-1-1/p,\,p}_{l-2}(\Gamma)} \Big) \\ \leq C \, \|f\|_{W^{0,\,p'}_{-l+2}(\mathbb{R}^N_+)} \Big(\|g_0\|_{W^{-1/p,\,p}_{l-2}(\Gamma)} + \|g_1\|_{W^{-1-1/p,\,p}_{l-2}(\Gamma)} \Big).$$

Hence T is continuous on $W^{0,\,p'}_{-l+2}(\mathbb{R}^N_+) \perp \mathscr{B}_{[2-l-N/p]}$, and according to Riesz representation theorem, there exists an unique $u \in W^{0,\,p}_{l-2}(\mathbb{R}^N_+)/\mathscr{B}_{[2-l-N/p]}$ such that $T(f) = \langle u, f\rangle_{W^{0,\,p}_{l-2}(\mathbb{R}^N_+)\times W^{0,\,p'}_{-l+2}(\mathbb{R}^N_+)}$. This means that u is a solution to problem $(\mathcal{Q})$ and $\mathscr{K}^{-2} = \mathscr{B}_{[2-l-N/p]}$. □

4.3. **Intermediate boundary conditions.** We also need for the sequel to solve $(\mathcal{P}^0)$ for the data $g_0 \in W^{1-1/p,\,p}_{l-1}(\Gamma)$ and $g_1 \in W^{-1/p,\,p}_{l-1}(\Gamma)$.

Theorem 4.5. *Let $l \in \mathbb{Z}$ such that*

$$\frac{N}{p'} \notin \{1, \ldots, l-1\} \quad \text{and} \quad \frac{N}{p} \notin \{1, \ldots, -l+1\}. \tag{31}$$

For any $g_0 \in W^{1-1/p,\,p}_{l-1}(\Gamma)$ and $g_1 \in W^{-1/p,\,p}_{l-1}(\Gamma)$ satisfying the compatibility condition (30), problem $(\mathcal{P}^0)$ has a solution $u \in W^{1,\,p}_{l-1}(\mathbb{R}^N_+)$, unique up to an element of $\mathscr{B}_{[2-l-N/p]}$, with the estimate

$$\inf_{q \in \mathscr{B}_{[2-l-N/p]}} \|u+q\|_{W^{1,\,p}_{l-1}(\mathbb{R}^N_+)} \;\leq\; C \Big(\|g_0\|_{W^{1-1/p,\,p}_{l-1}(\Gamma)} + \|g_1\|_{W^{-1/p,\,p}_{l-1}(\Gamma)} \Big).$$

Remark 4.6. We can give a very quick proof of this result by interpolation between the previous case and the regular case, *i.e.* $g_0 \in W^{2-1/p,\,p}_{l}(\Gamma)$ and $g_1 \in W^{1-1/p,\,p}_{l}(\Gamma)$. But the problem with this reasoning is that we must combine the critical values of hypotheses (9) and (20), and then we obtain two supplementary values with respect to (31). Thus we shall give a direct proof similar to the singular case, with however some new arguments. □

For any $l \in \mathbb{Z}$, we introduce the space

$$Y^{1,\,p}_{l,\,1}(\mathbb{R}^N_+) = \Big\{ v \in W^{1,\,p}_{l-1}(\mathbb{R}^N_+);\ \Delta^2 v \in W^{0,\,p}_{l+2,\,1}(\mathbb{R}^N_+) \Big\}.$$

It's a reflexive Banach space equipped with it's natural norm:

$$\|v\|_{Y^{1,\,p}_{l,\,1}(\mathbb{R}^N_+)} = \|v\|_{W^{1,\,p}_{l-1}(\mathbb{R}^N_+)} + \|\Delta^2 v\|_{W^{0,\,p}_{l+2,\,1}(\mathbb{R}^N_+)}.$$

We also define the subspace of $Y^{1,\,p}_{l,\,1}(\mathbb{R}^N_+)$

$$\mathring{Y}^{1,\,p}_{l,\,1}(\mathbb{R}^N_+) = \Big\{ v \in \mathring{W}^{1,\,p}_{l-1}(\mathbb{R}^N_+);\ \Delta^2 v \in W^{0,\,p}_{l+2,\,1}(\mathbb{R}^N_+) \Big\}.$$

Lemma 4.7. *Let $l \in \mathbb{Z}$. Under hypothesis (31), the space $\mathscr{D}\big(\overline{\mathbb{R}^N_+}\big)$ is dense in $Y^{1,\,p}_{l,\,1}(\mathbb{R}^N_+)$.*

Proof. Let P be an extension operator mapping $W^{1,p}_{l-1}(\mathbb{R}^N_+)$ into $W^{1,p}_{l-1}(\mathbb{R}^N)$. For every continuous linear form $T \in \big(Y^{1,p}_{l,1}(\mathbb{R}^N_+)\big)'$, there exists $(u_1, u_2) \in W^{-1,p'}_{-l+1}(\mathbb{R}^N) \times W^{0,p'}_{-l-2,\,-1}(\mathbb{R}^N_+)$ such that for any $v \in Y^{1,p}_{l,1}(\mathbb{R}^N_+)$,

$$\langle T, v\rangle = \langle u_1, Pv\rangle_{W^{-1,p'}_{-l+1}(\mathbb{R}^N)\times W^{1,p}_{l-1}(\mathbb{R}^N)} + \int_{\mathbb{R}^N_+} u_2\,\Delta^2 v\,\mathrm{d}x.$$

Moreover, since T depends only on v and not on the restriction of Pv to $\mathbb{R}^N_-$, the support of u_1 is contained in $\overline{\mathbb{R}^N_+}$.

