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BACK TO THE KELLER-OSSERMAN CONDITION FOR BOUNDARY BLOW-UP SOLUTIONS.

S. DUMONT, L. DUPAIGNE, O. GOUBET, V. RĂDULESCU

Abstract. This article is concerned with the existence, uniqueness and numerical approximation of boundary blow up solutions for elliptic PDE's as $\Delta u = f(u)$ where f satisfies the so-called Keller-Osserman condition. We characterize existence of such solutions for non-monotone f . As an example, we construct an infinite family of boundary blow up solutions for the equation $\Delta u = u^2(1+\cos u)$ on a ball. We prove uniqueness (on balls) when f is increasing and convex in a neighborhood of infinity and we discuss and perform some numerical computations to approximate such boundary blow-up solutions.

2000 AMS Mathematics Subject Classification. 35J60.

Key words. Elliptic equations, boundary blow-up, Keller-Osserman condition

In memory of Thomas Lachand-Robert

1. Introduction

1.1. **Setting the problem.** . Let f be a nonnegative function defined on $[0,+\infty[$ such that $f(0)=0$. We assume, for the sake of simplicity that f is a C^1 function. Considering Ω a smooth bounded domain of $\mathbb{R}^D$, $D \geq 1$, we seek $u > 0$ a smooth function such that

$$\begin{cases} \Delta u = f(u) \ in \ \Omega, \\ u = +\infty \ on \ \partial\Omega, \end{cases} \tag{1}$$

where the boundary condition is to be understood as :

$$\lim_{x \to x_0} u(x) = +\infty \quad \forall x_0 \in \partial\Omega.$$

To prove existence of such a boundary blow-up solution, it is classically assumed that f is a nondecreasing function with suitable growth rate at infinity, as demonstrated independently by Keller [10] and Osserman [14].

In this article, we study existence, asymptotic behaviour, uniqueness and numerical approximation of solutions of (1), when f may exhibit *non monotone* behaviour.

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1.1.1. *Existence results.* Existence of solutions of (1) is closely related to the following growth conditions : for $s \in [0, +\infty)$, let $F(s) = \int_0^s f(t)\,dt$ and define $\Phi : (0, +\infty) \to (0, +\infty]$ by

$$\Phi(\alpha) = \frac{1}{\sqrt{2}} \int_\alpha^\infty \frac{ds}{\sqrt{F(s) - F(\alpha)}} \ ,$$

where we let by convention $\Phi(\alpha) = +\infty$, whenever the integral is divergent or $F(s) = F(\alpha)$ on a set of positive measure.

Definition 1.1. We say that f satisfies the Keller-Osserman condition whenever

$$\exists \alpha > 0 \quad \Phi(\alpha) < \infty. \tag{2}$$

We say that f satisfies the Sharpened Keller-Osserman condition whenever

$$\liminf_{\alpha \to \infty} \Phi(\alpha) = 0. \tag{3}$$

Clearly, the Sharpened Keller-Osserman condition implies the classical one. Under fairly weak assumptions on the nonlinearity f, it turns out that both conditions are equivalent :

Proposition 1.2. *Assume there exist $\eta, T > 0$ such that $t \to \frac{F(t)}{t^\eta}$ is non-decreasing for $t > T$. Assume* (2) *holds for some $\alpha > 0$. Then,*

$$\lim_{\alpha \to \infty} \Phi(\alpha) = 0.$$

See the Appendix for a proof. In the special case $f(u) = u^2(1 + \cos u)$, $\limsup_{\alpha \to \infty} \Phi(\alpha) = +\infty$. However, (3) still holds (see the Appendix). This raises the following question :

Open Problem 1. *Does there exist a nonnegative C^1 function f such that* (2) *holds for some $\alpha > 0$ but $\liminf_{\alpha \to \infty} \Phi(\alpha) \in (0, +\infty]$?*

With these definitions in mind, our main results concerning existence read as follows :

Theorem 1.3. *f satisfies the Keller-Osserman condition if and only if* (1) *admits (at least) a positive boundary blow-up solution on* **some** *ball.*

Theorem 1.4. *f satisfies the Sharpened Keller-Osserman condition if and only if* (1) *admits (at least) a positive boundary blow-up solution on* **any** *(smooth bounded) domain Ω.*

In particular, Theorem 1.4 implies existence of boundary blow-up solutions for functions such as $f(u) = u^2(1 + \cos u)$.

1.1.2. *Asymptotic behaviour.* The blow-up rate of solutions of (1) is determined implicitely by the following theorem :

Theorem 1.5. *Assume Ω satisfies an interior and an exterior sphere condition at every point of its boundary. Assume* (3) *holds and let u denote any positive solution of* (1)*. Then,*

$$\lim_{x\to x_0} \frac{\displaystyle\int_{u(x)}^{\infty} \frac{dt}{\sqrt{2F(t)}}}{\delta(x)} = 1,$$

where $\delta(x) = \mathrm{dist}(x, \partial\Omega)$.

1.1.3. *Uniqueness results.* In view of the maximum principle, it seems natural to only assume that f is nondecreasing in order to obtain uniqueness. To the best of our knowledge, no proof (or counter-example) of such a statement has been given yet. Extra requirements such as the convexity of f or the monotony of $f(u)/u$ are needed in the proofs found in the literature (see e.g. [13]). When the domain is a ball, we relax such assumptions as follows:

Theorem 1.6. *Assume that f is nondecreasing and that f is convex in a neighborhood of $+\infty$. Assume also that f satisfies the Keller-Osserman condition* (2)*. Then on any ball $B(0, R)$ there exists a unique boundary blow up solution of* (1)*.*

Remark 1.7. The same result also holds if f is nondecreasing, $f(u)/u$ is nondecreasing in a neighbourhood of $+\infty$ and (2) holds.

In dimension $D = 1$, a necessary and sufficient condition for uniqueness can be derived. Namely, we have the following

Proposition 1.8. *Assume that f satisfies the Sharpened Keller-Osserman condition. Then* (1) *admits a unique solution on $\Omega =]-R, R[$ if and only if the equation*

$$\Phi(\alpha) = R$$

admits exactly one solution.

As a straightforward consequence, we obtain

Corollary 1.9. *Assume that f satisfies the Sharpened Keller-Osserman condition.* (1) *admits a unique solution on* **any** *domain $\Omega =]-R, R[$ if and only if $\Phi : (0, \infty) \to (0, \infty)$ is one-to-one.*

Remark 1.10. In particular, if f is nondecreasing, one can easily show that Φ is one-to-one.

2. Existence of boundary blow-up solutions

2.1. **Minimality Principle.** We restate the well-known sub and supersolution method (see [5] and [6]) and derive elementary but important corollaries.

Proposition 2.1. *Consider Ω a bounded domain of $\mathbb{R}^D$ such that all boundary points are regular, $f \in C(\mathbb{R})$ and $g \in C(\partial\Omega)$. Assume there exist two functions $\underline{u}$, $\overline{u} \in C(\bar{\Omega})$ such that $\underline{u} \leq \overline{u}$ and*

$$\begin{cases} \Delta\underline{u} \geq f(\underline{u}) \text{ in } \mathcal{D}'(\Omega), & (\text{resp. } \Delta\overline{u} \leq f(\overline{u}) \text{ in } \mathcal{D}'(\Omega)) \\ \underline{u} \leq g \text{ on } \partial\Omega, & (\text{resp. } \overline{u} \geq g \text{ on } \partial\Omega), \end{cases} \tag{4}$$

Then problem

$$\begin{cases} \Delta u = f(u) \text{ in } \mathcal{D}'(\Omega), \\ u = g \text{ on } \partial\Omega. \end{cases} \tag{5}$$

possesses at least one solution $u \in C(\bar{\Omega}, \mathbb{R})$ such that $\underline{u} \leq u \leq \overline{u}$.

Corollary 2.2. *(Minimality Principle) Make the same assumptions as in Proposition 2.1. Then there exists a unique solution $u \in C(\bar{\Omega})$ of (5) such that $\underline{u} \leq u$ and $u|_\omega \leq \overline{v}$ for any open subset ω of Ω and any function $\overline{v} \in C(\bar{\omega})$ satisfying*

$$\begin{cases} \Delta\overline{v} \leq f(\overline{v}) \text{ in } \mathcal{D}'(\omega), \\ \overline{v} \geq \underline{u} \text{ in } \omega, \\ \overline{v} \geq u \text{ on } \partial\omega. \end{cases} \tag{6}$$

We call u the minimal solution of (5) relative to $\underline{u}$.

A complete proof of Corollary 2.2 is given in the Appendix. We present here a short proof in the case where f is a locally Lipschitz function.
Proof:

Uniqueness : Let u_1, u_2 be two such solutions. Choosing $\omega = \Omega$ and $\overline{v} = u_2$ in the statement of Corollary 2.2, we conclude that $u_1 \leq u_2$. Reversing the roles of u_1 and u_2, we conclude that $u_1 = u_2$.

Existence :
Let $\Lambda = \sup_{[\min \underline{u}, \max \overline{u}]} |f'|$, $u_0 = \underline{u}$ and for $k \geq 1$, define $u_k \in C(\bar{\Omega})$ inductively by

$$\begin{cases} \Delta u_k - \Lambda u_k = f(u_{k-1}) - \Lambda u_{k-1} \text{ in } \mathcal{D}'(\Omega), \\ u_k = g \text{ on } \partial\Omega. \end{cases}$$

Then it is known that the sequence (u_k) is nondecreasing and converges to a solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ of (5), which satisfies in addition $\underline{u} \leq u \leq \overline{u}$.

Let $\overline{v} \in C(\bar{\omega})$ verify (6) and assume by contradiction that the set $\omega_1 := \{x \in \omega \ : \ \overline{v}(x) < u(x)\}$ is non empty. Clearly ω_1 is open. Working if necessary with a connected component of ω_1, we assume that ω_1 is connected. We prove by induction that $\overline{v} \geq u_k$ in ω_1 for all $k \in \mathbb{N}$. Passing to the limit as $k \to \infty$, we then obtain a contradiction with the definition of ω_1.

