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Cosets and genericity

Eric Jaligot

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Abstract

In a connected group of finite Morley rank in which the generic element belongs to a connected nilpotent subgroup, proper normalizing cosets of definable subgroups are not generous. We explain why this is true and what consequences this has on an abstract theory of Weyl groups in groups of finite Morley rank.

The only known infinite simple groups of finite Morley rank are the simple algebraic groups over algebraically closed fields and this is a motivation, among many others, for a classification project of these groups. It borrows ideas and technics from the Classification of the Finite Simple Groups but at the same time, and it is a highly nontrivial task, it provides a kind of simplified version of the finite case. This is mostly due to the existence of well-behaved notions of genericity and connectivity in the infinite case which unfortunately can hardly find finite analogs.

The present short technical note deals with a very specific topic concerning such arguments based on genericity in the case of infinite groups of finite Morley rank, showing at least similarities with algebraic groups in any case as far as a theory of Weyl groups is concerned. This applies also to certain non-algebraic configurations of groups of finite Morley rank, which may not be entirely unexpected.

In a simple algebraic group, maximal (algebraic) tori are conjugate and cover the group generically, with the Weyl group governing essentially the structure of the entire group. There are at least two abstract versions of tori in groups of finite Morley rank and which coincide at least in the case of a simple algebraic group, *decent tori* on the one hand and *Carter subgroups* on the other. The main caveat with these two more abstract notions is in both cases an unknown existence, more precisely the existence of a *nontrivial* decent torus on the one hand and the existence of a *generous* Carter subgroup on the other. In general “generous” for a subset of a group means “whose union of conjugates is generic in the group”, the typical property of tori in algebraic groups. With both notions there are conjugacy theorems in case of existence, or more precisely there are conjugacy theorems with nontrivial contents in case of existence. This gives a natural notion of Weyl group in each case, $N(T)/C^\circ(T)$ for some maximal decent torus T or $N(Q)/Q$ for some generous Carter subgroup Q . In any case and whatever the Weyl group is, it is finite and, as with classical Weyl groups

and algebraic tori in algebraic groups, its determination and its action on the underlying subgroup is fundamental in the abstract context.

As an element of the Weyl group is also a coset in the ambient group, it is then useful to get a fine description of such cosets, at least generically, even though recovering from such a description the structure and the action of the Weyl group is a particularly delicate task. In [CJ04] such arguments were developed intensively for this purpose, and this was highly influenced by one of the most critical aspects of the early work on the so-called “bad” groups of finite Morley rank [BN94, Theorem 13.3]. In this paper a pathological coset, whose representative is typically a Weyl element which should not exist, is usually shown to be both generous *and* nongenerous, and then the coset does not exist. This is the main protocol, sometimes referred to as “coset arguments”, for the limitation of the size of the Weyl group. Generosity is usually obtained by unexpected commutations between the Weyl elements and the underlying subgroup, and this generally depends on the specific configuration. It is certainly the pathological property in any case, and we shall prove here at a reasonable level of generality that the existing cosets should be nongenerous.

In particular we rearrange as follows the protocol of this paper in the light of further developments of [Jal06] concerning generosity.

Theorem 1 (Generix and the Cosets) *Let G be a group of finite Morley rank in which the generic element of G° belongs to a connected nilpotent subgroup and let H be a definable subgroup of G° . Then wH is not generous for any element w of G° normalizing H but not in H .*

The typical case one has in mind in Theorem 1 is the case in which $H = Q$ is a generous Carter subgroup. In particular the present paper can also be seen as an appendix of [Jal06] on the structure of groups of finite Morley rank with such a generous Carter subgroup, and more precisely as a follow-up to Section 3.3 in that paper.

The general idea of the protocol indicated above has been reused repeatedly in various contexts in a series of papers, most notably to get a fine description of p -torsion in terms of connected nilpotent subgroups of bounded exponent and of decent tori. Applied to the most natural kind of Weyl groups an application of the protocol, particularly smooth and typical since [CJ04], shows that centralizers of decent tori are connected in any connected group, implying in particular that the Weyl group $N(T)/C^\circ(T)$ attached to a decent torus T acts faithfully on T .

