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A locally compact quantum group of triangular matrices.

Pierre Fima* and Leonid Vainerman†

Dedicated to Professor M.L. Gorbachuk on the occasion of his 70-th anniversary.

Abstract

We construct a one parameter deformation of the group of 2×2 upper triangular matrices with determinant 1 using the twisting construction. An interesting feature of this new example of a locally compact quantum group is that the Haar measure is deformed in a non-trivial way. Also, we give a complete description of the dual C^* -algebra and the dual comultiplication.

1 Introduction

In [3, 14], M. Enock and the second author proposed a systematic approach to the construction of non-trivial Kac algebras by twisting. To illustrate it, consider a cocommutative Kac algebra structure on the group von Neumann algebra $M = \mathcal{L}(G)$ of a non commutative locally compact (l.c.) group G with comultiplication $\Delta(\lambda_g) = \lambda_g \otimes \lambda_g$ (here λ_g is the left translation by $g \in G$). Let us define on M another, "twisted", comultiplication $\Delta_\Omega(\cdot) = \Omega\Delta(\cdot)\Omega^*$, where Ω is a unitary from $M \otimes M$ verifying certain 2-cocycle condition, and construct in this way new, non cocommutative, Kac algebra structure on M . In order to find such an Ω , let us, following to M. Rieffel [10] and M. Landstad [8], take an inclusion $\alpha : L^\infty(\hat{K}) \rightarrow M$, where \hat{K} is the dual to some abelian subgroup K of G such that $\delta|_K = 1$, where $\delta(\cdot)$ is the module of G . Then, one lifts a usual 2-cocycle Ψ of $\hat{K} : \Omega = (\alpha \otimes \alpha)\Psi$. The main result of [3], [14] is that the integral by the Haar measure of G gives also the Haar measure of the deformed object. Recently P. Kasprzak studied the deformation of l.c. groups by twisting in [5], and also in this case the Haar measure was not deformed.

In [4], the authors extended the twisting construction in order to cover the case of non-trivial deformation of the Haar measure. The aim of the present paper is to illustrate this construction on a concrete example and to compute

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explicitly all the ingredients of the twisted quantum group including the dual C^* -algebra and the dual comultiplication. We twist the group von Neumann algebra $\mathcal{L}(G)$ of the group G of 2×2 upper triangular matrices with determinant 1 using the abelian subgroup $K = \mathbb{C}^*$ of diagonal matrices of G and a one parameter family of bicharacters on K . In this case, the subgroup K is not included in the kernel of the modular function of G , this is why the Haar measure is deformed. We compute the new Haar measure and show that the dual C^* -algebra is generated by 2 normal operators $\hat{\alpha}$ and $\hat{\beta}$ such that

$$\hat{\alpha}\hat{\beta} = \hat{\beta}\hat{\alpha} \quad \hat{\alpha}\hat{\beta}^* = q\hat{\beta}^*\hat{\alpha},$$

where $q > 0$. Moreover, the comultiplication $\hat{\Delta}$ is given by

$$\hat{\Delta}_t(\hat{\alpha}) = \hat{\alpha} \otimes \hat{\alpha}, \quad \hat{\Delta}_t(\hat{\beta}) = \hat{\alpha} \otimes \hat{\beta} \dot{+} \hat{\beta} \otimes \hat{\alpha}^{-1},$$

where $\dot{+}$ means the closure of the sum of two operators.

This paper is organized as follows. In Section 2 we recall some basic definitions and results. In Section 3 we present in detail our example computing all the ingredients associated. This example is inspired by [5], but an important difference is that in the present example the Haar measure is deformed in a non trivial way. Finally, we collect some useful results in the Appendix.

2 Preliminaries

2.1 Notations

Let $B(H)$ be the algebra of all bounded linear operators on a Hilbert space H , \otimes the tensor product of Hilbert spaces, von Neumann algebras or minimal tensor product of C^* -algebras, and Σ (resp., σ) the flip map on it. If H, K and L are Hilbert spaces and $X \in B(H \otimes L)$ (resp., $X \in B(H \otimes K)$, $X \in B(K \otimes L)$), we denote by X_{13} (resp., X_{12} , X_{23}) the operator $(1 \otimes \Sigma^*)(X \otimes 1)(1 \otimes \Sigma)$ (resp., $X \otimes 1$, $1 \otimes X$) defined on $H \otimes K \otimes L$. For any subset X of a Banach space E , we denote by $\langle X \rangle$ the vector space generated by X and $[X]$ the closed vector space generated by X . All l.c. groups considered in this paper are supposed to be second countable, all Hilbert spaces are separable and all von Neumann algebras have separable preduals.

Given a *normal semi-finite faithful* (n.s.f.) weight θ on a von Neumann algebra M (see [12]), we denote: $\mathcal{M}_\theta^+ = \{x \in M^+ \mid \theta(x) < +\infty\}$, $\mathcal{N}_\theta = \{x \in M \mid x^*x \in \mathcal{M}_\theta^+\}$, and $\mathcal{M}_\theta = \langle \mathcal{M}_\theta^+ \rangle$.

When A and B are C^* -algebras, we denote by $M(A)$ the algebra of the multipliers of A and by $\text{Mor}(A, B)$ the set of the morphisms from A to B .

2.2 G -products and their deformation

For the notions of an action of a l.c. group G on a C^* -algebra A , a C^* dynamical system (A, G, α) , a crossed product $G_\alpha \rtimes A$ of A by G see [9]. The crossed product has the following universal property:

For any C^* -covariant representation (π, u, B) of (A, G, α) (here B is a C^* -algebra, $\pi : A \rightarrow B$ a morphism, u is a group morphism from G to the unitaries of $M(B)$, continuous for the strict topology), there is a unique morphism $\rho \in \text{Mor}(G_\alpha \ltimes A, B)$ such that

$$\rho(\lambda_t) = u_t, \quad \rho(\pi_\alpha(x)) = \pi(x) \quad \forall t \in G, x \in A.$$

Definition 1 Let G be a l.c. abelian group, B a C^* -algebra, λ a morphism from G to the unitary group of $M(B)$, continuous in the strict topology of $M(B)$, and θ a continuous action of \hat{G} on B . The triplet (B, λ, θ) is called a G -product if $\theta_\gamma(\lambda_g) = \overline{\langle \gamma, g \rangle} \lambda_g$ for all $\gamma \in \hat{G}$, $g \in G$.

The unitary representation $\lambda : G \rightarrow M(B)$ generates a morphism :

$$\lambda \in \text{Mor}(C^*(G), B).$$

Identifying $C^*(G)$ with $C_0(\hat{G})$, one gets a morphism $\lambda \in \text{Mor}(C_0(\hat{G}), B)$ which is defined in a unique way by its values on the characters

$$u_g = (\gamma \mapsto \langle \gamma, g \rangle) \in C_b(\hat{G}) : \lambda(u_g) = \lambda_g, \quad \text{for all } g \in G.$$

One can check that λ is injective.

