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# Asymptotics of solutions of the Neumann problem in a domain with closely posed components of the boundary

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## Abstract

The Neumann problem for the Poisson equation is considered in a domain  $\Omega_\varepsilon \subset \mathbb{R}^n$  with boundary components posed at a small distance  $\varepsilon > 0$  so that in the limit, as  $\varepsilon \rightarrow 0^+$ , the components touch each other at the point  $\mathcal{O}$  with the tangency exponent  $2m \geq 2$ . Asymptotics of the solution  $u_\varepsilon$  and the Dirichlet integral  $\|\nabla_x u_\varepsilon; L^2(\Omega_\varepsilon)\|^2$  are evaluated and it is shown that main asymptotic term of  $u_\varepsilon$  and the existence of the finite limit of the integral depend on the relation between the spatial dimension  $n$  and the exponent  $2m$ . For example, in the case  $n < 2m - 1$  the main asymptotic term becomes of the boundary layer type and the Dirichlet integral has no finite limit. Some generalization are discussed and certain unsolved problems are formulated, in particular, non-integer exponents  $2m$  and tangency of the boundary components along smooth curves.

AMS subject classification: primary 35J25, secondary 46E35, 35J20.

## 1 Introduction

### 1.1 Formulation of the problem

Let  $\Gamma$  and  $\Gamma_0$  be smooth closed  $(n-1)$ -dimensional surfaces in the Euclidian space  $\mathbb{R}^n$ , mutually tangent at the point  $\mathcal{O}$ , the origin of the Cartesian coordinate system  $x = (y, z) \in \mathbb{R}^{n-1} \times \mathbb{R}$ . If  $\Omega$  is the domain whose boundary is the surface  $\Gamma$ , the set  $\Gamma_0 \setminus \mathcal{O}$  is included in  $\Omega$  and, in a cylindrical neighborhood  $\mathcal{U} = \mathbb{B}_R^{n-1} \times (-d, d)$  of the point  $\mathcal{O}$ , the surfaces  $\Gamma$  and  $\Gamma_0$  are given respectively by

$$z = -H_-(y) \text{ and } z = H_+(y) \quad (1.1)$$

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where the smooth functions  $H_{\pm}$  on the ball  $\mathbb{B}_R^{n-1} = \{y : r := |y| < R\}$  verify the relations

$$\begin{aligned} H(y) &:= H_-(y) + H_+(y) = r^{2m} H_0(\theta) + \tilde{H}(y), \\ H_{\pm}(0) &= 0, \quad \nabla_y H_{\pm}(0) = 0 \in \mathbb{R}^{n-1}, \quad \nabla_y^j \tilde{H}(0) = 0, \quad j = 0, \dots, 2m. \end{aligned} \quad (1.2)$$

Here  $m$  is an integer,  $r$  and  $\theta = r^{-1}y$  are the spherical coordinates in  $\mathbb{R}^{n-1}$ ,  $H_0$  is a positive function on the unit sphere  $\mathbb{S}^{n-2}$ , and  $\tilde{H}$  is a smooth function on the ball  $\mathbb{B}_R^{n-1}$  which, with all derivatives up to the order  $2m$  included, according to the last formula in (1.2), vanishes at the center of  $\mathbb{B}_R^{n-1}$ . In other words,  $\tilde{H}(y)$  is a small remainder term in comparison with the homogeneous polynomial  $\mathbf{H}(y) = r^{2m} H_0(\theta)$  in variables  $y = (y_1, \dots, y_{n-1})$  of order  $2m$ . Two typical geometrical situations are drawn on Figures 1 and 2; in particular in Fig. 1 the fixed surface  $\Gamma$  is not simply connected.

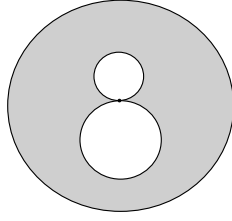


Fig. 1

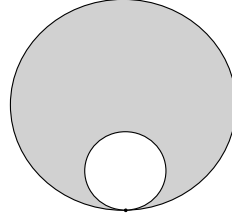


Fig. 2

Let  $\varepsilon$  be a small positive parameter, and

$$\Gamma_{\varepsilon} = \{x = (y, z) : (y, z - \varepsilon) \in \Gamma_0\}. \quad (1.3)$$

The domain bounded by the surfaces  $\Gamma$  and  $\Gamma_{\varepsilon}$  is denoted by  $\Omega_{\varepsilon}$  (cf. Figures 3 and 4 with Figures 1 and 2, respectively).

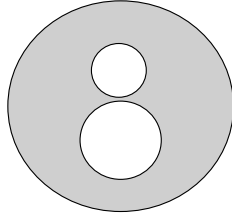


Fig. 3

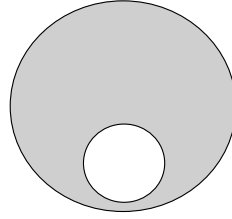


Fig. 4

The sizes  $R, d > 0$  and  $\varepsilon_0$  are chosen in such a way that for  $\varepsilon \in (0, \varepsilon_0]$  the intersection  $\Omega_{\varepsilon} \cap \mathcal{U}$  is defined by the formula

$$\Omega_{\varepsilon} \cap \mathcal{U} = \{(y, z) : y \in \mathbb{B}_R^{n-1}, \quad z \in \Upsilon_{\varepsilon}(y)\} \quad (1.4)$$

where  $\Upsilon_{\varepsilon}(y) := (-H_-(y), \varepsilon + H_+(y))$ . This part of the *singularly perturbed* domain  $\Omega_{\varepsilon}$  is called a *ligament*.

In the domain  $\Omega_{\varepsilon}$  with the ligament located near the point  $\mathcal{O}$ , the Neumann boundary value problem for the Poisson equation is considered

$$-\Delta_x u_{\varepsilon}(x) = f(\varepsilon, x), \quad x \in \Omega_{\varepsilon}, \quad (1.5)$$

$$\partial_{\nu} u_{\varepsilon}(x) = g(\varepsilon, y, z - \varepsilon), \quad x \in \Gamma_{\varepsilon}, \quad \partial_{\nu} u_{\varepsilon}(x) = 0, \quad x \in \Gamma. \quad (1.6)$$

Here  $\partial_\nu$  stands for the normal derivative along the outer normal,  $f$  and  $g$  are smooth functions on the sets  $[0, \varepsilon_0] \times \overline{\Omega}$  and  $[0, \varepsilon_0] \times \Gamma_0$ , respectively. Obviously, the right-hand sides of problem (1.5), (1.6) have to verify the compability condition

$$\int_{\Omega_\varepsilon} f(\varepsilon, x) dx + \int_{\Gamma_\varepsilon} g(\varepsilon, y, z - \varepsilon) ds_x = 0. \quad (1.7)$$

Such condition is satisfied, e.g., in the case of

$$f = 0, \quad g(\varepsilon, x) = G(x), \quad (1.8)$$

where  $G$  is a smooth function on  $\Gamma_0$  with null mean value and independent of the parameter  $\varepsilon$ . In the variable domain  $\Omega_\varepsilon$ , in general, condition (1.7) can be met for the elements  $f$  and  $g$  depending on  $\varepsilon$ . In (1.6) the boundary condition on the surface  $\Gamma$  is homogeneous, without loosing the generality, since a non-homogeneous boundary condition can be compensated by a smooth solution of the Neumann problem in the comprehensive domain  $\Omega$  which by definition includes  $\Omega_\varepsilon$  for all  $\varepsilon \in (0, \varepsilon_0]$ ; so in this way the nonhomogeneous boundary condition is moved from  $\Gamma$  to  $\Gamma_\varepsilon$ .

The main goal of the paper is the derivation of asymptotic formulae for solutions of problem (1.5), (1.6), as well as for the related shape functionals.

## 1.2 Boundary value problems in domains with thin ligaments

Singular perturbed problem (1.5), (1.6) enjoys physical interpretation, and asymptotics of its solution are constructed in a simplified case of the half plane with taken off a circle in paper [1] on the basis of the explicit formulae, obtained by an application of conformal mappings. The Dirichlet problem in the plane domain  $\Omega_\varepsilon$  and in some other type of domains with parts of boundaries put closely one to another, and also the asymptotics of the corresponding Dirichlet integrals, which describe, e.g., the capacities of electrical capacitors, are investigated in [2, 3] (we refer also to monograph [4, Ch. 14]). The asymptotics of the eigenvalues and the eigenfunctions for the Neumann spectral problem in  $\Omega_\varepsilon$  are determined in [5] and the asymptotics of the Dirichlet integral in [6].

The problem investigated in [1] can be considered as out-of-plane problem in the elasticity theory. Analogously, the plane problem of the elasticity theory is studied in [7] for a canonical domain in the form of the half plane with a circle taken off, on the basis of elastic potentials, and in [8] for the domains in much more general form by an applications of the asymptotic methods. The analogous three dimensional problem of elasticity is discussed in [8, 9].

The presented condensed review of the literature on the subject shows that, firstly, the only plane problems were investigated, in exclusion of [8, 9], where it is only performed formal asymptotic analysis, without proofs, and secondly, in the notation of (1.2), the case of  $m = 1$  is worked out. The latter fact is fundamental for the plane Neumann problem, actually for  $G \in C(\Gamma_0)$  and  $m = 1$ , the limit problem for  $\varepsilon = 0$  in the domain  $\Omega_0$  admits a solution with the finite Dirichlet integral; however for  $G(\mathcal{O}) \neq 0$  and  $m > 1$  such a solution does not exist (we refer the reader to [10, 11] and in the sequel to section 1§2). Therefore, the energy functional

$$E(u_\varepsilon; \Omega_\varepsilon) = \frac{1}{2} \int_{\Omega_\varepsilon} |\nabla_x u_\varepsilon(x)|^2 dx + \int_{\Omega_\varepsilon} f(\varepsilon, x) u_\varepsilon(x) dx + \int_{\Gamma_\varepsilon} g(\varepsilon, y, z - \varepsilon) u_\varepsilon(x) ds_x \quad (1.9)$$

admits the finite limit  $E(u_0; \Omega_0)$  for  $m = 1$ , and the main term of asymptotics for the solution  $u_\varepsilon$  is of regular type. In contrast, the case of  $m > 1$  provides no convergence (see Figure 5 where a ball is tangent to a paraboloid with the same curvature at  $\mathcal{O}$  so that  $m = 2$  in (1.1), (1.2)), and the main term of asymptotics becomes of the boundary layer type, localized in a small neighborhood of the point  $\mathcal{O}$ , and written in the *fast* variables

$$\zeta = \varepsilon^{-1} z, \quad \eta = \hbar^{-1} y, \quad \hbar = \varepsilon^{\frac{1}{2m}}. \quad (1.10)$$

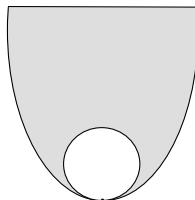


Fig. 5

In view of the Green formula, the quadratic functional (1.9), evaluated with the solutions of problem (1.5), (1.6), is equal to

$$-\frac{1}{2} \int_{\Omega_\varepsilon} |\nabla_x u_\varepsilon(x)|^2 dx = -\frac{1}{2} \int_{\Omega_\varepsilon} f(\varepsilon, x) u_\varepsilon(x) dx - \frac{1}{2} \int_{\Gamma_\varepsilon} g(\varepsilon, y, z - \varepsilon) u_\varepsilon(x) ds_x. \quad (1.11)$$

Let us observe, that according to [10, 11], the necessary and sufficient conditions for the non existence of any solution  $u_0 \in H^1(\Omega_0)$  for  $G(\mathcal{O}) \neq 0$  takes the form

$$n \leq 2m - 1. \quad (1.12)$$

The second feature of the singularly perturbed problem (1.5), (1.6) are related to the spatial dimensions of the domain. Indeed, in any case it is required to have the expansion of the solution  $u_0$  of the limit Neumann problem in the vicinity of the point  $\mathcal{O}$ , where two smooth components of the boundary are tangent each to other. For  $n = 2$  such a singularity of the boundary in the form of two separated peaks can be treated by an application of general results in [12, 13], and is included in the framework of the theory of elliptic boundary problems in the domains with piecewise smooth boundaries (see the key references, [14, 15, 16, 17], and also, e.g., monographs [18, 19]). For  $n \geq 3$  the singularity of the boundary in the vicinity of the point  $\mathcal{O}$  cannot be converted into a conical one and the general theory does not furnish an answer on the behavior of the solution  $u_0$  for  $x \rightarrow \mathcal{O}$ . The asymptotic decomposition of solutions in the vicinity of the point  $\mathcal{O}$  are constructed and justified only for some classes of equations in mathematical physics (see [20, 9, 21] and also [22, 23, 24, 25, 26, 27] for related problems in layered domains) and the analysis is performed with the so-called procedure of dimension reduction (see e.g., [28, 29]). The same procedure is used in section §3 to describe the phenomenon of boundary layer, stretching the coordinates in two different scales  $\varepsilon^{-1}$  and  $\hbar^{-1} = \varepsilon^{-1/2m}$  for transversal and longitudinal directions (see (1.10)) results in the second limit problem which becomes  $(n - 1)$ -dimensional

$$-\nabla_\eta \cdot (1 + \mathbf{H}(\eta)) \nabla_\eta w(\eta) = F(\eta), \quad \eta \in \mathbb{R}^{n-1}, \quad (1.13)$$

see also section 1§4. The behavior of the solution  $u_0$  with  $x \rightarrow \mathcal{O}$ , and of the solution  $w$  with  $\eta \rightarrow \infty$  is investigated in §2, with the particular attention paid to the case  $n \geq 3$  for the reasons explained above. The plane problem has some particular features in the algorithm of asymptotic construction presented in §3 and in the weighted trace inequality necessary for the justification of asymptotics and given in the next section of §1. Concluding remarks are presented in §4, where the possible generalization of the results are discussed and certain unsolved problems are mentioned.

### 1.3 The weighted Poincaré inequality

To derive an a priori estimate for solutions of problem (1.5), (1.6), we proceed with proving variants of one-dimensional Hardy's inequalities.

**Lemma 1** If  $U \in C_c^1[0, R]$ , the following inequality holds

$$\int_0^R (\hbar + r)^{2m} \mathbf{R}_n(r) |U(r)|^2 r^{n-2} dr \leq c_n \int_0^R (\hbar + r)^{2m} \left| \frac{dU(r)}{dr} \right|^2 r^{n-2} dr, \quad (1.14)$$

where  $n \geq 3$  and  $\mathbf{R}_3(r) = r^{-2} \left| \ln \frac{r}{R} \right|^{-2}$ ,  $\mathbf{R}_n(r) = r^{-2}$  for  $n \geq 4$ . In the case  $n = 2$  the inequality takes the form

$$\int_0^R (\hbar + r)^{2m-2} |U(r)|^2 dr \leq \frac{4}{(2m-1)^2} \int_0^R (\hbar + r)^{2m} \left| \frac{dU(r)}{dr} \right|^2 dr. \quad (1.15)$$

**Proof.** If  $n > 3$ , inequality (1.14) with the constant  $c_n = 4(n-3)^{-2}$  follows from the relation using the Newton-Leibnitz formula and the Cauchy-Bunyakowsky-Schwartz inequality:

$$\begin{aligned} \int_0^R (\hbar + r)^{2m} r^{n-4} |U(r)|^2 dr &= -2 \int_0^R (\hbar + r)^{2m} r^{n-4} \int_r^R \frac{dU}{d\rho}(\rho) U(\rho) d\rho dr \leq \\ &\leq 2 \int_0^R \left| \frac{dU}{d\rho}(\rho) \right| |U(\rho)| \int_0^\rho (\hbar + r)^{2m} r^{n-4} dr d\rho \leq \frac{2}{n-3} \int_0^R \left| \frac{dU}{d\rho}(\rho) \right| |U(\rho)| (\hbar + \rho)^{2m} \rho^{n-3} d\rho \leq \\ &\leq \frac{2}{n-3} \left( \int_0^R (\hbar + \rho)^{2m} \rho^{n-2} \left| \frac{dU}{d\rho}(\rho) \right|^2 d\rho \right)^{\frac{1}{2}} \left( \int_0^R (\hbar + \rho)^{2m} \rho^{n-4} |U(\rho)|^2 d\rho \right)^{\frac{1}{2}}. \end{aligned}$$

Similar calculations are sufficient in the case  $n = 3$ , namely,  $c_3 = 4$  and

$$\begin{aligned} \int_0^R (\hbar + r)^{2m} r^{-1} \left| \ln \frac{r}{R} \right|^{-2} |U(r)|^2 dr &\leq 2 \int_0^R \left| \frac{dU}{d\rho}(\rho) \right| |U(\rho)| \int_0^\rho (\hbar + r)^{2m} r^{-1} \left| \ln \frac{r}{R} \right|^{-2} dr d\rho \leq \\ &\leq 2 \int_0^R \left| \frac{dU}{d\rho}(\rho) \right| |U(\rho)| (\hbar + \rho)^{2m} \left| \ln \frac{r}{R} \right|^{-2} d\rho \leq \\ &\leq 2 \left( \int_0^R (\hbar + \rho)^{2m} \rho \left| \frac{dU}{d\rho}(\rho) \right|^2 d\rho \right)^{\frac{1}{2}} \left( \int_0^R (\hbar + \rho)^{2m} \rho^{-1} \left| \ln \frac{r}{R} \right|^{-2} |U(\rho)|^2 d\rho \right)^{\frac{1}{2}}. \end{aligned}$$

Finally, estimate (1.15) is obtained from

$$\begin{aligned} \int_0^R (\hbar + r)^{2m-2} |U(r)|^2 dr &\leq 2 \int_0^R \left| \frac{dU}{d\rho}(\rho) \right| |U(\rho)| \int_0^\rho (\hbar + r)^{2m-2} dr d\rho = \\ &\leq \frac{2}{2m-1} \int_0^R \left| \frac{dU}{d\rho}(\rho) \right| |U(\rho)| \left( (\hbar + \rho)^{2m-1} - \hbar^{2m-1} \right) d\rho \leq \frac{2}{2m-1} \int_0^R (\hbar + \rho)^m \left| \frac{dU}{d\rho}(\rho) \right| (\hbar + \rho)^{m-1} |U(\rho)| d\rho. \blacksquare \end{aligned}$$

We are ready to verify the *weighted* Poincaré and trace inequalities.

**Proposition 2** Let  $u_\varepsilon \in H^1(\Omega_\varepsilon)$  satisfy the orthogonality condition

$$\int_{\Omega_\varepsilon} u_\varepsilon(x) dx = 0. \quad (1.16)$$

The relationship

$$\begin{aligned} &\left\| \mathcal{R}_n u_\varepsilon; L^2(\Omega_\varepsilon) \right\|^2 + \left\| (\varepsilon + \rho^{2m})^{\frac{1}{2}} \mathcal{R}_n u_\varepsilon; L^2(\partial\Omega_\varepsilon) \right\|^2 + \\ &+ \left\| (\varepsilon + \rho^{2m})^{-1} (u_\varepsilon^+ - u_\varepsilon^-); L^2(\mathbb{B}_R^{n-1}) \right\|^2 \leq c \left\| \nabla_x u_\varepsilon; L^2(\Omega_\varepsilon) \right\|^2 \end{aligned} \quad (1.17)$$

is valid, where  $u_\varepsilon^+(y) = u_\varepsilon(y, \varepsilon + H_+(y))$ ,  $u_\varepsilon^-(y) = u_\varepsilon(y, -H_-(y))$ ,  $\rho(x) = |x|$ ,

$$\mathcal{R}_n(x) = \begin{cases} (\hbar + \rho(x))^{-1} & \text{as } n \neq 3, \\ (\hbar + \rho(x))^{-1} (1 + |\ln(\hbar + \rho(x))|)^{-1} & \text{as } n = 3, \end{cases} \quad (1.18)$$

$\hbar$  is the small parameter in (1.10), and the constant  $c$  depends on neither  $u_\varepsilon$ , nor  $\varepsilon \in (0, \varepsilon_0]$ .

**Proof.** By virtue of the completion, we, without any loss of generality, assume  $u_\varepsilon$  to be sufficiently smooth. Let  $\omega$  be a fixed domain in  $\Omega_\varepsilon \setminus \mathcal{U}$ ,  $\text{meas}_n \omega > 0$  (on Figure 6 the domain is overshadowed).

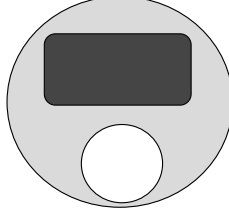


Fig. 6

We set

$$u_\varepsilon = v_\varepsilon + c_\varepsilon, \quad c_\varepsilon = (\text{meas}_n \omega)^{-1} \int_\omega u_\varepsilon(x) dx, \quad \int_\omega v_\varepsilon(x) dx = 0. \quad (1.19)$$

Clearly,  $\|\nabla_x v_\varepsilon; L^2(\Omega_\varepsilon)\| = \|\nabla u_\varepsilon; L^2(\Omega_\varepsilon)\|$ . Furthermore, by the orthogonality condition (1.19), the following variants of the Poincaré inequality are valid:

$$\begin{aligned} \|v_\varepsilon; L^2(\omega)\| &\leq c_\omega \|\nabla_x v_\varepsilon; L^2(\Omega(\varepsilon))\|, \\ \|v_\varepsilon; L^2(\Omega(\varepsilon))\| &\leq C(\Omega(\varepsilon), \omega) \|\nabla_x v_\varepsilon; L^2(\Omega(\varepsilon))\| + \|v_\varepsilon; L^2(\omega)\| \leq \\ &\leq C(\Omega(\varepsilon), \omega) (1 + c_\omega) \|\nabla_x v_\varepsilon; L^2(\Omega(\varepsilon))\|. \end{aligned} \quad (1.20)$$

The domain  $\Omega(\varepsilon) = \Omega_\varepsilon \setminus \overline{\mathbb{B}}_{R/2}^{n-1} \times [-d, d]$  depends *regularly* on the parameter  $\varepsilon$  and thus the coefficient  $C(\Omega(\varepsilon), \omega)$  can be taken independent of  $\varepsilon \in (0, \varepsilon_0]$  (cf. [30, §7.6.5]).