A much more delicate use of the protocol can be found in [CJ04, Proposition 6.17]. It is proved there, in a specific configuration, that the centralizer of a certain finite subgroup of a decent torus is connected, with then a much more restrictive faithful action of the Weyl group. As we think this special application of the protocol contains the main difficulty possibly inherent to the subject, we mostly refer to this example. As we will see below, the key point is that generosity is in general related to a finiteness property, as opposed to a uniqueness property. In particular reading this example first is the best

introduction to one of the aspects of what is done much more conceptually here, and at the same time reading the present paper first brings considerable light on the delicate computations done in that specific example.

Theorem 1 has general consequences on the action of the Weyl group on the underlying subgroup, again whatever these are. Back to the concrete example of a simple algebraic group, the maximal algebraic torus is a divisible abelian subgroup, and the Weyl group acts faithfully on it. The main corollary of Theorem 1 is a quite general form of this in the abstract context of groups of finite Morley rank.

Corollary 2 *Let G be a group of finite Morley rank in which the generic element of G° belongs to a connected nilpotent subgroup, H a definable connected generous subgroup, and w an element element of G° of finite order n normalizing H but not in H . Suppose that $\{h^n \mid h \in H\}$ is generic in H . Then $C_H(w) < H$.*

In the case of a simple algebraic group, the subgroup H in Corollary 2 is typically the maximal torus T and w a representative of finite order of a nontrivial element of the Weyl group, and in this case there are finitely many *proper* subtori of T and determined as $C_T^\circ(w)$ for such w 's. In the finite Morley rank case H may typically be a generous n -divisible Carter subgroup Q and w a lifting of finite order n of a nontrivial element of the Weyl group $N(Q)/Q$. One gets then, for instance if Q is a divisible abelian generous Carter subgroup as in Corollary 10 below, consequences qualitatively similar in the finite Morley rank case.

As for Theorem 1, the statement adopted in Corollary 2 is far more general than what it says about this typical cases.

1 Technicalities

Before passing to the proofs, we review briefly the background needed. Groups of finite Morley rank are equipped with a rudimentary notion of finite dimension on their definable sets, satisfying as axioms a few basic properties of the natural dimension of varieties in algebraic geometry over algebraically closed fields. By definable we mean definable by a first-order logic formula, possibly with parameters and possibly in quotients by definable equivalence relations. The dimension, or “rank”, of a definable set A is denoted by $\text{rk}(A)$.

The finiteness of the rank implies the descending chain condition on definable subgroups, and this naturally gives abstract versions of classical notions of the theory of algebraic groups:

- The *definable hull* of an arbitrary subset of the ambient group is the smallest definable subgroup containing that set. It is contained in the Zariski closure in the case of an algebraic group.

- The *connected component* G° of a group G of finite Morley rank is the smallest (normal) definable subgroup of finite index of G , and G is *connected* when $G = G^\circ$.

The main property of a connected group of finite Morley rank is that it cannot be partitioned into two definable *generic* subsets, that is two subsets of maximal rank. This is a necessary and sufficient condition by an early theorem of Cherlin. A fundamental corollary is that a connected group of finite Morley rank acting definably on a finite set must fix it pointwise, as orbits of elements are finite and their stabilizers are of finite index. We mention here a special application of this fundamental corollary.

Fact 3 *A connected group of finite Morley rank acting definably on a group H of finite Morley rank induces a trivial action on H/H° .*

Proof. As H° is definably characteristic in H , the acting group induces an action on H/H° , and we are then in presence of the action of a connected group on a finite set. \square

Following [Jal06], we say that a definable subset A of a group G of finite Morley rank is *generous* in G when the union A^G of its G -conjugates is generic in G .

We are not going to use the concepts of decent tori and of Carter subgroups, but as they were mentioned as examples in the introduction we briefly recall their definitions for the sake of completeness. A decent torus T of a group of finite Morley rank is a definable (connected) divisible abelian subgroup which coincides with the definable hull of its (divisible abelian) torsion subgroup, and a Carter subgroup Q is a definable connected nilpotent subgroup of finite index in its normalizer (and in particular it satisfies $Q = N^\circ(Q)$). Both types of subgroups exist in any group of finite Morley rank, which is trivial in the first case and follows in the second case from a graduated notion of unipotence on certain connected nilpotent subgroups, for which decent tori are precisely the first stones. By a theorem of Cherlin, maximal decent tori are conjugate in any group of finite Morley rank, which indeed follows from the fact that $C^\circ(T)$ is generous for any such decent torus T . By [Jal06], generous Carter subgroups are conjugate in any group of finite Morley rank.