The action θ is done by: $\theta_\gamma(\lambda(u_g)) = \theta_\gamma(\lambda_g) = \overline{\langle \gamma, g \rangle} \lambda_g = \lambda(u_g(\cdot - \gamma))$. Since the u_g generate $C_b(\hat{G})$, one deduces that:

$$\theta_\gamma(\lambda(f)) = \lambda(f(\cdot - \gamma)), \quad \text{for all } f \in C_b(\hat{G}).$$

The following definition is equivalent to the original definition by Landstad [8] (see [5]):

Definition 2 Let (B, λ, θ) be a G -product and $x \in M(B)$. One says that x verifies the Landstad conditions if

$$\begin{cases} (i) & \theta_\gamma(x) = x, \quad \text{for any } \gamma \in \hat{G}, \\ (ii) & \text{the application } g \mapsto \lambda_g x \lambda_g^* \text{ is continuous,} \\ (iii) & \lambda(f) x \lambda(g) \in B, \quad \text{for any } f, g \in C_0(\hat{G}). \end{cases} \quad (1)$$

The set $A \in M(B)$ verifying these conditions is a C^* -algebra called *the Landstad algebra* of the G -product (B, λ, θ) . Definition 2 implies that if $a \in A$, then $\lambda_g a \lambda_g^* \in A$ and the map $g \mapsto \lambda_g a \lambda_g^*$ is continuous. One gets then an action of G on A .

One can show that the inclusion $A \rightarrow M(B)$ is a morphism of C^* -algebras, so $M(A)$ can be also included into $M(B)$. If $x \in M(B)$, then $x \in M(A)$ if and only if

$$\begin{cases} (i) & \theta_\gamma(x) = x, \quad \text{for all } \gamma \in \hat{G}, \\ (ii) & \text{for all } a \in A, \text{ the application } g \mapsto \lambda_g x \lambda_g^* a \text{ is continuous.} \end{cases} \quad (2)$$

Let us note that two first conditions of (1) imply (2).

The notions of G -product and crossed product are closely related. Indeed, if (A, G, α) is a C^* -dynamical system with G abelian, let $B = G_\alpha \rtimes A$ be the crossed product and λ the canonical morphism from G into the unitary group of $M(B)$, continuous in the strict topology, and $\pi \in \text{Mor}(A, B)$ the canonical morphism of C^* -algebras. For $f \in \mathcal{K}(G, A)$ and $\gamma \in \hat{G}$, one defines $(\theta_\gamma f)(t) = \overline{\langle \gamma, t \rangle} f(t)$. One shows that θ_γ can be extended to the automorphisms of B in such a way that (B, \hat{G}, θ) would be a C^* -dynamical system. Moreover, (B, λ, θ) is a G -product and the associated Landstad algebra is $\pi(A)$. θ is called *the dual action*. Conversely, if (B, λ, θ) is a G -product, then one shows that there exists a C^* -dynamical system (A, G, α) such that $B = G_\alpha \rtimes A$. It is unique (up to a covariant isomorphism), A is the Landstad algebra of (B, λ, θ) and α is the action of G on A given by $\alpha_t(x) = \lambda_t x \lambda_t^*$.

Lemma 1 [5] *Let (B, λ, θ) be a G -product and $V \subset A$ be a vector subspace of the Landstad algebra such that:*

- $\lambda_g V \lambda_g^* \subset V$, for any $g \in G$,
- $\lambda(C_0(\hat{G})) V \lambda(C_0(\hat{G}))$ is dense in B .

Then V is dense in A .

Let (B, λ, θ) be a G -product, A its Landstad algebra, and Ψ a continuous bicharacter on \hat{G} . For $\gamma \in \hat{G}$, the function on \hat{G} defined by $\Psi_\gamma(\omega) = \Psi(\omega, \gamma)$ generates a family of unitaries $\lambda(\Psi_\gamma) \in M(B)$. The bicharacter condition implies:

$$\theta_\gamma(U_{\gamma_2}) = \lambda(\Psi_{\gamma_2}(\cdot - \gamma_1)) = \overline{\Psi(\gamma_1, \gamma_2)} U_{\gamma_2}, \quad \forall \gamma_1, \gamma_2 \in \hat{G}.$$

One gets then a new action θ^Ψ of \hat{G} on B :

$$\theta_\gamma^\Psi(x) = U_\gamma \theta(x) U_\gamma^*.$$

Note that, by commutativity of G , one has:

$$\theta_\gamma^\Psi(\lambda_g) = U_\gamma \theta(\lambda_g) U_\gamma^* = \overline{\langle \gamma, g \rangle} \lambda_g, \quad \forall \gamma \in \hat{G}, g \in G.$$

The triplet $(B, \lambda, \theta^\Psi)$ is then a G -product, called a *deformed G -product*.

2.3 Locally compact quantum groups [6], [7]

A pair (M, Δ) is called a (von Neumann algebraic) l.c. quantum group when

- M is a von Neumann algebra and $\Delta : M \rightarrow M \otimes M$ is a normal and unital $*$ -homomorphism which is coassociative: $(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta$ (i.e., (M, Δ) is a Hopf-von Neumann algebra).
- There exist n.s.f. weights φ and ψ on M such that

$$- \varphi \text{ is left invariant in the sense that } \varphi((\omega \otimes \text{id})\Delta(x)) = \varphi(x)\omega(1) \text{ for all } x \in \mathcal{M}_\varphi^+ \text{ and } \omega \in M_\psi^+,$$

- ψ is right invariant in the sense that $\psi((\text{id} \otimes \omega)\Delta(x)) = \psi(x)\omega(1)$ for all $x \in \mathcal{M}_\psi^+$ and $\omega \in M_*^+$.

Left and right invariant weights are unique up to a positive scalar.

Let us represent M on the GNS Hilbert space of φ and define a unitary W on $H \otimes H$ by

$$W^*(\Lambda(a) \otimes \Lambda(b)) = (\Lambda \otimes \Lambda)(\Delta(b)(a \otimes 1)), \quad \text{for all } a, b \in N_\phi.$$

Here, Λ denotes the canonical GNS-map for φ , $\Lambda \otimes \Lambda$ the similar map for $\varphi \otimes \varphi$. One proves that W satisfies the *pentagonal equation*: $W_{12}W_{13}W_{23} = W_{23}W_{12}$, and we say that W is a *multiplicative unitary*. The von Neumann algebra M and the comultiplication on it can be given in terms of W respectively as

$$M = \{(\text{id} \otimes \omega)(W) \mid \omega \in B(H)_*\}^{-\sigma\text{-strong}^*}$$

and $\Delta(x) = W^*(1 \otimes x)W$, for all $x \in M$. Next, the l.c. quantum group (M, Δ) has an antipode S , which is the unique σ -strongly* closed linear map from M to M satisfying $(\text{id} \otimes \omega)(W) \in \mathcal{D}(S)$ for all $\omega \in B(H)_*$ and $S(\text{id} \otimes \omega)(W) = (\text{id} \otimes \omega)(W^*)$ and such that the elements $(\text{id} \otimes \omega)(W)$ form a σ -strong* core for S . S has a polar decomposition $S = R\tau_{-i/2}$, where R (the unitary antipode) is an anti-automorphism of M and τ_t (the scaling group of (M, Δ)) is a strongly continuous one-parameter group of automorphisms of M . We have $\sigma(R \otimes R)\Delta = \Delta R$, so φR is a right invariant weight on (M, Δ) and we take $\psi := \varphi R$.