We now set  $w_\varepsilon = \chi v_\varepsilon$  where  $\chi$  is a smooth cut-off function such that  $\chi = 1$  on  $\Omega_\varepsilon \setminus \Omega(\varepsilon)$  and  $\chi = 0$  outside of  $\Omega_\varepsilon \cap \mathcal{U}$ . In view of (1.20), we evidently have

$$\|\nabla_x w_\varepsilon; L^2(\Omega_\varepsilon \cap \mathcal{U})\| \leq c_\chi \|\nabla_x v_\varepsilon; L^2(\Omega_\varepsilon)\| = c_\chi \|\nabla_x u_\varepsilon; L^2(\Omega_\varepsilon)\|. \quad (1.21)$$

Let us employ the decomposition

$$w_\varepsilon(x) = \overline{w}_\varepsilon(y) + w_\varepsilon^\perp(x), \quad \overline{w}_\varepsilon(y) = (\varepsilon + H(y))^{-1} \int_{-H_-(y)}^{\varepsilon + H_+(y)} w_\varepsilon(y, z) dz. \quad (1.22)$$

Since the integral of  $w_\varepsilon^\perp$  over the interval  $\Upsilon_\varepsilon(y) = (-H_-(y), \varepsilon + H_+(y))$  vanishes, the Poincaré inequality yields

$$\int_{\Omega_\varepsilon \cap \mathcal{U}} (\varepsilon + H(y))^{-2} |w_\varepsilon^\perp(x)|^2 dx \leq \pi^2 \int_{\Omega_\varepsilon \cap \mathcal{U}} |\partial_z w_\varepsilon^\perp(x)|^2 dx = \pi^2 \int_{\Omega_\varepsilon \cap \mathcal{U}} |\partial_z w_\varepsilon(x)|^2 dx. \quad (1.23)$$

Moreover, the standard way to prove trace inequalities (cf. [31]) leads to the relation

$$\begin{aligned} &\int_{\mathbb{B}_R^{n-1}} (\varepsilon + H(y))^{-1} \left( |w_\varepsilon^\perp(y, \varepsilon + H_+(y))|^2 + |w_\varepsilon^\perp(y, -H_-(y))|^2 \right) dy = \\ &= \int_{\Omega_\varepsilon \cap \mathcal{U}} (\varepsilon + H(y))^{-2} \frac{\partial}{\partial z} \left[ (2z + H_-(y) - \varepsilon - H_+(y)) + w_\varepsilon^\perp(y, z)^2 \right] dx \leq \\ &\leq \int_{\Omega_\varepsilon \cap \mathcal{U}} (\varepsilon + H(y))^{-1} \left( (\varepsilon + H(y))^{-1} |w_\varepsilon^\perp(y, z)| + 2 |\partial_z w_\varepsilon^\perp(y, z)| \right) |w_\varepsilon^\perp(y, z)| dx \leq \\ &\leq \int_{\Omega_\varepsilon \cap \mathcal{U}} \left( |\partial_z w_\varepsilon^\perp(y, z)|^2 + (\varepsilon + H(y))^{-1} |w_\varepsilon^\perp(y, z)|^2 \right) dx. \end{aligned} \quad (1.24)$$

On the other hand,

$$\begin{aligned}
& \int_{\mathbb{B}_R^{n-1}} (\varepsilon + H(y))^{-1} |w_\varepsilon(y, \varepsilon + H_+(y)) - w_\varepsilon(y, -H_-(y))|^2 dy = \\
& = \int_{\mathbb{B}_R^{n-1}} (\varepsilon + H(y))^{-1} |w_\varepsilon^\perp(y, \varepsilon + H_+(y)) - w_\varepsilon^\perp(y, -H_-(y))|^2 dy \leq \\
& \leq 2 \int_{\mathbb{B}_R^{n-1}} (\varepsilon + H(y))^{-1} |w_\varepsilon^\perp(y, \varepsilon + H_+(y))|^2 + |w_\varepsilon^\perp(y, -H_-(y))|^2 dy.
\end{aligned} \tag{1.25}$$

Let us process the component  $\bar{w}_\varepsilon$  in decomposition (1.22). We have

$$\begin{aligned}
\int_{\Omega_\varepsilon \cap \mathcal{U}} |\nabla_y w_\varepsilon(x)|^2 dx &= \int_{\Omega_\varepsilon \cap \mathcal{U}} |\nabla_y w_\varepsilon^\perp(x)|^2 dx + \int_{\mathbb{B}_R^{n-1}} (\varepsilon + H(y))^{-1} |\nabla_y \bar{w}_\varepsilon(y)|^2 dy + \\
&+ 2 \int_{\mathbb{B}_R^{n-1}} \nabla_y \bar{w}_\varepsilon(y) \cdot \int_{\Upsilon_\varepsilon(y)} \nabla_y w_\varepsilon^\perp(y, z) dz dy =: I_1 + I_2 + 2I_3.
\end{aligned} \tag{1.26}$$

By (1.2), the relation

$$c(\hbar + r)^{2m} \leq \varepsilon + H(y) \leq C(\hbar + r)^{2m}, \tag{1.27}$$

is valid with positive constants  $c$  and  $C$ , independent of  $\varepsilon \in (0, \varepsilon_0]$ . Thus,

$$\begin{aligned}
I_2 &\geq c \int_{\mathbb{B}_R^{n-1}} (\hbar + r)^{2m} |\nabla_y \bar{w}_\varepsilon(y)|^2 dy \geq c \int_0^R \int_{\mathbb{S}^{n-2}} (\hbar + r)^{2m} |\partial_r \bar{w}_\varepsilon(r\theta)|^2 r^{n-2} dr ds_\theta \geq \\
&\geq c \int_0^R \int_{\mathbb{S}^{n-2}} (\hbar + r)^{2m-2} (1 + \delta_{n,3} |\ln r|)^{-2} |\bar{w}_\varepsilon(r\theta)|^2 r^{n-2} dr ds_\theta = \\
&= c \int_{\mathbb{B}_R^{n-1}} (\hbar + r)^{2m-2} (1 + \delta_{n,3} |\ln(\hbar + r)|)^{-2} |\bar{w}_\varepsilon(y)|^2 dy.
\end{aligned} \tag{1.28}$$

Here we have applied Lemma 1 while observing that the weights on the right of (1.14) and (1.15) are larger than  $(\hbar + r)^{2m-2} (1 + \delta_{n,3} |\ln(\hbar + r)|)^{-2}$ .

It remains to estimate the integral  $I_3$  in (1.26). Since

$$\nabla_y \int_{\Upsilon_\varepsilon(y)} w_\varepsilon^\perp(y, z) dz = \int_{\Upsilon_\varepsilon(y)} \nabla_y w_\varepsilon^\perp(y, z) dz + \nabla_y H_+(y) w_\varepsilon^\perp(y, \varepsilon + H_+(y)) + \nabla_y H_-(y) w_\varepsilon^\perp(y, -H_-(y)), \tag{1.29}$$

we recall that the left-hand side of (1.29) vanishes due to (1.22) and, therefore, by (1.2) we obtain

$$\begin{aligned}
|I_3| &\leq \int_{\mathbb{B}_R^{n-1}} |\nabla_y \bar{w}_\varepsilon(y)| (|w_\varepsilon^\perp(y, \varepsilon + H_+(y))| + |w_\varepsilon^\perp(y, -H_-(y))|) dy \leq \\
&\leq \delta I_2 + C\delta^{-1} \int_{\mathbb{B}_R^{n-1}} (|w_\varepsilon^\perp(y, \varepsilon + H_+(y))|^2 + |w_\varepsilon^\perp(y, -H_-(y))|^2) dy,
\end{aligned} \tag{1.30}$$

where  $\delta > 0$  is arbitrary. The last integral has been estimated in (1.23), even with additional large factor  $(\varepsilon + H_+(y))^{-1}$  in the integrand. Thus, choosing  $\delta = \frac{1}{4}$  we derive from (1.26), (1.30) and (1.23) that

$$I_1 + I_2 \leq c \int_{\Omega_\varepsilon \cap \mathcal{U}} |\nabla_x w_\varepsilon(x)|^2 dx.$$

We now take into account that, first,  $\hbar + r$  and  $\hbar + \rho$  are equivalent infinitesimals on  $\Omega_\varepsilon \cap \mathcal{U}$  and, second,

$$\mathcal{R}_n(x) \leq c(\varepsilon + H_+(y))^{-1}, \quad \mathcal{R}_n(x) \leq c(\hbar + r)^{-1} (1 + \delta_{n,3} |\ln(\hbar + r)|)^{-1}, \quad x \in \Omega_\varepsilon \cap \mathcal{U}.$$

These ensure the inequality (1.17) with  $u_\varepsilon$  replaced by  $w_\varepsilon$ . Indeed, the necessary estimate for the first and the second terms on the right-hand side of (1.17) follow from (1.23), (1.28) and (1.24), (1.28),



respectively, while noticing that the integration in  $z$  of  $\bar{w}_\varepsilon$  over the interval  $\Upsilon_\varepsilon(y)$  brings the additional factor  $\varepsilon + H_+(y)$  but this factor is absent in the norms on the surfaces  $\Gamma_\varepsilon^+ = \Gamma_\varepsilon \cap \mathcal{U}$  and  $\Gamma_\varepsilon^- = \Gamma \cap \mathcal{U}$ . A bound for the norm of the differences of the traces  $w_\varepsilon|_{\Gamma_\varepsilon^\pm}$  is directly given by (1.25) because the difference is not influenced by  $\bar{w}_\varepsilon(y)$ . We emphasize that the Jacobians  $1 + |\nabla_y H_\pm|^2$  are bounded on  $\Gamma_\varepsilon^\pm$  and, hence, cannot spoil the estimates.

The weights in (1.17) and (1.18) are bounded in  $\Omega_\varepsilon \cap \mathcal{U}$  uniformly in  $\varepsilon \in (0, \varepsilon_0]$  and that is why the inequality (1.17) for  $w_\varepsilon$  together with relations (1.20) provide the inequality (1.17) for  $v_\varepsilon$ . By (1.19) and (1.16), we thus obtain

$$|c_\varepsilon| = \left| \int_{\Omega_\varepsilon} v_\varepsilon(x) dx \right| \leq c \|v_\varepsilon; L^2(\Omega_\varepsilon)\| \leq c \|\nabla_x u_\varepsilon; L^2(\Omega_\varepsilon)\|.$$

Since

$$\int_{\Omega_\varepsilon} \mathcal{R}_n(x)^2 dx + \int_{\partial\Omega_\varepsilon} (\varepsilon + \rho^{2m}) \mathcal{R}_n(x)^2 ds_x \leq \text{const}, \quad \varepsilon \in (0, \varepsilon_0],$$

the desired inequality (1.17) is proved. ■

## 2 Asymptotics of solutions of limit problems.

### 2.1 Asymptotics of unbounded energy solutions

Let us consider problem (1.5), (1.6) for  $\varepsilon = 0$ ,

$$-\Delta_x v_0(x) = f_0(x), \quad x \in \Omega_0, \quad \partial_\nu v_0(x) = g_0(x), \quad x \in \partial\Omega_0 \setminus \mathcal{O}. \quad (2.1)$$

Here  $\partial\Omega_0$  stands for the union of surfaces  $\Gamma_0$  and  $\Gamma$ ,  $f_0(x) = f(0, x)$  and  $g_0(x) = 0$  for  $x \in \Gamma$ , and  $g_0(x) = g(0, x)$  for  $x \in \Gamma_0$ . Passage to the limit  $\varepsilon \rightarrow 0$  in equality (1.7) leads to the compability condition

$$\int_{\Omega_0} f_0(x) dx + \int_{\partial\Omega_0} g_0(x) ds_x = 0. \quad (2.2)$$

However, problem (2.1) does not always admits a solution  $v_0$  in  $H^1(\Omega)$ . According to [10] the weighted trace inequality of the following Lemma shows that in the case of a smooth function  $g_0$  on  $\Gamma_0$ , for the existence of a solution of (2.1), in addition to (2.2) the following relations should be valid

$$\nabla_y^k g_0(\mathcal{O}) = 0, \quad k \leq m - \frac{n+1}{2}, \quad (2.3)$$

where  $\nabla_y^k g$  is the collection of derivatives of the function  $g$  of order  $k$  on the surface  $\Gamma_0$ .

**Lemma 3** *For any function  $v \in H^1(\Omega_0)$  the inequality*

$$\|\rho^{m-1}v; L^2(\partial\Omega_0)\| + \|\rho^{-1}v; L^2(\Omega_0)\| \leq c \|v; H^1(\Omega_0)\|, \quad (2.4)$$

*holds, where  $\rho = |x|$  and the constant  $c$  is independent of  $v$ .*

In section 3 §1 the approach of [32, 10] to prove inequality (2.4) was used to verify Proposition 2.

**Remark 4** *In the case of kissing balls (compare Figures 1 and 2 with Figures 3 and 4, respectively) we have  $m = 1$ , which implies that there are no additional conditions (2.3) for any  $n \geq 2$ , i.e., the solution  $v \in H^1(\Omega_0)$  readily exists. However, for the ball  $\mathbb{B}_{R^0}^n$  kissing from interior paraboloid (see Fig. 5) and having the same radius  $R^0$  of curvature at the top, the exponent  $m$  in formula (1.2) becomes 2. Therefore, the solvability of the most interesting cases of plane and three dimensional problems (2.1) is lost in the energy classes (we refer the reader to [10] for the details). ■*

If relations (2.3) are verified then  $\rho^{1-m}g_0 \in L^2(\partial\Omega)$  and, in view of Lemma 3, the right-hand side of the integral identity

$$(\nabla_x v_0, \nabla_x \Psi)_{\Omega_0} = (f_0, \Psi)_{\Omega_0} + (g_0, \Psi)_{\partial\Omega_0}, \quad \Psi \in C^\infty(\overline{\Omega_0} \setminus \mathcal{O}), \quad (2.5)$$

is a continuous functional on the space  $H^1(\Omega_0)$ . Thus, the Riesz representation theorem in Hilbert space combined with the Fredholm alternative ensures the solvability of the Neumann problem. Note that the embedding  $H^1(\Omega_0) \subset L^2(\Omega_0)$  is compact due to inequality (2.4).

In (2.5) and further  $(\cdot, \cdot)_\Xi$  denotes the scalar product in the Lebesgue space  $L^2(\Xi)$ .

**Lemma 5** *If  $\rho f_0 \in L^2(\Omega_0)$  and  $\rho^{1-m}g_0 \in L^2(\Gamma_0)$ , problem (2.1) admits a generalized solution  $v_0 \in H^1(\Omega_0)$  verifying integral identity (2.5) if and only if condition (2.2) is satisfied. Such a solution is defined up to an additive constant, and meets the inequality*

$$\|\nabla_x v_0; L^2(\Omega_0)\| \leq c(\|\rho f_0; L^2(\Omega_0)\| + \|\rho^{1-m}g_0; L^2(\Gamma_0)\|). \quad (2.6)$$

The solution with null mean value in the domain  $\Omega_0$  is unique and satisfies the estimate

$$(\|\rho^{m-1}v_0; L^2(\partial\Omega_0)\| + \|\rho^{-1}v_0; L^2(\Omega_0)\|) \leq c_0(\|\rho f_0; L^2(\Omega_0)\| + \|\rho^{1-m}g_0; L^2(\Gamma_0)\|).$$

In an artificial way, condition (2.3) or  $\rho^{1-m}g_0 \in L^2(\Gamma_0)$  can be ensured. Few unbounded energy terms in asymptotics of the solution can be constructed with the remainder in the class  $H^1(\Omega_0)$ . The corresponding procedure is described in [20] (see also [9, 21] for other problems in mathematical physics). We are going to recall briefly the procedure, namely, the way to compensate the right-hand sides of two special types. We check that additional discrepancies, resulting from asymptotic terms for the solution, are of the same kind, but are characterized by smaller growth rate or larger decay rate as  $x \rightarrow \mathcal{O}$ . In this way, in few iterations of the procedure, the main part of the asymptotics of the solution is generated, and the remainder is defined as the bounded energy solution of problem (2.1) with the corrected right-hand sides, the decay rate of which in the vicinity of the point  $\mathcal{O}$  is sufficiently high. Now the condition  $g_0 = 0$  is no longer required.

**Remark 6** *Owing to the Newton-Leibnitz formula*

$$v(y, H_+(y)) - v(y, -H_-(y)) = \int_{\gamma_0(y)} \partial_z v(y, z) dz;$$

and inequality (2.4), the restrictions imposed on the data of problem (2.1) can be changed for

$$\rho f_0 \in L^2(\Omega_0), \quad g_0 \in L_{loc}^2(\partial\Omega_0 \setminus \mathcal{O}),$$

$$\rho^{1-m}(g_0^+ + g_0^-) \in L^2(\mathbb{B}_R^{n-1}), \quad \rho^{1/2}(g_0^+ + g_0^-) \in L^2(\mathbb{B}_R^{n-1}),$$

where  $g_0^\pm(y) = g_0(y, \pm H(y))$  for  $y \in \mathbb{B}_R^{n-1}$ . We refer the reader to [10] and [33] for details. ■

The first kind of special right-hand sides is given by

$$f_0(x) = H(y)^{-2} r^\mu \mathbf{f}(\theta, \ln r, \zeta) + \dots, \quad g_0^\pm(y) = \left(1 + |\nabla_y H_\pm(y)|^2\right)^{-1/2} H(y)^{-1} r^\mu \mathbf{g}^\pm(\theta, \ln r) + \dots, \quad (2.7)$$

where  $\mu$  is a number,  $\mathbf{f}$  and  $\mathbf{g}^\pm$  are polynomials in  $\ln r$  and  $\zeta = H(y)^{-1}(z + H_-(y))$  while coefficients are smooth functions in  $\theta \in \mathbb{S}^{n-2}$ , and dots stand for the terms of lower order. Moreover,  $\mathbf{f}$  and  $\mathbf{g}^\pm$  are subjected to the orthogonality condition

$$\int_0^1 \mathbf{f}(\theta, \ln r, \zeta) d\zeta + \mathbf{g}^+(\theta, \ln r) + \mathbf{g}^-(\theta, \ln r) = 0. \quad (2.8)$$

By  $\zeta$  is denoted the *fast* variable, which describes the behaviour of the solution  $v_0$  on the degenerated ligament  $\Omega_0 \cap \mathcal{U}$ . For the variables  $y = (y_1, \dots, y_{n-1})$  the previous scale is retained and only the new notation  $\eta = (\eta_1, \dots, \eta_{n-1})$  is introduced. Since

$$\nabla_y = \nabla_\eta - H(\eta)^{-1} (\zeta \nabla_\eta H(\eta) - \nabla_\eta H_-(\eta)) \partial_\zeta, \quad \partial_z = H(\eta)^{-1} \partial_\zeta, \quad (2.9)$$

the Laplace operator in the curvilinear coordinates  $(\eta, \zeta)$  takes the form

$$\Delta_x = H(\eta)^{-2} \partial_\zeta^2 + \left[ \nabla_\eta - H(\eta)^{-1} (\zeta \nabla_\eta H(\eta) - \nabla_\eta H_-(\eta)) \partial_\zeta \right]^2. \quad (2.10)$$

The unit outer normal vectors  $\nu^\pm$  to the surfaces  $\Gamma^+ = \Gamma_0 \cap \mathcal{U}$  and  $\Gamma^- = \Gamma \cap \mathcal{U}$  are defined by

$$\nu^\pm(y) = \left( 1 + |\nabla_y H_\pm(y)|^2 \right)^{-1/2} (-\nabla_y H_\pm(y), \pm 1), \quad (2.11)$$

hence

$$\partial_{\nu^\pm} = \left( 1 + |\nabla_y H_\pm(y)|^2 \right)^{-1/2} (\pm \partial_z - \nabla_y H_\pm(y) \cdot \nabla_y), \quad (2.12)$$

and the central dot  $\cdot$  indicates the scalar product in  $\mathbb{R}^{n-1}$ . From (2.12) and (2.9), it follows that

$$\partial_{\nu^\pm} = \left( 1 + |\nabla_y H_\pm(y)|^2 \right)^{-1/2} H(\eta)^{-1} (\pm \partial_\zeta - \nabla_\eta H_\pm(\eta) \cdot (\nabla_\eta \mp \nabla_\eta H_\pm(\eta) \partial_\zeta)). \quad (2.13)$$

The main asymptotic parts of operators (2.10) and (2.13) are equal to  $H(\eta)^{-2} \partial_\zeta^2$  and  $H(\eta)^{-1} \partial_\zeta$ , respectively. Therefore, asymptotic solution which compensate for the terms separated in (2.7) becomes

$$r^\mu \mathbf{v}(\theta, \ln r, \zeta), \quad (2.14)$$

where the same notation as above is used. The angular part  $\mathbf{v}$  is a solution of Neumann problem for ordinary differential equation on the interval  $(0, 1)$

$$-\partial_\zeta^2 \mathbf{v}(\theta, \ln r, \zeta) = \mathbf{f}(\theta, \ln r, \zeta), \quad \zeta \in (0, 1), \quad \pm \partial_\zeta \mathbf{v}\left(\theta, \ln r, \frac{(1 \pm 1)}{2}\right) = \mathbf{g}^\pm(\theta, \ln r). \quad (2.15)$$

Relations (2.8) ensure the existence of a solution of (2.15); in addition, the orthogonality condition

$$\int_0^1 \mathbf{v}(\theta, \ln r, \zeta) d\zeta = 0, \quad (2.16)$$

makes it uniquely defined.

We determine the discrepancies left by the asymptotic solution (2.14). Let us recall that the conditions required in section 1§1 for functions  $H_\pm$ , describing the surfaces  $\Gamma^\pm$  (see, e.g., (1.2) and (1.1)), imply

$$\begin{aligned} H_\pm(y) &= \mathbf{H}_\pm(y) + O(r^{2m+1}), \quad \mathbf{H}(y) = \mathbf{H}_+(y) + \mathbf{H}_-(y) = r^{2m} H_0(\theta), \\ \pm \mathbf{H}_\pm(y) &= \sum_{p=2}^{2m} \mathbf{H}^p(y) \pm \mathbf{H}_\pm^{2m}(y), \end{aligned} \quad (2.17)$$

where  $\mathbf{H}^p$  and  $\mathbf{H}_\pm^{2m}$  are homogeneous polynomials of order  $p$  and  $2m$ , respectively. In this way, by (2.12) and (2.15), we obtain

$$\begin{aligned} r^\mu \mathbf{v}(\theta, \ln r, \zeta) |_{\zeta=(1\pm 1)/2} &- \left( 1 + |\nabla_y H_\pm(y)|^2 \right)^{-1/2} H(y)^{-1} \mathbf{g}^\pm(\theta, \ln r) = \\ &= - \left( 1 + |\nabla_y H_\pm(y)|^2 \right)^{-1/2} H(y)^{-1} \nabla_y H_\pm(y) \cdot (H(y) \nabla_y \mp \nabla_y H_\pm(y) \partial_\zeta) r^\mu \mathbf{v}(\theta, \ln r, \zeta) |_{\zeta=(1\pm 1)/2} \sim \\ &\sim \left( 1 + |\nabla_y H_\pm(y)|^2 \right)^{-1/2} H(y)^{-1} \sum_{p=2}^{\infty} r^{\mu+p} \mathbf{g}_p^\pm(\theta, \ln r). \end{aligned} \quad (2.18)$$

Here

$$r^{\mu+p} \mathbf{g}_p^\pm(\theta, \ln r) = \pm r^\mu \mathbf{h}_p(y) \partial_\zeta \mathbf{v}(\theta, \ln r, \zeta) |_{\zeta=(1\pm1)/2} \quad (2.19)$$

and  $\mathbf{h}_p(y)$  are the coefficients of the expansions of functions  $|\nabla_y H_\pm(y)|^2$ , obtained in the same way as in (2.17). Besides, the coefficients are independent of the sign  $\pm$  for  $p = 1, \dots, 2m-1$ .