We take this opportunity to mention the following corelation between decent tori and generous Carter subgroups.

Fact 4 *If Q is a generous Carter subgroup of a group of finite Morley rank, then $T \leq Q \leq C^\circ(T)$ for some maximal decent torus T , and $N(T) = C^\circ(T) \cdot N(Q)$.*

Proof. By [Jal06, Corollary 3.8], Q has a definable generic subset Q_{gen} generous in G° and such that Q is the *unique* G° -conjugate of Q containing g for any element $g \in Q_{\text{gen}}$. In particular $C^\circ(g) \leq N^\circ(Q) = Q$ for any such element g (see also [Jal06, Lemmma 3.10]). By Cherlin's theorem on decent tori, $C^\circ(T)$ is generous for any maximal decent torus T , and thus by connectedness one finds

an element g in Q_{gen} centralizing such a T . In particular $T \leq C^\circ(g) \leq Q$. Now Q has a unique maximal decent torus, which is central, by some early work of Nesin on the structure of connected nilpotent groups of finite Morley rank. In particular $Q \leq C^\circ(T)$.

It follows also that $N(Q) \leq N(T)$. Now Q is generous in $C^\circ(T)$ by [Jal06, Lemma 3.9] and thus the equality $N(T) = C^\circ(T) \cdot N(Q)$ follows from a Frattini Argument following the conjugacy of generous Carter subgroups in $C^\circ(T)$. \square

In presence of a *nontrivial* maximal decent torus T the Weyl group is naturally defined as in [CJ04, Theorem 1.8] as $N(T)/C^\circ(T)$, and in presence of a *generous* Carter subgroup Q it is defined as in [Jal06, §3.3] as $N(Q)/Q$. In the first case the original definition relied on a particular decent subtorus related to the prime $p = 2$ but since Cherlin’s proof of conjugacy of maximal decent tori it naturally takes this form. We also mention that the term “Weyl group” made his first appearance in [Nes89] in the context of bad groups of Morley rank 3, with all possible definitions equivalent in this case.

In Fact 4 we see that both notions essentially match, with however

$$N(T)/C^\circ(T) \simeq (N(Q)/Q)/(N_{C^\circ(T)}(Q)/Q)$$

isomorphic to a possibly proper quotient of $N(Q)/Q$, and thus a sharper notion with the second definition.

In general it seems there is no reason why an arbitrary group of finite Morley rank should contain a nontrivial decent torus, and there are a few hints according to which it should contain a generous Carter subgroup.

In Theorem 1 we use the sentence “a generic element has a certain property”. A model-theorist reader understands this as the fact that realizations of the generic type in a saturated elementary extension have that property, and can see in our proof that one uses the property of the generic element in only one particular such elementary extension. An algebraist reader can read it, and in practice this suffices, as the fact that the property is satisfied for all elements of a definable generic subset.

The assumption on the generic element in Theorem 1 is of course (much) weaker than the assumption of the existence of a generous Carter subgroup. It is known to be true, in the strong definable sense as above, in very specific contexts where the existence of a generous Carter subgroup is a priori not known. As such contexts also generally turn out to be important minimal cases in various inductive approaches to groups of finite Morley rank, it is relevant to state Theorem 1 in the present form.

2 Cosets and generosity

In our proof of Theorem 1 we are essentially going to reuse lines of arguments of [Jal06] for dealing with generosity, both for characterizing it and for applying it.

When working with generosity in very general contexts one has to inspect closely each conjugacy class of each individual element of the set considered. The reader can find in [Jal06, §2.2] such an analysis, done there for definable connected subgroups. Another approach for this analysis was then mentioned by Cherlin, with a more conceptual geometric proof giving also a few more rank equalities. We take here the opportunity to recast these computations in terms of permutation groups, not only because it generalizes naturally, but also as it certainly might be useful in this more general context.