By assumption, $\overline{v} \geq \underline{u} = u_0$ in ω_1. Given $k \geq 1$, assume that $\overline{v} \geq u_{k-1}$ in ω_1. In particular, we have that $\overline{v}(x) \in [\min \underline{u}, \max \overline{u}]$ for $x \in \omega_1$.

Observe that if $x \in \partial\omega_1$ then either $\overline{v}(x) = u(x)$, either $x \in \partial\omega$, whence $\overline{v}(x) \geq u(x)$. Since $u \geq u_k$, we conclude that $\overline{v} \geq u \geq u_k$ on $\partial\omega_1$. Hence,

$$\begin{cases} \Delta(\overline{v} - u_k) - \Lambda(\overline{v} - u_k) \leq f(\overline{v}) - f(u_{k-1}) - \Lambda(\overline{v} - u_{k-1}) \leq 0 \ in \ \mathcal{D}'(\omega_1) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overline{v} - u_k \geq 0 \ on \ \partial\omega_1. \end{cases}$$

By the (weak) Maximum Principle, $\overline{v} \geq u_k$ in ω_1.

Remark 2.3. Applying the Minimality Principle to (5) with nonlinearity $-f$, we also obtain the existence and uniqueness of a maximal solution relative to $\overline{u}$, defined in the straightforward way.

Remark 2.4. Assume $\Omega = B_R$ is a ball centered at the origin and g is a positive constant. If $\underline{u}$ is radial , one easily sees that the minimal solution u relative to $\underline{u}$ is radial : just apply the Minimality Principle 2.2 with $\overline{v}(x) = u(O(x))$, where $O \in \mathcal{O}_D$ is an arbitrary rotation of the Euclidean space. A well-known result of Gidas-Ni-Nirenberg [9] states that **any** solution $u > g$ is radially symmetric, provided f is e.g. locally Lipschitz.

Finally, letting $\phi(r) = u(x)$ for $r = |x|$, it follows from standard ODE theory that $\phi'(0) = 0$ and $\phi'(r) > 0$ in $(0, R)$.

Corollary 2.5. *(Minimality Principle for blow-up solutions) Let Ω be a bounded domain of $\mathbb{R}^D$ such that all boundary points are regular, $f \in C(\mathbb{R})$. Assume there exist a function $\underline{u} \in C(\bar{\Omega})$ such that $\Delta\underline{u} \geq f(\underline{u})$ in $\mathcal{D}'(\Omega)$ and a function $v \in C(\Omega)$ such that $\Delta v \leq f(v)$ in $\mathcal{D}'(\Omega)$, $\lim_{x\to x_0} v(x) = +\infty$ for all $x_0 \in \partial\Omega$ and $v \geq \underline{u}$.*

Then there exists a unique solution $u \in C(\Omega)$ of (1) such that $\underline{u} \leq u$ and $u|_\omega \leq \overline{v}$ for any open subset $\omega \subset \Omega$ and any $\overline{v} \in C(\omega)$ satisfying

$$\begin{cases} \Delta\overline{v} \leq f(\overline{v}) & in \ \mathcal{D}'(\omega), \\ \overline{v} \geq \underline{u} & in \ \omega, \\ \lim\limits_{x\to x_0} \overline{v}(x) = +\infty & for \ all \ x_0 \in \partial\omega. \end{cases} \tag{7}$$

We call u the minimal solution of (1) relative to $\underline{u}$.

For a proof, see the Appendix.

Remark 2.6. In contrast with Remark 2.3, there *does not* exist in general a maximal boundary blow-up solution of (1). See Section 6.3 for enlightening counter-examples.

2.2. **Existence of solutions on some ball.** In this section, we prove that (2) implies the existence of a boundary blow-up solution on some ball. First, we state and prove a useful technical lemma

Lemma 2.7. *Let $\phi \in C^2(0, R)$ be a nondecreasing function solving*

$$\phi'' + \frac{D-1}{r}\phi' = f(\phi) \quad in \ (0, R) \tag{8}$$

Then, given $0 < r_1 < r_2 < R$,

$$\frac{1}{\sqrt{2}}\int_{\phi(r_1)}^{\phi(r_2)} \frac{1}{\sqrt{F(s)-F(\phi(r_1))}}\,ds \geq \frac{1}{D-2} r_1 \left(1-\left(\frac{r_1}{r_2}\right)^{D-2}\right),$$

if $D \neq 2$ *and*

$$\frac{1}{\sqrt{2}}\int_{\phi(r_1)}^{\phi(r_2)} \frac{1}{\sqrt{F(s)-F(\phi(r_1))}}\,ds \geq r_1 \ln \frac{r_2}{r_1},$$

if $D = 2$.

Proof: for $r \in (r_1, r_2)$, (8) is equivalent to

$$\frac{d}{dr}\left(r^{D-1}\phi'\right) = r^{D-1} f(\phi). \tag{9}$$

Multiplying the above equation by $r^{D-1}\phi'$ and integrating between r_1 and r, we obtain

$$\begin{aligned}\frac{1}{2} r^{2D-2}\left(\phi'(r)\right)^2 &\geq \frac{1}{2}\left(r^{2D-2}\left(\phi'(r)\right)^2 - r_1^{2D-2}\left(\phi'(r_1)\right)^2\right)\\ &= \int_{r_1}^{r} t^{2D-2} f(\phi(t))\phi'(t)\,dt\\ &\geq r_1^{2D-2}\left(F(\phi(r)) - F(\phi(r_1))\right).\end{aligned}$$

So,

$$\frac{1}{\sqrt{2}} \frac{\phi'(r)}{\sqrt{F(\phi(r)) - F(\phi(r_1))}} \geq \left(\frac{r_1}{r}\right)^{D-1}.$$

Integrating the above equation between r_1 and r_2, we obtain the desired result. □

Assume now (2) holds for some $\alpha > 0$. If $D \neq 2$, assume temporarily that $\Phi(\alpha) < \frac{1}{|D-2|}$. Applying Proposition 2.1 with $\underline{u} = 0$ and $\overline{u} = \alpha$, let u be the minimal solution relative to $\underline{u}$ of

$$\begin{cases} \Delta u = f(u) \text{ in } B_1, \\ \quad u = \alpha \text{ on } \partial B_1. \end{cases}$$

Using Remark 2.4 and letting $\tilde{\alpha} = u(0)$, $\phi(r) := u(x)$ for $r = |x|$ solves (8), subject to the initial conditions $\phi(0) = \tilde{\alpha}$ and $\phi'(0) = 0$. ϕ can thus be extended on some maximal interval $(0, R)$. Assume temporarily that $R < \infty$. Then u is a boundary blow-up solution on B_R. Indeed, by definition of R, we must have either $\phi(R) = +\infty$ or $\phi'(R) = +\infty$. In the latter case, multiply (8) by ϕ' and integrate between 0 and r to obtain that $\frac{1}{2}(\phi')^2 \leq F(\phi)$. Hence $F(\phi(R)) = +\infty$, $\phi(R) = +\infty$ and u is a boundary blow-up solution. It remains to prove that $R < \infty$.

Assume by contradiction that $R = \infty$. Apply Lemma 2.7 between $r_1 = 1$ and $r_2 > 1$:

$$\Phi(\alpha) \geq \frac{1}{D-2}\left(1-\left(\frac{1}{r_2}\right)^{D-2}\right),$$

if $D \geq 3$ and

$$\Phi(\alpha) \geq \ln r_2,$$

if $D = 2$. Letting r_2 converge to ∞, we obtain a contradiction if either $D = 2$ or $\Phi(\alpha) < \frac{1}{|D-2|}$.

If $D \neq 2$ and $\Phi(\alpha) \geq \frac{1}{|D-2|}$, choose $K > 0$ so large that $\frac{1}{K}\Phi(\alpha) < \frac{1}{|D-2|}$. The above proof provides a boundary blow-up solution u of (1) on some ball B_R, when f is replaced by $K^2 f$. $\tilde{u}(x) := u(x/K)$ is then a boundary blow-up solution of (1) with nonlinearity f on B_{RK}. □

Remark 2.8. Let B be a ball of radius R and assume $\underline{u} \in C(\bar{B})$ is such that $\Delta \underline{u} \geq f(\underline{u})$ in B. Assume (2) holds for some $\alpha \geq \sup_B \underline{u}$ and let $\overline{u} = \alpha$. Using Proposition 2.1, let u be the minimal solution relative to $\underline{u}$ of

$$\begin{cases} \Delta u = f(u) \text{ in } B, \\ \quad u = \alpha \text{ on } \partial B. \end{cases}$$

Repeating the above proof, we conclude that u can be extended to a radially symmetric boundary blow-up solution on some ball $\tilde{B}$ of radius $\tilde{R} > R$, satisfying $u \geq \underline{u}$ in B.