In view of relations (2.10), (2.15) and (2.17) we have

$$\begin{aligned} & -\Delta_x r^\mu \mathbf{v}(\theta, \ln r, \zeta) - H(y)^{-2} r^\mu \mathbf{f}(\theta, \ln r, \zeta) = \\ & = \left[ \nabla_y - H(y)^{-1} (\zeta \nabla_y H(y) - \nabla_y H_-(y)) \partial_\zeta \right]^2 r^\mu \mathbf{v}(\theta, \ln r, \zeta) \sim \\ & \sim H(y)^{-2} \sum_{p=2}^{\infty} r^{\mu+p} \mathbf{f}_p(\theta, \ln r, \zeta) \end{aligned} \quad (2.20)$$

with

$$r^{\mu+p} \mathbf{f}_p(\theta, \ln r, \zeta) = -\mathbf{h}_p(y) \partial_\zeta^2 r^\mu \mathbf{v}(\theta, \ln r, \zeta), \quad p = 2, \dots, 2m-1. \quad (2.21)$$

According to (2.16) and (2.15), (2.8), terms (2.19) and (2.21) in formal series of (2.18) and (2.20) satisfy the orthogonality conditions of type (2.8), while the terms  $\mathbf{g}_p^\pm$  and  $\mathbf{f}_p$  for  $p \geq 2m$  do not. We set

$$\mathbf{F}_p(\theta, \ln r) = H_0(\theta)^{-1} \left[ \int_0^1 \mathbf{f}_p(\theta, \ln r, \zeta) d\zeta + \mathbf{g}_p^+(\theta, \ln r) + \mathbf{g}_p^-(\theta, \ln r) \right]. \quad (2.22)$$

Three functions  $\mathbf{f}_p$  and  $\mathbf{g}_p^\pm$ ,  $\mathbf{g}_p^\pm - H_0 \mathbf{F}_p$  verify the orthogonality condition. Thus, it remains to compensate discrepancy (2.22) in the Neumann condition on the upper surface  $\Gamma^+ = \Gamma_0 \cap \mathcal{U}$ .

The second kind of special right-hand sides is compatible with data (1.8) or (2.22), namely,

$$f_0(x) = \dots, \quad g_0^-(y) = \dots, \quad g_0^+(y) = \left(1 + |\nabla_y H_+(y)|^2\right)^{-1/2} \mathbf{H}(y) r^\mu \mathbf{F}(\theta, \ln r) + \dots \quad (2.23)$$

The asymptotic solution, which compensates for the terms given by (2.23), reads

$$r^{\mu+2-2m} \mathbf{V}(\theta, \ln r). \quad (2.24)$$

According to (2.10) and (2.12), we have

$$-\Delta_x r^{\mu+2-2m} \mathbf{V}(\theta, \ln r) = -\Delta_y r^{\mu+2-2m} \mathbf{V}(\theta, \ln r), \quad (2.25)$$

$$\partial_{\nu^\pm} r^{\mu+2-2m} \mathbf{V}(\theta, \ln r) = - \left(1 + |\nabla_y H_\pm(y)|^2\right)^{-1/2} \nabla_y H_\pm(y) \cdot \nabla_y r^{\mu+2-2m} \mathbf{V}(\theta, \ln r).$$

In view of (1.2) and (2.17) expressions (2.25) can be represented by formal series with the terms, analogous to (2.20) and (2.18),

$$H(y)^{-2} r^{\mu+p} \mathbf{F}_p^0(\theta, \ln r), \quad \left(1 + |\nabla_y H_\pm(y)|^2\right)^{-1/2} H(y)^{-1} r^{\mu+p} \mathbf{G}_p^\pm(\theta, \ln r). \quad (2.26)$$

Here  $p = 1, 2, \dots$  and

$$\mathbf{F}_q^0 = 0, \quad \mathbf{G}_q^\pm = \mathbf{G}_q^0 \quad \text{if } q = 1, \dots, 2m-1; \quad (2.27)$$

$$r^{\mu+2m} \mathbf{F}_{2m}^0(\theta, \ln r) = -\mathbf{H}^2(y) \Delta_y r^{\mu+2-2m} \mathbf{V}(\theta, \ln r), \quad (2.28)$$

$$r^{\mu+2m} \mathbf{G}_{2m}^\pm(\theta, \ln r) = r^{\mu+2m} \mathbf{G}_{2m}^0(\theta, \ln r) - \mathbf{H}(y) \nabla_y \mathbf{H}_\pm^{2m}(y) \cdot \nabla_y r^{\mu+2-2m} \mathbf{V}(\theta, \ln r).$$

It is clear that the orthogonality conditions (2.8) hold for quantities in (2.27). Thus the terms in (2.26) with the indices  $p = 1, \dots, 2m-1$ , form the special right-hand sides of the first kind, of lower order, since  $p \geq 1$ , compared to terms (2.7). On the other hand, in general, by expressions (2.28) the compability conditions are not true. The expressions are introduced in order to compensate for the special right-hand sides in (2.23) of the second kind. The quantity  $\mathbf{H}(y) r^\mu \mathbf{F}(\theta, \ln r)$  is of the same polynomial order  $r^{\mu+2m}$

as  $r^{\mu+2m}\mathbf{G}_{2m}^+(\theta, \ln r)$ . Accordingly, we take the difference of the second quantity and the first one and we subject this difference together with  $r^{\mu+2m}\mathbf{F}_{2m}^0(\theta, \ln r)$  and  $r^{\mu+2m}\mathbf{G}_{2m}^-(\theta, \ln r)$  to the orthogonality conditions (2.8). In other words, these right-hand sides become of the first kind artificially.

The compability condition takes the form

$$-\nabla_y \cdot \mathbf{H}(y) \nabla_y r^{\mu+2-2m} \mathbf{V}(\theta, \ln r) = r^\mu \mathbf{F}(\theta, \ln r), \quad y \in \mathbb{R}^{n-1} \setminus \{0\}, \quad (2.29)$$

which is but a degenerate elliptic differential equation on punctured space. In the following section it is proved that for each right-hand side  $\mathbf{F}$ , which is a polynomial with respect to  $\ln r$  and a smooth function of the variable  $\theta \in \mathbb{S}^{n-2}$ , equation (2.29) admits at least one solution (2.24), where  $\mathbf{V}$  is a polynomial in  $\ln r$  with smooth coefficients in  $\theta \in \mathbb{S}^{n-2}$ . In the sequel we also explain why the data and solutions are assumed to be polynomials in  $\ln r$ .

**Remark 7** For  $n = 2$  the set  $\mathbb{R}^{n-1} \setminus \{0\}$  in (2.29) is the union of two non connected rays  $\mathbb{R}_\pm$ , i.e., model problem (2.29) becomes a system of two ordinary differential equations. The observation has no influence on the result stated above. However, it makes sense to consider the particular case of  $n = 2$  separately, and we refer the reader to sections 3 §3 and 5 §4 for further results on the plane problem (see also Remark 16). ■

The right-hand sides of form (2.7) and (2.23) are compensated by solutions (2.14) and (2.24), which leave discrepancies of the same type, but of lower order. Hence, the procedure of formal asymptotics expansions is completed.

## 2.2 Degenerated model equation and the second limit problem

Let  $n \geq 3$ . Besides the equation in the punctured space introduced in section 1 §2

$$\mathbf{L}(y, \nabla_y) v(y) := -\nabla_y \cdot \mathbf{H}(y) \nabla_y v(y) = f(y), \quad y \in \mathbb{R}^{n-1} \setminus \{0\}, \quad (2.30)$$

we consider closely related equation (1.13), posed on the whole space  $\mathbb{R}^{n-1}$ , in particular for  $n = 2$ . First, we introduce the Kondratiev function spaces [14], denoted by  $\mathbf{V}_\beta^l(\mathbb{R}^{n-1})$  and  $V_\beta^l(\mathbb{R}^{n-1})$ , with the norms

$$\|v; \mathbf{V}_\beta^l(\mathbb{R}^{n-1})\| = \left( \sum_{k=0}^l \|r^{\beta-l+k} \nabla_y^k v; L^2(\mathbb{R}^{n-1})\|^2 \right)^{1/2}, \quad (2.31)$$

$$\|w; V_\beta^l(\mathbb{R}^{n-1})\| = \left( \sum_{k=0}^l \|(1+r)^{\beta-l+k} \nabla_y^k w; L^2(\mathbb{R}^{n-1})\|^2 \right)^{1/2}. \quad (2.32)$$

The first space is the completion of the linear set  $C_c^\infty(\mathbb{R}^{n-1} \setminus \{0\})$  of smooth functions with compact supports disjoint from the origin  $y = 0$  in the *homogeneous* norm (2.31). The space contains functions with singularities at the origin  $y = 0$ . Since the *non homogeneous* norm (2.32) includes non degenerate weight multiplier, the space  $V_\beta^l(\mathbb{R}^{n-1})$  is embedded into  $H_{loc}^l(\mathbb{R}^{n-1})$ . In addition, there is continuous embedding  $V_\beta^l(\mathbb{R}^{n-1}) \subset V_\gamma^p(\mathbb{R}^{n-1})$  for all  $l \geq p$  and  $\beta - l \geq \gamma - p$ , which becomes compact only for the strict inequalities  $l > p$  and  $\beta - l > \gamma - p$ . On the other hand, the continuous embedding  $\mathbf{V}_\beta^l(\mathbb{R}^{n-1}) \subset \mathbf{V}_\gamma^p(\mathbb{R}^{n-1})$  takes place if and only if  $l \geq p$  and  $\beta - l = \gamma - p$ , and never is compact.

We equip equations (2.30) and (1.13) with the mappings

$$v \in \mathbf{V}_\beta^{l+1}(\mathbb{R}^{n-1}) \mapsto \mathbf{A}_\beta^l v = \mathbf{L}v \in \mathbf{V}_{\beta-2m}^{l-1}(\mathbb{R}^{n-1}), \quad (2.33)$$

$$w \in V_\beta^{l+1}(\mathbb{R}^{n-1}) \mapsto A_\beta^l w = Lw \in V_{\beta-2m}^{l-1}(\mathbb{R}^{n-1}), \quad (2.34)$$

where  $l \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$  is the index of regularity and  $\beta \in \mathbb{R}$  is the weight index. The case of  $l = 0$  should be considered separately; in such the case the generalized solution  $v \in V_\beta^1(\mathbb{R}^{n-1})$  satisfies the integral identity (see [31])

$$(\mathbf{H} \nabla_y v, \nabla_y V)_{\mathbb{R}^{n-1}} = (f, V)_{\mathbb{R}^{n-1}} \quad (2.35)$$

with any test function  $V \in V_{2m-\beta}^1(\mathbb{R}^{n-1})$ , and a linear and continuous functional on the space  $V_{2m-\beta}^1(\mathbb{R}^{n-1})$  on the right, i.e.,  $f \in V_{\beta-2m}^{-1}(\mathbb{R}^{n-1})$ . Similar interpretation can be applied to the mapping  $A_\beta^1$ .

Both equations can be considered in the framework of general theory of elliptic problems in domains with conical point; we refer the reader to the key papers [14], [15], [16], [17], and to monographs [18], [19]. The punctured space  $\mathbb{R}^{n-1} \setminus \{0\}$  is regarded as a *complete cone* and in view of (1.2) the operator  $\mathbf{L}$  admits the representation

$$\mathbf{L}(y, \nabla_y) = r^{2m-2} \mathcal{L}(\theta, \nabla_\theta, r \partial_r). \quad (2.36)$$

Finally, operator (2.36) is the main part of the operator  $L(y, \nabla_y) = -\nabla_y \cdot (1 + \mathbf{H}(y)) \nabla_y$  at infinity.

For an application of the general theory mentioned above, the first step concerns the analysis of the spectral problem  $\mathcal{L}(\theta, \nabla_\theta, \lambda) \Phi(\theta) = 0$  on the unit sphere  $\mathbb{S}^{n-2}$ , which in the full form reads

$$-\nabla_\theta \cdot H_0(\theta) \nabla_\theta \Phi(\theta) = \lambda(\lambda + 2m + n - 3) H_0(\theta) \Phi(\theta), \quad \theta \in \mathbb{S}^{n-2}, \quad (2.37)$$

where  $\nabla_\theta$  stands for spherical gradient. We present two simple statements. The first follows from the theory of selfadjoint operators in Hilbert spaces (see, e.g., [34]), the second is a result borrowed from [14] (see also [18]) on polynomial solutions

$$v(y) = r^\lambda \Phi(\theta) \quad (2.38)$$

of the model problem in the cone  $\mathbb{K} = \mathbb{R}^{n-1} \setminus \{0\}$ , the case of  $n = 2$  is discussed in Remark 16;

**Lemma 8** *The spectral equation*

$$-\nabla_\theta \cdot H_0(\theta) \nabla_\theta \Phi(\theta) = \Lambda H_0(\theta) \Phi(\theta), \quad \theta \in \mathbb{S}^{n-2}, \quad (2.39)$$

*admits infinite number of eigenvalues*

$$0 = \Lambda_0 < \Lambda_1 \leq \Lambda_2 \leq \dots \leq \Lambda_k \leq \dots \rightarrow +\infty, \quad (2.40)$$

*which are listed in (2.40) taking into account their multiplicities. The corresponding eigenfunctions*

$$\|H_0; L^1(\mathbb{S}^{n-2})\|^{-1/2} = \Phi_0, \Phi_1, \Phi_2, \dots, \Phi_k, \dots \quad (2.41)$$

*are smooth on the sphere  $\mathbb{S}^{n-2}$  and can be normalized by the orthogonality conditions*

$$(H_0 \Phi_j, \Phi_k)_{\mathbb{S}^{n-2}} = \delta_{j,k}, \quad (2.42)$$

*where  $j, k \in \mathbb{N}_0$  and  $\delta_{j,k}$  is the Kronecker symbol.*

**Lemma 9** *If the solution of the homogeneous equation (2.30) is given by (2.38) and  $\Phi \neq 0$ , then  $\lambda$  is one of the numbers*

$$\lambda_k^\pm = \frac{1}{2} \left( 3 - n - 2m \pm \sqrt{(3 - n - 2m)^2 + 4\Lambda_k} \right), \quad (2.43)$$

*and  $\Phi$  is an eigenfunction of equation (2.39) corresponding to the eigenvalue  $\Lambda_k$ .*

We present also some results which follows from the general theory combined with Lemma 9. Such results are given in [14] and we indicate the corresponding statement in [18] to unify the references.

**Proposition 10** ([14], [18, Thm. 3.5.1 and Thm. 4.1.2]) *The mapping (2.33) is an isomorphism, and the mapping (2.34) is Fredholm if and only if the number  $l - \beta - (n - 3)/2$  is not in the set defined by (2.43).*

**Proposition 11** ([14], [18, Thm.3.5.6]) Let  $f \in \mathbf{V}_{\beta_1-2m}^{l-1}(\mathbb{R}^{n-1}) \cap \mathbf{V}_{\beta_2-2m}^{l-1}(\mathbb{R}^{n-1})$ ,  $\beta_1 < \beta_2$ , the numbers  $l + 2 - \beta_i - n$  do not belong to set (2.43) and  $v_i \in \mathbf{V}_{\beta_i}^{l+1}(\mathbb{R}^{n-1})$  are solutions of equation (2.30). Then

$$v_1(y) - v_2(y) = \sum c_k^\pm r^{\lambda_k^\pm} \Phi_k(\theta), \quad (2.44)$$

where the summation is performed over indices from set (2.43) which belong to the interval

$$\left( l - \beta_2 - \frac{1}{2}(n-3), l - \beta_1 - \frac{1}{2}(n-3) \right). \quad (2.45)$$

If interval (2.45) is free of the numbers  $\lambda_k^\pm$ , then the right-hand side in (2.44) is null and the solutions  $v_1$  and  $v_2$  coincide. The coefficients  $c_k^\pm$  in (2.44) admit the estimate

$$|c_k^\pm| \leq c \left( \|f; \mathbf{V}_{\beta_1-2m}^{l-1}(\mathbb{R}^{n-1})\| + \|f; \mathbf{V}_{\beta_2-2m}^{l-1}(\mathbb{R}^{n-1})\| \right). \quad (2.46)$$

**Proposition 12** ([14], [18, Lemma 3.5.11]) For arbitrary number  $\mu$  and polynomial  $\mathbf{F}$  in  $\ln r$  with smooth coefficients in  $\theta \in \mathbb{S}^{n-1}$ , equation (2.29) has solution (2.23), which is a smooth function of  $\theta$  and a polynomial in  $\ln r$ . If  $\mu + 2 - 2m$  is not included in set (2.43), then  $\deg \mathbf{V} = \deg \mathbf{F}$ , otherwise,  $\deg \mathbf{V} = 1 + \deg \mathbf{F}$  for  $\mu + 2 - 2m = \lambda_k^\pm$ .

**Remark 13** If  $\mu = 0$  and  $\mathbf{F}(\theta) = \text{const}$ , then for  $m = 1$  the index  $2 - 2m$  coincides with  $\lambda_0^+ = 0$  and a solution to problem (2.29) takes the form

$$C_1 \ln r + \Psi(\theta) + C_0 \quad (2.47)$$

where  $C_1$  is a constant, the function  $\Psi \in C^\infty(\mathbb{S}^{n-2})$  has the null mean value on the sphere  $\mathbb{S}^{n-2}$ , and  $C_0$  is an arbitrary constant. We point out that

$$0 = \lambda_0^+ \geq 2 - 2m > \lambda_0^- = 3 - n - 2m,$$

which means, in view of Proposition 12, that for  $m = 2, 3, \dots$  equation (2.29) with the constant right-hand side admits polynomial solution  $r^{2-2m}\Psi(\theta)$ . ■

Now we are in position to establish the unique solubility of equation (1.13)

**Theorem 14** Mapping (2.34) turns out to be an isomorphism if and only if

$$l - \beta - \frac{1}{2}(n-3) \in (\lambda_0^-, \lambda_0^+) = (3 - n - 2m, 0). \quad (2.48)$$

**Proof.** Since  $\mathbf{L}$  is the main part of the operator  $L$  at infinity, by Proposition 11 the dimension  $d$  of the kernel  $\ker A_\beta^l = \{v \in V_\beta^{l+1}(\mathbb{R}^{n-1}) : Lv = 0\}$  is independent of the index  $\beta$  under condition (2.48). From a general result in [15] (see also [18, Thm. 3.4.2]), the operators  $A_\beta^l$  and  $A_{\beta_*}^l$ , where  $\beta_* = 2l + 2m - \beta$ , turn out to be formally adjoint, which means that from Fredholm alternative we can deduce that subspaces  $\text{coker} A_\beta^l \approx \ker A_{\beta_*}^l$  are of the same dimension  $d$ . We point out that the indices  $\beta$  and  $\beta_*$  verify (2.48) simultaneously. We fix the weight index  $\beta_0 = l + m$  which is exactly in the middle of the interval in (2.48), and note that the inclusion  $v \in V_{\beta_0}^{l+1}(\mathbb{R}^{n-1}) \subset V_m^1(\mathbb{R}^{n-1})$  implies the convergence of integral  $\int_{\mathbb{R}^{n-1}} (1 + \mathbf{H}(y)) |\nabla_y v(y)|^2 dy$ . Therefore, the only solution of the homogeneous problem (1.13) in the class  $V_{l+m}^{l+1}(\mathbb{R}^{n-1})$  is a constant, but non-null constants are absent in the class, since by the condition  $l - \beta - (n-3)/2 > -\lambda_0^- = 0$  the integral  $\int_{\mathbb{R}^{n-1}} (1 + r)^{2(\beta-l-1)} dy$  is divergent. Hence,  $d = 0$  and, in view of Proposition 10 and Lemma 9, the sufficiency of condition (2.48) is proved.

The necessity is checked in the standard way (see e.g., [18, §4.1]), namely for  $\beta = l - \frac{1}{2}(n-3) - \lambda_0^\pm$  operator  $A_\beta^l$  loses Fredholm property by Proposition 10, and for  $l - \beta - (n-3)/2 < \lambda_0^-$  (resp. for  $l - \beta - (n-3)/2 > \lambda_0^+$ ) the kernel of operator  $A_\beta^l$  (resp. the kernel of operator  $A_{\beta_*}^l$  or cokernel of operator  $A_\beta^l$ ) contains a constant. ■

**Remark 15** *There is another way to prove Theorem 14. At the first step the existence of the unique generalized solution can be shown (see the integral identity (2.35)) in the energy class  $V_m^1(\mathbb{R}^{n-1})$ . In the second step, the regularity of the generalized solution can be improved, by the method [14]. Finally, the existence result is generalized to interval (2.48) by applying of Proposition 11 on the asymptotics of solutions to the model problem (2.30). ■*

**Remark 16** *If  $n = 2$  then (1.13) is an ordinary differential equation on the real axis  $\mathbb{R} \ni y$ , and the model homogeneous problem (2.30), which implies two equations (see Remark 7), admits two pairs of linearly independent solutions*

$$v_0^\pm(y) = 1, \quad v_1^\pm(y) = |y|^{1-2m}, \quad \pm y > 0.$$

*In other words,  $\lambda_0^+ = 0$  and  $\lambda_0^- = 1 - 2m$  are the only indices in (2.43). Theorem 14 is still valid, however instead of one constant and bounded solution for  $n \geq 3$  there are two linearly independent bounded solutions of homogeneous equation (1.13) for  $n = 2$  of the form*

$$W_0(\eta) = 1, \quad W_1(\eta) = \int_0^\eta \left(1 + \mathbf{H}(\eta)^2\right)^{-1} d\eta,$$

*where  $W_1$  is an odd function of the variable  $\eta \in \mathbb{R}$ . ■*

### 2.3 Asymptotics of bounded energy solution of the limit problem in vicinity of point $\mathcal{O}$

In section 1§2 it is shown how to construct *unbounded energy* components of solutions to problem (2.1) with *bad* right-hand sides. Such components form the main terms of asymptotics of the solution  $v_0$  with infinite Dirichlet integral. However, it is still not explained, how to construct the asymptotic remainder  $\tilde{v}_0 \in H^1(\Omega)$ . The same question is raised in the case of *good* right-hand sides. The answer is given in paper [20] in much more complicated situation, but for the convenience of the reader a simple and independent proof is presented for the problem under consideration. Beside that, owing to the algorithm of formal asymptotic construction in section 1§2, we can suppose that the right-hand sides enjoy the sufficient decay rate for  $x \rightarrow \mathcal{O}$ .

All arguments and results can be applied to the problem with  $n = 2$ ; however the existing results for the plane domains with peaks singularities on the boundaries are much more advanced and include e.g., further estimates in weighted Hölder and  $L^p$ -spaces along with the asymptotics for solutions with insufficient smoothness of data (we refer the reader to the papers [12, 13, 16] for results in this direction). In contrast, most of such questions are still open for the singularities associated with two tangent surfaces in  $\mathbb{R}^n$  with  $n \geq 3$ .