Given a permutation group (G, Ω) and a subset H of Ω , we denote by $N(H)$ and $C(H)$ the *setwise* and the *pointwise* stabilizer of H respectively, that is $G_{\{H\}}$ and $G_{(H)}$ in a usual permutation group theory notation, and by H^G the orbit of H under the action of G . Subsets of the form H^g for some g in G are also called *G -conjugates* of H . When considering the action of a group on itself by conjugation, as we will do below, these terminologies and notations are the usual ones, with $N(H)$ the *normalizer* and $C(H)$ the *centralizer* of H respectively.

We note that in this paper we work only with “exact” normalizers $N(H) = \{g \in G \mid H^g = H\}$, or “stabilizers”, as opposed to “generic stabilizers”, where the equality $H^g = H$ is understood up to a symmetric difference of lower rank.

Fact 5 [Jal06, Proposition 2.9] *Let (G, Ω) be a permutation group of finite Morley rank in which the Morley rank is additive, or a ranked permutation group, H a definable subset of Ω , and assume that for r between 0 and $\text{rk}(G/N(H))$ the definable set H_r , consisting of those elements of H belonging to a set of G -conjugates of H of rank exactly r , is nonempty. Then*

$$\text{rk}(H_r^G) = \text{rk}(G) + \text{rk}(H_r) - \text{rk}(N(H)) - r.$$

Proof. One may proceed exactly as in the geometric proof of [Jal06, Proposition 2.9]. In the natural geometry associated to this computation, points are the elements of Ω which are G -conjugate to those of H and lines are the G -conjugates of H . The set of flags is the set of couples (point, line) where the point belongs to the line, and one considers the subflag naturally associated to H_r . Projecting on the set of points one gets $\text{rk}(H_r^G) + r$ for the rank of this subflag, and similarly $\text{rk}(G/N(H)) + \text{rk}(H_r)$ by projecting on the set of lines. The equality follows.

In this proof we use essentially only two properties of the rank. The first one is a guarantee that the sets H_r considered are definable, which is true under all assumptions. This *definability of the rank* is a property of the Morley rank, or one of the axioms, sometimes called Borovik-Poizat axioms, for a treatment of ranked structures involving no more model theory than the notion of definable sets. The second one is the *additivity of the rank*, which guarantees the formulas as above for the rank of a set as the sum of the rank of its image by a definable function and of the rank of the fibers of that function when these are constant. This is usually considered as one of the Borovik-Poizat axioms. \square

In the context of a permutation group as in Fact 5, we may naturally say that the definable subset H of Ω is *generous* when H^G is generic in Ω . Of course

this matches with the usual definition in the case of the action of a group on itself by conjugation.

Continuing in the general context of permutation groups, Fact 5 has the following corollary for characterizing generosity.

Corollary 6 *Assume furthermore $\text{rk}(G) = \text{rk}(\Omega)$ and $\text{rk}(H) \leq \text{rk}(N(H))$ in Fact 5. Then H^G is generic in Ω if and only if $\text{rk}(H_0) = \text{rk}(N(H))$. In this case $\text{rk}(H_0) = \text{rk}(H) = \text{rk}(N(H))$, a generic element of Ω lies in only finitely many conjugates of H , and the same applies to a generic element of H .*

Proof. If H^G is generic in Ω , then one has for some r as in Fact 5 that H_r^G is generic in Ω , and then

$$0 \leq r = \text{rk}(H_r) - \text{rk}(N(H)) \leq \text{rk}(H) - \text{rk}(N(H)) \leq 0,$$

showing that all these quantities are equal to 0. In particular $r = 0$, and $\text{rk}(H_0) = \text{rk}(N(H))$. Conversely, if $\text{rk}(H_0) = \text{rk}(N(H))$, then $\text{rk}(H_0^G) = \text{rk}(G) = \text{rk}(\Omega)$ by Fact 5.

For our last statement, we also see with the above inequalities that $\text{rk}(H) = \text{rk}(N(H))$, and as H_0 and $N(H)$ have the same rank it follows that $\text{rk}(H_0) = \text{rk}(H) = \text{rk}(N(H))$. In particular the definable subset H_0 of H is generic in H , and together with the genericity of H_0^G in Ω this is exactly the meaning of our two last claims. \square

We stress the fact that under the circumstances of Corollary 6 the generosity of H is equivalent to the genericity of the *definable* sets H_0 and H_0^G in H and Ω respectively, so that working with these definable sets avoids troublesome saturation issues. At this point it is also worth mentioning that there are uniform bounds on finite sets throughout. This is one of the Borovik-Poizat axioms usually called *elimination of infinite quantifiers*, which gives uniform bounds on the cardinals of finite sets in uniformly definable families of sets. This is used on rather rare occasions, and could be used here also to see the definability of sets like H_0 in Fact 5 and Corollary 6: H_0 is exactly the set of elements contained in at most m distinct conjugates of H , for some *fixed* finite m . We do not use it as the definability of the rank amply suffices here, but this aspect can of course be kept in mind.