2.3. **Existence of solutions on small balls.** Assume (3) holds. By Theorem 1.3, (1) has a solution on some ball and we may define

$$R_0 := \inf\{R > 0 \ : \ (1) \text{ has a solution in } B_R\}$$

We assume by contradiction that $R_0 > 0$. Let (β_n) be a sequence of real numbers increasing to infinity and satisfying

$$\lim_{\beta_n \to \infty} \Phi(\beta_n) = 0.$$

Applying Proposition 2.1 with $\underline{u} = 0$ and $\overline{u} = \beta_n$, let u_n be the minimal solution relative to $\underline{u}$ of

$$\begin{cases} \Delta u_n = f(u_n) \text{ in } B_{R_0/2}, \\ \quad u_n = \beta_n \text{ on } \partial B_{R_0/2}. \end{cases}$$

By Remark 2.4, letting $\alpha_n = u_n(0)$, $\phi_n(r) := u_n(x)$ for $r = |x|$ solves (8) with initial conditions $\phi_n(0) = \alpha_n$ and $\phi_n'(0) = 0$. By definition of R_0, ϕ_n can be extended so that ϕ_n remains a solution of (8) in $(0, R_0)$. Now apply Lemma 2.7 with $r_1 = R_0/2$ and $r_2 = R_0$:

$$\Phi(\beta_n) \geq \frac{1}{D-2}\frac{R_0}{2}\left(1 - \left(\frac{1}{2}\right)^{D-2}\right),$$

if $D \geq 3$ and

$$\Phi(\beta_n) \geq \frac{R_0}{2}\ln(2),$$

if $D = 2$. Passing to the limit as $n \to \infty$, we obtain a contradiction in both cases. We have just proved that

$$\inf\{R > 0 \ : \ (1) \text{ has a solution in } B_R\} = 0. \tag{10}$$

Remark 2.9. Let B be a ball of radius R and assume $\underline{u} \in C(\bar{B})$ is such that $\Delta \underline{u} \geq f(\underline{u})$ in B. Using Remark 2.8 and working as above, one can show that $\inf\{\tilde{R} > R \ : \ (1) \text{ has a solution } u \text{ in } B_{\tilde{R}} \text{ such that } u \geq \underline{u} \text{ in } B\} = R$.

2.4. **Existence of solutions on smooth domains.** We assume here that (3) is valid. Applying Proposition 2.1 with $\underline{u} = 0$ and $\overline{u} = N$, $N \in \mathbb{N}$, let u_N be the minimal solution relative to $\underline{u}$ of (5) with $g \equiv N$. For $x \in \Omega$, choose a ball $B(x, r) \subset \Omega$ such that there exists a boundary blow-up solution u_r on $B(x, r)$. This is always possible since (10) holds. Applying the Minimality Principle 2.2 with $\bar{v} = u_r$, we conclude that $0 \leq u_N \leq u_r$ in $B(x, r)$. In particular, the sequence (u_N) is uniformly bounded in $B(x, r/2)$.

Let K denote an arbitrary compact subset of Ω. Covering K by finitely many balls $B(x_i, r_i/2)$, we conclude that (u_N) is uniformly bounded on K by a constant depending only on K and f. Applying the Minimality Principle 2.2 with $\bar{v} = u_{N+1}$, we can also infer that (u_N) is a nondecreasing sequence. Using these two facts and elliptic regularity, we conclude that (u_N) converges to a function u solving $\Delta u = f(u)$ in Ω.

Fix a point $x_0 \in \partial\Omega$ and an arbitrary sequence (x_k) in Ω converging to x_0. Then, since $u \geq u_N$,

$$\liminf_{k\to\infty} u(x_k) \geq \liminf_{k\to\infty} u_N(x_k) = N.$$

Letting N converge to infinity, we conclude that u is a boundary blow-up solution of (1) in Ω.

2.5. **Proof of Theorem 1.3 completed.** By Section 2.2, we know that if (2) holds, there exists a blow-up solution on some ball. Conversely, assume that $u > 0$ solves (1) on some ball B of radius R centered at the origin. By Corollary 2.5, we may always assume that u is the minimal solution relative to $\underline{u} = 0$ of (1). In particular u is radial and we define $\phi(r) = u(x)$ for $r = |x|$, so that ϕ solves (9) in $(0, R)$. Multiplying (9) by $r^{D-1}\phi'$ and integrating between 0 and r, we obtain

$$\frac{1}{2} r^{2D-2} \phi'(r)^2 = \int_0^r t^{2D-2} f(\phi(t))\phi'(t)\, dt \leq r^{2D-2} \left[F(\phi(r)) - F(\phi(0))\right].$$

Integrating once more between 0 and R,

$$0 \leq \int_0^R \frac{\phi'(r)}{\sqrt{2\left[F(\phi(r)) - F(\phi(0))\right]}}\, dr \leq R, \tag{11}$$

which implies (2) with $\alpha = \phi(0)$.

2.6. **Proof of Theorem 1.4 completed.** By Section 2.4, we know that if (3) holds, there exists a blow-up solution on any domain. Conversely, given $n \in \mathbb{N}$, assume that $u_n > 0$ solves (1) on the ball B of radius $1/n$ centered at the origin. By Corollary 2.5, we may always assume that u_n is the minimal solution relative to $\underline{u} = 0$. In particular u_n is radial. Let now $\beta_n = u_n(0)$. We claim that (β_n) is unbounded. Taking a subsequence if necessary, we then have that $\lim_n \beta_n = \infty$ and (3) follows from (11) applied with $R = 1/n$.

It remains to prove that (β_n) is unbounded. If not, up to a subsequence, (β_n) converges to some $\beta \geq 0$. By (11) applied with $R = 1/n$, we have

$$0 \leq \int_{\beta_n}^{\infty} \frac{dt}{\sqrt{2\left[F(t) - F(\beta_n)\right]}}\, dr \leq 1/n.$$

By Fatou's lemma, we conclude that

$$\int_{\beta}^{\infty} \frac{dt}{\sqrt{2\left[F(t) - F(\beta)\right]}}\, dr = 0,$$

which is not possible.

3. Asymptotic behaviour of solutions

3.1. **Blow-up rate of radially symmetric solutions.**

Proposition 3.1. *Assume that f satisfies the Keller-Osserman condition* (2). *Assume ϕ is a radially symmetric and monotone boundary blow-up solution on the unit ball. Then, for $r \sim 1$,*

$$\int_{\phi(r)}^{+\infty} \frac{dt}{\sqrt{F(t)}} \sim \sqrt{2}(1-r) \tag{12}$$

Proof: Multiplying (9) by $r^{D-1}\phi'$ and integrating by parts, we easily obtain that given $r \in (0,1)$,

$$\frac{(\phi')^2(r)}{2} = F(\phi(r)) - G_\phi(r), \tag{13}$$

where

$$G_\phi(r) = \frac{2D-2}{r} \int_0^r \left(\frac{s}{r}\right)^{2D-1} F(\phi(s))ds.$$

We claim that

$$G_\phi(r) = o(F(\phi(r))), \qquad \text{as } r \to 1. \tag{14}$$

Let indeed $\epsilon > 0$. Then, since F is nondecreasing,

$$\begin{aligned}\frac{G_\phi(r)}{F(\phi(r))} &= \frac{2D-2}{r} \int_0^{1-\epsilon} \left(\frac{s}{r}\right)^{2D-1} \frac{F(\phi(s))}{F(\phi(r))} ds \\ &\qquad + \frac{2D-2}{r} \int_{1-\epsilon}^{r} \left(\frac{s}{r}\right)^{2D-1} \frac{F(\phi(s))}{F(\phi(r))} ds. \\ &\leq C\frac{F(\phi(1-\epsilon))}{F(\phi(r))} \qquad + C\epsilon.\end{aligned}$$

Letting $r \to 1$ and then $\epsilon \to 0$, we obtain the desired result. Returning to (13), we obtain

$$1 - \frac{\phi'}{\sqrt{2F(\phi)}} = 1 - \left[1 - \frac{G_\phi}{F(\phi)}\right]^{1/2}.$$

Combining this with (14), it follows that for $r \sim 1$,

$$1 - \frac{\phi'}{\sqrt{2F(\phi)}} \sim \frac{G_\phi}{2F(\phi)}$$

and, integrating between r and 1,

$$(1-r) - \int_{\phi(r)}^{\infty} \frac{dt}{\sqrt{2F(t)}} \sim \int_r^1 \frac{G_\phi(s)}{2F(\phi(s))} ds = o(1),$$

which implies (12).

3.2. **Blow-up rate of solutions on smooth domains.** Let u be a blow-up solution on a domain Ω, which satisfies an interior and an exterior sphere condition at any boundary point. Fix $x_0 \in \partial\Omega$ and let $B_R \subset \Omega$ denote a small ball which is tangent to $\partial\Omega$ at x_0. Fix $\eta \in (0,1)$. Let $\underline{u} := u|_{B_{\eta R}}$. By Remark 2.8, there exists a radial boundary blow-up solution v defined on some ball $\tilde{B} \supset B_{\eta R}$, such that $v \geq \underline{u}$ in $B_{\eta R}$. Let $K > 0$ such that $K\tilde{B} = B_R$ and let $v_K(x) := v((x - x_1)/K + x_1)$, where x_1 is the center of B_R. Then v_K solves

$$\Delta v_K = \frac{1}{K^2} f(v_K) \ in \ B_R,$$
$$v_K = +\infty \ on \ \partial B_R.$$

Since $u(x_1) \leq v(x_1) = v_K(x_1)$, Proposition 3.1 implies that

$$K \int_{u(x_1)}^{+\infty} \frac{dt}{\sqrt{F(t)}} \geq K \int_{v_K(x_1)}^{+\infty} \frac{dt}{\sqrt{F(t)}} \sim \sqrt{2}R.$$

Letting $R \to 0$, we then have

$$K \liminf_{x \to x_0} \frac{\int_{u(x)}^{+\infty} \frac{dt}{\sqrt{F(t)}}}{\delta(x)} \geq \sqrt{2}.$$

By Remark 2.9, we may take K arbitrarily close to $1/\eta$. Also, $0 < \eta < 1$ was chosen arbitrarily, so letting $K, \eta \to 1$, we finally obtain

$$\liminf_{x \to x_0} \frac{\int_{u(x)}^{+\infty} \frac{dt}{\sqrt{F(t)}}}{\delta(x)} \geq \sqrt{2}.$$

Choose another ball $B_{R'} \subset \mathbb{R}^N \setminus \bar{\Omega}$ which is tangent to $\partial\Omega$ at x_0 and a concentric ball $B_{R''}$ with $R'' > R'$ so large that $\Omega \subset B_{R''}$. Finally, let $A = B_{R''} \setminus B_{R'}$. Let v denote the minimal boundary blow-up solution (relative to $\underline{u} = 0$) on A. By the Minimality Principle 2.5, we deduce that $u \geq v$ in Ω. Applying Proposition 3.1 (which still holds on an annulus) with v, we conclude that

$$\limsup_{x \to x_0} \frac{\int_{u(x)}^{+\infty} \frac{dt}{\sqrt{F(t)}}}{\delta(x)} \leq \limsup_{x \to x_0} \frac{\int_{v(x)}^{+\infty} \frac{dt}{\sqrt{F(t)}}}{\delta(x)} \leq \sqrt{2}.$$