Thanks to the Hahn-Banach theorem, it suffices to show that any T which vanishes on $\mathscr{D}\big(\overline{\mathbb{R}^N_+}\big)$ is actually zero on $Y^{1,p}_{l,1}(\mathbb{R}^N_+)$. Indeed for any $\Psi \in \mathscr{D}(\mathbb{R}^N)$, we have

$$\langle u_1, \Psi\rangle + \big\langle \tilde{u}_2, \Delta^2\Psi\big\rangle = \langle u_1, P\psi\rangle + \int_{\mathbb{R}^N_+} u_2\,\Delta^2\psi\,\mathrm{d}x = 0,$$

where $\tilde{u}_2$ is the extension by 0 of u_2 to $\mathbb{R}^N$ and $\psi = \Psi|_{\mathbb{R}^N_+}$. It follows that

$$\Delta^2\tilde{u}_2 = -u_1 \text{ in } \mathbb{R}^N.$$

Thus we have $\Delta^2\tilde{u}_2 \in W^{-1,p'}_{-l+1}(\mathbb{R}^N)$. Since $\tilde{u}_2 \in W^{0,p'}_{-l-2,\,-1}(\mathbb{R}^N_+)$, we also have $\Delta^2\tilde{u}_2 \perp \mathscr{P}^{\Delta^2}_{[2-l-N/p]}$. Now, thanks to Theorem 2.3, we can deduce that under hypothesis (31), we have $\tilde{u}_2 \in W^{3,p'}_{-l+1}(\mathbb{R}^N)$. It follows that $u_2 \in \overset{\circ}{W}{}^{3,p'}_{-l+1}(\mathbb{R}^N_+)$. By density of $\mathscr{D}(\mathbb{R}^N_+)$ in $\overset{\circ}{W}{}^{3,p'}_{-l+1}(\mathbb{R}^N_+)$, there exists a sequence $(\varphi_k)_{k\in\mathbb{N}} \subset \mathscr{D}(\mathbb{R}^N_+)$ such that $\varphi_k \to u_2$ in $\overset{\circ}{W}{}^{3,p'}_{-l+1}(\mathbb{R}^N_+)$. Thus for any $v \in Y^{1,p}_{l,1}(\mathbb{R}^N_+)$, we have

$$\begin{aligned}
\langle T, v\rangle &= \lim_{k\to\infty}\left\{\big\langle -\Delta^2\tilde{\varphi}_k, Pv\big\rangle + \int_{\mathbb{R}^N_+}\varphi_k\,\Delta^2 v\,\mathrm{d}x\right\}\\
&= \lim_{k\to\infty}\left\{-\int_{\mathbb{R}^N_+} v\,\Delta^2\varphi_k\,\mathrm{d}x + \int_{\mathbb{R}^N_+}\varphi_k\,\Delta^2 v\,\mathrm{d}x\right\}\\
&= 0.
\end{aligned}$$

Thus T is identically zero. □

Lemma 4.8. *Let $l \in \mathbb{Z}$. Under hypothesis (31), the mapping*

$$\gamma_1 : \mathscr{D}\big(\overline{\mathbb{R}^N_+}\big) \longrightarrow \mathscr{D}(\mathbb{R}^{N-1}),$$

can be extended to a linear continuous mapping

$$\gamma_1 : Y^{1,p}_{l,1}(\mathbb{R}^N_+) \longrightarrow W^{-1/p,\,p}_{l-1}(\Gamma),$$

and we have the following Green formula:

$$\begin{aligned}
&\forall v \in \overset{\circ}{Y}{}^{1,p}_{l,1}(\mathbb{R}^N_+),\ \forall\varphi \in \overset{\star}{W}{}^{3,p'}_{-l+1}(\mathbb{R}^N_+),\\
&\quad\big\langle\Delta^2 v, \varphi\big\rangle_{W^{0,p}_{l+2,\,1}(\mathbb{R}^N_+)\times W^{0,p'}_{-l-2,\,-1}(\mathbb{R}^N_+)} - \big\langle v, \Delta^2\varphi\big\rangle_{\overset{\circ}{W}{}^{1,p}_{l-1}(\mathbb{R}^N_+)\times W^{-1,p'}_{-l+1}(\mathbb{R}^N_+)}\\
&\qquad\qquad = -\big\langle\partial_N v, \Delta\varphi\big\rangle_{W^{-1/p,\,p}_{l-1}(\Gamma)\times W^{1/p,\,p'}_{-l+1}(\Gamma)}.
\end{aligned} \tag{32}$$