Let σ_t be the modular automorphism group of φ . There exist a number $\nu > 0$, called the scaling constant, such that $\psi \sigma_t = \nu^{-t} \psi$ for all $t \in \mathbb{R}$. Hence (see [13]), there is a unique positive, self-adjoint operator δ_M affiliated to M , such that $\sigma_t(\delta_M) = \nu^t \delta_M$ for all $t \in \mathbb{R}$ and $\psi = \varphi_{\delta_M}$. It is called the modular element of (M, Δ) . If $\delta_M = 1$ we call (M, Δ) unimodular. The scaling constant can be characterized as well by the relative invariance $\varphi \tau_t = \nu^{-t} \varphi$.

For the dual l.c. quantum group $(\hat{M}, \hat{\Delta})$ we have :

$$\hat{M} = \{(\omega \otimes \text{id})(W) \mid \omega \in B(H)_*\}^{-\sigma\text{-strong}^*}$$

and $\hat{\Delta}(x) = \Sigma W(x \otimes 1)W^* \Sigma$ for all $x \in \hat{M}$. A left invariant n.s.f. weight $\hat{\varphi}$ on \hat{M} can be constructed explicitly and the associated multiplicative unitary is $\hat{W} = \Sigma W^* \Sigma$.

Since $(\hat{M}, \hat{\Delta})$ is again a l.c. quantum group, let us denote its antipode by \hat{S} , its unitary antipode by \hat{R} and its scaling group by $\hat{\tau}_t$. Then we can construct the dual of $(\hat{M}, \hat{\Delta})$, starting from the left invariant weight $\hat{\varphi}$. The bidual l.c. quantum group $(\hat{\hat{M}}, \hat{\hat{\Delta}})$ is isomorphic to (M, Δ) .

M is commutative if and only if (M, Δ) is generated by a usual l.c. group $G : M = L^\infty(G)$, $(\Delta_G f)(g, h) = f(gh)$, $(S_G f)(g) = f(g^{-1})$, $\varphi_G(f) = \int f(g) dg$, where $f \in L^\infty(G)$, $g, h \in G$ and we integrate with respect to the left Haar measure dg on G . Then ψ_G is given by $\psi_G(f) = \int f(g^{-1}) dg$ and δ_M by the strictly positive function $g \mapsto \delta_G(g)^{-1}$.

$L^\infty(G)$ acts on $H = L^2(G)$ by multiplication and $(W_G \xi)(g, h) = \xi(g, g^{-1}h)$, for all $\xi \in H \otimes H = L^2(G \times G)$. Then $\hat{M} = \mathcal{L}(G)$ is the group von Neumann

algebra generated by the left translations $(\lambda_g)_{g \in G}$ of G and $\hat{\Delta}_G(\lambda_g) = \lambda_g \otimes \lambda_g$. Clearly, $\hat{\Delta}_G^{op} := \sigma \circ \hat{\Delta}_G = \hat{\Delta}_G$, so $\hat{\Delta}_G$ is cocommutative.

(M, Δ) is a Kac algebra (see [2]) if $\tau_t = \text{id}$, for all t , and δ_M is affiliated with the center of M . In particular, this is the case when $M = L^\infty(G)$ or $M = \mathcal{L}(G)$.

We can also define the C^* -algebra of continuous functions vanishing at infinity on (M, Δ) by

$$A = [(\text{id} \otimes \omega)(W) \mid \omega \in \mathcal{B}(H)_*]$$

and the reduced C^* -algebra (or dual C^* -algebra) of (M, Δ) by

$$\hat{A} = [(\omega \otimes \text{id})(W) \mid \omega \in \mathcal{B}(H)_*].$$

In the group case we have $A = C_0(G)$ and $\hat{A} = C_r(G)$. Moreover, we have $\Delta \in \text{Mor}(A, A \otimes A)$ and $\hat{\Delta} \in \text{Mor}(\hat{A}, \hat{A} \otimes \hat{A})$.

A l.c. quantum group is called compact if $\varphi(1_M) < \infty$ and discrete if its dual is compact.

2.4 Twisting of locally compact quantum groups [4]

Let (M, Δ) be a locally compact quantum group and Ω a unitary in $M \otimes M$. We say that Ω is a 2-cocycle on (M, Δ) if

$$(\Omega \otimes 1)(\Delta \otimes \text{id})(\Omega) = (1 \otimes \Omega)(\text{id} \otimes \Delta)(\Omega).$$

As an example we can consider $M = L^\infty(G)$, where G is a l.c. group, with Δ_G as above, and $\Omega = \Psi(\cdot, \cdot) \in L^\infty(G \times G)$ a usual 2-cocycle on G , i.e., a measurable function with values in the unit circle $\mathbb{T} \subset \mathbb{C}$ verifying

$$\Psi(s_1, s_2)\Psi(s_1s_2, s_3) = \Psi(s_2, s_3)\Psi(s_1, s_2s_3), \text{ for almost all } s_1, s_2, s_3 \in G.$$

This is the case for any measurable bicharacter on G .

When Ω is a 2-cocycle on (M, Δ) , one can check that $\Delta_\Omega(\cdot) = \Omega\Delta(\cdot)\Omega^*$ is a new coassociative comultiplication on M . If (M, Δ) is discrete and Ω is any 2-cocycle on it, then (M, Δ_Ω) is again a l.c. quantum group (see [1], finite-dimensional case was treated in [14]). In the general case, one can proceed as follows. Let $\alpha : (L^\infty(G), \Delta_G) \rightarrow (M, \Delta)$ be an inclusion of Hopf-von Neumann algebras, i.e., a faithful unital normal *-homomorphism such that $(\alpha \otimes \alpha) \circ \Delta_G = \Delta \circ \alpha$. Such an inclusion allows to construct a 2-cocycle of (M, Δ) by lifting a usual 2-cocycle of $G : \Omega = (\alpha \otimes \alpha)\Psi$. It is shown in [3] that if the image of α is included into the centralizer of the left invariant weight φ , then φ is also left invariant for the new comultiplication Δ_Ω .