This finishes the proof of Theorem 1.5. More can be said about the asymptotic behaviour of solutions provided F satisfies some extra growth assumption :

Lemma 3.2. *Let u, v denote two radially symmetric boundary blow-up solutions defined on the unit ball B. Assume there exist $\beta > 0$ and $M > 0$ such that $\frac{F(v)}{v^2} \geq \beta^2 \frac{F(u)}{u^2}$ whenever $M \leq u \leq v$. Then $u(r) \sim v(r)$ on ∂B.*

Proof: We recall from the proof of of Proposition 3.1 that for ϕ a radially symmetric boundary blow up solution and for $r \sim 1$,

$$(1-r) - \int_{\phi(r)}^{\infty} \frac{dt}{\sqrt{2F(t)}} \leq C \int_r^1 \frac{G_\phi(s)}{F(\phi(s))} ds \leq C \int_r^1 \frac{\phi(s)}{\sqrt{F(\phi(s))}} ds.$$

Using that $\frac{\dot{\phi(s)}}{\sqrt{2F(\phi(s))}} \sim 1$, we then obtain that, introducing $K(r)$ such that $1 - r = \int_{K(r)}^{\infty} \frac{dt}{\sqrt{2F(t)}}$,

$$\int_{K(r)}^{\phi(r)} \frac{dt}{\sqrt{2F(t)}} \leq C \int_{\phi(r)}^{\infty} \frac{t}{F(t)} dt. \tag{15}$$

Since F is increasing, we thus obtain

$$\left(1 - \frac{K(r)}{\phi(r)}\right) \leq C \frac{\sqrt{F(\phi(r))}}{\phi(r)} \int_{\phi(r)}^{\infty} \frac{t}{F(t)} dt. \tag{16}$$

Since $\frac{F(v)}{v^2} \geq \beta^2 \frac{F(u)}{u^2}$ for $u \leq v$ large enough, (16) implies that

$$\left(1 - \frac{K(r)}{\phi(r)}\right) \leq \frac{C}{\beta} \int_{\phi(r)}^{\infty} \frac{dt}{\sqrt{F(t)}}.$$

The classical Keller-Osserman (2) condition gives the result. □

Corollary 3.3. *Assume either that f is convex on some interval $[a, +\infty)$ or that $f(t)/t$ is nondecreasing on $[a, +\infty)$. Then the result of the previous lemma holds.*

Proof : Assume f is convex in $[a, +\infty)$ and let $G(t) = F(t+a) - F(a) - f(a)t$. Then $G(0) = G'(0) = 0$ and G' is convex in $\mathbb{R}^+$. So $G(t)/t^2$ is nondecreasing, i.e. $t \to \frac{F(t+a)-L(t)}{t^2}$ is nondecreasing, where $L(t) = F(a) + f(a)t$ is affine. Observe that $\lim_{t\to\infty} F(t)/t^2 = +\infty$ since the Keller-Osserman condition (2) implies

$$\frac{u}{2\sqrt{F(u)}} \leq \int_{u/2}^{u} \frac{dt}{\sqrt{F(t)}} = o(1) \qquad \text{as } u \to \infty.$$

It follows that there exists $\beta > 0$ such that $\frac{F(v)}{v^2} \geq \beta^2 \frac{F(u)}{u^2}$ for $u \leq v$ large enough and we may apply Lemma 3.2. The case where $f(t)/t$ is nondecreasing on $[a, +\infty)$ is similar, so we skip it.

4. The uniqueness result

Lemma 4.1. *Assume that f is nondecreasing on $[0,+\infty)$ and convex in a neighborhood of $+\infty$ (say $[a,+\infty)$). Consider two radially symmetric boundary blow-up solutions such that $u(r) \le v(r)$ on $B(0,R)$ then in fact $u = v$ everywhere.*

Proof: Set $R = 1$ for the sake of simplicity. Since f is nondecreasing, either $u(0) = v(0)$ (and then $u = v$ everywhere) or $u(r) < v(r)$ everywhere since $\partial(r^{D-1}(\dot v - \dot u)) = r^{D-1}(f(v) - f(u))$ so the map $r \to v(r) - u(r)$ is nondecreasing. Assume then $u(0) < v(0)$.

Let $\varepsilon > 0$. Consider the set $\omega_\varepsilon = \{r \in [0,1); \forall s < r,\ (1+\varepsilon)u(s) < v(s)\}$. If ε is small enough, $0 \in \omega_\varepsilon$. Due to lemma 3.2, $R = 1 \notin \omega_\varepsilon$ since $u \sim v$ close to the boundary. Then introduce $r_\varepsilon^0 = \sup \omega_\varepsilon$, which satisfies $0 < r_\varepsilon^0 < 1$. We now have

$$v(0) - u(0) \le v(r_\varepsilon^0) - u(r_\varepsilon^0) = \varepsilon u(r_\varepsilon^0). \tag{17}$$

Then either r_ε^0 converges towards $R = 1$ when $\varepsilon \to 0$ or, letting $\varepsilon \to 0$ $u(0) = v(0)$ and the proof is over.
Introduce now a such that f is convex on $[a,+\infty)$. Introduce R_0 such that $u(r) \ge a$ for $r \ge R_0$. Then for ε small enough $r_\varepsilon^0 > R_0$. Set $w(r) = (1+\varepsilon)u(r)$. Then, on the annulus $R_0 < s < r_\varepsilon^0$, using the convexity

$$\begin{aligned}\Delta(v - w) &= f(v) - (1+\varepsilon)f(u)\\ &\ge f(v) - (1+\varepsilon)(f(w) - f(a)\frac{u-a}{w-a} - (1+\varepsilon)f(a)\\ &\ge f(v) - f(w) + \frac{\varepsilon}{w-a}(af(w) - wf(a)).\end{aligned} \tag{18}$$

Observe now that the map $X \to \frac{Xf(a) - af(X)}{X-a}$ is majorized by some constant C for $X \ge a$. Then introducing χ that satisfies $-\Delta\chi = 1$ with homogeneous Dirichlet condition at $R = 1$ ($\chi(1) = 0$)

$$\Delta(v - w - C\varepsilon\chi) \ge 0 \tag{19}$$

and by the maximum principle, for any r in $R_0 < r < r_\varepsilon^0$

$$v(r) - w(r) - C\varepsilon\chi(r) \le \max(-C\varepsilon\chi(r_\varepsilon^0), v(R_0) - w(R_0) - C\varepsilon\chi(R_0)). \tag{20}$$

Then letting $\varepsilon \to 0$ we obtain that for any fixed r such that $R_0 < r < 1$,

$$v(r) - u(r) \le v(R_0) - u(R_0). \tag{21}$$

Since the map $r \to v(r) - u(r)$ is nondecreasing, this implies that $v(r) - u(r)$ is *constant* on $[R_0, 1)$. By standard ODE theory, this implies that v and u coincide everywhere on $[0,1]$. □

Remark 4.2. Since f is nondecreasing, there exists $\underline{U}, \overline{U}$ the minimal and the maximal boundary blow-up solutions of the problem (the latter can be obtained e.g. as the monotone limit of $u(R)$ as $R \to 1^-$, where $u(R)$ denotes the minimal boundary blow-up solution on B_R). Clearly both $\underline{U}$ and $\overline{U}$ are radial and they coincide by the previous lemma. Since any solution u of the problem must stay between $\underline{U}$ and $\overline{U}$, Theorem 1.6 follows. Alternatively, according to a result of Poretta-Veron [15], any boundary blow-up solution is radially symmetric if f is convex in a neighborhood of $+\infty$, whence again Theorem 1.6 follows from the previous lemma.

Remark 4.3. The previous lemma is still valid if we substitute the assumption $\frac{f(u)}{u}$ increasing in a neighborhood of infinity to the convexity assumption. Since the proofs are easier they are left as an exercice to the reader.

5. Discrete equations

We are concerned with finite difference approximations of (1) when $D = 1$ or $D = 2$ on a cube or a ball. After introducing some notation, we observe that both the maximum principle and the minimality principle extend to the case of finite difference operators. We conclude this section with some theoretical error estimates, assuming that f is a nondecreasing function.

5.1. **Finite differences.** To begin with, consider the interval $[-1, 1]$ or the unit square $[-1, 1]^2$. Consider a uniform grid Ω_h which mesh size is $h = \frac{1}{L}$ for some integer L. The nodes on the grid are respectively jh if D=1, $-L \le j \le L$, or (ih, jh) if $D = 2$, with $-L \le i, j \le L$.

The discrete Laplace operator is then defined on each point/node of the grid respectively by

$$(\Delta_h U)_j = \frac{1}{h^2}(-2U_j + U_{j+1} + U_{j-1}), \tag{22}$$

if $D = 1$ and

$$(\Delta_h U)_{i,j} = \frac{1}{h^2}(-4U_{i,j} + U_{i,j+1} + U_{i,j-1} + U_{i+1,j} + U_{i-1,j}), \tag{23}$$

if $D = 2$. In the above, U is a vector in $\mathbb{R}^{2L+1}$ (or $\mathbb{R}^{(2L+1)2}$ in 2D) which components are the $U_j \simeq u(jh)$ (or $U_{i,j} \simeq u(ih, jh)$). If $D = 2$, we then solve $\Delta_h U = f(U)$, i.e. $(\Delta_h U)_{i,j} = f(U_{i,j})$ for all interior nodes (i, j) and set $U_{\pm L,j} = U_{i,\pm L} = N$ at all boundary nodes, where N is a fixed large constant. We work accordingly when $D = 1$.