Proof. Since we always have the imbedding $W^{3,p'}_{-l+1}(\mathbb{R}^N_+) \hookrightarrow W^{0,p'}_{-l-2,\,-1}(\mathbb{R}^N_+)$, we can write the following Green formula:

$$\begin{aligned}\forall v \in \mathscr{D}\big(\overline{\mathbb{R}^N_+}\big),\ \forall \varphi \in \overset{\star}{W}{}^{3,p'}_{-l+1}(\mathbb{R}^N_+),\\ \int_{\mathbb{R}^N_+} \varphi\, \Delta^2 v \,\mathrm{d}x + \int_{\mathbb{R}^N_+} \nabla v.\nabla\Delta\varphi \,\mathrm{d}x = -\int_\Gamma \partial_N v\, \Delta\varphi \,\mathrm{d}x'.\end{aligned} \tag{33}$$

This implies that

$$\left|\int_\Gamma \partial_N v\, \Delta\varphi \,\mathrm{d}x'\right| \le \ \|v\|_{Y^{1,p}_{l,\,1}(\mathbb{R}^N_+)}\ \|\varphi\|_{W^{3,p'}_{-l+1}(\mathbb{R}^N_+)}.$$

For all $g \in W^{1-1/p',p'}_{-l+1}(\Gamma)$, thanks to Lemma 2.2, there exists a lifting function $\varphi_0 \in W^{3,p'}_{-l+1}(\mathbb{R}^N_+)$ such that $\varphi_0 = \partial_N\varphi_0 = 0$ on Γ and $\partial_N^2\varphi_0 = g$ on Γ, satisfying moreover

$$\|\varphi_0\|_{W^{3,p'}_{-l+1}(\mathbb{R}^N_+)} \le C\ \|g\|_{W^{1-1/p',p'}_{-l+1}(\Gamma)},$$

where C is a constant not depending on φ_0 and g. Then we have

$$\left|\int_\Gamma g\,\partial_N v \,\mathrm{d}x'\right| \ \le\ \|v\|_{Y^{1,p}_{l,\,1}(\mathbb{R}^N_+)}\ \|\varphi_0\|_{W^{3,p'}_{-l+1}(\mathbb{R}^N_+)} \ \le\ C\ \|v\|_{Y^{1,p}_{l,\,1}(\mathbb{R}^N_+)}\ \|g\|_{W^{1-1/p',p'}_{-l+1}(\Gamma)}.$$

Therefore

$$\|\gamma_1 v\|_{W^{-1/p,\,p}_{l-1}(\Gamma)} \ \le\ C\ \|v\|_{Y^{1,p}_{l,\,1}(\mathbb{R}^N_+)}.$$

Thus the linear mapping $\gamma_1 : v \longmapsto \partial_N v|_\Gamma$ defined on $\mathscr{D}\big(\overline{\mathbb{R}^N_+}\big)$ is continuous for the norm of $Y^{1,p}_{l,\,1}(\mathbb{R}^N_+)$. Since $\mathscr{D}\big(\overline{\mathbb{R}^N_+}\big)$ is dense in $Y^{1,p}_{l,\,1}(\mathbb{R}^N_+)$, γ_1 can be extended by continuity to a mapping $\gamma_1 \in \mathscr{L}\big(Y^{1,p}_{l,\,1}(\mathbb{R}^N_+);\ W^{-1/p,\,p}_{l-1}(\Gamma)\big)$.

By density of $\mathscr{D}\big(\overline{\mathbb{R}^N_+}\big)$ in $Y^{1,p}_{l,\,1}(\mathbb{R}^N_+)$, we can generalize the formula (33) to any $v \in Y^{1,p}_{l,\,1}(\mathbb{R}^N_+)$. Furthermore, thanks to the of $\mathscr{D}(\mathbb{R}^N_+)$ in $\overset{\circ}{W}{}^{1,p}_{l-1}(\mathbb{R}^N_+)$, we have for any $v \in \overset{\circ}{W}{}^{1,p}_{l-1}(\mathbb{R}^N_+)$ and $\varphi \in W^{3,p'}_{-l+1}(\mathbb{R}^N_+)$,

$$\langle \nabla v, \nabla\Delta\varphi\rangle_{W^{0,\,p}_{l-1}(\mathbb{R}^N_+)\times W^{0,\,p'}_{-l+1}(\mathbb{R}^N_+)} = \big\langle v, \Delta^2\varphi\big\rangle_{\overset{\circ}{W}{}^{1,\,p}_{l-1}(\mathbb{R}^N_+)\times W^{-1,\,p'}_{-l+1}(\mathbb{R}^N_+)}.$$

So we obtain the Green formula (32). □

Proof of Theorem 4.5. The first step is to reduce to zero the boundary condition on u in Problem $(\mathcal{P}^0)$. Let's consider the problem

$$(\mathcal{R}^0)\ \begin{cases}\Delta w = 0 & \text{in } \mathbb{R}^N_+,\\ w = g_0 & \text{on } \Gamma.\end{cases}$$

