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Cosets and genericity

Eric Jaligot

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In [CJ04] arguments pending on cosets and genericity were developed intensively for determining Weyl groups in groups of finite Morley rank, and this was strongly influenced by one of the essential contents of [Nes89]. In both papers a pathological coset is usually shown to be both generous *and* nongenerous, and then the coset does not exist. When the coset exists it should normally be nongenerous. This is what we shall see in this short paper, which can also be seen as an appendix of [Jal06] on the structure of groups of finite Morley rank with a generous Carter subgroup or satisfying even weaker generic covering properties.

As far as conjugates and ranks are concerned the fine analysis of conjugacy classes of [Jal06, §2.2] provided the following understanding of the situation, which we recast in terms of permutation groups here. Given a permutation group (G, Ω) and a subset H of Ω , we denote by $N(H)$ and $C(H)$ the setwise and the pointwise stabilizer of H respectively, that is $G_{\{H\}}$ and $G_{(H)}$ in a usual permutation group theory notation, and by H^G the orbit of H under the action of G . Subsets of the form H^g for some g in G are also called G -conjugates of H .

Fact 1 *Let (G, Ω) be a permutation group of finite Morley rank, H a definable subset of Ω , and assume that for r between 0 and $\text{rk}(G/N(H))$ the definable set H_r , consisting of those elements of H belonging to a set of G -conjugates of H of rank r , is nonempty. Then $\text{rk}(H_r^G) = \text{rk}(G) + \text{rk}(H_r) - \text{rk}(N(H)) - r$.*

Proof. One may proceed exactly as in the geometric proof of [Jal06, Proposition 2.9] mentioned by Cherlin for the analysis of fusion of [Jal06, §2.2]. In the natural geometry associated to this computation, points are the elements of Ω G -conjugate to those of H and lines are the G -conjugates of H . The set of flags is the set of couples (point, line) where the point belongs to the line, and one considers the subflag naturally associated to H_r . Projecting on the set of points one gets $\text{rk}(H_r^G) + r$ for the rank of this subflag, and similarly $\text{rk}(G/N(H)) + \text{rk}(H_r)$ by projecting on the set of lines. The equality follows. \square

Corollary 2 *Assume furthermore $\text{rk}(G) = \text{rk}(\Omega)$ and $\text{rk}(H) \leq \text{rk}(N(H))$ in Fact 1. Then H^G is generic in Ω if and only if $\text{rk}(H_0) = \text{rk}(N(H))$. In this case a generic element of Ω , and a generic element of H as well, is in finitely many conjugates of H .*

Proof. If H^G is generic in Ω , then one has for some r that $0 \leq r = \text{rk}(H_r) - \text{rk}(N(H)) \leq \text{rk}(H) - \text{rk}(N(H)) \leq 0$, hence $r = 0$ and $\text{rk}(H_0) = \text{rk}(H) = \text{rk}(N(H))$. Conversely, if $\text{rk}(H_0) = \text{rk}(N(H))$, then $\text{rk}(H_0^G) = \text{rk}(G)$. \square

Our last statement is clear. \square

We say that H is *generous* when the equivalent conditions of Corollary 2 are satisfied. Showing generosity in the circumstances of Corollary 2 is thus *equivalent* to showing that a generic element of the considered set is in finitely many conjugates of the set¹.

The following theorem evacuates most cosets as in [CJ04], where it was usually proved locally and/or on demand in more specific cases, and records precious information on the remaining cosets.

Theorem 3 (Generix and the Cosets) *Let G be a group of finite Morley rank in which the generic element of G° belongs to a connected nilpotent subgroup and let H be a definable subgroup of G° . Then $H \setminus H^\circ$ is not generous in G .*

Proof. Assume the contrary. We may assume G connected. By assumption there is an element w in $H \setminus H^\circ$ such that wH° is generous in G . By Corollary 2, $\text{rk}(wH^\circ) = \text{rk}(N(wH^\circ))$. A generic element g of G is, up to conjugacy, a generic element of wH° , and the element g is contained in only finitely many conjugates of wH° . By assumption g also belongs to a connected nilpotent subgroup Q , which may be assumed to be definable. We have $g \in wH^\circ \cap Q$, and we may also assume w in Q , replacing w by a suitable representative of the coset wH° in Q if necessary (for example g). Then $wH^\circ \cap Q = w(H^\circ \cap Q)$. Notice that w still has finite order modulo $H^\circ \cap Q$.