It is standard to prove that the matrix Δ_h has positive inverse, i.e. the entries of the inverse matrix are positive. Therefore, the maximum principle is valid (see [4]). Actually, if U satisfies $\Delta_h U \le 0$ on the interior nodes of the grid and $U \ge 0$ on the boundary, then $U \ge 0$ everywhere. Here and throughout this section we write $U \ge 0$ iff $U_{i,j} \ge 0$ for all (i, j) nodes of the grid. We shall use the same notation for B a matrix: $B \ge 0$ iff the entries of B are all non negative.

When working on the unit ball, we use a slightly modified scheme. Focussing on radially symmetric functions, we approximate the equation

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) = f \qquad \text{for } r \in (0,1). \tag{24}$$

Discretize $[0,1]$ by setting $Lh = 1$, $0 \le j \le L$. At $j = L$ set $U_L = N$ (boundary condition). For $0 < j < L$ solve

$$\frac{1}{r}\partial(r\partial u) \simeq \frac{1}{jh}D^+(jhD^-U) = \frac{-2U_j + U_{j+1} + U_{j-1}}{h^2} + \frac{U_{j+1} - U_j}{jh^2} = F_j, \tag{25}$$

where $(D^+U)_j = \frac{U_{j+1}-U_j}{h}$, $(D^-U)_j = \frac{U_j - U_{j-1}}{h}$. It remains to define the equation at $j = 0$. For that purpose, we use the symmetry property $u(h) = u(-h)$ and the approximation $\frac{\dot{u}(0)}{0} = \ddot{u}(0)$ to set

$$\frac{4}{h^2}(-U_0 + U_1) = F_0.$$

This approximation of the Laplace operator satisfies the maximum principle. Indeed, it can be easily checked that if $F_j \ge 0$ then $j \to U_j$ is increasing. The maximum principle follows promptly.

5.2. **Computing an approximation.** We aim to solve the following problem

$$\begin{aligned} \Delta_h u_h &= f(u_h) \ in\ \Omega, \\ u_h &= N \ on\ \partial\Omega, \end{aligned} \tag{26}$$

for N large enough.

We expect that u_h is an approximation for u, the minimal boundary blow up solution corresponding to $\underline{u} = 0$. As in Proposition 2.1, u_h is obtained by monotone iteration, starting from the discrete subsolution 0. We claim that Proposition 2.1 and Corollary 2.2 are valid for the finite difference approximation. The proof follows the guidelines of the continuous case and is left as an exercice to the reader.

An approximation of the solution to problem (26) is recursively obtained by the following discrete iterative scheme:

Consider $u^k \in \mathbb{R}^{(2L+1)D}$, where $L = \frac{1}{h}$, recursively defined by

$$u^0 = 0 \qquad \text{and for } k \ge 0,$$

$$\begin{cases} (\Delta_h - \Lambda_N Id)u^{k+1} = f(u^k) - \Lambda_N u^k & in\ \Omega_h \\ u^{k+1} = N \qquad on\ \partial\Omega_h \end{cases} \tag{27}$$

where $\Lambda_N = \sup_{[0,N]} f'$.

Therefore the error between u, the minimal boundary blow up solution, and u^k the kth iterate of (27) can be split as follows:

$$I_h(u) - u^k = I_h(u - u_N) + (I_h(u_N) - u_h) + (u_h - u^k) \tag{28}$$

where I_h is the interpolation operator defined by $I_h(u)_i = u(x_i)$ when $D = 1$ (respectively by $I_h(u)_a = u(a)$ when $D = 2$ for a node $a = (ih, jh)$ on the grid), and u_N is the solution of

$$\begin{cases} \Delta u = f(u) \ in \ \Omega, \\ \quad u = N \ on \ \partial\Omega. \end{cases} \tag{29}$$

5.3. **Error estimate.** Throughout this section, we assume that f is a convex increasing function and that $\Omega = [-1, 1]^D$. We first bound from above the rate of convergence of the algorithm (27).

Lemma 5.1. *Let $u^k \in \mathbb{R}^{(2L+1)D}$ be given by (27) and $u_h = \lim_{k\to\infty} u^k$. Then, there exist constants $C = C(D)$, $\mu = \mu(D) > 0$ and $\lambda = \lambda(f) < \Lambda_N$ such that*

$$\|u^k - u_h\|_{\ell^\infty} \leq C \left(1 - \frac{\lambda}{\Lambda_N}\right)^k \min\left(1, \frac{1}{h^{D/2}}\left[1 - \frac{\mu}{\Lambda_N} + \frac{\mu^2}{\Lambda_N^2}\right]^k\right) \|u^0 - u_h\|_{\ell^\infty} \tag{30}$$

Proof : In the sequel let us denote by a a node of the grid (that is $a = ih$ in 1D or $a = (ih, jh)$ in 2D). (u^k) is a nondecreasing sequence in $\mathbb{R}^{(2L+1)D}$ (i.e. $u_a^k \leq u_a^{k+1}$ for each node a). By the mean value theorem, there exists $\theta_a \in (u_a^k, (u_h)_a)$ such that

$$f(u_a^k) - f((u_h)_a) = f'(\theta_a)(u_a^k - (u_h)_a). \tag{31}$$

Therefore

$$\left|\left((u_h)_a - \frac{f((u_h)_a)}{\Lambda_N}\right) - \left((u^k)_a - \frac{f((u^k)_a)}{\Lambda_N}\right)\right| \leq (1 - \frac{\lambda}{\Lambda_N})((u_h)_a - u_a^k), \tag{32}$$

where $\lambda = \inf f'$.
On the other hand

$$u_h - u^k = \left(Id - \frac{\Delta_h}{\Lambda_N}\right)^{-1} \left(\left(u_h - \frac{f(u_h)}{\Lambda_N}\right) - \left(u^k - \frac{f(u^k)}{\Lambda_N}\right)\right), \tag{33}$$

where $f(u)$ denotes the vector which components are $f(u)_a = f(u_a)$. The key argument is to observe that the matrix $Id - \frac{\Delta_h}{\Lambda_N}$ satisfies the maximum principle. Therefore,

$$0 \leq u_h - u^{k+1} \leq \left(1 - \frac{\lambda}{\Lambda_N}\right) \left(Id - \frac{\Delta_h}{\Lambda_N}\right)^{-1} (u_h - u^k), \tag{34}$$

where inequalities hold component by component. We thus obtain that

$$\|u_h - u^k\|_{\ell^\infty} \leq \left(1 - \frac{\lambda}{\Lambda_N}\right)^k \left\|\left(Id - \frac{\Delta_h}{\Lambda_N}\right)^{-k}\right\|_{\mathcal{L}(\ell^\infty)} \|u_h - u^0\|_{\ell^\infty}. \tag{35}$$

On the one hand, the maximum principle implies that

$$\left\| \left(Id - \frac{\Delta_h}{\Lambda_N} \right)^{-k} \right\|_{\mathcal{L}(\ell^\infty)} \leq C, \tag{36}$$

for some constant C depending only on the dimension D.

On the other hand, since the spectrum of $-\Delta_h$ lies in a segment $[\mu, \frac{\mu}{h^2}]$ (see [4]) and $\frac{1}{\sqrt{(2/h+1)D}} \| \cdot \|_{\ell^2} \leq \| \cdot \|_{\ell^\infty} \leq \| \cdot \|_{\ell^2}$ in $\mathbb{R}^{(2L+1)D}$, we get

(37)

$$||(Id - \frac{\Delta_h}{\Lambda_N})^{-k}||_{\mathcal{L}(\ell^\infty)} \leq Ch^{-D/2} ||(Id - \frac{\Delta_h}{\Lambda_N})^{-k}||_{\mathcal{L}(\ell^2)} = Ch^{-D/2} \left(\frac{1}{1 + \frac{\mu}{\Lambda_N}} \right)^k.$$

Using that $\frac{1}{1+\frac{\mu}{\Lambda_N}} \leq 1 - \frac{\mu}{\Lambda_N} + \frac{\mu^2}{\Lambda_N^2}$ and collecting (35), (36) and (37), the proof is over. □

We now provide an upper bound for $u_h - I_h(u_N)$.

Lemma 5.2. *Assume f is convex. Let u_h be the solution of (26) and u_N be the solution of (29). Then,*

$$\|u_h - I_h(u_N)\|_{\ell^\infty} \leq Ch^2 \alpha(N, f) \tag{38}$$

where $\alpha(N, f) = \|u_N^{(4)}\|_{L^\infty}$ is a constant depending only on N and f.

Proof:

For the sake of simplicity we will denote $I_h(u_N)$ by u_N, this introduces no confusion.

We write down the proof for the 2D problem, leaving the 1D case as an exercise to the reader. Let $a = (ih, jh)$ be a node on the grid. By the mean value theorem, for each node a, there exist ξ, η in $\mathbb{R}^2$ such that

$$|\xi - a| < h \text{ and } |\eta - a| < h, \tag{39}$$

and

$$\Delta_h(u_N)_a - (\Delta u_N)_a = ch^2 (\frac{\partial^4 u}{\partial x^4}(\xi) + \frac{\partial^4 u}{\partial y^4}(\eta)). \tag{40}$$

Therefore

$$\begin{cases} \Delta_h(u_h - u_N)_a = f(u_h)_a - f(u_N)_a - ch^2 (\frac{\partial^4 u}{\partial x^4}(\xi) + \frac{\partial^4 u}{\partial y^4}(\eta)) \ in \ \Omega_h \\ (u_h - u_N)_a = 0 \ on \ \partial\Omega_h \end{cases} \tag{41}$$

Consider $\omega = u_h - u_N$, then working as in the previous lemma we obtain

(42)

$$w_a = \left[(\Delta_h + \Lambda_N)^{-1} \left(Id - D \left(\frac{f'(\theta)}{\Lambda_N} \right) \right) \Lambda_N \right] ch^2 \left(\frac{\partial^4 u}{\partial x^4}(\xi) + \frac{\partial^4 u}{\partial y^4}(\eta) \right),$$

where $\theta_a \in (u_N(a), (u_h)_a)$.