In particular, let G be a non commutative l.c. group and K a closed abelian subgroup of G . By Theorem 6 of [11], there exists a faithful unital normal *-homomorphism $\hat{\alpha} : \mathcal{L}(K) \rightarrow \mathcal{L}(G)$ such that

$$\hat{\alpha}(\lambda_g^K) = \lambda_g, \quad \text{for all } g \in K, \quad \text{and} \quad \hat{\Delta} \circ \hat{\alpha} = (\hat{\alpha} \otimes \hat{\alpha}) \circ \hat{\Delta}_K,$$

where λ^K and λ are the left regular representation of K and G respectively, and $\hat{\Delta}_K$ and $\hat{\Delta}$ are the comultiplications on $\mathcal{L}(K)$ and $\mathcal{L}(G)$ respectively. The composition of $\hat{\alpha}$ with the canonical isomorphism $L^\infty(\hat{K}) \simeq \mathcal{L}(K)$ given by the Fourier transformation, is a faithful unital normal *-homomorphism $\alpha : L^\infty(\hat{K}) \rightarrow \mathcal{L}(G)$ such that $\Delta \circ \alpha = (\alpha \otimes \alpha) \circ \Delta_{\hat{K}}$, where $\Delta_{\hat{K}}$ is the comultiplication on $L^\infty(\hat{K})$. The left invariant weight on $\mathcal{L}(G)$ is the Plancherel weight for which

$$\sigma_t(x) = \delta_G^{it} x \delta_G^{-it}, \quad \text{for all } x \in \mathcal{L}(G),$$

where δ_G is the modular function of G . Thus, $\sigma_t(\lambda_g) = \delta_G^{it}(g) \lambda_g$ or

$$\sigma_t \circ \alpha(u_g) = \alpha(u_g(\cdot - \gamma_t)),$$

where $u_g(\gamma) = \langle \gamma, g \rangle$, $g \in G, \gamma \in \hat{G}$, γ_t is the character K defined by $\langle \gamma_t, g \rangle = \delta_G^{-it}(g)$. By linearity and density we obtain:

$$\sigma_t \circ \alpha(F) = \alpha(F(\cdot - \gamma_t)), \quad \text{for all } F \in L^\infty(\hat{K}).$$

This is why we do the following assumptions. Let (M, Δ) be a l.c. quantum group, G an abelian l.c. group and $\alpha : (L^\infty(G), \Delta_G) \rightarrow (M, \Delta)$ an inclusion of Hopf-von Neumann algebras. Let φ be the left invariant weight, σ_t its modular group, S the antipode, R the unitary antipode, τ_t the scaling group. Let $\psi = \varphi \circ R$ be the right invariant weight and σ'_t its modular group. Also we denote by δ the modular element of (M, Δ) . Suppose that there exists a continuous group homomorphism $t \mapsto \gamma_t$ from \mathbb{R} to G such that

$$\sigma_t \circ \alpha(F) = \alpha(F(\cdot - \gamma_t)), \quad \text{for all } F \in L^\infty(G).$$

Let Ψ be a continuous bicharacter on G . Notice that $(t, s) \mapsto \Psi(\gamma_t, \gamma_s)$ is a continuous bicharacter on \mathbb{R} , so there exists $\lambda > 0$ such that $\Psi(\gamma_t, \gamma_s) = \lambda^{ist}$. We define:

$$u_t = \lambda^{i\frac{t^2}{2}} \alpha(\Psi(\cdot, -\gamma_t)) \quad \text{and} \quad v_t = \lambda^{i\frac{t^2}{2}} \alpha(\Psi(-\gamma_t, \cdot)).$$

The 2-cocycle equation implies that u_t is a σ_t -cocycle and v_t is a σ'_t -cocycle. The Connes' Theorem gives two n.s.f. weights on M , φ_Ω and ψ_Ω , such that

$$u_t = [D\varphi_\Omega : D\varphi]_t \quad \text{and} \quad v_t = [D\psi_\Omega : D\psi]_t.$$

The main result of [4] is as follows:

Theorem 1 *(M, Δ_Ω) is a l.c. quantum group with left and right invariant weight φ_Ω and ψ_Ω respectively. Moreover, denoting by a subscript or a superscript Ω the objects associated with (M, Δ_Ω) one has:*

- $\tau_t^\Omega = \tau_t$,
- $\nu_\Omega = \nu$ and $\delta_\Omega = \delta A^{-1} B$,

- $\mathcal{D}(S_\Omega) = \mathcal{D}(S)$ and, for all $x \in \mathcal{D}(S)$, $S_\Omega(x) = uS(x)u^*$.

Remark that, because Ψ is a bicharacter on G , $t \mapsto \alpha(\Psi(\cdot, -\gamma_t))$ is a representation of \mathbb{R} in the unitary group of M and there exists a positive self-adjoint operator A affiliated with M such that

$$\alpha(\Psi(\cdot, -\gamma_t)) = A^{it}, \quad \text{for all } t \in \mathbb{R}.$$

We can also define a positive self-adjoint operator B affiliated with M such that

$$\alpha(\Psi(-\gamma_t, \cdot)) = B^{it}.$$

We obtain :

$$u_t = \lambda^{i\frac{t^2}{2}} A^{it}, \quad v_t = \lambda^{i\frac{t^2}{2}} B^{it}.$$

Thus, we have $\varphi_\Omega = \varphi_A$ and $\psi_\Omega = \psi_B$, where φ_A and ψ_B are the weights defined by S. Vaes in [13].

One can also compute the dual C^* -algebra and the dual comultiplication. We put:

$$L_\gamma = \alpha(u_\gamma), \quad R_\gamma = JL_\gamma J, \quad \text{for all } \gamma \in \hat{G}.$$

From the representation $\gamma \mapsto L_\gamma$ we get the unital $*$ -homomorphism $\lambda_L : L^\infty(G) \rightarrow M$ and from the representation $\gamma \mapsto R_\gamma$ we get the unital normal $*$ -homomorphism $\lambda_R : L^\infty(G) \rightarrow M'$. Let \hat{A} be the reduced C^* -algebra of (M, Δ) . We can define an action of \hat{G}^2 on \hat{A} by

$$\alpha_{\gamma_1, \gamma_2}(x) = L_{\gamma_1} R_{\gamma_2} x R_{\gamma_2}^* L_{\gamma_1}^*.$$

Let us consider the crossed product C^* -algebra $B = \hat{G}^2 \ltimes_\alpha \hat{A}$. We will denote by λ the canonical morphism from \hat{G}^2 to the unitary group of $M(B)$ continuous in the strict topology on $M(B)$, $\pi \in \text{Mor}(\hat{A}, B)$ the canonical morphism and θ the dual action of G^2 on B . Recall that the triplet $(\hat{G}^2, \lambda, \theta)$ is a \hat{G}^2 -product. Let us denote by $(\hat{G}^2, \lambda, \theta^\Psi)$ the \hat{G}^2 -product obtained by deformation of the \hat{G}^2 -product $(\hat{G}^2, \lambda, \theta)$ by the bicharacter $\omega(g, h, s, t) := \overline{\Psi(g, s)}\Psi(h, t)$ on G^2 .

The dual deformed action θ^Ψ is done by

$$\theta_{(g_1, g_2)}^\Psi(x) = U_{g_1} V_{g_2} \theta_{(g_1, g_2)}(x) U_{g_1}^* V_{g_2}^*, \quad \text{for any } g_1, g_2 \in G, x \in B,$$

where $U_g = \lambda_L(\Psi_g^*)$, $V_g = \lambda_R(\Psi_g)$, $\Psi_g(h) = \Psi(h, g)$.