From now on we consider only the action of a group on itself by conjugation. We note that the rank is then always additive, so that Fact 5 and Corollary 6 can be applied freely.

As G and Ω are the same in this case, the extra assumption $\text{rk}(G) = \text{rk}(\Omega)$ is then automatically satisfied in the characterization of generosity of Corollary 6. The second assumption $\text{rk}(H) \leq \text{rk}(N(H))$ is not satisfied in general, but an interesting case in which it is satisfied is the case in which H has the form $x\Gamma$ where Γ is a definable subgroup of G and x is an element of G normalizing Γ : in this case $\Gamma \leq N(x\Gamma)$ and thus $\text{rk}(x\Gamma) = \text{rk}(\Gamma) \leq \text{rk}(N(x\Gamma))$. In fact one sees in this case that $N(x\Gamma)$ is exactly the preimage in $N(\Gamma)$ of $C_{N(\Gamma)/\Gamma}(x \bmod \Gamma)$.

All cosets considered in this paper are of this type, and we will make full use of Corollary 6 when considering the generosity of such cosets in the rest of the paper.

We insist again on the fact that the characterisation of Corollary 6 is in this case essentially the genericity of H_0 in H (in addition to $\text{rk}(H) = \text{rk}(N(H))$), and thus the fact that only finitely many conjugates of H pass through a generic element of H . In general, and we would like to say with probability almost one, there is not uniqueness. It may be seen by considering the generic element g of a connected algebraic group. It lies in a maximal torus T , which lies in a generous Borel subgroup B . As we saw in the proof of Fact 4, T is the unique of its conjugates containing g , but there are several conjugates of B containing g (and permuted by the Weyl group $N(T)/T$).

In most applications of the general protocol for computing Weyl groups in groups of finite Morley rank there is a uniqueness property, and then rank computations follow more or less immediately from the presence of *disjoint* unions. We refer for example to [CJ04, 3.3-3.4], which was essentially extracted from the original works on bad groups [BN94, Theorem 13.3, Claim (d)]. In general one can use only finiteness instead of uniqueness for generosity, as just illustrated. The reader can find in [CJ04, Proposition 6.17] a concrete application of the protocol for Weyl groups which uses finiteness only. More precisely, this finiteness property related to the generic element of the unexpected coset considered there can be tracked in the particularly delicate preparatory sequence 6.13-6.16, and more specifically 3.16, in that paper.

With this background on the characterisation of generosity we are now ready to prove Theorem 1. Recall that G is a group of finite Morley rank in which the generic element of G° is in a connected nilpotent subgroup, H is a definable subgroup of G° , w is an element of G° in $N(H) \setminus H$, and we want to show that wH is not generous.

Proof of Theorem 1.

Assume towards a contradiction wH generous in G . As G is a finite union of translates of G° , we may assume G connected.

We may freely apply Corollary 6 to the coset wH , as commented after that corollary. It follows that $\text{rk}(wH) = \text{rk}(N(wH))$ on the one hand, and on the other hand that wH has a definable generic subset, generous in G , and all of whose elements can lie in only finitely many conjugates of wH . In this sense, a generic element g of G is up to conjugacy a generic element of wH , and contained in only finitely many conjugates of wH . Of course, $N(wH) \leq N(H)$, and in fact $N(wH)$ is the preimage in $N(H)$ of $C_{N(H)/H}(w \bmod H)$. As H , wH , and $N(wH)$ have the same rank,

$$N^\circ(wH) = H^\circ.$$

In particular one sees also that w has finite order modulo H .