Thanks to (8), we know that $\mathscr{B}_{[2+l-N/p']} = \Pi_D \mathscr{A}^{\Delta}_{[l-N/p']} \oplus \Pi_N \mathscr{N}^{\Delta}_{[l-N/p']}$, thus the compatibility condition (30) on $(\mathcal{P}^0)$ implies

$$\forall r \in \mathscr{A}^{\Delta}_{[l-N/p']},\quad \langle g_0, \partial_N r\rangle_{W^{1-1/p,\,p}_{l-1}(\Gamma)\times W^{-1/p',\,p'}_{-l+1}(\Gamma)} = 0,$$

which is the compatibility condition on $(\mathcal{R}^0)$. Thus problem $(\mathcal{R}^0)$ admits a solution $w \in W^{1,p}_{l-1}(\mathbb{R}^N_+)$ under hypothesis (31) (see [5], Theorem 3.1 & 3.2). It follows that $w \in Y^{1,p}_{l,\,1}(\mathbb{R}^N_+)$, and thus $\gamma_1 w = h_1 \in W^{-1/p,\,p}_{l-1}(\Gamma)$. Let's set $v = u - w$, then problem $(\mathcal{P}^0)$ is equivalent to the following

$$\Delta^2 v = 0 \ \text{ in } \mathbb{R}^N_+,\quad v = 0 \ \text{ and } \ \partial_N v = g_1 - h_1 \ \text{ on } \Gamma. \tag{34}$$

Let $\mathscr{K}^{-1}$ denote the kernel of the operator associated to this problem. We can observe that Problem (34) is equivalent to the formulation:

$$(\mathcal{Q}) \quad \left\{ \begin{array}{l} \text{Find } v \in \overset{\circ}{Y}{}^{1,\,p}_{l,\,1}(\mathbb{R}^N_+)/\mathscr{K}^{-1} \text{ such that for any } \varphi \in \overset{\star}{W}{}^{3,\,p'}_{-l+1}(\mathbb{R}^N_+), \\ \left\langle v, \Delta^2\varphi \right\rangle_{\overset{\circ}{W}{}^{1,\,p}_{l-1}(\mathbb{R}^N_+)\times W^{-1,\,p'}_{-l+1}(\mathbb{R}^N_+)} \;=\; \langle g_1 - h_1, \Delta\varphi\rangle_\Gamma\,, \end{array} \right.$$

where we have used the Green formula (32) of Lemma 4.8.

Now, let's solve Problem $(\mathcal{Q})$. For any $f \in W^{-1,\,p'}_{-l+1}(\mathbb{R}^N_+) \perp \mathscr{B}_{[2-l-N/p]}$, according to Theorem 3.1, with $m = 1$, $-l$ instead of l and exchanging p and p', the problem

$$(\mathcal{P}^\star) \quad \begin{cases} \Delta^2 z = f & \text{in } \mathbb{R}^N_+, \\ z = 0 & \text{on } \Gamma, \\ \partial_N z = 0 & \text{on } \Gamma, \end{cases}$$

admits an unique solution $z \in W^{3,\,p'}_{-l+1}(\mathbb{R}^N_+)/\mathscr{B}_{[2+l-N/p']}$, under hypothesis (31). Moreover, v satisfies the estimate

$$\|z\|_{W^{3,\,p'}_{-l+1}(\mathbb{R}^N_+)/\mathscr{B}_{[2+l-N/p']}} \;\leq\; C\,\|f\|_{W^{-1,\,p'}_{-l+1}(\mathbb{R}^N_+)}.$$

Consider the linear form $T : \; f \longmapsto \langle g_1 - h_1, \Delta z\rangle_\Gamma$. We can show that it is continuous on $W^{-1,\,p'}_{-l+1}(\mathbb{R}^N_+) \perp \mathscr{B}_{[2-l-N/p]}$. Then, according to the Riesz representation theorem, there exists an unique $v \in \overset{\circ}{W}{}^{1,\,p}_{l-1}(\mathbb{R}^N_+)/\mathscr{B}_{[2-l-N/p]}$ such that $T(f) = \langle v, f\rangle_{\overset{\circ}{W}{}^{1,\,p}_{l-1}(\mathbb{R}^N_+)\times W^{-1,\,p'}_{-l+1}(\mathbb{R}^N_+)}$. This means that v is a solution to Problem $(\mathcal{Q})$ and $\mathscr{K}^{-1} = \mathscr{B}_{[2-l-N/p]}$. □

5. **Other boundary conditions.** The last part of this study is devoted to the biharmonic equation with other kinds of boundary conditions. These results will be useful in a forthcoming work on the Stokes problem with different types of boundary conditions.

5.1. **First case.** The biharmonic equation with boundary conditions on u and Δu

$$(\mathcal{Q}) \quad \begin{cases} \Delta^2 u = f & \text{in } \mathbb{R}^N_+, \\ u = g_0 & \text{on } \Gamma, \\ \Delta u = g_1 & \text{on } \Gamma. \end{cases}$$