Considering Ψ_g as an element of \hat{G} , we get a morphism from G to \hat{G} , also noted Ψ , such that $\Psi(g) = \Psi_g$. With these notations, one has $U_g = u_{(\Psi(-g), 0)}$ and $V_g = u_{(0, \Psi(g))}$. Then the action θ^Ψ on $\pi(\hat{A})$ is done by

$$\theta_{(g_1, g_2)}^\Psi(\pi(x)) = \pi(\alpha_{(\Psi(-g_1), \Psi(g_2))}(x)). \quad (3)$$

Let us consider the Landstad algebra A^Ψ associated with this \hat{G}^2 -product. By definition of α and the universality of the crossed product we get a morphism

$$\rho \in \text{Mor}(B, \mathcal{K}(H)), \quad \rho(\lambda_{\gamma_1, \gamma_2}) = L_{\gamma_1} R_{\gamma_2} \quad \text{et} \quad \rho(\pi(x)) = x. \quad (4)$$

It is shown in [4] that $\rho(A^\Psi) = \hat{A}_\Omega$ and that ρ is injective on A^Ψ . This gives a canonical isomorphism $A^\Psi \simeq \hat{A}_\Omega$. In the sequel we identify A^Ψ with \hat{A}_Ω . The comultiplication can be described in the following way. First, one can show that, using universality of the crossed product, there exists a unique morphism $\Gamma \in \text{Mor}(B, B \otimes B)$ such that:

$$\Gamma \circ \pi = (\pi \otimes \pi) \circ \hat{\Delta} \quad \text{and} \quad \Gamma(\lambda_{\gamma_1, \gamma_2}) = \lambda_{\gamma_1, 0} \otimes \lambda_{0, \gamma_2}.$$

Then we introduce the unitary $\Upsilon = (\lambda_R \otimes \lambda_L)(\tilde{\Psi}) \in \text{M}(B \otimes B)$, where $\tilde{\Psi}(g, h) = \Psi(g, gh)$. This allows us to define the *-morphism $\Gamma_\Omega(x) = \Upsilon \Gamma(x) \Upsilon^*$ from B to $\text{M}(B \otimes B)$. One can show that $\Gamma_\Omega \in \text{Mor}(A^\Psi, A^\Psi \otimes A^\Psi)$ is the comultiplication on A^Ψ .

Note that if $M = \mathcal{L}(G)$ and K is an abelian closed subgroup of G , the action α of K^2 on $C_0(G)$ is the left-right action.

3 Twisting of the group of 2×2 upper triangular matrices with determinant 1

Consider the following subgroup of $SL_2(\mathbb{C})$:

$$G := \left\{ \begin{pmatrix} z & \omega \\ 0 & z^{-1} \end{pmatrix}, z \in \mathbb{C}^*, \omega \in \mathbb{C} \right\}.$$

Let $K \subset G$ be the subgroup of diagonal matrices in G , i.e. $K = \mathbb{C}^*$. The elements of G will be denoted by (z, ω) , $z \in \mathbb{C}^*$, $\omega \in \mathbb{C}$. The modular function of G is

$$\delta_G((z, \omega)) = |z|^{-2}.$$

Thus, the morphism $(t \mapsto \gamma_t)$ from \mathbb{R} to $\widehat{\mathbb{C}^*}$ is given by

$$\langle \gamma_t, z \rangle = |z|^{2it}, \quad \text{for all } z \in \mathbb{C}^*, t \in \mathbb{R}.$$

We can identify $\widehat{\mathbb{C}^*}$ with $\mathbb{Z} \times \mathbb{R}_+^*$ in the following way:

$$\mathbb{Z} \times \mathbb{R}_+^* \rightarrow \widehat{\mathbb{C}^*}, \quad (n, \rho) \mapsto \gamma_{n, \rho} = (re^{i\theta} \mapsto e^{i \ln r \ln \rho} e^{in\theta}).$$

Under this identification, γ_t is the element $(0, e^t)$ of $\mathbb{Z} \times \mathbb{R}_+^*$. For all $x \in \mathbb{R}$, we define a bicharacter on $\mathbb{Z} \times \mathbb{R}_+^*$ by

$$\Psi_x((n, \rho), (k, r)) = e^{ix(k \ln \rho - n \ln r)}.$$

We denote by (M_x, Δ_x) the twisted l.c. quantum group. We have:

$$\Psi_x((n, \rho), \gamma_t^{-1}) = e^{ixtn} = u_{e^{ixt}}((n, \rho)).$$

In this way we obtain the operator A_x deforming the Plancherel weight:

$$A_x^{it} = \alpha(u_{e^{ixt}}) = \lambda_{(e^{ixt}, 0)}^G.$$

In the same way we compute the operator B_x deforming the Plancherel weight:

$$B_x^{it} = \lambda_{(e^{-ixt}, 0)}^G = A_x^{-it}.$$

Thus, we obtain for the modular element :

$$\delta_x^{it} = A_x^{-it} B_x^{it} = \lambda_{(e^{-2itx}, 0)}^G.$$

The antipode is not deformed. The scaling group is trivial but, if $x \neq 0$, (M_x, Δ_x) is not a Kac algebra because δ_x is not affiliated with the center of M . Let us look if (M_x, Δ_x) can be isomorphic for different values of x . One can remark that, since $\Psi_{-x} = \Psi_x^*$ is antisymmetric and Δ is cocommutative, we have $\Delta_{-x} = \sigma \Delta_x$, where σ is the flip on $\mathcal{L}(G) \otimes \mathcal{L}(G)$. Thus, $(M_{-x}, \Delta_{-x}) \simeq (M_x, \Delta_x)^{\text{op}}$, where "op" means the opposite quantum group. So, it suffices to treat only strictly positive values of x . The twisting deforms only the comultiplication, the weights and the modular element. The simplest invariant distinguishing the (M_x, Δ_x) is then the specter of the modular element. Using the Fourier transformation in the first variable, one has immediately $\text{Sp}(\delta_x) = q_x^{\mathbb{Z}} \cup \{0\}$, where $q_x = e^{-2x}$. Thus, if $x \neq y$, $x > 0, y > 0$, one has $q_x^{\mathbb{Z}} \neq q_y^{\mathbb{Z}}$ and, consequently, (M_x, Δ_x) and (M_y, Δ_y) are non isomorphic.

We compute now the dual C^* -algebra. The action of K^2 on $C_0(G)$ can be lifted to its Lie algebra \mathbb{C}^2 . The lifting does not change the result of the deformation (see [5], Proposition 3.17) but simplify calculations. The action of \mathbb{C}^2 on $C_0(G)$ will be denoted by ρ . One has

$$\rho_{z_1, z_2}(f)(z, \omega) = f(e^{z_2 - z_1} z, e^{-(z_1 + z_2)} \omega). \quad (5)$$