By assumption, a generic element g of G also belongs to a connected nilpotent subgroup Q , and as taking definable hulls does not affect connectedness

and nilpotence of subgroups in group of finite Morley rank we may assume Q definable. (For simplicity, we may assume that the property of the generic element is satisfied for all elements of a definable generic subset of G . As this property is in general not first-order, the model-theorist reader will mechanically pass to a saturated elementary extension where realisations of the generic type have the property, noticing that going up to the elementary extension does not affect our current assumptions. For the last point, one may use the elimination of infinite quantifiers to get a definition of the generic subset of wH as in the previous paragraph which does not require further parameters in the elementary extension.)

Using the connectedness of G , one concludes from the two preceding paragraphs that a generic element g of G is (up to conjugacy) in wH and in only finitely many of its conjugates on the one hand, and in a definable connected nilpotent subgroup Q on the other hand, and we will get a contradiction.

As $g \in wH \cap Q$, we may also assume w in Q , replacing the original representative w of the coset wH by a representative in Q in necessary. This is possible as we may take g . Then

$$wH \cap Q = w(H \cap Q).$$

Notice that w still has finite order modulo $H \cap Q$ as the original w had that property modulo H . The group $\langle w \rangle(H \cap Q)$ is in particular definable, and $(H \cap Q)^\circ$ is exactly its connected component. From now on we concentrate on the definable subgroup $\langle w \rangle(H \cap Q)$ of Q , and to its normalizer in Q .

$N_Q^\circ(\langle w \rangle(H \cap Q))$ acts by conjugation on the definable subgroup $\langle w \rangle(H \cap Q)$. By Fact 3, it induces a trivial action on this group modulo its connected component, that is $(H \cap Q)^\circ$. This means that it normalizes each coset of $(H \cap Q)^\circ$ in $\langle w \rangle(H \cap Q)$. In particular, $N_Q^\circ(\langle w \rangle(H \cap Q))$ normalizes the (possibly larger) coset $w(H \cap Q)$.

At this point one can use an argument similar to the one used in [Jal06, Fundamental Lemma 3.3]. We denote by X the set of elements of $w(H \cap Q)$ contained in only finitely many conjugates of wH . We note that the set X is not empty, as it contains the generic element g . We also note that the subset X of wH can be contained in only finitely many conjugates of wH , as it contains the element g which has this property. As $N_Q^\circ(\langle w \rangle(H \cap Q))$ normalizes $w(H \cap Q)$, it also normalizes X , and thus it permutes by conjugation the conjugates of wH containing X . We are now in presence of the definable action of a connected group on a finite set, and it follows that it has a trivial action, or in other words that $N_Q^\circ(\langle w \rangle(H \cap Q))$ normalizes each of these finitely many conjugates of wH containing X . In particular, it normalizes wH .

Hence

$$N_Q^\circ(\langle w \rangle(H \cap Q)) \leq N^\circ(wH) = H^\circ,$$

as noticed earlier, and the definable connected subgroup $N_Q^\circ(\langle w \rangle(H \cap Q))$ of Q then satisfies

$$N_Q^\circ(\langle w \rangle(H \cap Q)) \leq (H^\circ \cap Q)^\circ \leq (H \cap Q)^\circ.$$

But as $(H \cap Q)^\circ$ is exactly the connected component of $\langle w \rangle(H \cap Q)$, this inclusion shows that $\langle w \rangle(H \cap Q)$ has finite index in its normalizer in Q . Now definable subgroups of infinite index of nilpotent groups of finite Morley rank are of infinite index in their normalizers. This is a classical finite Morley rank version of the normalizer condition in finite nilpotent groups. One finds thus that $\langle w \rangle(H \cap Q)$ has finite index in Q , and by connectedness of the latter one gets

$$Q = \langle w \rangle(H \cap Q).$$

As $(H \cap Q)$ now has finite index in Q , one gets similarly

$$Q = (H \cap Q).$$

At this point one gets a contradiction, either by noticing that w has been pushed inside H , or that g has been pushed outside Q . \square

Theorem 1 has the following typical application.