Theorem 5.1. *Let $l \in \mathbb{Z}$. Under hypothesis (9) and for any $f \in W^{-1,\,p}_l(\mathbb{R}^N_+)$, $g_0 \in W^{3-1/p,\,p}_l(\Gamma)$ and $g_1 \in W^{1-1/p,\,p}_l(\Gamma)$ satisfying the compatibility condition*

$$\begin{array}{l} \forall \varphi \in \mathscr{A}^{\Delta^2}_{[1+l-N/p']}, \\ \qquad \langle f, \varphi\rangle_{W^{-1,\,p}_l(\mathbb{R}^N_+)\times \overset{\circ}{W}{}^{1,\,p'}_{-l}(\mathbb{R}^N_+)} - \langle g_1, \partial_N\varphi\rangle_\Gamma - \langle g_0, \partial_N\Delta\varphi\rangle_\Gamma = 0, \end{array} \tag{35}$$

problem $(\mathcal{Q})$ has a solution $u \in W^{3,\,p}_l(\mathbb{R}^N_+)$, unique up to an element of $\mathscr{A}^{\Delta^2}_{[3-l-N/p]}$, with the estimate

$$\inf_{q\in\mathscr{A}^{\Delta^2}_{[3-l-N/p]}} \|u+q\|_{W^{3,\,p}_l(\mathbb{R}^N_+)} \;\leq\; C\left(\|f\|_{W^{-1,\,p}_l(\mathbb{R}^N_+)} + \|g_0\|_{W^{3-1/p,\,p}_l(\Gamma)} + \|g_1\|_{W^{1-1/p,\,p}_l(\Gamma)}\right).$$

5.1.1. *The kernel.* We must characterize the kernel of the operator

$$(\Delta^2, \gamma_0, \gamma_0\Delta) : W_l^{3,\,p}(\mathbb{R}^N_+) \;\longrightarrow\; W_l^{-1,\,p}(\mathbb{R}^N_+) \times W_l^{3-1/p,\,p}(\Gamma) \times W_l^{1-1/p,\,p}(\Gamma).$$

Since $\dfrac{N}{p} \notin \{1, \ldots, -l\}$, we know that $\mathscr{P}_{[3-l-N/p]} \subset W_l^{3,\,p}(\mathbb{R}^N_+)$. Let u be a function of this kernel and set

$$\tilde{u}(x', x_N) \;=\; \begin{cases} u(x', x_N) & \text{if } x_N \geq 0, \\ -u(x', -x_N) & \text{if } x_N < 0. \end{cases}$$

Thus we have $\tilde{u} \in \mathscr{S}'(\mathbb{R}^N)$ and we show that $\Delta^2\tilde{u} = 0$ in $\mathbb{R}^N$. We can deduce that $\tilde{u}$, and consequently u, is a polynomial. By identification in the half-space $x_N < 0$, we obtain that u is odd with respect to x_N. Conversely it is clear that any polynomial u odd with respect to x_N verifies $u = \Delta u = 0$ on Γ. Furthermore $u \in W_l^{3,\,p}(\mathbb{R}^N_+)$ implies that its degree is at the most $[3 - l - N/p]$. So we have characterized this kernel as the space of biharmonic polynomials, odd with respect to x_N, of degree smaller than or equal to $[3-l-N/p]$. We denote it by $\mathscr{A}^{\Delta^2}_{[3-l-N/p]}$.

5.1.2. *The compatibility condition.* Problem $(\mathcal{Q})$ admits a solution u in $W_l^{3,\,p}(\mathbb{R}^N_+)$ only if the compatibility condition (35) is satisfied, where $\langle g_1, \partial_N\varphi\rangle_\Gamma$ denotes the duality bracket $\langle g_1, \partial_N\varphi\rangle_{W_l^{1-1/p,\,p}(\Gamma)\times W_{-l}^{-1/p',\,p'}(\Gamma)}$, and $\langle g_0, \partial_N\Delta\varphi\rangle_\Gamma$ the duality bracket $\langle g_0, \partial_N\Delta\varphi\rangle_{W_l^{3-1/p,\,p}(\Gamma)\times W_{-l}^{-2-1/p',\,p'}(\Gamma)}$.

Note that if $l \leq 0$, then $\mathscr{A}^{\Delta^2}_{[1+l-N/p']} = \{0\}$ and thus there is no compatibility condition. Let's now remark that if $\varphi \in \mathscr{A}^{\Delta^2}_{[1+l-N/p']}$, then $\varphi \in W^{2,\,p'}_{-l+1}(\mathbb{R}^N_+)$ and thus $\partial_N\varphi|_\Gamma \in W^{1-1/p',\,p'}_{-l+1}(\Gamma) \hookrightarrow W^{-1/p',\,p'}_{-l}(\Gamma)$. But we also have $\varphi \in W^{4,\,p'}_{-l+3}(\mathbb{R}^N_+)$ and thus $\partial_N\Delta\varphi \in W^{1-1/p',\,p'}_{-l+3}(\Gamma) \hookrightarrow W^{-2-1/p',\,p'}_{-l}(\Gamma)$. This gives a sense to (35). As in our first study of Problem $(\mathcal{P})$ (see [7]), we can verify that these imbeddings hold under hypothesis (9) for $l \geq 1$ and we can prove in a similar fashion the necessity of condition (35).