Let  $v_0 \in H^1(\Omega)$  be a solution to problem (2.1), or equivalently, to identity (2.5). We select the test function  $\psi(y)$ , which equals to  $\Psi(y)$  on the degenerate ligament  $\Omega_0 \cap \mathcal{U}$ , and to zero otherwise, with  $\Psi \in C_c^\infty(\mathbb{B}_R^{n-1} \setminus \{0\})$ . Then identity (2.5) takes the form

$$\begin{aligned} \int_{\mathbb{B}_R^{n-1}} \nabla_y \Psi(y) \cdot \int_{\Upsilon_0(y)} \nabla_y v_0(y, z) dz dy &= \int_{\mathbb{B}_R^{n-1}} \Psi(y) \int_{\Upsilon_0(y)} f_0(y, z) dz dy + \\ &+ \sum_{\pm} \int_{\mathbb{B}_R^{n-1}} \Psi(y) g_0^\pm(y) \left(1 + |\nabla_y H_\pm(y)|^2\right)^{1/2} dy. \end{aligned} \quad (2.49)$$

In the same way as (1.29) we obtain

$$\int_{\Upsilon_0(y)} \nabla_y v_0(y, z) dz = \nabla_y H(y) \bar{v}_0(y) - \sum_{\pm} \int_{\mathbb{B}_R^{n-1}} v_0(y, \pm H_\pm(y)) \nabla_y H_\pm(y), \quad (2.50)$$

where  $\bar{v}_0(y) = H(y)^{-1} \int_{\Upsilon_0(y)} v_0(y, z) dz$  (compare with formula (1.22) for  $z = 0$ ).



**Lemma 17** *If  $r^{-\sigma} \partial_z v_0 \in L_2(\Omega_0 \cap \mathcal{U})$  for some  $\sigma \geq 0$  then*

$$\|r^{-\sigma-m} (\bar{v}_0(\cdot) - v_0(\cdot, \pm H_{\pm}(\cdot)))\|; L^2(\mathbb{B}_R^{n-1})\| \leq c \|r^{-\sigma} \partial_z v_0; L^2(\Omega_0 \cap \mathcal{U})\|. \quad (2.51)$$

**Proof** By the Newton-Leibniz formula

$$v_0(y, \mathbf{z}) - v_0(y, -H_-(y)) = \int_{-H_-(y)}^{\mathbf{z}} \partial_z v_0(y, z) dz$$

and by integration over  $\Upsilon_0(y) \ni \mathbf{z}$ , we have

$$\begin{aligned} H(y) (\bar{v}_0(y) - v_0(y, -H_-(y))) &= \int_{-H_-(y)}^{H_+(y)} \int_{-H_-(y)}^{\mathbf{z}} \partial_z v_0(y, z) dz d\mathbf{z} = \\ &= \int_{-H_-(y)}^{H_+(y)} (H_+(y) - z) \partial_z v_0(y, z) dz. \end{aligned} \quad (2.52)$$

Multiplying equality (2.52), after taking square of both sides, by  $r^{-2\sigma} H(y)^{-3}$ , and integrating over  $\mathbb{B}_R^{n-1} \ni y$  lead to

$$\begin{aligned} &\int_{\mathbb{B}_R^{n-1}} r^{-2\sigma} H(y)^{-1} |\bar{v}_0(y) - v_0(y, -H_-(y))|^2 dy \leq \\ &\leq \int_{\mathbb{B}_R^{n-1}} r^{-2\sigma} H(y)^{-3} \left| \int_{\Upsilon_0(y)} (H_+(y) - z) \partial_z v_0(y, z) dz \right|^2 dy \leq \\ &\leq c \int_{\mathbb{B}_R^{n-1}} r^{-2\sigma} \int_{\Upsilon_0(y)} |\partial_z v_0(y, z)|^2 dy dz. \end{aligned}$$

To complete the proof, it suffices to make the change  $-H_- \mapsto H_+$  in the calculation. ■

Taking into account (2.50), we rewrite (2.49) in the form

$$\int_{\mathbb{B}_R^{n-1}} \mathbf{H}(y) \nabla_y \bar{v}_0(y) \nabla_y \Psi(y) dy = \int_{\mathbb{B}_R^{n-1}} \mathcal{F}_0(y) \Psi(y) dy + \int_{\mathbb{B}_R^{n-1}} \mathcal{F}_1(y) \cdot \nabla_y \Psi(y) dy \quad (2.53)$$

where

$$\begin{aligned} \mathcal{F}_1(y) &= (\mathbf{H}(y) - H(y)) \nabla_y \bar{v}_0(y) + \sum_{\pm} (v_0(y, \pm H_{\pm}(y)) - \bar{v}_0(y)) \nabla_y H_{\pm}(y), \\ \mathcal{F}_0(y) &= \int_{\Upsilon_0(y)} f_0(y, z) dz + \sum_{\pm} g_0^{\pm}(y) \left(1 + |\nabla_y H_{\pm}(y)|^2\right)^{1/2}. \end{aligned} \quad (2.54)$$

**Theorem 18** *Let  $\sigma$  and  $\tau$  be two positive numbers, such that  $\tau \in (\sigma, 1 + \sigma]$  and the interval*

$$\left( \sigma - m - \frac{1}{2}(n-3), \tau - m - \frac{1}{2}(n-3) \right) \quad (2.55)$$

*contains (compare with (2.45)) the exponents  $\lambda_k^{\pm}, \dots, \lambda_{k+\kappa-1}^{\pm}$  and the interval ends  $\sigma - m - \frac{1}{2}(n-3)$ ,  $\tau - m - \frac{1}{2}(n-3)$  are not in set (2.43). Let  $v_0 \in H^1(\Omega)$  be a solution to problem (2.1) with the right-hand sides*

$$\rho^{1-\tau} f_0 \in L^2(\Omega_0), \quad \rho^{1-\tau-m} g_0 \in L^2(\partial\Omega_0) \quad (2.56)$$

*and, in addition,*

$$\rho^{-\sigma} \nabla_y v_0 \in L^2(\Omega_0), \quad \rho^{-1-\sigma} v_0 \in L^2(\Omega_0). \quad (2.57)$$

*Then the representation*

$$v_0(x) = \chi(y) \sum_{p=0}^{\kappa-1} c_p r^{\lambda_{k+p}^+} \Phi_{k+p}(\theta) + \tilde{v}_0(x), \quad (2.58)$$

is valid where  $\chi$  is a smooth cut-off function, which vanishes outside the degenerate ligament and is equal to one on  $\Omega_0 \cap \mathcal{U}$  for  $r < R/2$ ,  $c_k, \dots, c_{k+\kappa-1}$  are constants,  $\tilde{v}_0$  is the remainder with  $\rho^{-\tau} \nabla_x \tilde{v}_0 \in L^2(\Omega_0)$ ,  $\rho^{-1-\tau} \tilde{v}_0 \in L^2(\Omega_0)$ . Furthermore, the estimate

$$\sum_{p=0}^{\kappa-1} |c_p| + \|\rho^{-\tau} \nabla_x \tilde{v}_0; L^2(\Omega_0)\| + \|\rho^{-1-\tau} \tilde{v}_0; L^2(\Omega_0)\| \leq c\mathcal{N}, \quad (2.59)$$

holds true where  $\mathcal{N}$  stands for the sum of norms of functions (2.56) and (2.57) in the indicated spaces.

**Proof.** Asymptotic formula (2.58) and estimate (2.56) are local, namely, taking into account that  $\rho > c > 0$  for  $x \in \Omega_0 \setminus \mathcal{U}$ , the required properties of the remainder outside a vicinity of the point  $\mathcal{O}$  follow from inclusions (2.57). Therefore, after multiplication of  $v_0$  by an appropriate cut-off function  $\chi$  we assume that the solution  $v_0$  as well as the right-hand sides  $f_0$  and  $g_0$  vanish outside the set  $\overline{\Omega}_0 \setminus \mathcal{U}$ . By definition of the mean value  $\bar{v}_0$ , inclusions (2.57) yield

$$\rho^{-1-\sigma+m} \bar{v}_0 \in L^2(\mathbb{B}_R^{n-1}), \quad \rho^{-\sigma+m} \nabla_x \bar{v}_0 \in L^2(\mathbb{B}_R^{n-1}), \quad (2.60)$$

i.e.,  $\bar{v}_0 \in \mathbf{V}_{-\sigma-m}^1(\mathbb{B}_R^{n-1})$  is a solution of variational problem (2.53). In addition, in view of (1.2) and (2.51), respectively, we obtain

$$\begin{aligned} (\mathbf{H} - H) \nabla_y \bar{v}_0 &\in \mathbf{V}_{-\sigma-m-1}^0(\mathbb{B}_R^{n-1}), \\ (v_0(\cdot, \pm H_{\pm}) - \bar{v}_0) \nabla_y H_{\pm} &\in \mathbf{V}_{-\sigma-m-1}^0(\mathbb{B}_R^{n-1}). \end{aligned}$$

Beside that, by (2.55), we have

$$\int_{\Upsilon_0(y)} f_0(\cdot, z) dz \in \mathbf{V}_{-\tau-m+1}^0(\mathbb{B}_R^{n-1}),$$

which, owing to the assumed inequality  $-\sigma-1 \leq -\tau$ , means that expressions (2.54) satisfy the inclusions

$$\mathcal{F}_0 \in \mathbf{V}_{-\tau-m+1}^0(\mathbb{B}_R^{n-1}), \quad \mathcal{F}_1 \in \mathbf{V}_{-\tau-m}^0(\mathbb{B}_R^{n-1}). \quad (2.61)$$

In other words, the right-hand side (2.53) is a continuous linear functional on the space  $\mathbf{V}_{\tau+m}^1(\mathbb{B}_R^{n-1})$ . Thus, all assumptions of Proposition 11 are verified, for the choice of exponents  $\beta_1 = -\tau+m$ ,  $\beta_2 = -\sigma+m$  and  $l = 0$ . As a result, taking into account the required bounds on the indices  $\sigma$  and  $\tau$ , asymptotic formula (2.44) is transformed into

$$\bar{v}_0(y) = \chi(y) \sum_{p=0}^{\kappa-1} c_p r^{\lambda_{k+p}^+} \Phi_{k+p}(\theta) + \tilde{v}_0(x). \quad (2.62)$$

We point out that Proposition 11 covers two singular points of the complete cone  $\mathbb{K} = \mathbb{R}^{n-1} \setminus \{0\}$ , namely, the origin and the point at infinity. At the same time, formula (2.62) delivers the decomposition of the function  $\bar{v}_0$  for  $r \rightarrow 0$ , therefore, we can include a cut-off function  $\chi$  which is equal to one in the ball  $\mathbb{B}_{R/2}^{n-1}$  and null outside the ball  $\mathbb{B}_R^{n-1}$ . In this way the remainder (2.62) is still included in the space  $\mathbf{V}_{-\tau-m}^1(\mathbb{B}_R^{n-1})$  and for  $r > R$ . Beside that, inequality (2.46) and Proposition 10 imply the estimate

$$\sum_{p=0}^{\kappa-1} |c_p| + \|\tilde{v}_0; \mathbf{V}_{-\tau-m}^1(\mathbb{B}_R^{n-1})\| \leq c\mathcal{N}. \quad (2.63)$$

Note that the norms of functions (2.60) and (2.61) do not exceed  $c\mathcal{N}$ .

Since

$$\|\rho^{-\tau} \nabla_x \tilde{v}_0; L^2(\Omega_0)\| + \|\rho^{-1-\tau} \tilde{v}_0; L^2(\Omega_0)\| \leq c \|\tilde{v}_0; \mathbf{V}_{-\tau-m}^1(\mathbb{B}_R^{n-1})\|, \quad (2.64)$$

it remains to analyse the difference  $v_0^\perp(y, z) = v_0(y, z) - \bar{v}_0(y)$ , subject to the orthogonality condition

$$\int_{\mathcal{U}_0(y)} v_0^\perp(y, z) dz = 0, \quad y \in \mathbb{B}_R^{n-1}. \quad (2.65)$$

To this end, we introduce the continuous weight function

$$P(y) = \begin{cases} (1 + \mathbf{t}r^{-1})^\tau, & \text{if } r > \mathbf{r}, \\ (1 + \mathbf{t}\mathbf{r}^{-1})^\tau, & \text{if } r \leq \mathbf{r}, \end{cases} \quad (2.66)$$

where  $\mathbf{t}$  and  $\mathbf{r}$  are small positive parameters. Derivatives of functions (2.66) vanish on the ball  $\mathbb{B}_{\mathbf{r}}^{n-1}$  and satisfy

$$|\nabla_y P(y)| \leq \tau \mathbf{t} r^{-2} P(y). \quad (2.67)$$

We select the test function  $\psi = P^2 v_0^\perp \in H^1(\Omega_0 \cap \mathcal{U})$  in the integral identity (2.5), and recall that  $v_0 = 0$  in the exterior of  $\Omega_0 \cap \mathcal{U}$ . Denote  $V_0 = P v_0$ , and respectively  $V_0^\perp = P v_0^\perp$ ,  $\bar{V}_0 = P \bar{v}_0$ . We then obtain

$$\begin{aligned} (Pf_0, V_0^\perp)_{\Omega_0} + \sum_{\pm} (Pg_0^\pm, V_0^\perp)_{\Gamma^\pm} &= (\nabla_x v_0, \nabla_x (P^2 v_0^\perp))_{\Omega_0} = \\ &= (P \nabla_x v_0, \nabla_x V_0^\perp)_{\Omega_0} + (P \nabla_y v_0, V_0^\perp P^{-1} \nabla_y P)_{\Omega_0} = \\ &= (\nabla_x V_0, \nabla_x V_0^\perp)_{\Omega_0} - (V_0 P^{-1} \nabla_y P, \nabla_y V_0^\perp)_{\Omega_0} + \\ &\quad + (\nabla_y V_0, V_0^\perp P^{-1} \nabla_y P)_{\Omega_0} - (V_0 P^{-1} \nabla_y P, V_0^\perp P^{-1} \nabla_y P)_{\Omega_0}. \end{aligned} \quad (2.68)$$

The orthogonality condition (2.65) leads to

$$\begin{aligned} (V_0 P^{-1} \nabla_y P, V_0^\perp P^{-1} \nabla_y P)_{\Omega_0} &= \|V_0^\perp P^{-1} \nabla_y P; L^2(\Omega_0 \cap \mathcal{U})\|^2, \\ (\nabla_y V_0, V_0^\perp P^{-1} \nabla_y P)_{\Omega_0} &= (\nabla_y V_0^\perp, V_0^\perp P^{-1} \nabla_y P)_{\Omega_0}. \end{aligned} \quad (2.69)$$

Taking into account formula (2.50), we find

$$\begin{aligned} |(\nabla_y \bar{V}_0, \nabla_y V_0^\perp)_{\Omega_0}| &= \left| \int_{\mathbb{B}_R^{n-1}} \nabla_y \bar{V}_0(y) \int_{\mathcal{U}_0(y)} \nabla_y V_0^\perp(y, z) dz dy \right| \leq \\ &\leq c \int_{\mathbb{B}_R^{n-1}} |\nabla_y \bar{V}_0(y)| r \sum_{\pm} |V_0^\perp(y, \pm H_\pm(y))| dy, \\ |(\bar{V}_0 P^{-1} \nabla_y P, \nabla_y V_0^\perp)_{\Omega_0}| &\leq c \int_{\mathbb{B}_R^{n-1}} |\bar{V}_0(y)| \frac{|\nabla_y P(y)|}{P(y)} r \sum_{\pm} |V_0^\perp(y, \pm H_\pm(y))| dy. \end{aligned} \quad (2.70)$$

Note that the multiplier  $r$  appears in (2.70) due to the estimate  $|\nabla_y H_\pm(y)| \leq cr$  (see (1.2)). We use also the Poincaré inequality combined with the trace inequality, which are valid by condition (2.65) and the relation  $H(y) \geq cr^{2m}$ ,  $c > 0$ :

$$\int_{\Omega_0 \cap \mathcal{U}} r^{-4m} |V_0^\perp(x)|^2 dx + \sum_{\pm} \int_{\mathbb{B}_R^{n-1}} r^{-2m} |V_0^\perp(y, \pm H_\pm(y))|^2 dy \leq c \|\nabla_x V_0^\perp; L^2(\Omega_0 \cap \mathcal{U})\|^2. \quad (2.71)$$

Collecting formulae (2.68)-(2.71), we derive that

$$\begin{aligned} &\|\nabla_x V_0^\perp; L^2(\Omega_0 \cap \mathcal{U})\|^2 - \|V_0^\perp P^{-1} \nabla_y P; L^2(\Omega_0 \cap \mathcal{U})\|^2 = \\ &= (Pf_0, V_0^\perp)_{\Omega_0 \cap \mathcal{U}} + \sum_{\pm} (Pg_0^\pm, V_0^\perp)_{\Gamma^\pm} + (\bar{V}_0 P^{-1} \nabla_y P, \nabla_y V_0^\perp)_{\Omega_0 \cap \mathcal{U}} - (\nabla_y \bar{V}_0, \nabla_y V_0^\perp)_{\Omega_0 \cap \mathcal{U}} \leq \\ &\leq c \|\nabla_x V_0^\perp; L^2(\Omega_0 \cap \mathcal{U})\| \left\{ \|r^{2m} Pf_0; L^2(\Omega_0 \cap \mathcal{U})\| + \sum_{\pm} \|r^m Pg_0^\pm; L^2(\Gamma^\pm)\| + \right. \\ &\quad \left. + \|r^{1+m} \bar{v}_0 \nabla_y P; L^2(\mathbb{B}_R^{n-1})\| + \|r^m P \nabla_y \bar{v}_0; L^2(\mathbb{B}_R^{n-1})\| \right\}. \end{aligned} \quad (2.72)$$

Inclusions (2.56), (2.60), formulae (2.66), (2.67) for the weight function  $P$ , and also the relations

$$\begin{aligned} 1 - \tau \leq 2m - \tau, \quad 1 - \tau - m \leq m - \tau, \quad -1 - \sigma - m \leq -\tau - m \leq 1 + m - \tau - 2, \\ -\sigma - m \leq 1 - \tau - m \leq m - \tau, \end{aligned}$$

allow us to compare weight exponents of  $r$  in (2.56), (2.60) and (2.72), and show that the sum of norms in the curly brackets in (2.72) is bounded by the quantity  $c(\mathbf{t})$ , depending on the parameter  $\mathbf{t}$  but independent of the parameter  $\mathbf{r}$  in the definition of  $P$ . Beside that, by inequalities (2.67) and (2.71), we obtain

$$\begin{aligned} \|V_0^\perp P^{-1} \nabla_y P; L^2(\Omega_0 \cap \mathcal{U})\| &\leq c\mathbf{t} \|r^{-2} V_0^\perp; L^2(\Omega_0 \cap \mathcal{U})\| \leq \\ &\leq c\mathbf{t} \|r^{-2m} V_0^\perp; L^2(\Omega_0 \cap \mathcal{U})\| \leq c\mathbf{t} \|\nabla_x V_0^\perp; L^2(\Omega_0 \cap \mathcal{U})\|. \end{aligned}$$

Therefore, the quantity  $\mathbf{t}$  can be fixed sufficiently small, such that the left-hand side of (2.72) becomes greater than a half of the squared norm  $\|\nabla_x V_0^\perp; L^2(\Omega_0 \cap \mathcal{U})\|$ . Dividing estimate (2.72), without the middle part, by the latter norm, passage to the limit  $\mathbf{r} \rightarrow 0^+$  leads to

$$\|\nabla_x \left( (1 + \mathbf{t}r^{-1})^\tau v_0^\perp \right); L^2(\Omega_0 \cap \mathcal{U})\| \leq c\mathcal{N}.$$

Therefore, by the application of inequality (2.71), we obtain

$$\|r^{-2m-\tau} v_0^\perp; L^2(\Omega_0 \cap \mathcal{U})\| + \|r^{-\tau} \nabla_x v_0^\perp; L^2(\Omega_0 \cap \mathcal{U})\| \leq c\mathcal{N}. \quad (2.73)$$

This inequality, in fact, implies the required estimate for the component  $v_0^\perp$ , i.e., inequality (2.59) follows from (2.63), (2.64) and (2.73). ■

Theorem 18 gives the main terms of the required expansion of bounded energy solution to problem (2.1). By combination of the formal procedure to construct the formal asymptotics from Section 1§2 with iterative application of Theorem 18, one can determine the full asymptotic expansion of the solution  $v_0$  in the case when the right-hand sides  $f_0$  and  $g_0$  admit power series expansions.