The group \mathbb{C} is self-dual, the duality is given by

$$(z_1, z_2) \mapsto \exp(i\text{Im}(z_1 z_2)).$$

The generators u_z , $z \in \mathbb{C}$, of $C_0(\mathbb{C})$ are given by

$$u_z(w) = \exp(i\text{Im}(zw)), \quad z, w \in \mathbb{C}.$$

Let $x \in \mathbb{R}$. We will consider the following bicharacter on \mathbb{C} :

$$\Psi_x(z_1, z_2) = \exp(ix\text{Im}(z_1 \bar{z}_2)).$$

Let B be the crossed product C^* -algebra $\mathbb{C}^2 \rtimes C_0(G)$. We denote by $((z_1, z_2) \mapsto \lambda_{z_1, z_2})$ the canonical group homomorphism from G to the unitary group of $M(B)$, continuous for the strict topology, and $\pi \in \text{Mor}(C_0(G), B)$ the canonical homomorphism. Also we denote by $\lambda \in \text{Mor}(C_0(G^2), B)$ the morphism given by the representation $((z_1, z_2) \mapsto \lambda_{z_1, z_2})$. Let θ be the dual action of \mathbb{C}^2 on B . We have, for all $z, w \in \mathbb{C}$, $\Psi_x(w, z) = u_{x\bar{z}}(w)$. The deformed dual action is given by

$$\theta_{z_1, z_2}^{\Psi_x}(b) = \lambda_{-x\bar{z}_1, x\bar{z}_2} \theta_{z_1, z_2}(b) \lambda_{-x\bar{z}_1, x\bar{z}_2}^*. \quad (6)$$

Recall that

$$\theta_{z_1, z_2}^{\Psi_x}(\lambda(f)) = \theta_{z_1, z_2}(\lambda(f)) = \lambda(f(\cdot - z_1, \cdot - z_2)), \quad \forall f \in C_b(\mathbb{C}^2). \quad (7)$$

Let \hat{A}_x be the associated Landstad algebra. We identify \hat{A}_x with the reduced C^* -algebra of (M_x, Δ_x) . We will now construct two normal operators affiliated with \hat{A}_x , which generate \hat{A}_x . Let a and b be the coordinate functions on G , and $\alpha = \pi(a)$, $\beta = \pi(b)$. Then α and β are normal operators, affiliated with B , and one can see, using (5), that

$$\lambda_{z_1, z_2} \alpha \lambda_{z_1, z_2}^* = e^{z_2 - z_1} \alpha, \quad \lambda_{z_1, z_2} \beta \lambda_{z_1, z_2}^* = e^{-(z_1 + z_2)} \beta. \quad (8)$$

We can deduce, using (6), that

$$\theta_{z_1, z_2}^{\Psi_x}(\alpha) = e^{x(\bar{z}_1 + \bar{z}_2)} \alpha, \quad \theta_{z_1, z_2}^{\Psi_x}(\beta) = e^{x(\bar{z}_1 - \bar{z}_2)} \beta. \quad (9)$$

Let T_l and T_r be the infinitesimal generators of the left and right shift respectively, i.e. T_l and T_r are normal, affiliated with B , and

$$\lambda_{z_1, z_2} = \exp(i\text{Im}(z_1 T_l)) \exp(i\text{Im}(z_2 T_r)), \quad \text{for all } z_1, z_2 \in \mathbb{C}.$$

Thus, we have:

$$\lambda(f) = f(T_l, T_r), \quad \text{for all } f \in C_b(\mathbb{C}^2).$$

Let $U = \lambda(\Psi_x)$, we define the following normal operators affiliated with B :

$$\hat{\alpha} = U^* \alpha U, \quad \hat{\beta} = U \beta U^*.$$

Proposition 1 *The operators $\hat{\alpha}$ and $\hat{\beta}$ are affiliated with \hat{A}_x and generate \hat{A}_x .*

Proof. First let us show that $f(\hat{\alpha}), f(\hat{\beta}) \in M(\hat{A}_t)$, for all $f \in C_0(\mathbb{C})$. One has, using (7):

$$\begin{aligned} \theta_{z_1, z_2}^{\Psi_x}(U) &= \lambda(\Psi_x(\cdot - z_1, \cdot - z_2)) \\ &= U e^{ix\text{Im}(-\bar{z}_2 T_l)} e^{ix\text{Im}(\bar{z}_1 T_r)} \Psi_x(z_1, z_2) \\ &= U \lambda_{-x\bar{z}_2, x\bar{z}_1} \Psi_x(z_1, z_2). \end{aligned}$$

Now, using (9) and (8), we obtain:

$$\theta_{z_1, z_2}^{\Psi_x}(\hat{\alpha}) = \hat{\alpha}, \quad \theta_{z_1, z_2}^{\Psi_x}(\hat{\beta}) = \hat{\beta}, \quad \text{for all } z_1, z_2 \in \mathbb{C}.$$

Thus, for all $f \in C_0(\mathbb{C})$, $f(\hat{\alpha})$ and $f(\hat{\beta})$ are fixed points for the action θ^{Ψ_x} . Let $f \in C_0(\mathbb{C})$. Using (8) we find:

$$\begin{aligned} \lambda_{z_1, z_2} f(\hat{\alpha}) \lambda_{z_1, z_2}^* &= U^* f(e^{z_2 - z_1} \alpha) U, \\ \lambda_{z_1, z_2} f(\hat{\beta}) \lambda_{z_1, z_2}^* &= U^* f(e^{-(z_1 + z_2)} \beta) U. \end{aligned} \quad (10)$$

Because f is continuous and vanish at infinity, the applications

$$(z_1, z_2) \mapsto \lambda_{z_1, z_2} f(\hat{\alpha}) \lambda_{z_1, z_2}^* \quad \text{and} \quad (z_1, z_2) \mapsto \lambda_{z_1, z_2} f(\hat{\beta}) \lambda_{z_1, z_2}^*$$

are norm-continuous and $f(\hat{\alpha}), f(\hat{\beta}) \in M(\hat{A}_x)$, for all $f \in C_0(\mathbb{C})$.

Taking in mind Proposition 4 (see Appendix), in order to show that $\hat{\alpha}$ is affiliated with \hat{A}_x , it suffices to show that the vector space \mathcal{I} generated by $f(\hat{\alpha})a$, with $f \in C_0(\mathbb{C})$ and $a \in \hat{A}_x$, is dense in \hat{A}_x . Using (10), we see that \mathcal{I} is globally invariant under the action implemented by λ . Let $g(z) = (1 + \bar{z}z)^{-1}$. As $\lambda(C_0(\mathbb{C}^2))U = \lambda(C_0(\mathbb{C}^2))$, we can deduce that the closure of $\lambda(C_0(\mathbb{C}^2))g(\hat{\alpha})\hat{A}_x\lambda(C_0(\mathbb{C}^2))$ is equal to

$$\left[\lambda(C_0(\mathbb{C}^2))(1 + \alpha^*\alpha)^{-1}U^*\hat{A}_x\lambda(C_0(\mathbb{C}^2)) \right].$$

As the set $U^*\hat{A}_x\lambda(C_0(\mathbb{C}^2))$ is dense in B and α is affiliated with B , the set $\lambda(C_0(\mathbb{C}^2))(1 + \alpha^*\alpha)^{-1}U^*\hat{A}_x\lambda(C_0(\mathbb{C}^2))$ is dense in B . Moreover, it is included in $\lambda(C_0(\mathbb{C}^2))\mathcal{I}\lambda(C_0(\mathbb{C}^2))$, so $\lambda(C_0(\mathbb{C}^2))\mathcal{I}\lambda(C_0(\mathbb{C}^2))$ is dense in B . We conclude, using Lemma 1, that \mathcal{I} is dense in \hat{A}_x . One can show in the same way that $\hat{\beta}$ is affiliated with \hat{A}_x .