We therefore obtain

$$\|w_a\|_{\ell^\infty} \leq \frac{1}{\Lambda_N}\frac{4}{1+\frac{4}{\Lambda_N}}\|\Delta_h^{-1}\|_{\mathcal{L}(\ell^\infty)}Ch^2\|u_N^{(4)}\|_{L^\infty}, \tag{43}$$

where $\|u_N^{(4)}\|_{L^\infty} = \max_{|\alpha|\leq 4}(\|\partial^\alpha u_N\|_{L^\infty})$. □

Remark 5.3. When $D = 1$ and $f(u) = u^p$ with $p \geq 2$, the constant $\alpha(N, f)$ is given by

$$\alpha(N, f) = \|u_N^{(4)}\|_{L^\infty} = \frac{p(3p-1)}{p+1}N^{2p-1}.$$

5.4. **Error estimate for**$\|u-u_N\|$**.** Assume here that the Sharpened Keller-Osserman condition (3) is valid. Consider then a sequence (α_N) such that $\Phi(\alpha_N)$ converges towards 0. Consider the minimal solution u_N of

$$\begin{aligned} \Delta u_N &= f(u_N) \quad \text{in } \Omega = B(0,1), \\ u_N &= \alpha_N \quad \text{on } \partial\Omega. \end{aligned} \tag{44}$$

Then one may wonder how u_N approximates the minimal boundary blow up solution u defined on the unit ball. We first state a qualitative result

Proposition 5.4. *There exists R_N such that u_N is a boundary blow up solution on $B(0, R_N)$. Moreover*

$$R_N - 1 \sim \frac{1}{\sqrt{2}}\int_{\alpha_N}^{+\infty}\frac{dt}{\sqrt{F(t)}}. \tag{45}$$

Remark 5.5. This proposition shows that when we plot the approximation u_N, we plot in fact a boundary blow-up solution on a ball that is close to the unit ball.

Let us proceed to the proof in the case where $D \geq 3$. The cases $D = 1, 2$ are very similar and then omitted. Assume that u_N extends to $\mathbb{R}^D$. Then by Lemma 2.7

$$\frac{1}{\sqrt{2}}\int_{u_N(1)}^{u_N(r)}\frac{dt}{\sqrt{F(t)-F(u_N(1))}} \geq \frac{1}{D-2}(1-(\frac{1}{r})^{D-2}). \tag{46}$$

Here we have used that u_N is radially symmetric. Therefore since $u_N(1) = \alpha_N$, $\Phi(\alpha_N) \geq \frac{1}{D-2}(1-(\frac{1}{r})^{D-2})$. $N \to +\infty$ leads to a contradiction. The estimate (45) comes from Proposition 3.1. □

Remark 5.6. Observe that if $f(u) = u^p$, then $u_N(r) = R_N^{-\frac{2}{p-1}}u(rR_N)$. In that case, $0 \leq u(r) - u_N(r) \leq C(R_N - 1)(\dot{u}(r) + u(r))$.The inequality is sharp for some numerical constant C. To prove the estimate in a more general context, we need extra hypotheses.

Definition 5.7. Consider $g : [0, +\infty) \to [0, +\infty)$ a function. We say that g is *strongly increasing* if the function

$$\rho(\lambda) = \inf_{u \geq 0} \frac{g(\lambda u)}{g(u)}$$

is a C^1 increasing function on $[1, +\infty)$ that satisfies $\dot{\rho}(1) \neq 0$.

A strongly increasing function is increasing in the usual sense. $g(u) = u^p$, $p > 0$ is strongly increasing. $g(u) = \ln(u + 1)$ is not.
We now state and prove

Proposition 5.8. *Assume that* $\frac{f(u)}{u}$ *is strongly increasing. Then*

$$0 \leq u(r) - u_N(r) \leq C(R_N - 1)(\dot{u}(r) + u(r)).$$

Proof: $v(r) = \lambda u_N(rR_N)$ is a blow up function on the unit ball. We have

$$\Delta v = \lambda R_N^2 f\left(\frac{v}{\lambda}\right) \leq \frac{R_N^2}{\rho(\lambda)} f(v). \tag{47}$$

For N large enough, we choose λ_N close to 1 such that $\rho(\lambda_N) = R_N^2$. Then v is a blow-up supersolution to (1). Since u is the minimal blow up solution, then $u(r) \leq v(r)$. Therefore

$$\begin{aligned} 0 \leq u(r) - u_N(r) \leq u(r) - \frac{1}{\lambda_N} v(\frac{r}{R_N}) \leq \\ (R_N - 1)\dot{u}(r) + (1 - \frac{1}{\lambda_N}) u(rR_N^{-1}) \leq C(R_N - 1)(\dot{u}(r) + u(r)), \end{aligned} \tag{48}$$

since $(1 - \frac{1}{\lambda_N}) \leq 2\frac{R_N - 1}{\dot{\rho}(1)}$. $\square$

6. Numerical computations

In this section, we present some numerical results obtained with our method of approximation.

Remark 6.1. At this stage, we would like to point out that our method is self-contained, and does not use the knowledge of the boundary blow up behavior of the solution. In fact, as in [11], one can introduce another approximate problem such as taking $\Omega_\varepsilon \subset \Omega$ where $dist(\Omega, \Omega_\varepsilon) \leq \varepsilon$, and solve the problem

$$\begin{cases} \Delta u_\varepsilon = f(u_\varepsilon) \ in \ \Omega_\varepsilon, \\ u_\varepsilon(x) = K(x) \ on \ \partial\Omega_\varepsilon, \end{cases} \tag{49}$$

where

$$\frac{1}{\sqrt{2}} \int_{K(x)}^{+\infty} \frac{dt}{\sqrt{F(t)}} = dist(x, \partial\Omega).$$

We discuss our numerical results successively on three examples:

- $f(u) = u^2$ (i.e. $\frac{f(u)}{u}$ is increasing),
- $f(u) = u^2(2 + \cos u)$,
- $f(u) = u^2(1 + \cos u)$.

6.1. $f(u) = u^2$**.** Since $\frac{f(u)}{u}$ is increasing, on any domain we have a unique boundary blow up solution (see e.g. [1] and references therein).

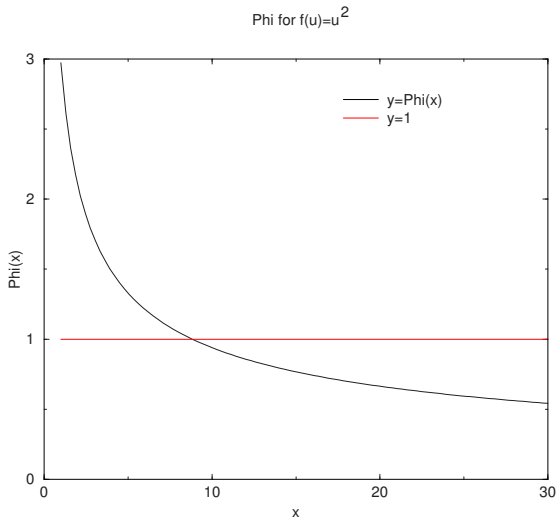

FIGURE 1. Φ when $f(u) = u^2$

We see in figure (1) that Φ is a strictly decreasing function. On figure (2) below some boundary blow up solutions obtained respectively on $[-1, 1]$, $[-1, 1]^2$, $B(0, 1) \subset \mathbb{R}^2$ (we plot $i \mapsto U(ih, 0)$ in 2D).

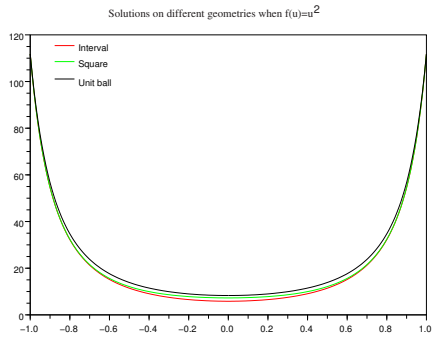

FIGURE 2. Solutions on the interval, the square and the disk

For the sake of completness, we plot in 1D the values of $u_h(0)$ for several values of h and we compare to theorical results (see Proposition 5.8).

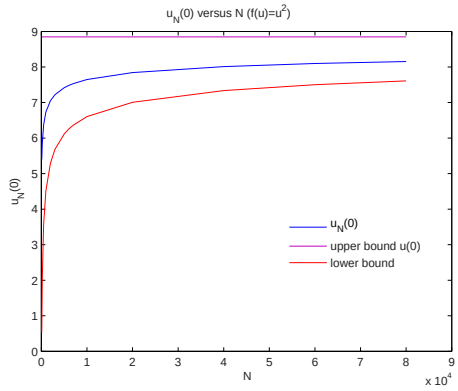

FIGURE 3. Convergence of $u_h(0)$ to $u(0)$ versus h

Remark 6.2. The value of $u(0)$ is obtained by solving

$$\frac{1}{\sqrt{2}}\int_{u(0)}^{+\infty}\frac{dt}{\sqrt{F(t)-F(u(0))}}=1.$$

6.2. $f(u)=u^2(2+\cos u)$**.** This function f satisfies the Sharpened Keller-Osserman condition (3) (see the Appendix for a proof in a similar case).

Figure (4) below shows that $\Phi(\alpha)$ tends to 0 when α tends to $+\infty$ (then the Sharpened Keller-Osserman condition is valid).