### 3 Construction of asymptotics

#### 3.1 Super-critical case.

Let  $f_0 \in L^2(\Omega_0)$ ,  $g_0 \in C(\Gamma_0)$  and

$$g(\mathcal{O}) \neq 0, \quad n < 2m - 1, \quad n \geq 2. \quad (3.1)$$

Relations (3.1) deny condition (2.3) for the existence of the solution  $v_0$  to the first limit problem (2.1). Consequently, the main asymptotic term of the solution  $u_\varepsilon \in H^1(\Omega_\varepsilon)$  of the singular perturbed problem (1.5), (1.6) becomes of the boundary layer type and it is described in the rapid variables (1.10). We accept the following asymptotic ansatz

$$u_\varepsilon(x) = \chi_0(y) \hbar^{2-2m} w_0(\hbar^{-1}y) + \hat{u}_\varepsilon(x) - \hat{c}_\varepsilon \quad (3.2)$$

where  $\chi_0 \in C_c^\infty(\mathbb{B}_R^{n-1})$  is a cut-off function such that  $\chi_0(y) = 1$  for  $r < \frac{R}{4}$  and  $\chi_0 \chi = \chi_0$  with the cut-off function  $\chi$  introduced in Theorem 18. Note that we use the notation  $\chi_0(y)$  for the function which is non-null only on the ligament  $\Omega_\varepsilon \cap \mathcal{U}$ . Furthermore, in (3.2) we have  $\hbar = \varepsilon^{\frac{1}{2m}}$ ,  $\hat{u}_\varepsilon$  is the asymptotic remainder to be estimated and  $w_0$  implies a solution to the second limit problem (1.13) with the right-hand side

$$F(\eta) = g_0(\mathcal{O}). \quad (3.3)$$

The constant function (3.3) belongs to the space  $V_{\beta-2m}^{l-1}(\mathbb{R}^{n-1})$  (see (2.32)) with any smoothness exponent  $l \in \mathbb{N}$  and the weight exponent

$$\beta = -\delta + l + 2m - \frac{n+1}{2} \quad (3.4)$$

where  $\delta$  is positive and small. Hence, Theorem 14 supplies the equation (1.13), (3.3) with a unique solution  $w_0 \in V_\beta^{l+1}(\mathbb{R}^{n-1})$  because, with  $\delta \in (0, 2m - 2)$ , exponent (3.4) verifies requirement (2.48):

$$l - \beta - \frac{n-3}{2} = \delta - 2m + 2.$$

**Remark 19** *The constant*

$$\widehat{c}_\varepsilon = \hbar^{2-2m} \int_{\mathbb{B}_R^{n-1}} \chi_0(y) w_0(\hbar^{-1}y) dy$$

is chosen such that the function  $\widehat{u}_\varepsilon$  inherits the orthogonality condition (1.16) from  $u_\varepsilon$ . Note that  $\widehat{c}_\varepsilon = O(\hbar^{2-2m+n-1})$  because

$$\int_{\mathbb{R}^{n-1}} |w_0(\eta)| d\eta \leq \|w_0; V_\beta^{l+1}(\mathbb{R}^{n-1})\| \left( \int_{\mathbb{B}_R^{n-1}} (1 + |\eta|)^{2(l+1-\beta)} d\eta \right)^{1/2}$$

and the last integral converges while  $\delta \in (0, 2m - n - 1)$  in (3.4) (see (3.1)). ■

**Lemma 20** *The solution  $w_0$  admits the asymptotic form*

$$w_0(\eta) = X(\eta) \mathbf{V}_0(\eta) + \widetilde{w}_0(\eta) \quad (3.5)$$

where  $X \in C^\infty(\mathbb{R}^{n-1})$  is a cut-off function such that  $X(\eta) = 1$  for  $|\eta| > 2$  and  $X(\eta) = 0$  for  $|\eta| < 1$ ,  $\mathbf{V}_0(\eta) := r^{2-2m} \Psi(\theta)$  is the power-law solution of the model equation (2.29) with  $\mu = 0$  and  $F(\theta, \ln r) = g_0(0)$  (cf. Remark 13) while the remainder  $\widetilde{w}_0$  falls into  $V_\gamma^{l+1}(\mathbb{R}^{n-1})$  with any  $l \in \mathbb{N}$  and

$$\gamma < \min \left\{ l + 4m - \frac{n+1}{2}, l + 2m + \frac{n-3}{2} \right\}. \quad (3.6)$$

**Proof.** Outside a neighborhood of the coordinate origin  $y = 0$ , the function  $\widetilde{w}_0 = w_0 - X|\eta|^{2-2m} \Psi$  satisfies the equation

$$\mathbf{L}(\eta, \nabla_\eta) \widetilde{w}_0(\eta) = (\mathbf{L}(\eta, \nabla_\eta) - L(\eta, \nabla_\eta)) w_0(\eta) = \Delta_\eta w_0(\eta)$$

while  $\Delta_\eta w_0 \in V_\beta^{l-1}(\mathbb{R}^{n-1})$ . Comparing formulas (3.4) and (3.6), we see that it is possible to choose  $\delta > 0$  in (3.4) such that  $\Delta_\eta w_0 \in V_{\gamma-2m}^{l-1}(\mathbb{R}^{n-1})$ . Hence, the conditions in Proposition 11 are satisfied while interval (2.45) lays inside the interval  $(\lambda_0^-, \lambda_0^+)$  in (2.48) and, therefore, is free of exponents (2.43). This means that  $\widetilde{w}_0 \in V_\gamma^{l+1}(\mathbb{R}^{n-1})$ . ■

By (2.12), right hand sides in the problem (1.5), (1.6) for the remainder  $\widehat{u}_\varepsilon$ ,

$$\begin{aligned} -\Delta_x \widehat{u}_\varepsilon(x) &= f(\varepsilon, x) + \widehat{f}(\varepsilon, x), & x \in \Omega_\varepsilon, \\ \partial_\nu \widehat{u}_\varepsilon(x) &= g(\varepsilon, y, z - \varepsilon) - \chi_0(y) g(0, \mathcal{O}) + \widehat{g}^+(\varepsilon, x), & x \in \Gamma_\varepsilon, \\ \partial_\nu \widehat{u}_\varepsilon(x) &= \widehat{g}^-(\varepsilon, x), & x \in \Gamma, \end{aligned} \quad (3.7)$$

take the form

$$\begin{aligned} \widehat{f}(\varepsilon, x) &= \hbar^{2-2m} \Delta_y (\chi_0(y) w_0(\hbar^{-1}y)), \\ \widehat{g}^-(\varepsilon, x) &= \hbar^{2-2m} \left( 1 + |\nabla_y H_-(y)|^2 \right)^{-1/2} \nabla_y H_-(y) \cdot \nabla_y (\chi_0(y) w_0(\hbar^{-1}y)), \\ \widehat{g}^+(\varepsilon, x) &= \chi(y) g(0, \mathcal{O}) + \hbar^{2-2m} \left( 1 + |\nabla_y H_+(y)|^2 \right)^{-1/2} \nabla_y H_+(y) \cdot \nabla_y (\chi_0(y) w_0(\hbar^{-1}y)). \end{aligned} \quad (3.8)$$

Note that supports of functions (3.8) belong to  $\Omega_\varepsilon \cap \mathcal{U}$ ,  $\Gamma_\varepsilon^-$  and  $\Gamma_\varepsilon^+$ , respectively. We multiply the equation in (3.7) with  $\widehat{u}_\varepsilon(x)$ , integrate over  $\Omega_\varepsilon$  taking the boundary conditions into account. As a

result, we obtain

$$\begin{aligned} \|\nabla_x \widehat{u}_\varepsilon; L^2(\Omega_\varepsilon)\|^2 &= \left( \int_{\Omega_\varepsilon} f \widehat{u}_\varepsilon dx + \int_{\Gamma_\varepsilon} (g - \chi_0 g(0, \mathcal{O})) \widehat{u}_\varepsilon ds_x \right) + \\ &+ \left( \int_{\Omega_\varepsilon \cap \mathcal{U}} \widehat{f} \widehat{u}_\varepsilon dx + \sum_{\pm} \int_{\Gamma_\varepsilon^\pm} \widehat{g}^\pm \widehat{u}_\varepsilon ds_x \right) =: J_0(\widehat{u}_\varepsilon) + J_1(\widehat{u}_\varepsilon). \end{aligned} \quad (3.9)$$

To process the expression  $J_1(\widehat{u}_\varepsilon)$ , we employ decomposition (1.22) for  $\chi \widehat{u}_\varepsilon$ , namely,

$$\chi(y) \widehat{u}_\varepsilon(x) = \bar{u}_\varepsilon(y) + u_\varepsilon^\perp(x), \quad \bar{u}_\varepsilon(y) = (\varepsilon + H_+(y))^{-1} \int_{\Upsilon_\varepsilon(y)} \widehat{u}_\varepsilon(y, z) dz. \quad (3.10)$$

By estimates (1.23), (1.24) and representation (3.5), we obtain

$$\begin{aligned} |J_2(u_\varepsilon^\perp)| &\leq c \|\nabla_x \widehat{u}_\varepsilon; L^2(\Omega_\varepsilon)\| \times \\ &\times \left( \int_{\Omega_\varepsilon \cap \mathcal{U}} (\varepsilon + H(y))^2 |\widehat{f}(\varepsilon, x)|^2 dx + \sum_{\pm} \int_{\Gamma_\varepsilon^\pm} (\varepsilon + H(y)) |\widehat{g}^\pm(\varepsilon, x)|^2 ds_x \right)^{\frac{1}{2}} \leq \\ &\leq c \|\nabla_x \widehat{u}_\varepsilon; L^2(\Omega_\varepsilon)\| \times |g(0, \mathcal{O})| + \hbar^{2-2m} \int_{\mathbb{B}_R^{n-1}} (\varepsilon + H(y)) \left( (\varepsilon + H(y))^2 \hbar^{-4} |\nabla_\eta^2 w_0(\eta)|^2 + \right. \\ &\quad \left. + \hbar^{-2} r^2 |\nabla_\eta w_0(\eta)|^2 + r^2 |w_0(\eta)|^2 dy \right)^{\frac{1}{2}} \leq \\ &\leq c \hbar^{2-2m} \|\nabla_x \widehat{u}_\varepsilon; L^2(\Omega_\varepsilon)\| \left( \hbar^{2-2m} + \sum_{j=0}^3 \left\{ \int_0^R (\hbar + r)^{2m+2j} \hbar^{-2j} \left(1 + \frac{r}{\hbar}\right)^{2(2-2m-j)} r^{n-2} dr + \right. \right. \\ &\quad \left. \left. + \int_{\mathbb{B}_R^{n-1}} (\hbar + r)^{2m+2j} \hbar^{-2j} |\nabla_\eta^j \tilde{w}_0(\eta)|^2 dy \right\} \right) \leq \\ &\leq c \hbar^{1-m+\frac{n+1}{2}} \|\nabla_x \widehat{u}_\varepsilon; L^2(\Omega_\varepsilon)\|. \end{aligned} \quad (3.11)$$

This calculation needs an additional explanation. First, we have used the evident relations in the ball  $\mathbb{B}_R^{n-1}$

$$(\varepsilon + H(y))^2 \leq c(\hbar + r)^4, \quad r^2 \leq (\hbar + r)^2, \quad r^2 < c$$

in order to estimate the sum of integrals in the braces. Second, the integral over  $r \in (0, R)$  has emerged from the power-law solution  $|\eta|^{2-2m} \Psi(\theta)$  in (3.5) and has been computed directly. Third, in the integral over the ball  $\mathbb{B}_R^{n-1}$  the change  $y \mapsto \eta$  has been performed and the inclusion  $\tilde{w}_0 \in V_{l+1+m}^{l+1}(\mathbb{R}^{n-1})$  has been applied with the exponent  $\gamma = l + 1 + m$  satisfying (3.6).

For the component  $\bar{u}_\varepsilon$  in (3.10), we, by (3.8), have

$$\begin{aligned} J_1(\bar{u}_\varepsilon) &= \hbar^{2-2m} \int_{\mathbb{B}_R^{n-1}} \bar{u}_\varepsilon(y) (\varepsilon + H(y)) \Delta_y \chi_0(y) w_0(\hbar^{-1}y) dy + \\ &+ \int_{\mathbb{B}_R^{n-1}} \bar{u}_\varepsilon(y) \chi_0(y) g(0, \mathcal{O}) \left(1 + |\nabla_y H_+(y)|^2\right)^{1/2} dy + \\ &- \hbar^{2-2m} \sum_{\pm} \int_{\mathbb{B}_R^{n-1}} \bar{u}_\varepsilon(y) \nabla_y H_\pm(y) \cdot \nabla_y \chi_0(y) w_0(\hbar^{-1}y) dy = \\ &= \int_{\mathbb{B}_R^{n-1}} \bar{u}_\varepsilon(y) \chi_0(y) (\hbar^{2-2m} \nabla_y \cdot (\varepsilon + \mathbf{H}(y)) \nabla_y w_0(\hbar^{-1}y) g(0, \mathcal{O})) dy + \\ &+ \hbar^{2-2m} \int_{\mathbb{B}_R^{n-1}} \bar{u}_\varepsilon(y) \nabla_y \cdot (H(y) - \mathbf{H}(y)) \nabla_y (\chi_0(y) w_0(\hbar^{-1}y)) dy + \\ &+ \hbar^{2-2m} \int_{\mathbb{B}_R^{n-1}} \bar{u}_\varepsilon(y) [\nabla_y \cdot (\varepsilon + \mathbf{H}(y)) \nabla_y, \chi_0(y)] w_0(\hbar^{-1}y) dy \end{aligned} \quad (3.12)$$

where  $[A, B] = AB - BA$  stands for the commutator of the operators  $A$  and  $B$ . The first integral in (3.12) vanishes because  $w_0$  is a solution to equation (1.13) with the right-hand side (3.3) (recall the change of variables (1.10)). Moreover, according to (1.2) modulo of the second and third integrals in (3.12) do not exceed

$$\begin{aligned} & c\hbar^{2-2m} \int_{\mathbb{B}_R^{n-1}} (\hbar + r)^{m-1} |\bar{u}_\varepsilon(y)| \sum_{j=0}^2 (\hbar + r)^{m+j} \hbar^{-j} |\nabla_y^j w_0(\hbar^{-1}y)| dy \leq \\ & \leq c\hbar^{2-2m} \left\| (\hbar + r)^{m-1} (1 + \delta_{n,3} |\ln(\hbar + r)|)^{-1} \bar{u}_\varepsilon; L^2(\mathbb{B}_R^{n-1}) \right\| \times \\ & \times (1 + \delta_{n,3} |\ln(\hbar + r)|) \sum_{j=0}^2 \left( \int_{\mathbb{B}_R^{n-1}} (\hbar + r)^{2(m+j)} \hbar^{-2j} |\nabla_\eta^j w_0(\hbar^{-1}y)|^2 dy \right)^{\frac{1}{2}}. \end{aligned}$$

The last integrals can be estimated in the same way as similar integrals in (3.12). Thus,

$$|J_1(\bar{u}_\varepsilon)| \leq c\hbar^{1-m+\frac{n+1}{2}} (1 + \delta_{n,3} |\ln(\hbar + r)|) \|\nabla_x \bar{u}_\varepsilon; L^2(\Omega_\varepsilon)\|. \quad (3.13)$$

**Theorem 21** *Let the right-hand sides of problem (1.5), (1.6) satisfy the compatibility condition (1.7) and the estimate*

$$\begin{aligned} & \|\mathcal{R}_n^{-1} f; L^2(\Omega_\varepsilon)\| + \left\| (\varepsilon + \rho^{2m})^{-\frac{1}{2}} \mathcal{R}_n^{-1} (g - \chi_0 g(0, \mathcal{O})); L^2(\Gamma_\varepsilon) \right\| \leq \\ & \leq c \left\{ 1 + \hbar^{1-m+\frac{n+1}{2}} (1 + \delta_{n,3} |\ln \hbar|) \right\}. \end{aligned} \quad (3.14)$$

Then a solution  $u_\varepsilon$  of the problem verifies the relation

$$\|\nabla_x (u_\varepsilon - \hbar^{2-2m} \chi_0 w_0(\hbar^{-1} \cdot)); L^2(\Omega_\varepsilon)\| \leq c \left\{ 1 + \hbar^{1-m+\frac{n+1}{2}} (1 + \delta_{n,3} |\ln \hbar|) \right\}. \quad (3.15)$$

where the constant  $c$  does not depend on the small parameter.

**Proof** is adjusted by relations (3.8) and (3.11), (3.13), (3.14) together with inequality (1.17). ■

Computing the Dirichlet integral of the asymptotic term detached in (3.2) while taking into account the inclusion  $w_0 \in V_{l+m}^{l+1}(\mathbb{R}^{n-1})$  ( $\delta = m - \frac{n+1}{2} > 0$  in (3.4)), we get

$$\begin{aligned} & \hbar^{4-4m} \int_{\Omega_\varepsilon} |\nabla_x \chi_0(y) w_0(\hbar^{-1}y)|^2 dx = \hbar^{4-4m} \int_{\mathbb{B}_R^{n-1}} (\varepsilon + H(y)) |\nabla_y \chi_0(y) w_0(\hbar^{-1}y)|^2 dy = \\ & = \hbar^{-2m+n+1} \left( \int_{\mathbb{R}^{n-1}} (1 + \mathbf{H}(\eta)) |\nabla_\eta w_0(\eta)|^2 d\eta + O(\hbar(1 + \delta_{2m,n+2} |\ln \hbar|)) \right). \end{aligned} \quad (3.16)$$

Note that  $\ln \hbar$  appears in the remainder when the integral  $\int_{\mathbb{R}^{n-1}} (1 + |\eta|)^{2m+1} |\eta|^{2(2-2m)} d\eta$  is diverging.

Due to requirement (3.1), the exponent  $-2m + n + 1$  is negative, however the exponent  $1 - m + \frac{n+1}{2}$  can be of any sign. That is why we have included the summand 1 into the majorant of (3.14) and, therefore, it has appeared in (3.15) as well. In particular, under the assumption (1.8), hypothesis (3.14) holds true: the left-hand side of (3.14) does not exceed

$$\begin{aligned} & c \int_{\Gamma_0} (\hbar + \rho)^{-2m+2} (1 + \delta_{n,3} |\ln(\hbar + \rho)|)^2 |G(x) - G(0)|^2 ds_x \leq \\ & \leq c \left( \int_{\Gamma_0 \setminus \mathcal{U}} \dots ds_x + \int_{\Gamma_0 \cap \mathcal{U}} \dots ds_x \right) \leq \\ & \leq c \left( 1 + \int_0^R (\hbar + r)^{-2m+2} (1 + \delta_{n,3} |\ln(\hbar + r)|)^2 r^2 r^{n-2} dr \right) \leq \\ & \leq c \left( 1 + \hbar^{1-m+\frac{n+1}{2}} (1 + \delta_{n,3} |\ln \hbar|) \right). \end{aligned}$$

In any case  $\|\nabla_x \widehat{u}_\varepsilon; L^2(\Omega_\varepsilon)\| \leq c\hbar^{1/2}\hbar^{-m+(n-1)/2}$  and, indeed, Theorem 21 shows that  $\hbar^{2-2m}w_0(\hbar^{-1}y)$  is the main asymptotic term of the solution  $u_\varepsilon$  to the singularly perturbed problem (1.5), (1.6), since  $\hbar^{2-2m}\|\nabla_x(\chi w_0); L^2(\Omega_\varepsilon)\| = O(\hbar^{-m+(n-1)/2})$  under condition (3.1).

**Corollary 22** *In the case (1.8), (3.1) the energy functional (1.9) takes the asymptotic form*

$$E(u_\varepsilon; \Omega_\varepsilon) = -\frac{1}{2}\hbar^{-2m+n+1} \int_{\mathbb{R}^{n-1}} (1 + \mathbf{H}(\eta)) |\nabla_\eta w_0(\eta)|^2 d\eta + O(\hbar^{-2m+n+2} (1 + \delta_{2m, n+2} |\ln \hbar|)) \quad (3.17)$$

and, thus, it is infinitely large as  $\hbar \rightarrow 0^+$ . In (3.17)  $w_0$  is the solution in  $V_\beta^{l+1}(\mathbb{R}^{n-1})$  of equation (1.13) with the right-hand side  $F(\eta) = G(\mathcal{O})$ . ■

### 3.2 Sub-critical case for $n \geq 3$

Let us assume that

$$n > 2m - 1, \quad n > 2. \quad (3.18)$$

Furthermore, for the right-hand sides  $f_0$  and  $g_0$  of the first limit problem (2.1) the orthogonality conditions (2.2) are satisfied, as well as the inclusions

$$f_0 \in C^{0, \alpha}(\Omega), \quad g_0 \in C^{1, \alpha}(\Gamma_0), \quad \alpha \geq \frac{1}{2}, \quad (3.19)$$

where  $\Omega$  is the domain which is bounded by the surface  $\Gamma$  and contains the set  $\Gamma_0 \setminus \mathcal{O}$ . Recall that the Hölder space  $C^{l, \alpha}(\Xi)$  with the regularity indices  $l \in \mathbb{N}_0$  and  $\alpha \in (0, 1)$  has the standard norm

$$\|U; C^{l, \alpha}(\Xi)\| = \sum_{k=0}^l \sup_{x \in \Xi} |\nabla_x^k U(x)| + \sup_{x, \mathbf{x} \in \Xi} |x - \mathbf{x}|^{-\alpha} |\nabla_x^l U(x) - \nabla_{\mathbf{x}}^l U(\mathbf{x})|. \quad (3.20)$$

Since  $f_0$  is a bounded function,  $\rho f_0 \in L^2(\Omega_0)$ . Moreover, from conditions (3.18) and (3.19) it follows that  $\rho^{1-m}g_0 \in L^2(\Gamma_0)$ , which can be combined with Lemma 3 in order to deduce the solvability of problem (2.1). The solution  $v_0 \in H^1(\Omega_0)$  is unique under the condition

$$\int_{\Omega_0} v_0(x) dx = 0. \quad (3.21)$$

We note that by the second inclusion in (3.19) the remainder in expansion

$$g_0(x) = g_0(\mathcal{O}) + y \cdot \nabla_y g_0(\mathcal{O}) + \widetilde{g}_0(x) \quad (3.22)$$

satisfies the estimate  $|\widetilde{g}_0(x)| \leq cr^{1+\alpha}$ . Thus, the inclusion  $\rho^{1-\tau-m}\widetilde{g}_0 \in L^2(\partial\Omega_0)$  is valid for any exponent

$$\tau < \alpha - m + \frac{n+3}{2} \quad (3.23)$$

(compare with (3.19)). Beside that, relations (3.23) and (3.19) show that  $\rho^{1-\tau}f_0 \in L^2(\Omega_0)$ . Therefore, the asymptotic procedure in section 2§2 and Theorem 18 allow to determine asymptotics of the solution  $v_0$  with the remainder  $\widetilde{v}_0$ , which enjoys the inclusions  $\rho^{-\tau}\nabla_x \widetilde{v}_0, \rho^{-1-\tau}\widetilde{v}_0 \in L^2(\Omega_0)$ . To this end, we need some simplifications of the asymptotic expansion, which is assumed to be a one-term expansion in the present section, and two-term in section 4§3 (in the critical case). In order to simplify the presentation and avoid the repetition of complicated arguments, we consider in the present section just the two-term expansion

$$v_0(x) = \chi(y)(\mathbf{V}_0(y) + \mathbf{V}_1(y)) + \widetilde{v}_0(x). \quad (3.24)$$

Here

$$\mathbf{V}_0(y) = \begin{cases} r^{2-2m}\Psi(\theta) & \text{for } m \geq 2, \\ C_0 + \Psi(\theta) + C_1 \ln r & \text{for } m = 1, \end{cases} \quad (3.25)$$



$$|\mathbf{V}_1(y)| + r |\nabla_y \mathbf{V}_1(y)| \leq cr^\mu, \quad \mu = \begin{cases} 3-2m & \text{for } m \geq 2, \\ \min\{\lambda_1^+, 1-\delta_1\} & \text{for } m = 1, \end{cases} \quad (3.26)$$

$$\rho^{-1-\tau}\tilde{v}_0, \quad \rho^{-\tau}\nabla_x \tilde{v}_0 \in L^2(\Omega_0), \quad \tau = \mu + \delta + m + \frac{n-3}{2}, \quad (3.27)$$

where  $\Psi$  is a smooth function on the unit sphere  $\mathbb{S}^{n-2}$ ,  $C_0$  and  $C_1$  are constants (see Remark 13, by normalization (3.21) the constant  $C_0$  is fixed),  $\lambda_1^+$  is the first positive exponent in (2.43), and  $\delta_1, \delta$  are positive numbers. We point out that, in view of the choice (3.26) of the index  $\mu$ , the number  $\tau$  from (3.27) verifies relation (3.23).