5.1.3. *Proof of Theorem 5.1.* Let's first consider the Dirichlet problem

$$\Delta v = f \;\text{ in } \mathbb{R}^N_+, \qquad v = g_1 \;\text{ on } \Gamma,$$

which admits a solution $v \in W_l^{1,\,p}(\mathbb{R}^N_+)$, if the following compatibility condition is satisfied (see [5], Theorem 3.1 and Theorem 3.2):

$$\forall \vartheta \in \mathscr{A}^{\Delta}_{[1+l-N/p']}, \quad \langle f, \vartheta\rangle_{W_l^{-1,\,p}(\mathbb{R}^N_+)\times \mathring{W}^{1,\,p'}_{-l}(\mathbb{R}^N_+)} - \langle g_1, \partial_N\vartheta\rangle_\Gamma = 0. \tag{36}$$

Then, we must solve the second Dirichlet problem

$$\Delta u = v \;\text{ in } \mathbb{R}^N_+, \qquad u = g_0 \;\text{ on } \Gamma,$$

which admits a solution $u \in W_l^{3,\,p}(\mathbb{R}^N_+)$, if the following compatibility condition is satisfied (see [5], Corollary 3.4):

$$\forall \psi \in \mathscr{A}^{\Delta}_{[-1+l-N/p']}, \quad \langle v, \psi\rangle_{W_l^{1,\,p}(\mathbb{R}^N_+)\times W_{-l}^{-1,\,p'}(\mathbb{R}^N_+)} - \langle g_0, \partial_N\psi\rangle_\Gamma = 0. \tag{37}$$

Now, let's show that the compatibility condition (35) of problem $(\mathcal{Q})$ implies the conditions (36) and (37). Condition (35) must be satisfied for any $\varphi \in \mathscr{A}^{\Delta^2}_{[1+l-N/p']}$, thus for any $\vartheta \in \mathscr{A}^{\Delta}_{[1+l-N/p']}$, and then it is reduced to

$$\langle f, \vartheta\rangle_{W_l^{-1,\,p}(\mathbb{R}^N_+)\times \mathring{W}^{1,\,p'}_{-l}(\mathbb{R}^N_+)} - \langle g_1, \partial_N\vartheta\rangle_\Gamma = 0,$$

i.e. precisely the condition (36). Now, note that by (35), v satisfies

$$\forall \varphi \in \mathscr{A}^{\Delta^2}_{[1+l-N/p']},$$
$$\langle \Delta v, \varphi \rangle_{W_l^{-1,\,p}(\mathbb{R}^N_+)\times \mathring{W}_{-l}^{1,\,p'}(\mathbb{R}^N_+)} - \langle v, \partial_N \varphi \rangle_\Gamma - \langle g_0, \partial_N \Delta \varphi \rangle_\Gamma = 0.$$

It remains to write for such a φ, the Green formula

$$\langle \Delta v, \varphi \rangle_{W_l^{-1,\,p}(\mathbb{R}^N_+)\times \mathring{W}_{-l}^{1,\,p'}(\mathbb{R}^N_+)} = \langle v, \Delta \varphi \rangle_{W_l^{1,\,p}(\mathbb{R}^N_+)\times W_{-l}^{-1,\,p'}(\mathbb{R}^N_+)} + \langle v, \partial_N \varphi \rangle_\Gamma ,$$

to deduce the condition

$$\forall \varphi \in \mathscr{A}^{\Delta^2}_{[1+l-N/p']}, \quad \langle v, \Delta \varphi \rangle_{W_l^{1,\,p}(\mathbb{R}^N_+)\times W_{-l}^{-1,\,p'}(\mathbb{R}^N_+)} - \langle g_0, \partial_N \Delta \varphi \rangle_\Gamma = 0.$$

If we finally remark that any $\psi \in \mathscr{A}^{\Delta}_{[-1+l-N/p']}$ can be written $\psi = \Delta\varphi$ with $\varphi \in \mathscr{A}^{\Delta^2}_{[1+l-N/p']}$, we exactly find the condition (37).

Remark 5.2. Problem $(\mathcal{Q})$ is ill-posed for $f \in W_l^{-2,\,p}(\mathbb{R}^N_+)$. However, if $f = 0$, we can consider less regular boundary conditions g_0 and g_1. □

5.1.4. *Regularity of solutions to Problem* $(\mathcal{Q})$. To complete Theorem 5.1, we can give a result for different types of data.

Theorem 5.3. *Let $l \in \mathbb{Z}$ and $m \geq 1$ be two integers and assume that*

$$\frac{N}{p'} \notin \{1, \dots, l+1\} \quad \text{and} \quad \frac{N}{p} \notin \{1, \dots, -l-m\}. \tag{38}$$

For any $f \in W^{m-1,\,p}_{m+l}(\mathbb{R}^N_+)$, $g_0 \in W^{m+3-1/p,\,p}_{m+l}(\Gamma)$ and $g_1 \in W^{m+1-1/p,\,p}_{m+l}(\Gamma)$ satisfying the compatibility condition (35), problem $(\mathcal{Q})$ has a solution $u \in W^{m+3,\,p}_{m+l}(\mathbb{R}^N_+)$, unique up to an element of $\mathscr{A}^{\Delta^2}_{[3-l-N/p]}$, with the estimate

$$\inf_{q \in \mathscr{A}^{\Delta^2}_{[3-l-N/p]}} \|u+q\|_{W^{m+3,\,p}_{m+l}(\mathbb{R}^N_+)} \leq$$
$$C\left(\|f\|_{W^{m-1,\,p}_{m+l}(\mathbb{R}^N_+)} + \|g_0\|_{W^{m+3-1/p,\,p}_{m+l}(\Gamma)} + \|g_1\|_{W^{m+1-1/p,\,p}_{m+l}(\Gamma)} \right).$$

It can be readily checked that the kernel is unchanged under the hypothesis $\dfrac{N}{p} \notin \{1, \dots, -l-m\}$. We also keep the compatibility condition (35) and the proof of the existence of a solution is similar to that employed for Theorem 5.1 by means of the regularity result for the two Dirichlet problems.