Now, let us show that $\hat{\alpha}$ and $\hat{\beta}$ generate \hat{A}_x . By Proposition 5, it suffices to show that

$$\mathcal{V} = \left\langle f(\hat{\alpha})g(\hat{\beta}), f, g \in C_0(\mathbb{C}) \right\rangle$$

is a dense vector subspace of \hat{A}_x . We have shown above that the elements of \mathcal{V} satisfy the two first Landstad's conditions. Let

$$\mathcal{W} = \left[\lambda(C_0(\mathbb{C}^2))\mathcal{V}\lambda(C_0(\mathbb{C}^2)) \right].$$

We will show that $\mathcal{W} = B$. This proves that the elements of \mathcal{V} satisfy the third Landstad's condition, and then $\mathcal{V} \subset \hat{A}_x$. Then (10) shows that \mathcal{V} is globally invariant under the action implemented by λ , so \mathcal{V} is dense in \hat{A}_x by Lemma 1. One has:

$$\mathcal{W} = \left[xU^*f(\alpha)U^2g(\beta)U^*y, f, g \in C_0(\mathbb{C}), x, y \in \lambda(C_0(\mathbb{C}^2)) \right].$$

Because U is unitary, we can substitute x with xU and y with Uy without changing \mathcal{W} :

$$\mathcal{W} = \left[xf(\alpha)U^2g(\beta)y, f, g \in C_0(\mathbb{C}), x, y \in \lambda(C_0(\mathbb{C}^2)) \right].$$

Using, for all $f \in C_0(\mathbb{C})$, the norm-continuity of the application

$$(z_1, z_2) \mapsto \lambda_{z_1, z_2}f(\alpha)\lambda_{z_1, z_2}^* = e^{z_2 - z_1}\alpha,$$

one deduces that

$$\begin{aligned} & \left[f(\alpha)x, f \in C_0(\mathbb{C}), x \in \lambda(C_0(\mathbb{C}^2)) \right] \\ &= \left[xf(\alpha), f \in C_0(\mathbb{C}), x \in \lambda(C_0(\mathbb{C}^2)) \right]. \end{aligned}$$

In particular,

$$\mathcal{W} = \left[f(\alpha)xU^2g(\beta)y, f, g \in C_0(\mathbb{C}), x, y \in \lambda(C_0(\mathbb{C}^2)) \right].$$

Now we can commute $g(\beta)$ and y , and we obtain:

$$\mathcal{W} = [f(\alpha)xU^2yg(\beta), f, g \in C_0(\mathbb{C}), x, y \in \lambda(C_0(\mathbb{C}^2))].$$

Substituting $x \mapsto xU^*$, $y \mapsto U^*y$, one has:

$$\mathcal{W} = [f(\alpha)xyg(\beta), f, g \in C_0(\mathbb{C}), x, y \in \lambda(C_0(\mathbb{C}^2))].$$

Commuting back $f(\alpha)$ with x and $g(\beta)$ with y , we obtain:

$$\mathcal{W} = [xf(\alpha)g(\beta)y, f, g \in C_0(\mathbb{C}), x, y \in \lambda(C_0(\mathbb{C}^2))] = B.$$

This concludes the proof. ■

We will now find the commutation relations between $\hat{\alpha}$ and $\hat{\beta}$.

Proposition 2 *One has:*

1. α et $T_l^* + T_r^*$ strongly commute and $\hat{\alpha} = e^{x(T_l^* + T_r^*)}\alpha$.
2. β et $T_l^* - T_r^*$ strongly commute and $\hat{\beta} = e^{x(T_l^* - T_r^*)}\beta$.

Thus, the polar decompositions are given by :

$$\begin{aligned} Ph(\hat{\alpha}) &= e^{-ixIm(T_l + T_r)} Ph(\alpha), & |\hat{\alpha}| &= e^{xRe(T_l + T_r)} |\alpha|, \\ Ph(\hat{\beta}) &= e^{-ixIm(T_l - T_r)} Ph(\beta), & |\hat{\beta}| &= e^{xRe(T_l - T_r)} |\beta|. \end{aligned}$$

Moreover, we have the following relations:

1. $|\hat{\alpha}|$ and $|\hat{\beta}|$ strongly commute,
2. $Ph(\hat{\alpha})Ph(\hat{\beta}) = Ph(\hat{\beta})Ph(\hat{\alpha})$,
3. $Ph(\hat{\alpha})|\hat{\beta}|Ph(\hat{\alpha})^* = e^{4x}|\hat{\beta}|$,
4. $Ph(\hat{\beta})|\hat{\alpha}|Ph(\hat{\beta})^* = e^{4x}|\hat{\alpha}|$.

Proof. Using (8), we find, for all $z \in \mathbb{C}$:

$$e^{iIm(z(T_l^* + T_r^*))}\alpha e^{-iIm(z(T_l^* + T_r^*))} = \lambda_{-\bar{z}, -\bar{z}}\alpha\lambda_{-\bar{z}, -\bar{z}}^* = e^{-\bar{z} + \bar{z}}\alpha = \alpha.$$

Thus, $T_l^* + T_r^*$ and α strongly commute. Moreover, because $e^{ixImT_lT_l^*} = 1$, one has:

$$\hat{\alpha} = e^{-ixImT_lT_r^*}\alpha e^{ixImT_lT_r^*} = e^{-ixImT_l(T_l + T_r)^*}\alpha e^{ixImT_l(T_l + T_r)^*}.$$

We can now prove the point 1 using the equality $e^{-ixImT_l\omega}\alpha e^{ixImT_l\omega} = e^{x\omega}\alpha$, the preceding equation and the fact that $T_l^* + T_r^*$ and α strongly commute. The proof of the second assertion is similar and the polar decompositions follows. From (8) we deduce :

$$\begin{aligned}
e^{-ix\text{Im}(T_r-T_l)}\alpha e^{ix\text{Im}(T_r-T_l)} &= e^{-2x}\alpha, \\
e^{ix\text{Im}(T_l+T_r)}\beta e^{-ix\text{Im}(T_l+T_r)} &= e^{-2x}\beta, \\
e^{ix\text{Re}(T_r-T_l)}\alpha e^{-ix\text{Re}(T_r-T_l)} &= e^{2ix}\alpha, \\
e^{ix\text{Re}(T_l+T_r)}\beta e^{-ix\text{Re}(T_l+T_r)} &= e^{-2ix}\beta.
\end{aligned}$$

It is now easy to prove the last relations from the preceding equations and the polar decompositions. \blacksquare

We can now give a formula for the comultiplication.

Proposition 3 *Let $\hat{\Delta}_x$ be the comultiplication on \hat{A}_x . One has:*

$$\hat{\Delta}_x(\hat{\alpha}) = \hat{\alpha} \otimes \hat{\alpha}, \quad \hat{\Delta}_x(\hat{\beta}) = \hat{\alpha} \otimes \hat{\beta} + \hat{\beta} \otimes \hat{\alpha}^{-1}.$$

Proof. Using the Preliminaries, we have that $\hat{\Delta}_x = \Upsilon\Gamma(\cdot)\Upsilon^*$, where

$$\Upsilon = e^{ix\text{Im}T_r \otimes T_l^*}$$

and Γ is given by

- $\Gamma(T_l) = T_l \otimes 1, \Gamma(T_r) = 1 \otimes T_r;$
- Γ restricted to $C_0(G)$ is equal to the comultiplication Δ_G .