**Remark 23** *The second asymptotic term  $\mathbf{V}_1$  in expansion (3.24) is designed to compensate for the linear term in (3.22), and also for the main term of discrepancy, generated in problem (2.1) by the expression  $\mathbf{V}_0(y)$ . If  $m > 1$ , the term  $\mathbf{V}_1$  is of the form  $r^{3-2m}\Psi_1(\theta)$ , since the number  $3-2m$  belongs to the segment  $(3-n-2m, 0)$  and, therefore, does not coincide with any of exponents in (2.43) (compare with Proposition 12). Moreover, asymptotic terms of the first kind (2.14) take the form  $r^{2m}\mathbf{v}(\theta, \zeta)$ , and can be included in the asymptotic remainder (3.26).*

*The case  $m = 1$  is separated. First, the expression  $\chi(y)r^{2m}\mathbf{v}(\theta, \zeta)$ , which has only bounded gradient, does not verify inclusions (3.27) with the exponent  $\tau = 3-2m+0+m+(n-3)/2 = -1+(n+3)/2$ , thus in (3.26) the exponent  $\mu = 3-2m = 1$  is reduced to  $1-\delta_1$  (in this particular case we have  $\mathbf{V}_1 = 0$ ). Second, the exponent  $\lambda_1^+$  can be located in the segment  $(0, 1)$  (see examples in section 4§4), and this possibility is also taken into account in formula (3.26), in such a case it follows that  $\mathbf{V}_1 = c_1 r^{\lambda_1^+} \phi_1(\theta)$  with  $\lambda_1^+ \in (0, 1)$ . ■*

For the solutions of the second limit problem (1.13) with the right-hand side (3.3) Lemma 20 applies. However for  $m = 1$  (such a possibility is excluded in section 1§3 by condition (3.1)) it is necessary to define

$$\mathbf{V}_0(\hbar^{-1}\eta) = C_0 + \Psi(\theta) + C_1 \ln(\hbar^{-1}|\eta|) \quad (3.28)$$

(compare with (3.25)). The existence of the solution  $w_0$ , which satisfies requirements (3.5), (3.28), follows from Theorem 14, since we have

$$(L - \mathbf{L})(X\mathbf{V}_0) = \Delta_\eta(X\mathbf{V}_0) \in V_{\beta+2}^{l-1}(\mathbb{R}^{n-1}) \quad (3.29)$$

with arbitrary weight index  $\beta < l+2-(n+1)/2$ , in particular, satisfying condition (2.48). Such a solution is determined up to an additive constant, which is fixed to  $C_0 - C_1 \ln \hbar$  in order to equalize expressions (3.28) and (3.25), however, the dependence of solution  $w_0$  on the parameter  $\ln \hbar$  is not indicated explicitly.

In any case the remainder  $\tilde{w}_0$  belongs to the space  $V_\gamma^{l+1}(\mathbb{R}^{n-1})$  with the weight index (3.6).

By the local estimates of solutions to elliptic boundary value problems [35], a solution  $v_0$  of problem (2.1) with the right-hand sides (3.19) belongs to the Hölder class  $C_{loc}^{2,\alpha}(\Omega_0 \setminus \mathcal{O})$ , i.e.,  $v_0 \in C^{2,\alpha}(\Omega_0 \setminus \mathbb{B}_R^n)$  with any  $R > 0$ . There exists an extension  $V_0 \in C_{loc}^{2,\alpha}(\Omega_0 \setminus \mathcal{O})$  of the function  $v_0$  onto the domain  $\Omega$ , bounded by the surface  $\Gamma$  and containing the set  $\Gamma_\varepsilon \setminus \mathcal{O}$  for all  $\varepsilon \in [0, \varepsilon_0)$  (see Figures 1-4). Since  $\nabla_x V_0 \in C^{1,\alpha}(\Omega_0 \setminus \mathcal{U}')$ , where  $\mathcal{U}' = \mathbb{B}_{R/2}^{n-1} \times (-d, d) \subset \mathcal{U}$ , and the surfaces  $\Gamma_\varepsilon \setminus \mathcal{U}'$  and  $\Gamma_0 \setminus \mathcal{U}'$  are close to each other at the distance  $O(\varepsilon)$ , the relation

$$|\partial_\nu V_0(y, z) - \partial_\nu V_0(y, z - \varepsilon)| \leq c\varepsilon, \quad (y, z) \in \Gamma_\varepsilon \setminus \mathcal{U}', \quad (3.30)$$

is valid and the term  $\partial_\nu V_0(y, z - \varepsilon)$  in the above inequality implies the right-hand side  $g_\varepsilon(y, z - \varepsilon)$  of boundary condition in (2.1). Moreover,  $-\Delta_x V_0(x) = f_0(x)$ ,  $x \in \Omega_0$  and  $\Delta_x V_0 \in C^{0,\alpha}(\Omega_0 \setminus \mathcal{U}')$  so that

$$\begin{aligned} |\Delta_x V_0(x) + f_0(x)| &\leq c\varepsilon^\alpha, \quad x \in \Omega_\varepsilon \setminus (\Omega_0 \cap \mathcal{U}'), \\ \Delta_x V_0(x) + f_0(x) &= 0, \quad x \in \Omega_0 \setminus (\Omega_\varepsilon \cap \mathcal{U}'). \end{aligned} \quad (3.31)$$

We point out that by a known variant of the Hardy's inequality (see, e.g., [28, Lemma 1.2.4]), the following inequality is valid:

$$\|u_\varepsilon; L^2(\Omega_\varepsilon \setminus (\Omega_0 \cup \mathcal{U}'))\| \leq c\varepsilon^{\frac{1}{2}} \|u_\varepsilon; H^1(\Omega_\varepsilon)\|. \quad (3.32)$$

Note that the set  $\Omega_\varepsilon \setminus (\Omega_0 \cup \mathcal{U}')$  has the small "thickness"  $O(\varepsilon)$ .

Global asymptotic approximation of the solution  $u_\varepsilon$  to the singularly perturbed problem (1.5), (1.6) is taken in the form

$$U_\varepsilon(x) = \mathcal{X}_0(x) V_0(x) + \chi_0(y) \hbar^{2-2m} w_0(\hbar^{-1}y) + \chi_0(y) (1 - \chi_0^h(y)) (\mathbf{V}_1(y) + \tilde{v}_0(y, \mathfrak{z})), \quad (3.33)$$

where  $\mathcal{X}_0$  is a cut-off function, which is equal to  $1 - \chi_0(y)$  for  $x \in \Omega_\varepsilon \cap \mathcal{U}$  and one outside the ligament  $\Omega_\varepsilon \cap \mathcal{U}$ . Furthermore, the new variable

$$\mathfrak{z} = (\varepsilon + H(y))^{-1} (zH(y) - \varepsilon H_-(y)) \quad (3.34)$$

belongs to the segment  $\Upsilon_\varepsilon(y)$  when  $z \in \Upsilon_0(y)$ . In other words, the function

$$x \longmapsto \tilde{\mathbf{v}}_0(y, z) = \tilde{v}_0(y, \mathfrak{z}) \quad (3.35)$$

is defined on the ligament  $\Omega_\varepsilon \cap \mathcal{U}$  of the positive thickness, however the function  $x \longmapsto \tilde{v}_0(y, z)$  is defined only on the degenerate ligament  $\Omega_0 \cap \mathcal{U}$ . In addition, function (3.35) has a singularity at  $y = 0$ , which is smoothened in (3.33) after multiplication by the cut-off function  $\chi_0^h(y) = 1 - \chi_0(\hbar^{-1}y)$ . We point out again that the cut-off function  $\chi_0^h$  does not vanish only on the ligament.

Set  $\hat{c}_\varepsilon = \text{meas}_n(\Omega_\varepsilon)^{-1} \int_{\Omega_\varepsilon} U_\varepsilon(x) dx$ . The expression  $\hat{u}_\varepsilon = u_\varepsilon - U_\varepsilon + \hat{c}_\varepsilon$  meets the orthogonality condition (1.16), since  $u_\varepsilon$  is normalized. Moreover,  $\hat{u}_\varepsilon$  verifies the integral identity

$$\begin{aligned} (\nabla_x \hat{u}_\varepsilon, \nabla_x \psi)_{\Omega_\varepsilon} &= (f, \psi)_{\Omega_\varepsilon} + (g, \psi)_{\Gamma_\varepsilon} - (\nabla_x U_\varepsilon, \nabla_x \psi)_{\Omega_\varepsilon} = \\ &= (f - \mathcal{F}_\varepsilon^0, \psi)_{\Omega_\varepsilon} + (g - \mathcal{G}_\varepsilon^+, \psi)_{\Gamma_\varepsilon} - (\mathcal{G}_\varepsilon^-, \psi)_{\Gamma_0} - (\mathcal{F}_\varepsilon^1, \nabla_x \psi)_{\Omega_\varepsilon}, \end{aligned} \quad (3.36)$$

where

$$\begin{aligned} \mathcal{F}_\varepsilon^1(x) &= \nabla_x \{ \chi_0(y) (1 - \chi_0^h(y)) (\tilde{\mathbf{v}}_0(y, \varsigma) + \mathbf{V}_1(y) + \mathbf{V}_0(y)) \}, \\ \mathcal{F}_\varepsilon^0(x) &= -\Delta_x (\mathcal{X}_0(x) V_0(x)) - \hbar^{2-2m} \Delta_y \{ \chi_0(y) (w_0(\hbar^{-1}y) - X(\hbar^{-1}y) \mathbf{V}_0(\hbar^{-1}y)) \}, \\ \mathcal{G}_\varepsilon^\pm(x) &= \partial_\nu (\mathcal{X}_0(x) V_0(x)) + \hbar^{2-2m} \partial_\nu \{ \chi_0(y) (w_0(\hbar^{-1}y) - X(\hbar^{-1}y) \mathbf{V}_0(\hbar^{-1}y)) \} \end{aligned} \quad (3.37)$$

and the cut-off function  $X$  in (3.5) is such that  $X(\eta) = 1 - \chi_0(\eta)$ .

Now, we transform the expression in (3.37). Then we shall estimate the corresponding terms in (3.36), after that, with the replacement  $\psi = \hat{u}_\varepsilon$  we shall estimate the asymptotic remainder  $\hat{u}_\varepsilon$ .

First, we observe that, owing to sufficiently strong decay of the remainder  $\tilde{w}_0 \in V_\gamma^{l+1}(\mathbb{R}^{n-1})$  in expansion (3.5) of the solution of the second limit problem (see formula (3.6)), an analysis of the quantity

$$\mathbf{J}_0 = \hbar^{2-2m} (\Delta_y (\chi_0 \tilde{w}_0), \psi)_{\Omega_\varepsilon} - \hbar^{2-2m} (\partial_\nu (\chi_0 \tilde{w}_0), \psi)_{\partial\Omega_\varepsilon}, \quad (3.38)$$

where  $\tilde{w}_0 = w_0 - X V_0$  (see (3.5)) can be performed in exactly the same way as in section 1§3. This leads to the estimate

$$|\mathbf{J}_0| \leq c \hbar^{-m + \frac{n+1}{2}} (1 + \delta_{n,3} |\ln \hbar|). \quad (3.39)$$

We note that by requirement (3.18) the exponent  $-m + \frac{(n+1)}{2}$  of the parameter  $\hbar$  in the majorant (3.36) is at least  $1/2$ .

Let us consider the last term in (3.36) rewritten in the form

$$\begin{aligned} (\mathcal{F}_\varepsilon^1, \nabla_x \psi)_\Omega &= -(\chi_0(\tilde{\mathbf{v}}_0 + \mathbf{V}_1 + \mathbf{V}_0) \nabla_x \chi_0^h, \nabla_x \psi)_{\Omega_\varepsilon \cap \mathcal{U}} + \\ &+ (\nabla_x (\chi_0(\tilde{\mathbf{v}}_0 + \mathbf{V}_1 + \mathbf{V}_0)), \psi \nabla_x \chi_0^h)_{\Omega_\varepsilon \cap \mathcal{U}} + (\nabla_x (\chi_0(\tilde{\mathbf{v}}_0 + \mathbf{V}_1 + \mathbf{V}_0)), \nabla_x ((1 - \chi_0^h) \psi))_{\Omega_\varepsilon \cap \mathcal{U}} =: \\ &=: \mathbf{J}_1 + \mathbf{J}_2 + \mathbf{J}_3. \end{aligned} \quad (3.40)$$

The support of derivatives of the cut-off function  $\chi_0^{\hbar}(y) = \chi_0(\hbar^{-1}y)$  is located in the set  $\Xi_\varepsilon = \{x \in \overline{\Omega_\varepsilon} \cap \mathcal{U} : C \geq \hbar^{-1}|y| \geq c > 0\}$  where  $\rho = O(\hbar)$  and

$$z - \mathfrak{z} = \varepsilon \frac{z + H_-(y)}{\varepsilon + H(y)} = O(\varepsilon). \quad (3.41)$$

Therefore, taking into account equality (1.17) for  $\psi$ , inclusions (3.27) for  $\tilde{v}_0$  and formulae (3.25), (3.26) for  $\mathbf{V}_0$  and  $\mathbf{V}_1$ , we find that

$$\begin{aligned} |\mathbf{J}_1| + |\mathbf{J}_2| &\leq c\hbar^{-1} \left( \|\nabla_x \psi; L^2(\Omega_\varepsilon)\| + (1 + \delta_{n,3} |\ln \hbar|) \|\mathcal{R}_n \psi; L^2(\Omega_\varepsilon)\| \right) \cdot \\ &\cdot \left( (\text{meas}_n \Xi_\varepsilon)^{\frac{1}{2}} \sup_{x \in \Xi_\varepsilon} \sum_{i=0}^1 (|\mathbf{V}_i(y)| + r |\nabla_y \mathbf{V}_i(y)|) + \hbar^{1+\tau} \sum_{i=0}^1 \|\rho^{-\tau-1+i} \nabla_x^i \tilde{v}_0; L^2(\Omega)\| \right) \leq \\ &\leq c\hbar^{-1} \hbar^{m+\frac{n-1}{2}} \hbar^{2-2m} (1 + \delta_{m,1} |\ln \hbar|) (1 + \delta_{n,3} |\ln \hbar|) \|\nabla_x \psi; L^2(\Omega_\varepsilon)\|. \end{aligned} \quad (3.42)$$

Let us explain the above calculations. The factor  $\hbar^{-1}$  comes out from the differentiation of the cut-off function  $\chi_0^{\hbar}$ , the measure satisfies  $\text{meas}_n \Xi_\varepsilon = O(\hbar^{2m+n-1})$ , the logarithms are taken from relations (1.18) and (3.25), and the exponent  $1+\tau$  of the multiplier  $c\hbar^{1+\tau} = \sup_{x \in \Xi_\varepsilon} \rho(x)^{1+\tau}$  equals to  $\mu + \delta + m + (n-3)/2$  by (3.27) and is greater than  $3/2$  by (3.26) and (3.18). We observe that the resulting exponent of  $\hbar$  in majorant (3.42), as before, is equal to  $-m + (n+1)/2 \geq 1/2$ .

Let us consider the last term  $\mathbf{J}_3$  in (3.40). The main idea becomes to replace  $\mathbf{J}_3$  by the expression

$$(\nabla_x (\chi_0 (\tilde{v}_0 + \mathbf{V}_1 + \mathbf{V}_0)), \nabla_x ((1 - \chi_0^{\hbar}) \mathfrak{y}))_{\Omega_0 \cap \mathcal{U}}, \quad (3.43)$$

where the function

$$x \longmapsto y(y, z) = \psi(y, \mathbf{z}) \quad (3.44)$$

is defined on the degenerate ligament  $\Omega_0 \cap \mathcal{U}$ , because the variable

$$\mathbf{z} = H(y)^{-1} (z (\varepsilon + H(y)) \varepsilon H_-(y)) \quad (3.45)$$

takes values in the segment  $\Upsilon_0(y)$  for  $z \in \Upsilon_\varepsilon(y)$  (compare with variable (3.34), used for a similar reason).

The Jacobian of the change of variables  $(y, z) \mapsto (y, \mathbf{z})$  is equal to  $H(y)^{-1}(\varepsilon + H(y))$ . Thus, it is uniformly bounded with respect to  $\varepsilon \in (0, \varepsilon_0]$ , and for  $|y| > c\hbar > 0$  (we recall the properties of the cut-off function  $1 - \chi_0^{\hbar}$  in (3.43)) and it is approximatively equal to one at a distance from the point  $\mathcal{O}$ . Denote  $\mathfrak{V}(y, z) = V(y, \mathfrak{z})$  and  $\mathfrak{W}(y, z) = W(y, \mathfrak{z})$ ; then

$$\begin{aligned} \partial_z \mathfrak{V}(y, z) - \partial_z V(y, z)|_{z=\mathfrak{z}} &= \left( (\varepsilon + H(y))^{-1} H(y) - 1 \right) \partial_z V(y, z)|_{z=\mathfrak{z}} = -\varepsilon (\varepsilon + H(y))^{-1} \partial_z V(y, z)|_{z=\mathfrak{z}}, \\ \partial_z \mathfrak{W}(y, z) - \partial_z V(y, z)|_{z=\mathbf{z}} &= \left( H(y)^{-1} (\varepsilon + H(y)) - 1 \right) \partial_z W(y, z)|_{z=\mathbf{z}} = \varepsilon H(y)^{-1} \partial_z W(y, z)|_{z=\mathbf{z}}. \end{aligned} \quad (3.46)$$

Similarly

$$\begin{aligned} \left| \nabla_y \mathfrak{V}(y, z) - \nabla_y V(y, z)|_{z=\mathfrak{z}} \right| &\leq c \frac{\varepsilon r}{\varepsilon + H(y)} \left| \partial_z V(y, z)|_{z=\mathfrak{z}} \right|, \\ \left| \nabla_y \mathfrak{W}(y, z) - \nabla_y W(y, z)|_{z=\mathbf{z}} \right| &\leq c \frac{\varepsilon r}{H(y)} \left| \partial_z W(y, z)|_{z=\mathbf{z}} \right|. \end{aligned} \quad (3.47)$$

The differences from the left-hand sides in (3.47), respectively, equal to

$$\begin{aligned} &\varepsilon \frac{z \nabla_y H(y) + \varepsilon \nabla_y H_-(y) - H_-(y) \nabla_y H(y) + H(y) \nabla_y H_-(y)}{(\varepsilon + H(y))^2} \partial_z V(y, z)|_{z=\mathfrak{z}}, \\ &-\varepsilon \frac{z \nabla_y H(y) - H(y) \nabla_y H_-(y) + H_-(y) \nabla_y H(y)}{H(y)^2} \partial_z W(y, z)|_{z=\mathbf{z}}. \end{aligned}$$

The factor  $r$  appears in estimates (3.47) due to the relation  $|\nabla_y H_\pm(y)| \leq cr$  which follows from (1.2). The multipliers on the right-hand sides of (3.47) are only of order  $O(\hbar)$  for  $|y| \geq c\hbar$ , however the multipliers

in (3.46) turn out to be small only far from the point  $\mathcal{O}$ . In other words, the proposed change of variables has small influence on the derivatives in  $y$ , but it is not the case for the derivatives in  $z$ . The latter observation is also valid for the change of variables in integrals, since  $H(y)^{-1}(\varepsilon + H(y))^{-1} = \varepsilon H(y)^{-1}$ . We are save due to the infinitely small weight factor as  $r \rightarrow 0^+$  which are present in all integrals.

We now estimate the following differences of scalar products :

$$\begin{aligned} \mathbf{J}_3 &= (\nabla_y (\chi_0 \mathbf{V}_0), \nabla_y ((1 - \chi_0^{\hbar}) \psi))_{\Omega_\varepsilon \cap \mathcal{U}} - (\nabla_y (\chi_0 \mathbf{V}_0), \nabla_y ((1 - \chi_0^{\hbar}) \eta))_{\Omega_0 \cap \mathcal{U}}, \\ \mathbf{J}_5 &= (\nabla_x (\chi_0 \tilde{\mathbf{v}}_0), \nabla_x ((1 - \chi_0^{\hbar}) \psi))_{\Omega_\varepsilon \cap \mathcal{U}} - (\nabla_x (\chi_0 \tilde{\mathbf{v}}_0), \nabla_x ((1 - \chi_0^{\hbar}) \eta))_{\Omega_0 \cap \mathcal{U}}. \end{aligned} \quad (3.48)$$

In a similar way as for  $\mathbf{J}_3$ , the expression  $\mathbf{J}_4$  with  $\mathbf{V}_1$  on the place of  $\mathbf{V}_0$  has a smaller majorant, in view of the better behavior of  $\mathbf{V}_1(y)$  for  $r \rightarrow 0^+$ , compared to  $\mathbf{V}_0$  (cf. (3.26) and (3.25)).

Formulae (3.47) and (3.25), (1.17) yield

$$\begin{aligned} |\mathbf{J}_3| &\leq c \int_{\Omega_\varepsilon \cap \mathcal{U}} |\nabla_y (\chi_0(y) \mathbf{V}_0(y))| \frac{\varepsilon r}{H(y)} |1 - \chi_0^{\hbar}(y)| |\partial_z \psi(y, z)| \frac{\varepsilon}{H(y)} dy dz \leq \\ &\leq c (1 + \delta_{n,3} |\ln \hbar|) \|\nabla_x \psi; L^2(\Omega_\varepsilon)\| (1 + \delta_{m,1} |\ln \hbar|) \left( \varepsilon^4 \int_{c\hbar}^R H(y)^{-4} (\varepsilon + H(y)) r^2 r^{2(1-2m)} r^{n-2} \right)^{\frac{1}{2}} dr \leq \\ &\leq c (1 + \delta_{n,3} |\ln \hbar|) (1 + \delta_{m,1} |\ln \hbar|) \hbar^{-m + \frac{n+3}{2}} \|\nabla_x \psi; L^2(\Omega_\varepsilon)\|. \end{aligned} \quad (3.49)$$

We point out that, in view of the weighted Poincaré inequality from Proposition 2, the estimates are valid

$$\begin{aligned} \|\nabla_x ((1 - \chi_0^{\hbar}) \psi); L^2(\Omega_\varepsilon)\| &\leq c (\|\nabla_x \psi; L^2(\Omega_\varepsilon)\| + \hbar^{-1} \|\psi; L^2(\text{supp } |\nabla_x \chi_0^{\hbar}|)\|) \leq \\ &\leq c (\|\nabla_x \psi; L^2(\Omega_\varepsilon)\| + (1 + \delta_{n,3} |\ln \hbar|) \|\mathcal{R}_n \psi; L^2(\Omega_\varepsilon)\|) \leq c (1 + \delta_{n,3} |\ln \hbar|) \|\nabla_x \psi; L^2(\Omega_\varepsilon)\|, \end{aligned} \quad (3.50)$$

which has been already applied in the derivation of (3.49).