5.2. **Second case.** The biharmonic equation with boundary conditions on $\partial_N u$ and $\partial_N \Delta u$

$$(\mathcal{R}) \quad \begin{cases} \Delta^2 u = f & \text{in } \mathbb{R}^N_+, \\ \partial_N u = g_0 & \text{on } \Gamma, \\ \partial_N \Delta u = g_1 & \text{on } \Gamma. \end{cases}$$

Theorem 5.4. *Let $l \in \mathbb{Z}$ and assume that*

$$\frac{N}{p'} \notin \{1, \dots, l\} \quad \text{and} \quad \frac{N}{p} \notin \{1, \dots, -l+1\}. \tag{39}$$

For any $f \in W^{0,\,p}_l(\mathbb{R}^N_+)$, $g_0 \in W^{2-1/p,\,p}_{l-1}(\Gamma)$ and $g_1 \in W^{-1/p,\,p}_{l-1}(\Gamma)$ satisfying the compatibility condition

$$\forall \varphi \in \mathscr{N}^{\Delta^2}_{[l-N/p']}, \quad \langle f, \varphi \rangle_{W_l^{0,\,p}(\mathbb{R}^N_+)\times W_{-l}^{0,\,p'}(\mathbb{R}^N_+)} + \langle g_1, \varphi \rangle_\Gamma + \langle g_0, \Delta \varphi \rangle_\Gamma = 0, \tag{40}$$

problem $(\mathcal{R})$ *has a solution* $u \in W^{3,\,p}_{l-1}(\mathbb{R}^N_+)$*, unique up to an element of* $\mathscr{N}^{\Delta^2}_{[4-l-N/p]}$*, with the estimate*

$$\inf_{q\in\mathscr{N}^{\Delta^2}_{[4-l-N/p]}} \|u+q\|_{W^{3,\,p}_{l-1}(\mathbb{R}^N_+)} \;\le\; C\left(\|f\|_{W^{0,\,p}_{l}(\mathbb{R}^N_+)} + \|g_0\|_{W^{2-1/p,\,p}_{l-1}(\Gamma)} + \|g_1\|_{W^{-1/p,\,p}_{l-1}(\Gamma)}\right).$$

5.2.1. *The kernel.* We must characterize the kernel of the operator

$$(\Delta^2, \gamma_1, \gamma_1\Delta) : W^{3,\,p}_{l-1}(\mathbb{R}^N_+) \;\longrightarrow\; W^{-1,\,p}_{l-1}(\mathbb{R}^N_+) \times W^{2-1/p,\,p}_{l-1}(\Gamma) \times W^{-1/p,\,p}_{l-1}(\Gamma).$$

Let u be a function of this kernel and set

$$\tilde{u}(x', x_N) \;=\; \begin{cases} u(x', x_N) & \text{si } x_N \ge 0, \\ u(x', -x_N) & \text{si } x_N < 0. \end{cases}$$

Here again, $\tilde{u} \in \mathscr{S}'(\mathbb{R}^N)$ and we show that $\Delta^2\tilde{u} = 0$ in $\mathbb{R}^N$. We can deduce that $\tilde{u}$, and consequently u, is a polynomial. By identification in the half-space $x_N < 0$, we obtain that u is even with respect to x_N. Conversely it is clear that any polynomial u even with respect to x_N verifies $\partial_N u = \partial_N \Delta u = 0$ on Γ. Furthermore $u \in W^{3,\,p}_{l-1}(\mathbb{R}^N_+)$ implies that its degree is at the most $[4 - l - N/p]$. So we have characterized this kernel as the space of biharmonic polynomials, even with respect to x_N, of degree smaller than or equal to $[4-l-N/p]$. We denote it by $\mathscr{N}^{\Delta^2}_{[4-l-N/p]}$.

5.2.2. *The compatibility condition.* Problem $(\mathcal{R})$ admits a solution u in $W^{3,\,p}_{l-1}(\mathbb{R}^N_+)$ only if the compatibility condition (40) is satisfied, where $\langle g_1, \varphi\rangle_\Gamma$ denotes the duality bracket $\langle g_1, \varphi\rangle_{W^{-1/p,\,p}_{l-1}(\Gamma)\times W^{1-1/p',\,p'}_{-l+1}(\Gamma)}$, and $\langle g_0, \Delta\varphi\rangle_\Gamma$ the duality bracket $\langle g_0, \Delta\varphi\rangle_{W^{2-1/p,\,p}_{l-1}(\Gamma)\times W^{-1-1/p',\,p'}_{-l+1}(\Gamma)}$.

The arguments are exactly the same as in the other cases.