$N_Q^\circ(\langle w \rangle(H^\circ \cap Q))$ normalizes the coset $w(H^\circ \cap Q)$, as connected groups of finite Morley rank acting definably on finite sets fix them pointwise. Now one can argue as in [Jal06, Fundamental Lemma 3.3]: if one denotes by X the set of elements of $w(H^\circ \cap Q)$ contained in finitely many conjugates of wH° , then $N_Q^\circ(\langle w \rangle(H^\circ \cap Q))$ permutes by conjugation these finitely many conjugates, and again by connectedness this group acting on a finite set must fix it pointwise. Notice that X is not empty, as it contains the generic element g . Hence $N_Q^\circ(\langle w \rangle(H^\circ \cap Q))$ normalizes wH° . In particular it normalizes H° . As H° , wH° , and $N(wH^\circ)$ have the same rank, $N^\circ(wH^\circ) = H^\circ$. One gets thus $N_Q^\circ(\langle w \rangle(H^\circ \cap Q)) \leq (H^\circ \cap Q)^\circ$, and in particular $\langle w \rangle(H^\circ \cap Q)$ has finite index in its normalizer in Q . By a classical normalizer condition in infinite connected nilpotent groups of finite Morley rank one finds $Q = \langle w \rangle(H^\circ \cap Q)$. By connectedness of Q one gets $Q = (H^\circ \cap Q)$. Now w must be in H° , and this is a contradiction. \square

¹In trivial cases there is uniqueness. To date, the only known nontrivial case corresponds to the sequence of arguments 6.13-6.17 in [CJ04], and a posteriori this explains why the production of the two single corresponding pages took almost a third of the time devoted to this whole paper.

The following lemma recasts some conclusions of [CJ04] somehow in their original content.

Lemma 4 *Let G be a group of finite Morley rank, H a definable connected generous subgroup of G , w an element of G° in $N(H) \setminus H$ such that wH is not generous. Then*

- *A generic element of wH is in infinitely many conjugates of wH .*
- *If one denotes for a generic element x of wH by $\hat{H}(x)$ the intersection of all $[\langle w \rangle H]^g$, $g \in G$, such that $x \in [wH]^g$, (a definable subgroup containing x and depending on x and wH), then $\hat{H}(x)$ contains no generic element of H*

(Here the notation “ \hat{H} ” has nothing to do with the subgroup H , but rather means a relative definable “ H ”ull.)

Proof. As $N^\circ(H) = H$ by generosity of H and Corollary 2, the first item follows from the nongenerosity of wH by Fact 1 and Corollary 2.

$\hat{H}(x)$ is definable by descending chain condition on definable subgroups. For the second point we simply notice that x is in infinitely many conjugates of wH and that a generic element of H is in finitely many conjugates of H by Fact 1 and Corollary 2 and the generosity of H . \square

Lemma 5 *Let G be a group of finite Morley rank, H a definable connected generous subgroup, and w an element of G° of finite order n normalizing H and such that wH is not generous in G . Suppose that the generic element h of H has the property that h^n is also generic in H . Then $C_H(w) < H$.*

Proof. Assume towards a contradiction $C_H(w) = H$. A generic element of wH has the form wh for some element h generic in H . Now $(wh)^n = h^n$ is generic in H by assumption. But the definable hull of wh contains h^n , and this is a contradiction to Lemma 4. \square

This has the following consequence.

Corollary 6 *Let G be a group of finite Morley rank, n a natural number, H a definable connected generous subgroup with the property that, for h generic in H , h is in a connected nilpotent subgroup of H and h^n is also generic in H , and assume w is an element of G° of finite order n normalizing H without being inside. Then $C_H(w) < H$.*

Proof. It suffices to apply Theorem 3 and Lemma 5. \square

References

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