Now, using formulae (3.46), (3.47), (3.50) and inclusion (3.27), and also relations (3.26) and (3.6) for the indices  $\mu$  and  $\gamma$ , we obtain

$$\begin{aligned} |\mathbf{J}_5| &\leq c \|\rho^{-\tau} \nabla_x (\chi_0 \tilde{\mathbf{v}}_0); L^2(\Omega_0 \cap \mathcal{U})\| (1 + \delta_{n,3} |\ln \hbar|) \|\nabla_x \psi; L^2(\Omega_\varepsilon)\| \sup_{y \in \mathbb{B}_R^{n-1} \setminus \mathbb{B}_{c\hbar}^{n-1}} \left( \frac{\varepsilon \rho^\tau}{H(y)} \right) \leq \\ &\leq c (1 + \delta_{n,3} |\ln \hbar|) \hbar^{-m + \frac{n+3}{2}} \|\nabla_x \psi; L^2(\Omega_\varepsilon)\|. \end{aligned} \quad (3.51)$$

Therefore, with the error defined by the majorant in estimate (3.49), not smaller than all remaining errors, the scalar product  $\mathbf{J}_3$  in (3.40) can be replaced by the scalar product (3.43), which by obvious reasons (in particular, owing to  $\chi \chi_0 = \chi_0$ ) coincides with

$$\mathbf{J}_6 = (-\Delta_x (\chi_0 \tilde{\mathbf{v}}_0), (1 - \chi_0^{\hbar}) \eta)_{\Omega_0 \cap \mathcal{U}} + \sum_{\pm} (\partial_\nu (\chi_0 \tilde{\mathbf{v}}_0), (1 - \chi_0^{\hbar}) \eta)_{\Gamma^\pm \cap \mathcal{U}}. \quad (3.52)$$

We return to the analysis of the first three terms on the right-hand side of (3.36), we have already rewritten two of them as follows:

$$\begin{aligned} \mathbf{J}_7 &= (f, \psi)_{\Omega_\varepsilon} + (g, \psi)_{\Gamma_\varepsilon}, \\ \mathbf{J}_8 &= (-\Delta_x (\mathcal{X}_0 V_0), \psi)_{\Omega_\varepsilon} + (\partial_\nu (\mathcal{X}_0 V_0), \psi)_{\partial \Omega_\varepsilon}. \end{aligned} \quad (3.53)$$

Let  $\Psi$  be the extension of the function  $\psi \in H^1(\Omega_\varepsilon)$  over the set  $\Omega_0 \setminus (\Omega_\varepsilon \cup \mathcal{U}')$  in the Sobolev class  $H^1$ , with the estimate

$$\|\Psi; H^1(\Omega_0 \setminus \mathcal{U}')\| \leq c \|\nabla_x \psi; L^2(\Omega_\varepsilon)\|. \quad (3.54)$$

We introduce the function

$$\psi_1(x) = \mathcal{X}_0(x) \Psi(x) + \chi_0(y) \eta(y, z), \quad (3.55)$$

defined in the domain  $\Omega_0$  and observe that

$$\|\psi_1 - \Psi; L^2(\Omega_0 \cap (\mathcal{U} \setminus \mathcal{U}'))\| \leq c\varepsilon \|\nabla_x \psi; L^2(\Omega_\varepsilon)\|. \quad (3.56)$$

Indeed, the arguments of functions  $\psi$  and  $\mathfrak{y}$  differ a little on the set  $\Omega_0 \cap (\mathcal{U} \setminus \mathcal{U}')$  owing to the relation  $z - \mathbf{z} = \varepsilon H(y)^{-1}(z + H_-(y))$  which follows from (3.45), and in addition, on the set  $\Omega_0 \cap (\mathcal{U} \setminus \mathcal{U}')$  the functions  $\psi$  and  $\Psi$  coincide. In other words, with an admissible error, the cut-off functions  $\chi_0$  and  $\mathcal{X}_0$  can be summed in expressions  $\mathbf{J}_6$  and  $\mathbf{J}_8$  (see (3.52) and (3.53)); here we note that the cut-off functions  $\chi_0$  and  $\mathcal{X}_0$  form a partition of unity in  $\Omega_\varepsilon$ . In this way, in view of relations (3.31) (3.32) and formula (3.24), we find that

$$\begin{aligned} |\mathbf{J}_6 + \mathbf{J}_8 + \mathbf{J}_9| &\leq c\varepsilon, \\ \mathbf{J}_9 &= ((1 - \chi_0^{\hbar}) f_0, \psi_1)_{\Omega_0} + ((1 - \chi_0^{\hbar}) g_0, \psi_1)_{\Gamma_0}. \end{aligned} \quad (3.57)$$

Finally, replacing  $\psi_1$  by  $\psi$  in (3.57) and applying the same reasoning as above, we have

$$|\mathbf{J}_9 - (f_0, \psi)_{\Omega_\varepsilon} - (g, \psi)_{\Gamma_\varepsilon}| \leq c \max \left\{ \hbar, (1 + \delta_{n,3} |\ln \hbar|) \hbar^{-m + \frac{n+3}{2}} \right\} \|\nabla_x \psi; L^2(\Omega_\varepsilon)\| \quad (3.58)$$

where the function  $f_0$  is defined on the domain  $\Omega_\varepsilon$ ,

$$f_0(\varepsilon, x) = \mathcal{X}_0(x) F_0(x) + \chi_0(y) f_0(y, \mathfrak{z}). \quad (3.59)$$

Here  $F_0$  is an extension of  $f_0$  to  $\Omega$  in the class  $C^{0,\alpha}(\overline{\Omega} \setminus \mathcal{O})$ , and  $\mathfrak{z}$  is variable (3.34). In derivation of (3.58) it is taken into account the following: first, by the definition of functions  $\psi$  and  $\mathfrak{y}$  their traces coincide on the surfaces  $\Gamma_\varepsilon \cap \mathcal{U}$  and  $\Gamma_0 \cap \mathcal{U}$ , second, the functions (3.59) and  $f_0$  are bounded, and, third, by the trace inequality in Proposition 2 we have

$$\begin{aligned} \left| \int_{\Gamma_\varepsilon \cap \mathcal{U}} \chi_0^{\hbar} g_0 \psi_1 ds_x \right| &\leq c(1 + \delta_{n,3} |\ln \hbar|) \|\nabla_x \psi; L^2(\Omega_\varepsilon)\| \cdot \left( \int_0^{c\hbar} (\varepsilon + r^{2m})^{-1} r^2 r^{n-2} dr \right)^{\frac{1}{2}} \leq \\ &\leq c(1 + \delta_{n,3} |\ln \hbar|) \hbar^{-m + \frac{n+1}{2}} \|\nabla_x \psi; L^2(\Omega_\varepsilon)\|. \end{aligned}$$

Furthermore,

$$\left| \int_{\Omega_\varepsilon \cap \mathcal{U}} \chi_0^{\hbar} f_0 \psi dx \right| \leq c \hbar^{m + \frac{n-1}{2}} \|\psi; L^2(\Omega_\varepsilon)\|.$$

Now, we are in position to present the main result.

**Theorem 24** Assume that the right-hand sides of problem (1.5), (1.6) verify orthogonality condition (1.7) and the relation

$$\begin{aligned} \|\mathcal{R}_n^{-1}(f - f_0); L^2(\Omega_\varepsilon)\| + \|(\varepsilon + \rho^{2m})^{-\frac{1}{2}} \mathcal{R}_n^{-1}(g - g_0); L^2(\Gamma_\varepsilon)\| &\leq \\ &\leq c \hbar^{\frac{1}{2}} \max \left\{ \hbar^{\frac{1}{2}}, (1 + \delta_{n,3} |\ln \hbar|) (1 + \delta_{m,1} |\ln \hbar|) \hbar^{-m + \frac{n}{2}} \right\}, \end{aligned} \quad (3.60)$$

where  $f_0$  and  $g_0$  are the right-hand sides (3.19) of the limit problem (2.1), and the function  $f_0$  is defined by (3.59). Then the solution  $u_\varepsilon$  and its asymptotic approximation (3.33) are related by the inequality

$$\|\nabla_x(u_\varepsilon - U_\varepsilon); L^2(\Omega_\varepsilon)\| \leq c \hbar^{\frac{1}{2}} \max \left\{ \hbar^{\frac{1}{2}}, (1 + \delta_{n,3} |\ln \hbar|) (1 + \delta_{m,1} |\ln \hbar|) \hbar^{-m + \frac{n}{2}} \right\}, \quad (3.61)$$

where the constant  $c$  is independent of the parameter  $\varepsilon \in (0, \varepsilon_0]$  and  $\hbar = \varepsilon^{1/2m}$ .

**Proof.** Inequality (3.61) follows from estimates (3.39), (3.42), (3.49), (3.51), (3.57), and assumption (3.60). ■

**Remark 25** In the right-hand sides of the estimates given by Theorem 24, there are infinitesimal small terms of different orders. If  $n > 2m + 1$  and  $\varepsilon^{-m + (n+1)/2} \geq \varepsilon^{3/2}$  then the majorant  $c\hbar^1$  (see e.g., (3.58)) dominates all other majorants. If  $n = 2m$ , then the case  $n = 3, m = 1$  is excluded and there is the quantity  $c\hbar^{1/2}$  on the right-hand side of (3.61). Finally, for  $n = 2m + 1$  on the right-hand side of (3.61)

there is  $c\hbar$  if  $n \neq 3$ , and  $c\hbar(1 + |\ln \hbar|)^2$  in the most interesting case of  $n = 3$  and  $m = 1$  (kissing balls in Figures 1-4). The result can be formulated as follows: the majorant in (3.61) takes the form  $c\mathcal{E}$ , where

$$\mathcal{E} = \begin{cases} \hbar^{\frac{1}{2}} & \text{for } n = 2m, \\ \hbar & \text{for } n \geq 2m + 1, \quad n + m > 4, \\ \hbar^{\frac{1}{2}} (1 + |\ln \hbar|)^2 & \text{for } n = 3, \quad m = 1. \end{cases} \quad \blacksquare \quad (3.62)$$

We emphasize that in situation (1.8) the left-hand side of relation (3.60) is null, i.e., the corresponding requirement is verified.

**Corollary 26** Assume that the right-hand sides of problem (1.5), (1.6) take form (1.8) and  $G \in C^{1,\alpha}(\Gamma_0)$ ,  $\alpha > 1/2$ . Then energy functional (1.9) satisfies the relation

$$|E(u_\varepsilon; \Omega_\varepsilon) - E(v_0; \Omega_0)| \leq c\mathcal{E}, \quad (3.63)$$

where  $\mathcal{E}$  is defined by (3.62),  $v_0$  is the solution of the limit problem (2.1) with the right-hand sides  $f_0 = 0$ ,  $g_0 = 0$  on  $\Gamma$  and  $g_0 = G$  on  $\Gamma_0$ .

**Proof** In view of formula (1.11) we have

$$\begin{aligned} E(u_\varepsilon; \Omega_\varepsilon) &= -\frac{1}{2} \int_{\Gamma_\varepsilon} G(y, z - \varepsilon) u_\varepsilon(x) ds_x = -\frac{1}{2} \int_{\Gamma_\varepsilon} G(y, z - \varepsilon) U_\varepsilon(x) ds_x + I_\varepsilon, \\ |I_\varepsilon| &= \frac{1}{2} \left| \int_{\Gamma_\varepsilon} G(u_\varepsilon - U_\varepsilon) ds_x \right| \leq c \|(\varepsilon + \rho^{2m}) \mathcal{R}_n(u_\varepsilon - U_\varepsilon); L^2(\Gamma_\varepsilon)\| \times \\ &\quad \times \left( \int_{\Gamma_\varepsilon} (\varepsilon + \rho^{2m})^{-1} \mathcal{R}_n(x)^2 ds_x \right)^{\frac{1}{2}} \leq c\mathcal{E}. \end{aligned}$$

We observe that the last integral is bounded by a constant independent of  $\varepsilon$  due to assumption (3.18) and definition (1.18). The trace on  $\Gamma_\varepsilon$  of the approximate solution equals to

$$\begin{aligned} v_0(y, z - \varepsilon) + \mathcal{X}_0(x) V_0(x) - \mathcal{X}_0(y, z - \varepsilon) v_0(y, z - \varepsilon) + \chi_0(\varepsilon^{-1}y) v_0(y, z - \varepsilon) + \\ + \hbar^{2-2m} \chi_0(y) (w_0(\hbar^{-1}y) - X(\hbar^{-1}y) \mathbf{V}_0(\hbar^{-1}y)). \end{aligned} \quad (3.64)$$

In order to complete the proof we make use of the following two estimates. First, we have

$$|\mathcal{X}_0(x) V_0(x) - \mathcal{X}_0(y, z - \varepsilon) v_0(y, z - \varepsilon)| \leq c\varepsilon$$

by the continuous differentiability on  $\bar{\Omega} \setminus \mathcal{U}'$  of the functions  $v_0, V_0$  and  $\mathcal{X}_0$ . Second,

$$\int_{\Gamma_0} |\chi_0(\hbar^{-1}y) v_0(y, z)| ds_x \leq c \left( \int_{\Gamma_0} \rho^{2m-2} |v_0(x)|^2 ds_x \right)^{\frac{1}{2}} \left( \int_0^{c\hbar} r^{2-2m} r^{n-2} dr \right)^{\frac{1}{2}} \leq c\hbar^{-m+\frac{n+1}{2}}.$$

The required estimate for the integral of the last term in (3.64) can be shown in the same way as (3.16) and (3.38), (3.39) by an application of Lemma 20. Finally,  $-\frac{1}{2} \int_{\Gamma_0} G v_0 ds_x = E(v_0, \Omega_0)$  by the Green formula.  $\blacksquare$

### 3.3 Subcritical case for $n = 2$

If the first inequality of (3.18) is verified and  $n = 2$  then  $m = 1$ . In such case there are some certain particularities in the construction of asymptotics for the solution to problem (1.5), (1.6). However the justification of asymptotics is performed in the exactly same way as it has been done in section 2§3. On the other hand, the plane problems with ligaments are analysed in details in [5] and [6]. That is why we here restrict ourselves to the construction of formal asymptotics only.

General results of [12], [13] or the framework developed in section §2 taking into account the procedure presented in Remark 7 shows that the solution of the first limit problem (2.1) with the right-hand sides (3.19) allows the expansion

$$v_0(y, z) = c_0^\pm - H_0^{-1} g_0(\mathcal{O}) \ln |y| + O(|y|), \quad \Omega_0 \cap \mathcal{U} \ni y = y_1 \rightarrow 0^\pm. \quad (3.65)$$

The orthogonality condition (3.21) implies the uniqueness of the solution  $v_0$  as well as of the constants  $c_\pm$ , which do not coincide by any reason (note that for  $n \geq 3$  the value of the function  $v_0$  at the point  $\mathcal{O}$  is uniquely determined). To *glue* asymptotics coming from two peaks resulting from the decomposition of degenerated ligament  $\Omega_0 \cap \mathcal{U}$ , we employ the general solution

$$-(2H_0)^{-1} g_0(\mathcal{O}) \ln(1 + H_0 \eta^2) + c_0 + H_0^{-\frac{1}{2}} c_1 \arctan(H_0^{\frac{1}{2}} \eta) \quad (3.66)$$

(compare with (3.41)) of the second limit problem

$$-\partial_\eta(1 + H_0 \eta^2) \partial_\eta w_0(\eta) = g_0(\mathcal{O}), \quad \eta \in \mathbb{R}. \quad (3.67)$$

We, readily, have

$$w_0(\eta) = -H_0^{-1} g_0(\mathcal{O}) \ln |\eta| - \frac{g_0(\mathcal{O})}{2H_0} \ln H_0 + c_0 \pm \frac{c_1 \pi}{2\sqrt{H_0}} + O(|\eta|^{-1}). \quad (3.68)$$

Therefore, taking into account (1.10) with  $\hbar = \sqrt{\varepsilon}$ , for matching the asymptotics (3.65) and (3.68), it is necessary to find the constants  $c_0$  and  $c_1$  by solving the system of two algebraic equations

$$-\frac{g_0(\mathcal{O})}{2H_0} \ln \frac{H_0}{\varepsilon} + c_0 \pm \frac{c_1 \pi}{2\sqrt{H_0}} = c_\pm. \quad (3.69)$$

We observe that the parameter  $\ln \varepsilon$  appears by the change of variables, i.e. the stretching of coordinates  $y \mapsto \eta = \varepsilon^{-1/2} y$ . As a result we have

$$c_0 = \frac{1}{2}(c_+ - c_-) + \frac{g_0(\mathcal{O})}{2H_0} \ln \frac{H_0}{\varepsilon}, \quad c_1 = \frac{\sqrt{H_0}}{\pi}(c_+ - c_-). \quad (3.70)$$

For given constants (3.70), the solution (3.66) of equation (3.67) is obtained in a precise form, thus we need to repeat all arguments given in previous section in order to apply Theorem 24. However, we restrict ourselves to a result on asymptotics of energy functional (1.9), similar to Corollary 26.

**Proposition 27** *Assume that  $n = 2$ ,  $m = 1$  and relation (1.21) is verified. Then the asymptotic formula for the energy functional*

$$E(u_\varepsilon; \Omega_\varepsilon) = E(u_0; \Omega_0) + O\left(\varepsilon^{\frac{1}{2}}(1 + |\ln \varepsilon|)\right) = -\frac{1}{2} \int_{\Gamma_0} G(x) v_0(x) ds_x + O\left(\varepsilon^{\frac{1}{2}}(1 + |\ln \varepsilon|)\right), \quad (3.71)$$

*is valid, where  $v_0 \in H^1(\Omega_0)$  is the solution of the first limit problem (2.1) with the right-hand sides  $f_0 = 0$  and  $g_0 = G$  on  $\Gamma_0$ ,  $g_0 = 0$  on  $\Gamma$ .*

### 3.4 Critical case

The plane problem is automatically excluded from considerations in the critical case

$$n = 2m - 1, \quad (3.72)$$

The peculiarity of the case (3.72) is non existence of solutions for both the limit problems (2.1) and (1.13) in the energy classes  $H^1(\Omega)$  and  $\mathbf{V}_m^1(\mathbb{R}^{n-1})$ , respectively. However, if the solutions  $v_0$  and  $w_0$  are found out and their asymptotics are matched for  $x \rightarrow \mathcal{O}$  and  $\eta \rightarrow \infty$ , respectively, then the



asymptotic constructions and the analysis of asymptotic remainders looks pretty the same as explained in section 2§3 (actually, for such a reason we have detached two asymptotic terms in (3.31)). Indeed, in definition (3.33) of the approximate solution  $U_\varepsilon$  of problem (1.5), (1.6) all singular terms are multiplied by appropriate cut-off functions, and the justification of asymptotics is performed by an application of weighted inequality (1.17), which covers the critical case (3.72) as well, and operates mainly with the exponents of power solutions. Some differences come out when evaluating the discrepancies, in particular for the energy functional, therefore, we pay attention to this question.

In accordance with the procedure described in section 1§2, the first limit problem admits the solution  $v_0 = \chi_0 \mathbf{V}_0 + \widehat{v}_0$ , where  $\mathbf{V}_0(y) = r^{2-2m} \Psi(\theta)$  is a power solution of model problem (2.30) with the right-hand side  $f(y) = g_0(\mathcal{O})$  (see (3.25) and Remark 13), with the *energy* remainder  $\widehat{v}_0 \in H^1(\Omega_0)$ . The results of section 3§2 provide the expansion of the remainder in the vicinity of the point  $\mathcal{O}$ , in such a way that for the solution  $v_0$  formulas (2.25) - (2.28) are still valid.

The solution  $w_0$  of second limit problem (1.13) with right-hand side (3.3) is still of the form (3.5) (see the comments to formula (3.29)).

When we compare the result obtained in the critical case with that of section 2§3, we see that the estimation of expression (3.38) is straightforward; however in critical case (3.72), inequality (3.39) is not satisfactory, since its right-hand side turns out not to be infinitesimal as  $\varepsilon \rightarrow 0^+$ . Therefore,  $\widetilde{w}_0$  is a solution to problem (1.13) with the right-hand side (3.29). We proceed as follows: the change of variables  $\eta \mapsto y = \hbar \eta$  and the scalar multiplication in  $L^2(\Omega_\varepsilon)$  by  $\psi$  result in transformation of the right-hand side in such a way, that it can be included in expression (3.40) and simplifies with the discrepancies generated by asymptotic term  $\mathbf{V}_0$  under gradient  $\nabla_y$  commutation with the cut-off function  $1 - \chi_0^\hbar(y) = X(\hbar^{-1}y)$ . In this way, the majorant in (3.40) is defined by the subsequent asymptotic term  $\mathbf{V}_1$  (see (3.24) and (3.25)), thus, it is given by

$$c\hbar^{-1}\hbar^{m+\frac{n-1}{2}}\hbar^\mu(1+\delta_{n,3}|\ln \hbar|) = c\hbar(1+\delta_{n,3}|\ln \hbar|)$$

The exponent of the parameter  $\hbar$  in the right-hand sides of inequalities (3.49), (3.51) and (3.58) is also equal to one, by relation (3.72).

**Theorem 28** *The conclusion of Theorem 24 remains true in the critical case  $2m = n - 1$  provided that the majorants in inequalities (3.60) and (3.61) are made equal to  $c\hbar(1 + \delta_{n,3}|\ln \hbar|)$ .*

The singularity  $O(r^{2-2m}) = O(r^{-n+1})$  of the integrand (see (3.24)) causes the divergence of the integral

$$\int_{\Gamma_0} v_0(x) g_0(x) ds_x$$

Indeed, for the set  $\Gamma_0(\delta) = \Gamma_0 \setminus (\mathbb{B}_\delta^{n-1} \times (-d, d))$  we obtain in view of (3.24) - (3.27) the equality

$$\int_{\Gamma_0(\delta)} v_0(x) g_0(x) ds_x = \ln \frac{1}{\delta} g_0(\mathcal{O}) \int_{\mathbb{S}^{n-2}} \Psi(\theta) ds_\theta + I_0 + o(1), \text{ for } \delta \rightarrow 0^+. \quad (3.73)$$

In a similar way, by (3.5) we have

$$\begin{aligned} \hbar^{2-2m} \int_{\Gamma_0 \setminus \Gamma_0(\delta)} g_0(x) w_0(\hbar^{-1}y) ds_x &= \int_{\mathbb{B}_{\delta/\hbar}^{n-1}} g_0(\mathcal{O}) w_0(\eta) d\eta + o(1) = \\ &= \ln \frac{\delta}{\hbar} g_0(\mathcal{O}) \int_{\mathbb{S}^{n-2}} \Psi(\theta) ds_\theta + I_\infty + o(1), \text{ for } \delta \rightarrow 0^+. \end{aligned} \quad (3.74)$$

Structure (3.33) of approximate solution  $U_\varepsilon$  and theorem 28 show that  $u_\varepsilon(x) \sim v_0(x)$  far from the point  $\mathcal{O}$  (outer asymptotic expansion in the framework of method of matched asymptotic expansions; see, e.g., [36, 37, 38]) and  $u_\varepsilon(x) \sim \hbar^{2-2m} w_0(\hbar^{-1}y)$  on the ligament in the vicinity of the point  $\mathcal{O}$  (inner asymptotic expansion). In this way, the combination of relations (3.73) and (3.74) with  $\delta = \hbar^{1/2}$  results in the following asymptotic formula for the energy functional, with the justification by an application of Theorem 28 along the lines of procedure in section 2§3.