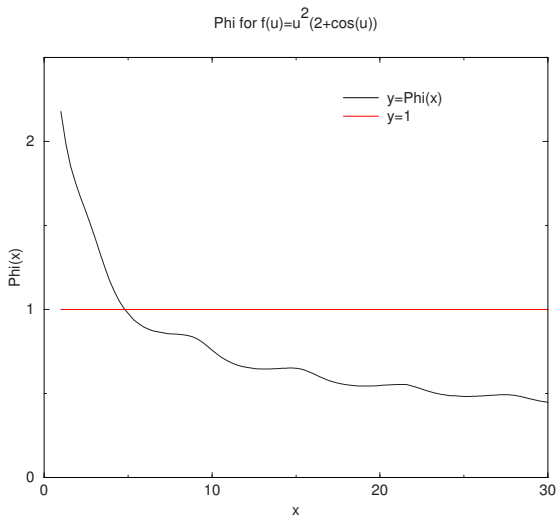

FIGURE 4. Φ when $f(u)=u^2(2+\cos(u))$

But Φ is not a decreasing function ; for instance for $c\approx 0.49$ there exist $\alpha\neq\beta$ such that $\Phi(\alpha)=\Phi(\beta)=c$. Therefore, at least in 1D, uniqueness does not hold.

Remark : The uniqueness result for $f(u) = u^2(2 + \cos u)$ in $B(0,1) \subset \mathbb{R}^D$ is still an open question for $D \geq 2$.

We now plot an approximation of the minimal boundary blow-up radial solution on $B(0,1) \subset \mathbb{R}^2$.

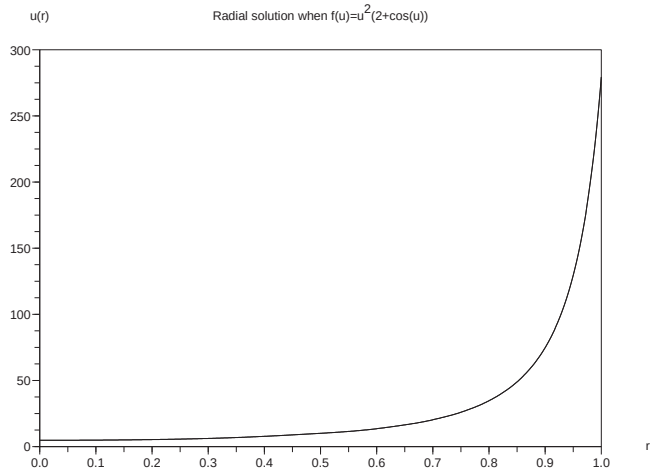

FIGURE 5. Solution on the disk when $f(u) = u^2(2 + \cos u)$

6.3. $f(u) = u^2(1 + \cos u)$**.** We first state

Proposition 6.3. *The function $f(u) = u^2(1+\cos u)$ satisfies the Sharpened Keller-Osserman condition* (3)*. Moreover,*

$$\lim_{\alpha \to (2k+1)\pi} \Phi(\alpha) = +\infty.$$

For a proof, see the Appendix (Proposition 7.1).

Figure (6) shows that for any domain, there exist an infinite sequence of boundary blow up solutions.

When $D = 1$, this follows from the fact that $\Phi(\alpha) = R$ admits an infinite number of solutions α. When $D \geq 2$, fix an integer m and observe that $\alpha_m = (2m + 1)\pi$ is a subsolution. Let u_m denote the minimal boundary blow-up solution relative to $\underline{u} = \alpha_m$. Infinitely many u_m's must be distinct. Indeed, choose m_1 such that $\alpha_{m_1} > u_0(0)$. Then $u_{m_1}(0) \geq \alpha_{m_1} > u_0(0)$. Repeating this process inductively yields infinitely many distinct solutions u_{m_k}.

We plot in figure 7 approximations of different boundary blow-up solutions on the interval $[-1, 1]$.

Figure 8 shows two radial approximations of different boundary blow up solutions on the unit ball in $\mathbb{R}^2$.

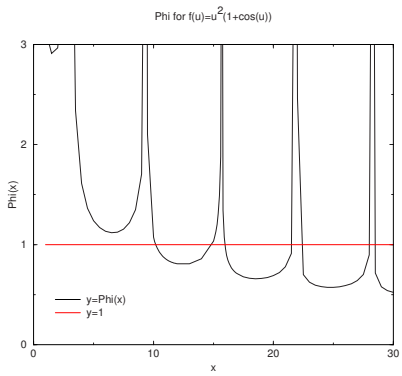

FIGURE 6. Φ when $f(u) = u^2(1 + \cos(u))$

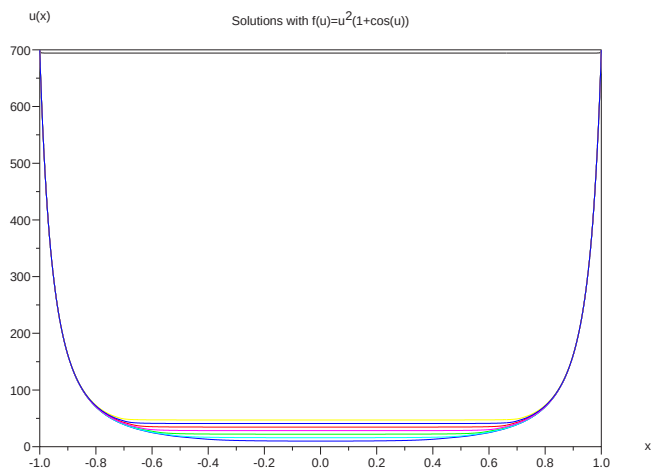

FIGURE 7. Solutions when $f(u) = u^2(1 + \cos u)$

7. APPENDIX

7.1. **Regarding the two forms of the Keller-Osserman condition.** We present the proof of Proposition 1.2 and consider the special case $f(u) = u^2(1 + \cos u)$.

Proof of Proposition 1.2. Given $\epsilon \in (0, 1)$, we recall that $(1 - \epsilon)^{-1/2} = \sum_{n=0}^{\infty} a_n \epsilon^n$, where

$$a_n = \frac{1}{n!} \frac{1 \cdot 3 \cdots (2n-1)}{2^n} = \frac{(2n)!}{4^n (n!)^2} \sim \frac{C}{\sqrt{n}}. \tag{50}$$

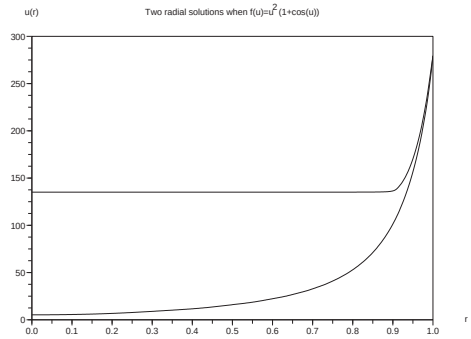

FIGURE 8. Solutions on the disk when $f(u) = u^2(1 + \cos u)$

By Tonelli's theorem we deduce that given $\alpha > T$

$$\sqrt{2}\Phi(\alpha) = \int_\alpha^\infty \frac{dt}{(F(t) - F(\alpha))^{1/2}} = \sum_{n=0}^\infty a_n F(\alpha)^n \int_\alpha^\infty \frac{dt}{F(t)^{n+1/2}}. \tag{51}$$

Choose $N \in \mathbb{N}$ so large that $(N + 3/2)\eta > 1$. Since F is nondecreasing, we have on the one hand

$$\sum_{n=0}^N a_n F(\alpha)^n \int_\alpha^\infty \frac{dt}{F(t)^{n+1/2}} \le \left(\sum_{n=0}^N a_n\right) \int_\alpha^\infty \frac{dt}{F(t)^{1/2}}. \tag{52}$$

On the other hand, since $F(t)/t^\eta$ is nondecreasing for $t > \alpha$,

(53)

$$\begin{aligned}
\sum_{n=N+1}^\infty a_n F(\alpha)^n \int_\alpha^\infty \frac{dt}{F(t)^{n+1/2}} &\le \sum_{n=N+1}^\infty a_n \frac{\alpha^{\eta(n+1/2)}}{F(\alpha)^{n+1/2}} F(\alpha)^n \int_\alpha^\infty \frac{dt}{t^{\eta(n+1/2)}} \\
&\le \left(\sum_{n=N+1}^\infty a_n \frac{1}{\eta(n+1/2) - 1}\right) \frac{\alpha}{\sqrt{F(\alpha)}} \\
&\le C \frac{\alpha}{\sqrt{F(\alpha)}},
\end{aligned}$$

where we used (50) in the last inequality. Now,

$$\frac{\alpha}{\sqrt{F(\alpha)}} \le 2 \int_{\alpha/2}^\alpha \frac{dt}{\sqrt{F(t)}} = o(1).$$

Collecting the above inequality, (51), (52) and (53), we obtain the desired result by letting $\alpha \to \infty$. □

Proposition 7.1. *Let $f(u) = u^2(1 + \cos u)$. Then,*

$$\limsup_{\alpha\to\infty} \Phi(\alpha) = \infty,$$
$$\liminf_{\alpha\to\infty} \Phi(\alpha) = 0.$$

Proof. Set $\alpha = (2k+1)\pi$. For t close to α, $F(t) - F(\alpha) \sim \alpha^2(t-\alpha)^2$. Therefore $\Phi(\alpha) = +\infty$. In particular, $\limsup_{\alpha\to\infty} \Phi(\alpha) = \infty$. To compute the $\liminf$, introduce $\varepsilon > 0$. Set $\alpha = 2k\pi$ and assume that k is large enough with respect to ε to ensure that $\alpha\varepsilon \geq \frac{\pi}{2}$ and that for all $u \geq \alpha$

$$F(u) \geq \frac{1}{1+\varepsilon}(\frac{u^3}{3} + 2u). \tag{54}$$

On the one hand

$$\int_\alpha^{\alpha+\frac{\pi}{2}} \frac{dt}{(F(t)-F(\alpha))^{1/2}} \leq \int_\alpha^{\alpha+\frac{\pi}{2}} \frac{dt}{\alpha(t-\alpha)^{1/2}} \leq C/\alpha. \tag{55}$$

On the other hand

$$\int_{\alpha+\frac{\pi}{2}}^{\alpha(1+\varepsilon)} \frac{dt}{(F(t)-F(\alpha))^{1/2}} \leq \int_\alpha^{\alpha(1+\varepsilon)} \frac{dt}{(F(\alpha+\frac{\pi}{2})-F(\alpha))^{1/2}} \leq C\varepsilon. \tag{56}$$

Finally, introducing $G(u) = \frac{u^3}{3} + 2u$,

$$\int_{\alpha(1+\varepsilon)}^{+\infty} \frac{dt}{(F(t)-F(\alpha))^{1/2}} \leq \int_{\alpha(1+\varepsilon)}^{+\infty} \frac{dt}{(\frac{1}{1+\varepsilon}G(t)-G(\alpha))^{1/2}} \tag{57}$$

$$\leq (1+\varepsilon)\int_\alpha^{+\infty} \frac{ds}{(G(s)-G(\alpha))^{1/2}}. \tag{58}$$

This last integral converges to 0 when α converges to $+\infty$, by applying Proposition 1.2 to G. Therefore, using Chasles theorem

$$\limsup_{\alpha\to+\infty} \Phi(\alpha) \leq 0 + \sqrt{2}C\varepsilon + 0. \tag{59}$$

$\varepsilon \to 0$ completes the proof of the proposition.