Corollary 7 *Let G be a group of finite Morley rank in which the generic element of G° belongs to a connected nilpotent subgroup and let H be a definable subgroup of G° . Then $H \setminus H^\circ$ is not generous in G and H° is generous in G , and in fact in any definable subgroup containing it.*

Proof. As $H \setminus H^\circ$ is a finite union of cosets of H° normalizing H° , the first claim follows from Theorem 1. Now H° must be generous in G , and our last claim is [Jal06, Lemma 3.9]. \square

3 Cosets and actions

As stressed in the introduction already, recovering the action of a Weyl group on its underlying subgroup from weak information on the elements of the corresponding cosets is a particularly delicate task. Corollary 2 is however a quite general result of faithfulness following merely from the nongenerosity provided by Theorem 1. The rest of this paper is devoted to the proof of Corollary 2, or rather of what we see as the most interesting intermediary steps.

The most general situation is that of a definable connected generous subgroup H , and we want to examine the action of $N(H)/H$ on H , and much more generally the action on H of elements w in $N(H)$. Typically, H may be a generous Carter subgroup, with then $N(H)/H$ the natural Weyl group, and w a representative of any coset of H in $N(H)$.

We note that a definable generous subgroup H always satisfies

$$N^\circ(H) = H^\circ$$

by Corollary 6, and is in particular of finite index in its normalizer. We note also that there is a basic result of lifting of torsion in group of finite Morley rank,

implying in particular that any element of finite order of $N(H)/H$ lifts to an element of $N(H)$ of finite order (and where the primes involved in both primary decompositions are the same). In particular taking an element of finite order n as in Corollary 2 is a particularly low cost operation. We will conclude our paper with more comments on the other assumption in Corollary 2 concerning the n -th powers of elements of H .

The following lemma can be seen as the natural continuation of [CJ04, Lemma 3.4] with the present much better understanding of generosity as a finiteness property as opposed to a uniqueness property. It is the finest correlation one can get between generic elements of the coset wH and generic elements of H in the typical situation where the conclusion of Theorem 1 holds. At least it shows, we think, the real power of the method.

Lemma 8 *Let G be a group of finite Morley rank, H a definable generous subgroup of G , and w an element in $N(H) \setminus H$ such that $\langle w \rangle H \setminus H$ is not generous. Then*

- (1) *The coset wH has a definable subset $[wH]_{\text{gen}}$, whose complement is non-generic in wH , and all of whose elements are in infinitely many conjugates of wH .*
- (2) *The subgroup H has a definable generic subset H_{gen} such that, for any x in $[wH]_{\text{gen}}$, the subgroup of $\langle w \rangle H$ containing x and defined as*

$$\bigcap_{g \in G, x \in [wH]^g} [\langle w \rangle H]^g$$

has an empty intersection with $(H_{\text{gen}})^G$.

Proof. As $N^\circ(H) = H^\circ$ by generosity of H and Corollary 6, $\text{rk}(wH) = \text{rk}(N(wH))$, and the first claim follows from the nongenerosity of wH by Corollary 6. Again we remark that the sets provided by Corollary 6 are definable.

Now one can apply Corollary 6 to $\langle w \rangle H$ also. The generosity of $\langle w \rangle H$ (following that of H) then gives a definable subset $[\langle w \rangle H]_0$, generic in $\langle w \rangle H$ and all of whose elements can lie in only finitely many conjugates of $\langle w \rangle H$. If that set had a nongeneric intersection with H , then it would have a generic intersection with one of the proper cosets of H in $\langle w \rangle H$, say $w'H$. As all elements lying in this intersection would be contained in only finitely many conjugates of $w'H$, as contained in only finitely many conjugates of $\langle w \rangle H$ and all normalizers are finite modulo H° , Corollary 6 would give the generosity of $w'H$, a contradiction to the assumption that $\langle w \rangle H \setminus H$ is not generous. One may thus consider a generic element of H as an element of $H_{\text{gen}} := H \cap [\langle w \rangle H]_0$, and thus with the property that it is in only finitely many conjugates of $\langle w \rangle H$.

Consider now x generic in wH in the sense of the first claim, i.e., such that x is in infinitely many conjugates of wH . The intersection of all subgroups as considered as in our second claim is a subgroup of $\langle w \rangle H$. It is contained in infinitely many conjugates $\langle w \rangle H$, again as all normalizers are finite modulo H° .

Hence it contains no conjugates of an element in H_{gen} , as such an element is contained in only finitely many conjugates of $\langle w \rangle H$. \square

We mention, parenthetically, that the subgroup as in Lemma 8 (2) containing the generic element x of wH is normalized by $C(x)$ and definable by descending chain condition on definable subgroups. We also note that the nongenerosity of wH implies, in any saturated elementary extension, that the canonical extension of wH contains no realizations of the generic types.