Define $R = \Upsilon\Gamma(U^*)$. One has $\Delta_x(\hat{\alpha}) = R(\alpha \otimes \alpha)R^*$. Thus, it is sufficient to show that $(U \otimes U)R$ commute with $\alpha \otimes \alpha$. Indeed, in this case, one has

$$\hat{\Delta}_x(\hat{\alpha}) = R(\alpha \otimes \alpha)R^* = (U^* \otimes U^*)(U \otimes U)R(\alpha \otimes \alpha)R^*(U^* \otimes U^*)(U \otimes U) = \hat{\alpha} \otimes \hat{\alpha}.$$

Let us show that $(U \otimes U)R$ commute with $\alpha \otimes \alpha$. From the equality $U = e^{ix\text{Im}T_l T_r^*}$, we deduce that

$$\Gamma(U^*) = e^{-ix\text{Im}T_l \otimes T_r^*}, \quad U \otimes U = e^{ix\text{Im}(T_l T_r^* \otimes 1 + 1 \otimes T_l T_r^*)}.$$

Thus, $R = e^{-ix\text{Im}(T_r^* \otimes T_l + T_l \otimes T_r^*)}$ and

$$(U \otimes U)R = e^{ix\text{Im}(T_l T_r^* \otimes 1 + 1 \otimes T_l T_r^* - T_r^* \otimes T_l - T_l \otimes T_r^*)}.$$

Notice that

$$T_l T_r^* \otimes 1 + 1 \otimes T_l T_r^* - T_r^* \otimes T_l - T_l \otimes T_r^* = (T_l \otimes 1 - 1 \otimes T_l)(T_r^* \otimes 1 - 1 \otimes T_r^*).$$

Thus, it suffices to show that $T_l \otimes 1 - 1 \otimes T_l$ and $T_r^* \otimes 1 - 1 \otimes T_r^*$ strongly commute with $\alpha \otimes \alpha$. This follows from the equations

$$\begin{aligned}
&e^{i\text{Im}z(T_r^* \otimes 1 - 1 \otimes T_r^*)}(\alpha \otimes \alpha)e^{-i\text{Im}z(T_r^* \otimes 1 - 1 \otimes T_r^*)} \\
&= (\lambda_{0, -\bar{z}} \otimes \lambda_{0, \bar{z}})(\alpha \otimes \alpha)(\lambda_{0, -\bar{z}} \otimes \lambda_{0, \bar{z}})^* \\
&= e^{-\bar{z}} e^{\bar{z}} \alpha \otimes \alpha = \alpha \otimes \alpha, \quad \forall z \in \mathbb{C}
\end{aligned}$$

and

$$\begin{aligned}
& e^{i\text{Im}z(T_1 \otimes 1 - 1 \otimes T_1)}(\alpha \otimes \alpha)e^{-i\text{Im}z(T_1 \otimes 1 - 1 \otimes T_1)} \\
&= (\lambda_{z,0} \otimes \lambda_{-z,0})(\alpha \otimes \alpha)(\lambda_{z,0} \otimes \lambda_{-z,0})^* \\
&= e^{-z}e^z\alpha \otimes \alpha = \alpha \otimes \alpha, \quad \forall z \in \mathbb{C}.
\end{aligned}$$

Put $S = \Upsilon\Gamma(U)$. One has:

$$\hat{\Delta}_x(\hat{\beta}) = S(\alpha \otimes \beta + \beta \otimes \alpha^{-1})S^* = S(\alpha \otimes \beta)S^* + S(\beta \otimes \alpha^{-1})S^*.$$

As before, we see that it suffices to show that $(U \otimes U^*)S$ commutes with $\alpha \otimes \beta$ and that $(U^* \otimes U)S$ commutes with $\beta \otimes \alpha^{-1}$, and one can check this in the same way. \blacksquare

Let us summarize the preceding results in the following corollary (see [16, 5] for the definition of commutation relation between unbounded operators):

Corollary 1 *Let $q = e^{8x}$. The C^* -algebra \hat{A}_x is generated by 2 normal operators $\hat{\alpha}$ and $\hat{\beta}$ affiliated with \hat{A}_x such that*

$$\hat{\alpha}\hat{\beta} = \hat{\beta}\hat{\alpha} \quad \hat{\alpha}\hat{\beta}^* = q\hat{\beta}^*\hat{\alpha}.$$

Moreover, the comultiplication $\hat{\Delta}_x$ is given by

$$\hat{\Delta}_x(\hat{\alpha}) = \hat{\alpha} \otimes \hat{\alpha}, \quad \hat{\Delta}_x(\hat{\beta}) = \hat{\alpha} \otimes \hat{\beta} + \hat{\beta} \otimes \hat{\alpha}^{-1}.$$

Remark. One can show, using the results of [4], that the application $(q \mapsto W_q)$ which maps the parameter q to the multiplicative unitary of the twisted l.c. quantum group is continuous in the σ -weak topology.

4 Appendix

Let us cite some results on operators affiliated with a C^* -algebra.

Proposition 4 *Let $A \subset \mathcal{B}(H)$ be a non degenerated C^* -subalgebra and T a normal densely defined closed operator on H . Let \mathcal{I} be the vector space generated by $f(T)a$, where $f \in C_0(\mathbb{C})$ and $a \in A$. Then:*

$$(T\eta A) \Leftrightarrow \left(\begin{array}{l} f(T) \in M(A) \text{ for any } f \in C_0(\mathbb{C}) \\ \text{et } \mathcal{I} \text{ is dense in } A \end{array} \right).$$

Proof. If T is affiliated with A , then it is clear that $f(T) \in M(A)$ for any $f \in C_0(\mathbb{C})$, and that \mathcal{I} is dense in A (because \mathcal{I} contains $(1+T^*T)^{-\frac{1}{2}}A$). To show the converse, consider the $*$ -homomorphism $\pi_T : C_0(\mathbb{C}) \rightarrow M(A)$ given by $\pi_T(f) = f(T)$. By hypothesis, $\pi_T(C_0(\mathbb{C}))A$ is dense in A . So, $\pi_T \in \text{Mor}(C_0(\mathbb{C}), A)$ and $T = \pi_T(z \mapsto z)$ is then affiliated with A . \blacksquare

Proposition 5 *Let $A \subset \mathcal{B}(H)$ be a non degenerated C^* -subalgebra and T_1, T_2, \dots, T_N normal operators affiliated with A . Let us denote by \mathcal{V} the vector space generated by the products of the form $f_1(T_1)f_2(T_2) \dots f_N(T_N)$, with $f_i \in C_0(\mathbb{C})$. If \mathcal{V} is a dense vector subspace of A , then A is generated by T_1, T_2, \dots, T_N .*

Proof. This follows from Theorem 3.3 in [15]. ■

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