7.2. **Minimality Principle.** We present here the proof of Corollary 2.2 and Corollary 2.5.

Proof of Corollary 2.2. Let $\underline{u}, \overline{u}$ be the sub and supersolution given in the statement of Proposition 2.1. Let $(I, >)$ denote the set of all finite families containing $\overline{u}$ of supersolutions of (5) which stay above $\underline{u}$, ordered by inclusion : $i \in I$ if there exist $n \in \mathbb{N}$ and supersolutions $\overline{v}_k \in C(\bar{\Omega})$, $1 \leq k \leq n$ (i.e. (4) holds when $\overline{u}$ is replaced by $\overline{v}_k$) with $\overline{v}_k \geq \underline{u}$, such that $i = \{\overline{u}, \overline{v}_1, \ldots, \overline{v}_n\}$.

I is non-empty since $\{\overline{u}\} \in I$. I is filtrating increasing, i.e. if $i_1, i_2 \in I$ there exists $i_3 \in I$ such that $i_3 > i_1, i_2$ (take e.g. $i_3 = i_1 \cup i_2$). We prove that given $i = \{\overline{u}, \overline{v}_1, \ldots, \overline{v}_n\} \in I$ there exists a solution $u_i \in C(\bar{\Omega})$ of (5) such that $u_i \leq \overline{v}$ for all $\overline{v} \in i$. Let indeed u^0 denote the solution given by

Proposition 2.1. Following [6], since $\underline{u} \leq u^0 \leq \overline{u}$, u^0 is also a solution of (5), when f is replaced by the truncation $f^0 \in C(\bar{\Omega} \times \mathbb{R})$ defined by

$$f^0(x,u) = \begin{cases} f(\underline{u}(x)) & \text{if } u < \underline{u}(x), \\ f(u) & \text{if } \underline{u}(x) \leq u \leq \overline{u}(x), \\ f(\overline{u}(x)) & \text{if } u > \overline{u}(x). \end{cases}$$

In [6], the authors prove that in fact *any* solution u of (5) with nonlinearity f^0 satisfies $\underline{u} \leq u \leq \overline{u}$ (and solves the problem with the original nonlinearity f). For convenience, we reproduce here the argument of Clément and Sweers : take any solution u of (5) with nonlinearity f^0. Assume by contradiction that $\Omega_+ := \{x \in \Omega \ : \ u(x) > \overline{u}(x)\}$ is non-empty. Working if necessary on a connected component of Ω_+, we may also assume that Ω_+ is connected. For $x \in \partial\Omega_+$, either $u(x) = \overline{u}(x)$ or $x \in \partial\Omega$, so that $u(x) = g(x) \leq \overline{u}(x)$. Hence,

$$\begin{cases} \Delta(\overline{u} - u) \leq f(\overline{u}) - f^0(x,u) = 0 \ in \ \mathcal{D}'(\Omega_+), \\ \overline{u} - u \geq 0 \ on \ \partial\Omega_+. \end{cases}$$

By the (weak) Maximum Principle, $\overline{u} \geq u$ in Ω_+, which is a contradiction. Hence, $u \leq \overline{u}$ and we can prove similarly that $u \geq \underline{u}$.

Define now the truncation $f^1 \in C(\bar{\Omega} \times \mathbb{R})$ of f^0 associated to $\overline{v}_1$ by :

$$f^1(x,u) = \begin{cases} f^0(x,\underline{u}(x)) & \text{if } u < \underline{u}(x), \\ f^0(x,u) & \text{if } \underline{u}(x) \leq u \leq \overline{v}_1(x), \\ f^0(x,\overline{v}_1(x)) & \text{if } u > \overline{v}_1(x). \end{cases}$$

Clearly, $\underline{u}$ and $\overline{v}_1$ are a sub and a super solution of (5) with nonlinearity f^1. Applying Proposition 2.1 (which still holds for non-autonomous nonlinearities, see [6]), we can thus construct a solution u^1 of (5) with nonlinearity f^1, satisfying $\underline{u} \leq u^1 \leq \overline{v}_1$. Clearly, u^1 is a solution of the problem with nonlinearity f^0 and, as we mentioned earlier, we must have $u^1 \leq \overline{u}$. Repeating the process inductively, we obtain a solution $u_i := u^n$ such that $\underline{u} \leq u_i \leq \overline{u}, \overline{v}_1, \ldots, \overline{v}_n$.

Note that u_i may not be unique. Nevertheless, using the Axiom of Choice on the set of all such solutions, we can construct a well-defined generalized sequence $(u_i)_{i \in I}$, contained in the set K of all solutions u satisfying $\underline{u} \leq u \leq \overline{u}$.

By standard elliptic estimates, K is a compact subset of $C(\bar{\Omega})$ so there exists a generalized subsequence $(u_{\phi(j)})_{j \in J}$ converging to a solution u of (5).

Choose now an arbitrary supersolution $\overline{v} \geq \underline{u}$ and let $i_1 := \{\overline{v}, \overline{u}\} \in I$. Given $\epsilon > 0$, let $j_0 \in J$ such that $j > j_0 \Longrightarrow \|u_{\phi(j)} - u\|_\infty < \epsilon$. Also choose $j_1 \in J$ such that $j > j_1 \Longrightarrow \phi(j) > i_1$. Finally pick $j_3 > j_1, j_2$. Then, for $j > j_3$,

$$u \leq \|u_{\phi(j)} - u\|_\infty + u_{\phi(j)} \leq \epsilon + \overline{v}.$$

Letting $\epsilon \to 0$, we conclude that $u \leq \overline{v}$ for any supersolution $\overline{v} \geq \underline{u}$. Clearly, u is the unique such solution.

It remains to prove that given any subdomain ω and any function $\overline{v} \in C(\bar{\omega})$ satisfying (6), $u \leq \overline{v}$. Fix such a function $\overline{v}$ and define $h^k \in C(\bar{\Omega} \times \mathbb{R})$, $k = 0, 1$, by

$$h^0(x,t) = \begin{cases} f(\underline{u}(x)) & \text{if } t < \underline{u}(x), \\ f(t) & \text{if } \underline{u}(x) \leq t \leq u(x), \\ f(u(x)) & \text{if } t > u(x). \end{cases}$$

and

$$h^1(x,t) = \begin{cases} h^1(x, \underline{u}(x)) & \text{if } t < \underline{u}(x), \\ h^1(x,t) & \text{if } x \in \Omega \setminus \omega \text{ or if } x \in \omega \text{ and } \underline{u}(x) \leq t \leq \overline{v}(x), \\ h^1(x, \overline{v}(x)) & \text{if } x \in \omega \text{ and } t > \overline{v}(x). \end{cases}$$

Working as before, we may solve (5) with nonlinearity h^1 and obtain a solution $\tilde{u}$ of (5) with nonlinearity f such that $\underline{u} \leq \tilde{u} \leq u$ and $\tilde{u}|_\omega \leq \overline{v}$. Since $\tilde{u}$ is a (super)solution of (5), we also have $u \leq \tilde{u}$. Hence $u|_\omega = \tilde{u}|_\omega \leq \overline{v}$.

Proof of Corollary 2.5. Clearly, there exists a most one such solution. Let now N denote any integer larger than $\|\underline{u}\|_{L^\infty(\Omega)}$ and let u_N denote the minimal solution relative to $\underline{u}$ of

$$\begin{cases} \Delta u_N = f(u_N) \ in \ \mathcal{D}'(\Omega), \\ u_N = N \ on \ \partial\Omega. \end{cases}$$

Using the Minimality Principle 2.2, one can easily show that the sequence (u_N) is nondecreasing and that $u_N \leq N$. Take any (smooth) open set $\omega \subset\subset \Omega$ such that $v \geq N$ on $\partial\omega$. By the Minimality Principle 2.2 again, we conclude that $v|_\omega \geq u_N|_\omega$. Since this holds for any such ω, we conclude that $v \geq u_N$ in Ω. In particular the sequence (u_N) is bounded on compact subsets of Ω and using elliptic regularity, we conclude that (u_N) converges to a blow-up solution u of (1) such that $\underline{u} \leq u \leq v$.

Now take $\omega \subset \Omega$ open and $\overline{v} \in C(\omega)$ satisfying (7). Take $\tilde{\omega} \subset\subset \omega$ such that $\overline{v} \geq N$ on $\partial\tilde{\omega}$. Applying the Minimality Principle 2.2, we have that $u_N|_{\tilde{\omega}} \leq \overline{v}|_{\tilde{\omega}}$. Again, since $\tilde{\omega} \subset\subset \omega$ is arbitrary, we conclude that $u_N|_\omega \leq \overline{v}$. Letting $N \to \infty$ yields the desired inequality. □

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