5.2.3. *Proof of Theorem 5.4.* We solve this case in the same way that the precedent, but this time by two successive Neumann problems.

$$\Delta v = f \;\text{ in } \mathbb{R}^N_+, \qquad \partial_N v = g_1 \;\text{ on } \Gamma,$$

which admits a solution $v \in W^{1,\,p}_{l-1}(\mathbb{R}^N_+)$, if the following compatibility condition is satisfied (see [6], Theorem 3.7):

$$\forall \vartheta \in \mathscr{N}^{\Delta}_{[l-N/p']}, \quad \langle f, \vartheta\rangle_{W^{0,\,p}_{l}(\mathbb{R}^N_+)\times W^{0,\,p'}_{-l}(\mathbb{R}^N_+)} + \langle g_1, \vartheta\rangle_\Gamma = 0. \tag{41}$$

Then

$$\Delta u = v \;\text{ in } \mathbb{R}^N_+, \qquad \partial_N u = g_0 \;\text{ on } \Gamma,$$

which admits a solution $u \in W^{3,\,p}_{l-1}(\mathbb{R}^N_+)$, if the following compatibility condition is satisfied (see [7], Theorem 2.8):

$$\forall \psi \in \mathscr{N}^{\Delta}_{[-2+l-N/p']}, \quad \langle v, \psi\rangle_{W^{1,\,p}_{l-1}(\mathbb{R}^N_+)\times W^{-1,\,p'}_{-l+1}(\mathbb{R}^N_+)} + \langle g_0, \psi\rangle_\Gamma = 0. \tag{42}$$

Here again the compatibility condition (40) of problem $(\mathcal{R})$ implies the conditions (41) and (42). On the one hand condition (40) must be satisfied for any $\vartheta \in \mathscr{N}^{\Delta}_{[l-N/p']}$, and that gives (41).

On the other hand if we introduce the equations $\Delta v = f$ in $\mathbb{R}^N_+$ and $\partial_N v = g_1$ on Γ, in condition (40); with the Green formula

$$\langle \Delta v, \varphi\rangle_{W^{-1,\,p}_{l-1}(\mathbb{R}^N_+)\times W^{1,\,p'}_{-l+1}(\mathbb{R}^N_+)} = \langle v, \Delta\varphi\rangle_{W^{1,\,p}_{l-1}(\mathbb{R}^N_+)\times W^{-1,\,p'}_{-l+1}(\mathbb{R}^N_+)} - \langle \partial_N v, \varphi\rangle_\Gamma,$$

and the remark that any $\psi \in \mathscr{N}^{\Delta}_{[-2+l-N/p']}$ can be written $\psi = \Delta\varphi$ with $\varphi \in \mathscr{N}^{\Delta^2}_{[l-N/p']}$, then we obtain (42).

5.2.4. *Regularity of solutions to Problem* $(\mathcal{R})$. To complete Theorem 5.4, we can give a result for different types of data.

Theorem 5.5. *Let $l \in \mathbb{Z}$ and $m \in \mathbb{N}$. Under hypothesis (11), for any $f \in W^{m,\,p}_{m+l}(\mathbb{R}^N_+)$, $g_0 \in W^{m+3-1/p,\,p}_{m+l}(\Gamma)$ and $g_1 \in W^{m+1-1/p,\,p}_{m+l}(\Gamma)$ satisfying the compatibility condition (40), problem $(\mathcal{R})$ admits a solution $u \in W^{m+4,\,p}_{m+l}(\mathbb{R}^N_+)$, unique up to an element of $\mathscr{N}^{\Delta^2}_{[4-l-N/p]}$, with the estimate*

$$\inf_{q\in\mathscr{N}^{\Delta^2}_{[4-l-N/p]}} \|u+q\|_{W^{m+4,\,p}_{m+l}(\mathbb{R}^N_+)} \;\le\; C\left(\|f\|_{W^{m,\,p}_{m+l}(\mathbb{R}^N_+)} + \|g_0\|_{W^{m+3-1/p,\,p}_{m+l}(\Gamma)} + \|g_1\|_{W^{m+1-1/p,\,p}_{m+l}(\Gamma)}\right).$$

It can be readily checked that the kernel is unchanged under the hypothesis $\dfrac{N}{p} \notin \{1,\dots,-l-m\}$. We also keep the compatibility condition (40) and the proof for the existence is similar to that employed in Theorem 5.4 by means of the regularity result for the two Neumann problems.

5.3. **Third case.** Consider now the problem

$$\Delta^2 u = f \;\text{ in } \mathbb{R}^N_+, \qquad \Delta u = g_0 \;\text{ and }\; \partial_N \Delta u = g_1 \;\text{ on } \Gamma.$$

Let's first note that these boundary conditions do not satisfy the complementing condition by Agmon-Douglis-Nirenberg (see [2]). Thus this problem is ill-posed. Indeed, if we set $v = \Delta u$, we obtain

$$\Delta v = f \;\text{ in } \mathbb{R}^N_+, \qquad v = g_0 \;\text{ and }\; \partial_N v = g_1 \;\text{ on } \Gamma,$$

i.e. a Laplace equation with both Dirichlet and Neumann boundary conditions.

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