**Corollary 29** Assume that under the assumption of Theorem 28 the right-hand sides of problem (1.5), (1.6) take form (1.8). Then the energy functional enjoys the property

$$E(u_\varepsilon; \Omega_\varepsilon) = -\frac{1}{2} |\ln \varepsilon| |G(\mathcal{O})| \int_{\mathbb{S}^{n-2}} \Psi_0(\theta) ds_\theta - \frac{1}{2} (I_0 + I_\infty) + O(\hbar(1 + \delta_{n,3} |\ln \varepsilon|)),$$

where  $\Psi_0$  is the angular part of the power solution  $r^{2-2m}\Psi_0(\theta)$  of model equation (2.30) with the right-hand side  $f(y) = 1$ , and the quantities  $I_0, I_\infty$  are taken from relations (3.73) and (3.74). ■

## 4 Generalizations and conclusions

### 4.1 Geometrical forms

To describe the limit domain  $\Omega_0$  in section 1§1 we suppose that  $\Gamma_0 \setminus \mathcal{O}$  is a simply connected part of the boundary  $\partial\Omega_0$ , i.e., the surfaces  $\Gamma_0$  and  $\Gamma$  has only one common point  $\mathcal{O}$  (see Figures 3-5). All asymptotic constructions in section §3 are of local nature, and are still valid for simply connected set  $\partial\Omega \setminus \mathcal{O}$ , for example for three dimensional ash-tray drawn in Figure 7.

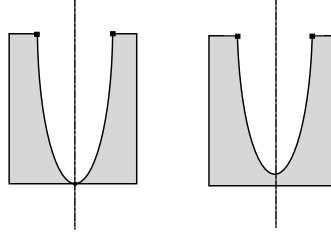


Fig. 7

In addition the boundary admits perturbations and in the vicinity of  $(n-2)$ -dimensional edge  $\Sigma$  (on the rotation axis in Figure 7 the edges  $\Sigma$  and  $\Sigma_\varepsilon$  are transformed into points, designated by symbols ■). The asymptotics of solutions for such perturbations of piecewise smooth boundary are analyzed in details in monograph [4]. We point out that it is not difficult to construct almost diffeomorphism of a neighborhood of the edge  $\Sigma_\varepsilon$  onto the neighborhood of the edge  $\Sigma$ , which maps  $\Sigma_\varepsilon$  onto  $\Sigma$ ; hence the perturbation of the boundary can be considered as *regular*.

Owing to local character, the asymptotic constructions from section §3 could be relatively easily adapted to the domains designated in Figure 8

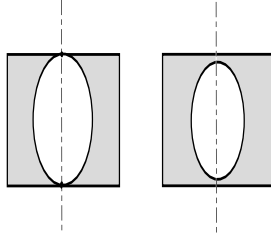


Fig. 8

(axis of the ellipsoid, originally tangent to the basis of spherical cylinder, is reduced  $1 - \varepsilon$  times). The boundary of  $\Omega_0$  may have more than two singular points of the described type.

More complex singular perturbations of the boundary are still admissible, for example caverns or bulges of diameter  $O(\varepsilon)$  on the surfaces  $\Gamma_\varepsilon$  and  $\Gamma$  in the  $c\varepsilon$ -neighborhood of the point  $\mathcal{O}$  (Figure 9).

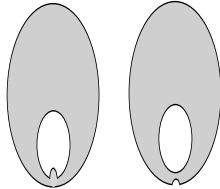


Fig. 9

Furthermore, the uniform stretching of coordinates  $x \mapsto \xi = \varepsilon^{-1}x$  (stretching of coordinates in (1.10) is not uniform, and uses two scales of order  $\varepsilon$  and  $\hbar = \varepsilon^{1/2m}$ ) leads to the third Neumann limit problem in unbounded domain  $\Pi$ ; the domain in the exterior of a ball  $\mathbb{B}_R^n$  coincides with the layer  $\mathbb{R}^{n-1} \times (0, 1)$  (with the strip  $\mathbb{R} \times (0, 1)$  for  $n = 2$ ). Asymptotic behavior at infinity of solutions of similar problems is investigated in [22, 23, 24, 25, 26, 27]; the derivation of supplementary limit problem is given in [8].

## 4.2 Construction and justification of asymptotics

In section §3 we restrict ourselves to main asymptotics terms, which is sufficient for analysis of the energy functional (1.9). The analysis and procedure of section §2 are general to derive the full asymptotic expansions of solutions of the limit problems (2.1) and (1.13) for  $x \rightarrow \mathcal{O}$  and  $\eta \rightarrow \infty$ , respectively. Therefore, we are in position to construct full asymptotic expansions of solutions  $u_\varepsilon$ . To this end we can apply either the method of compound asymptotic expansions (see [39, 4], and others) or the method of matched asymptotic expansions (see [36, 37, 38, 40], and others). We point out that in view of variable coefficients of differential operator  $L(\eta, \nabla_\eta)$  in (1.13), in order to construct the lower order terms of asymptotics in the framework of compound asymptotic expansions, the procedure of rearrangement of discrepancies should be applied (see monograph [4, Ch. 2 and 5], where the procedure is applied for some specific problems).

In §3 the estimate of asymptotic remainders are obtained in Dirichlet integral metrics (or in the weighted Sobolev space when using Proposition 2). The estimates can be derived in the scales of weighted spaces generated by the  $L^p$ - or Hölder norms (see [16]). *A priori* estimates for solutions of problems (1.5), (1.6) or (2.1) are derived from inequalities (1.17) or (2.4) by means of the decomposition of the domain  $\Omega_\varepsilon$  or  $\Omega_0$  into *almost the same* cells which are refined when approaching the point  $\mathcal{O}$  (compare with, e.g., [20] and [33]).

## 4.3 Shape sensitivity analysis

As it is shown in [41], the mentioned in the previous section weighted estimates of asymptotic remainders are very useful for analysis of general shape functionals. Let us consider the simplest examples

$$\mathfrak{F}_p^0(u_\varepsilon; \Omega_\varepsilon) = \int_{\Omega_\varepsilon} |u_\varepsilon(x)|^p dx, \quad \mathfrak{F}_q^1(u_\varepsilon; \Omega_\varepsilon) = \int_{\Omega_\varepsilon} |\nabla_x u_\varepsilon(x)|^q dx, \quad (4.1)$$

where  $p, q \in [1, \infty)$ . Asymptotic constructions, recalled in section §2, imply that within of application of conditions (1.8) the solution  $v_0$  to problem (2.1) leaves in the space  $L^p(\Omega_0)$  provided

$$p < \frac{2m + n - 1}{2(m - 1)} \quad (4.2)$$

(if  $m = 1$ , the ratio is by definition equal to  $\infty$ ). In this way, for the first functional in (4.1) and for  $m > 1$ , sub-critical, critical, and super-critical situations are characterized by the following relations, respectively,

$$p < \frac{2m+n-1}{2(m-1)}, \quad p = \frac{2m+n-1}{2(m-1)}, \quad p > \frac{2m+n-1}{2(m-1)}, \quad (4.3)$$

and for  $m = 1$  the situation is always sub-critical. The bounds in (4.3) differ from those given in §3. For the second functional in (4.1) conditions analogous to (4.3) are of the form

$$q < \frac{2m+n-1}{2m-1}, \quad q = \frac{2m+n-1}{2m-1}, \quad q > \frac{2m+n-1}{2m-1}. \quad (4.4)$$

For  $q = 2$ , in the case of the Dirichlet integral  $\mathfrak{F}_2^1(u_\varepsilon; \Omega_\varepsilon)$ , the bounds in (4.4) are exactly the same as in section §3.

For the surface shape functional (compare with [41])

$$\mathfrak{F}_p^\Gamma(u_\varepsilon; \Omega_\varepsilon) = \int_{\Gamma_\varepsilon} |u_\varepsilon(x)|^p ds_x, \quad (4.5)$$

we have the following conditions

$$p < \frac{n-1}{2(m-1)}, \quad p = \frac{n-1}{2(m-1)}, \quad p > \frac{n-1}{2(m-1)}. \quad (4.6)$$

By equality (1.11) the formulae (4.4) for  $q = 2$  and (4.5) for  $p = 1$  are equivalent.

We have performed only formal and preliminary analysis of shape functionals. Investigation of asymptotic properties of shape functionals in forms (4.1), (4.5), and of more involved shape functionals is still an open problem.

#### 4.4 Rotational symmetry

For kissing balls with the radii  $R_\pm$  (Figures 3 and 4) the function  $\mathbf{H} = r^2 H_0(\theta)$  from (1.2) equals

$$\frac{R_- + R_+}{2R_- R_+} r^2, \quad \text{kissing from outside}, \quad (4.7)$$

$$\frac{R_- - R_+}{2R_- R_+} r^2, \quad \text{kissing from inside } (R_- > R_+). \quad (4.8)$$

In Figure 5 the ball of radius  $R$  is tangent to the paraboloid given by

$$z = \frac{r^2}{2R} \quad (4.9)$$

with the curvature  $R^{-1}$  at the point  $\mathcal{O}$ . The Taylor formula for the function  $h \mapsto (1+h)^{1/2}$  yields

$$R - \sqrt{R^2 - r^2} = R - R \left( 1 - \frac{1}{2} \frac{r^2}{R^2} - \frac{1}{8} \frac{r^4}{R^4} + O\left(\frac{r^6}{R^6}\right) \right) = \left( \frac{r^2}{2R^2} - \frac{1}{8} \frac{r^4}{R^3} + O\left(\frac{r^6}{R^5}\right) \right),$$

i.e., the ball is inside of the paraboloid (4.9), and, beside that,  $m = 2$  and  $\mathbf{H}(y) = (8R^3)r^4$ . For exterior tangency of the ball and of the paraboloid, formula (4.7) is valid. The same formula in the limit for  $R_- \rightarrow \infty$  applies to the ball sitting on a hyperplane.

In the case of rotational symmetry the function  $H_0$  is constant and in (2.40) there is the sequence of eigenvalues of the Laplace-Beltrami operator  $\tilde{\Delta}_\theta$  on the unit ball  $\mathbb{S}^{n-2}$ . In addition, in the case of  $n \geq 3$  the eigenvalue

$$\tilde{\Lambda}_p = p(p+n-3) \quad (4.10)$$

is of multiplicity  $\kappa_p = (n-3+2p) \frac{(n-4+p)!}{(n-3)!p!}$ , where  $q! = 1 \cdot 2 \cdot \dots \cdot (q-1) \cdot q$  and  $0! = 1, (-1)! = \frac{1}{2}$ . The eigenvalue (4.10) is repeated  $\kappa_p$  times in the sequence (2.40). In particular,

$$\begin{aligned} \kappa_0 &= 1, \quad \kappa_p = 2, \quad p = 1, 2, \dots \quad \text{for } n = 3, \\ \kappa_q &= 1 + 2q, \quad q = 0, 1, 2, \dots \quad \text{for } n = 4. \end{aligned}$$

The exponents  $\lambda_k^\pm$  of power solutions can be determined by (2.43) taking into account the multiplicity, and for the first positive among them, which is present in (3.26) we obtain

$$\lambda_1^+ = \frac{1}{2} \left( 3 - n - 2m + \sqrt{(3 - n - 2m)^2 + 4(n-2)} \right).$$

In particular, for the kissing balls (Figures 3 and 4)

$$\begin{aligned} \lambda_1^+ &= -1 + \sqrt{2} \in \left( \frac{1}{3}, \frac{1}{2} \right), \quad \text{for } n = 3, \\ \lambda_1^+ &= \frac{1}{2} (-3 + \sqrt{17}) \in \left( \frac{1}{2}, 1 \right), \quad \text{for } n = 4, \end{aligned}$$

and for interior tangency of the ball with the paraboloid (Figure 5)

$$\begin{aligned} \lambda_1^+ &= -2 + \sqrt{10} \in (1, 2), \quad \text{for } n = 3, \\ \lambda_1^+ &= \frac{1}{2} (-5 + \sqrt{33}) \in \left( \frac{1}{4}, \frac{1}{3} \right), \quad \text{for } n = 4. \end{aligned}$$

The solution  $w_0$  of the second limit problem (1.13) with the right-hand side  $F(\eta) = G(\mathcal{O})$  keeps the rotational symmetry in the case of  $\mathbf{H}(y) = H_0 r^{2m}$ , where  $H_0$  is a constant. Indeed, equation (1.13) takes the form

$$-r^{2-n} \partial_r r^{n-2} \partial_r w_0(r) = G(\mathcal{O}), \quad r \in \mathbb{R}^+,$$

therefore,

$$\begin{aligned} \partial_r w_0(r) &= -G(\mathcal{O}) \frac{r}{n-1} (1 + H_0 r^{2m})^{-1}, \\ w_0(r) &= G(\mathcal{O}) \frac{1}{n-1} \int_r^\infty (1 + H_0 \mathbf{r}^{2m})^{-1} \mathbf{r} d\mathbf{r} \quad \text{for } m > 1, \\ w_0(r) &= \frac{1}{n-1} \frac{1}{2H_0} \ln(1 + H_0 \rho^2) + \text{const} \quad \text{for } m = 1 \end{aligned} \tag{4.11}$$

(see Remark 13 for the logarithm presence in the above formulae). The first formula in (4.11) shows that in super-critical case  $n < 2m - 1$  (cf. section 1§3) the energy functional (1.9) (or the Dirichlet integral, cf. (1.11)) gets according to (3.17) the asymptotics

$$\begin{aligned} E(u_\varepsilon; \Omega_\varepsilon) &= -\frac{1}{2} \hbar^{-2m+n+1} \text{meas}_{n-2}(\mathbb{S}^{n-2}) \frac{G(\mathcal{O})^2}{(n-1)^2} \int_0^\infty (1 + H_0 r^{2m})^{-1} r^n dr + \\ &+ O(\hbar^{-2m+n+2} (1 + \delta_{2m,n+2} |\ln \hbar|)). \end{aligned}$$

In the same way one can predict the asymptotic formulae for shape functionals (4.1) and (4.5) in the super-critical cases (see (4.3), (4.4) and (4.6)). For instance, if the inverse inequality (4.3) is valid, then the following relation holds

$$\mathfrak{F}_q^1(u_\varepsilon; \Omega_\varepsilon) \sim \hbar^{n-(2m-1)(q-1)} \frac{|G(\mathcal{O})|^q}{(n-1)^q} \text{meas}_{n-2}(\mathbb{S}^{n-2}) \int_0^\infty (1 + H_0 r^{2m})^{1-q} r^{n-2+q} dr. \tag{4.12}$$

The power exponent of small parameter  $\hbar$  in (4.1) is negative.

## 4.5 Loosing the connectivity for the limit set

The limit passage  $\varepsilon \rightarrow +0$  transforms the plane ( $n = 2$ ) domain, designated in Figure 7 in the union of two domains  $\Omega_0^{left}$  and  $\Omega_0^{right}$  with the boundary cusps, i.e., the limit problem (2.1) contains two independent Neumann problems, which need two compability conditions  $I_\alpha = 0$ , where  $\alpha = right, left = r, l$  and

$$I^\alpha = \int_{\Omega_0^\alpha} f_0(x) dx + \int_{\partial\Omega_0^\alpha} g_0(x) ds_x. \quad (4.13)$$

The additional difficulty is related to the compability conditions. It is easy to see that the orthogonality condition (2.2) follows from the compatibility condition (1.7) for the singularly perturbed problem (1.5), (1.6), and implies the relation

$$I^r = -I^l = I, \quad (4.14)$$

but by no means the equality  $I = 0$ , which is necessary for the solvability of two parts of the first limit problem in the Sobolev class  $H^1(\Omega_0^\alpha)$ . We refer the reader to paper [5] for details and to section §3 for justification of asymptotics. For the sake of simplicity we here restrict ourselves to the case  $m = 1$ , and derive the explicit formulae. For  $m > 1$  some evident changes are required in the asymptotic procedure. It is interesting that, in contrast to section 3§3, the Dirichlet integral loses the finite limit for  $\varepsilon \rightarrow 0^+$ . However the lost of boundedness has other reason than in section 1§3 and has no relation to the values of  $g_0$  at the point  $\mathcal{O}$ .

The limit problems (2.1) in the domains  $\Omega_0^l$  and  $\Omega_0^r$  admit the solutions

$$v_0^\alpha(x) = \chi(y) I H_0^{-1} y^{-1} + \widehat{v}_0^\alpha(x), \quad \widehat{v}_0^\alpha \in H^1(\Omega_0^\alpha), \quad \alpha = l, r. \quad (4.15)$$

This fact comes out from the general results of [12, 13, 15]. In order to verify the result in the simple situation of the Neumann problem for the Poisson equation we point out two features. First, the function  $x \mapsto Y(x) = \chi(y)y^{-1}$  leaves in the homogeneous problem (2.1) a discrepancy with *sufficiently good* behavior in the vicinity of the point  $\mathcal{O}$ , i.e., such that the problem admits a solution in the space  $H^1(\Omega_0^\alpha)$ . Second, the required condition for solvability of the non homogeneous problem (2.1) can be assured by the choice of a factor at  $Y(x)$  in singular solution (4.14) due to the calculation

$$\begin{aligned} - \int_{\Omega_0^r} \Delta_x Y(x) dx + \int_{\partial\Omega_0^r} \partial_\nu Y(x) ds_x &= \lim_{\delta \rightarrow 0^+} \left( - \int_{\Omega_0^r(\delta)} \Delta_x Y(x) dx + \int_{(\partial\Omega_0^r)(\delta)} \partial_\nu Y(x) ds_x \right) = \\ &= \lim_{\delta \rightarrow 0^+} \int_{-H_-(\delta)}^{H_+(\delta)} \partial_y Y(y, z) dz = \lim_{\delta \rightarrow 0^+} - \frac{1}{\delta^2} \int_{-H_-(\delta)}^{H_+(\delta)} dz = -H_0. \end{aligned} \quad (4.16)$$

Here,  $\Omega_0^r(\delta) = \{x \in \Omega_0^r : y > \delta\}$  is the domain with blunted peak, and  $(\partial\Omega_0^r)(\delta) = \{x \in \partial\Omega_0^r : y > \delta\}$ . In the same way the left domain  $\Omega_0^l$  is considered, with the obvious difference, which is related to the direction of interior normal to the segment  $\{x : y = -\delta, -H_-(\delta) < z < H_+(\delta)\}$  on the boundary  $\partial\Omega_0^l$ , and results in the change of sign in three last terms in expressions (4.16)

Solutions (4.14) of the first limit problem in  $\Omega_0^\alpha$  are matched with the solution  $\varepsilon^{-1/2} w_0(\varepsilon^{-1/2} y)$  of the homogeneous equation (3.67), i.e., the second limit problem. The required solution (3.66) and enjoys the following asymptotics for  $\eta \rightarrow \infty$ :

$$w_0(\eta) = -I H_0^{-\frac{1}{2}} \arctan\left(H_0^{\frac{1}{2}} \eta\right) = \mp I H_0^{-\frac{1}{2}} \left( \frac{\pi}{2} - \frac{1}{H_0^{\frac{1}{2}} |\eta|} + O(|\eta|^{-3}) \right). \quad (4.17)$$

Due to the presence of the constant terms  $I(4H_0)^{-1/2}\pi$  in expansions (4.17), the asymptotic ansätze in  $\Omega_\varepsilon^\alpha$  have to be changed for

$$\begin{aligned} u_\varepsilon(x) &= -\frac{\pi I}{2\sqrt{H_0}} \varepsilon^{-\frac{1}{2}} + v_0(x) + \dots \quad \text{in } \Omega_\varepsilon^r, \\ u_\varepsilon(x) &= \frac{\pi I}{2\sqrt{H_0}} \varepsilon^{-\frac{1}{2}} + v_0(x) + \dots \quad \text{in } \Omega_\varepsilon^l. \end{aligned} \quad (4.18)$$

It is clear that the presence in (4.18) of large but constant terms has influence neither on the differential equation, nor on the Neumann boundary conditions but the terms effect the behavior of the energy functional as  $\varepsilon \rightarrow 0^+$ ,

$$E(u_\varepsilon; \Omega_\varepsilon) = -\frac{1}{2} \int_{\Omega_\varepsilon} f(\varepsilon, x) u_\varepsilon(x) dx - \frac{1}{2} \int_{\partial\Omega_\varepsilon} g(\varepsilon, x) u_\varepsilon(x) ds_x. \quad (4.19)$$

The following result, provided by the asymptotic anzatz (4.18) (the contribution of boundary layer is of order  $O(1)$ ), can be justified by the same scheme as that of section 2§3, and furnishes the asymptotics of functional (4.19).

**Proposition 30** *Let  $\Omega_\varepsilon$  be the domain designated in Figure 7,  $f = 0$  and  $g(x, \varepsilon) = g_0(x)$ , the support of the function  $g_0$  is included in the domain  $\partial\Omega_\varepsilon \cap \partial\Omega$ , i.e., on the fixed part of the boundary of singularly perturbed domain, and  $g_0$  is of the null mean value. Then the following asymptotic formula is valid*

$$E(u_\varepsilon; \Omega_\varepsilon) = -\frac{\pi}{2} (\varepsilon H_0)^{-\frac{1}{2}} \left( \int_{\partial\Omega_0^r} g_0(x) ds_x \right)^2 + O(1 + |\ln \varepsilon|), \quad \varepsilon \rightarrow 0^+. \quad \blacksquare$$

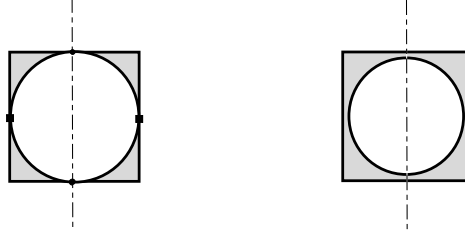


Fig. 10

The decomposition of the domain  $\Omega_\varepsilon$  in the limit  $\varepsilon \rightarrow 0^+$  into non-connected components can be achieved in multidimensional case, in Figure 10 there is a ball of radius  $R$  in the space  $\mathbb{R}^3$  which is located inside of the cylinder of the same radius and of the height  $2R$ , and the perturbation of the domain is performed by decreasing of the radius to  $R - \varepsilon$ . In such case for the limit domain appears cuspidal edges (denoted by the symbol  $\blacksquare$  on the profile of symmetry, in Figure 10). The question on asymptotic structure of the solution  $u_\varepsilon$  to problem (1.5), (1.6), and the behavior of functional (1.9) for  $\varepsilon \rightarrow 0^+$  in such a singularly perturbed domain is a fully open problem. We can forecast, that the solutions of the first limit problems enjoy the singularities of order  $O(|z|^{-1})$ , distributed along the edges, and the density of distribution can be found by a solution of a supplementary equation on the edge, which results from the matching of the exterior and interior asymptotic expansions of solutions. The same effects appear in slightly different situations described in [42, 43].

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