In general, an element x of a coset wH has the form $x = wh$ for some h in H and taking powers one gets

$$(wh)^n = w^n h^{w^{n-1}} h^{w^{n-2}} \dots h$$

for any natural number n , some useful formulas when considering torsion ([CJ04, §3.3]). Assuming additionally that the element w of $N(H)$ has finite order n , which can be done in a quite general way as explained above, one has

$$(wh)^n = h^n$$

in the easiest case in which w and h commute. This way to plug the element wh of wH into the n -th power of the element h of H can be combined as follows to the full force of the pure genericity argument of Lemma 8.

Lemma 9 *Let G be a group of finite Morley rank, H a definable connected generous subgroup, and w an element of finite order n in $N(H) \setminus H$ and such that $\langle w \rangle H \setminus H$ is not generous in G . Suppose that $\{h^n \mid h \in H\}$ is generic in H . Then $C_H(w) < H$.*

Proof. Let H_{gen} denote the definable generic subset of H provided by Lemma 8 (2), and $\phi : h \mapsto h^n$ the definable map from H to H consisting of taking n -th powers. By connectedness of H , our assumption forces the genericity of $H_{\text{gen}} \cap \phi(H)$ in H , and its preimage $\phi^{-1}(H_{\text{gen}} \cap \phi(H))$ must then also be generic in H . Now $w\phi^{-1}(H_{\text{gen}} \cap \phi(H))$ is generic in wH , and one finds an element $x = wh$ in the subset $[wH]_{\text{gen}}$ provided by Lemma 8 (1), for some h in $\phi^{-1}(H_{\text{gen}} \cap \phi(H))$.

Now $h^n \in H_{\text{gen}}$, and assuming towards a contradiction $C_H(w) = H$ one also has $h^n = (wh)^n = x^n$. In particular one gets

$$h^n \in H_{\text{gen}} \cap \langle x \rangle$$

in this case, a contradiction to Lemma 8 (2). \square

At this point, Corollary 2 simply follows from Theorem 1 and Lemma 9. \square

The reader may probably already have mentioned the following special case of Corollary 2 typical of an algebraic group.

Corollary 10 *Let G be a connected group of finite Morley rank with an abelian generous Carter subgroup Q , and assume Q p -divisible for any prime p dividing the order of $N(Q)/Q$. Then Q has finitely proper definable subgroups, corresponding to all subgroups of the form $C_Q(w)$ for w varying in $N(Q) \setminus Q$, and with a canonical definition as the centers of proper cyclic extensions of Q in $N(Q)$. In particular $N(Q)/Q$ acts faithfully on Q .*

Proof. Let w in $N(Q) \setminus Q$. By lifting of torsion one may pick up an element w' of finite order n in the definable hull of w , with all primes involved in n dividing the order of $N(Q)/Q$. As Q is p -divisible for all such primes, it is n -divisible, and in particular $Q^n = Q$. Now $C_Q(w') < Q$ by Corollary 2. We have shown that $C_Q(w) \leq C_Q(w') < Q$ for any element w in $N(Q) \setminus Q$.

The fact that there are finitely many possibilities for such subgroups $C_Q(w)$ follows from their alternative definitions as

$$C_Q(w) = Z(\langle w \rangle Q)$$

and from the fact that $N(Q)/Q$ is finite. For a canonical definition of such subgroups, one may then take $Z(\langle w \rangle Q)$, with w varying in $N(Q) \setminus Q$. \square

As we said already, the possibility to choose an element w of finite order n in Corollary 2 is always guaranteed by lifting of torsion.

If in Corollary 10 the Carter subgroup Q is not p -divisible for some prime p , then it is in particular *nondivisible* and it has to contain a nontrivial definable subgroup of bounded exponent by some early work of Nesin on the structure of connected nilpotent groups of finite Morley rank. In this situation, and similarly if Q is *nonabelian*, some uniqueness theorems may apply in certain cases. Such theorems, which can be seen in parallel to those resulting from the Bender Method in finite group theory, are known for the analogs of finite Morley rank of the finite groups encountered in the Feit-Thompson Theorem or more generally in the classification of N -groups of Thompson.

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