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Laboratoire EM2C, CNRS UPR 288, Ecole Centrale Paris, Grande Voie des Vignes, Chatenay-Malabry
Courriel: frederique.laurent@em2c.ecp.fr; marc.massot@em2c.ecp.fr

Abstract. *In this paper, we investigate the existence of mono-dimensional traveling wave of a coupled thermo-diffusive–kinetic model describing the propagation of a plane flame in a polydisperse spray. We first present the model which bridges the gap between the models used for realistic numerical simulations and the ones used for mathematical studies; it couples a thermo-diffusive model for the gaseous phase with a one-step finite rate chemistry with a kinetic equation describing the vaporization of a polydisperse spray, the vaporization rate of which depends on the local gas temperature and fuel mass fraction. Such a mixed hyperbolic-parabolic system of equations offers two difficulties as compared to previous works: first there is no spatial diffusion on the spray variables and second, if we look for a traveling wave solution, the resulting system of equation is not a system of ODE's but a system of ODE's coupled to a partial differential equation. In order to tackle these difficulties, we introduce some spatial diffusion, which can be interpreted as the gas turbulent agitation, and discretize the PDE in the size phase space using a multi-fluid model. We can then use the topological degree for elliptic systems in unbounded domains, and deduce the existence of the wave for the modified and discretized system. We finally pass to the limit in zero spatial mass diffusion, subsequently, in the droplet size discretization step, and conclude with the existence of a traveling wave for the initial system. It is a first step in the study of the qualitative properties of polydispersped spray flames.*

1 Introduction

The propagation of plane premixed gaseous flames has been the subject of a very wide literature in the past 50 years since the early works of Zeldovich and Frank-Kamenetskii [40], [41] who introduced the principles of high activation energy asymptotics. The problem has been tackled using a great variety of tools ranging from direct numerical simulations, asymptotics, to topological degree theory in order to prove the existence of traveling waves for reaction diffusion partial differential equations. Having physics, chemistry and mathematics as various breeding grounds, the theory of flame propagation has provided fundamental studies in order to capture the qualitative behavior of homogeneous flames and can be considered to have reached maturity.

Such a conclusion can not be drawn at all for two-phase flames. One can find articles published mainly in the physics literature for both laminar and turbulent flames [30], [29], [15], [17], [31], [26], [20]. The main purposes of these papers is either to develop new models of turbulent spray combustion or to characterize the spray flame essential features (flame velocity, structure...). It is then difficult to precisely extract from the complexity of the parameter space the relevant ones. On the mathematical side however, few studies have been devoted to the subject since the structure of the system of partial differential equations is essentially more difficult to tackle. A very recent study has been devoted to the subject [4] for monodisperse sprays with a simplified model for the vaporization thus leading to an interesting flame stability investigation. It was written at the same time as the present work and can be considered as complementary. In the present paper, we aim at bridging the gap between the models used in order to simulate realistic flames [1], [26], [22] and simplified vaporization models from which too much physics has been extracted, and at conducting a mathematical analysis of the existence of traveling waves for the obtained fluid-kinetic model. This model is obtained using the usual thermo-diffusive model with a one-step finite rate chemistry where the fuel mass fraction equation admits a source term coming from the mass exchange between the phases. Since it can have a strong effect on flame structures [26], [31], the polydispersion in size of the spray is considered in the model. The spray satisfies either an advection equation in the size phase space, the velocity of which depends nonlinearly on the local gas temperature and mass fraction and is related to the rate of vaporization, or a generalized Fokker-Planck equation which is associated to a local turbulent agitation of zero mean value.

Numerous studies have been devoted to the propagation of plane gaseous flames, as mono-dimensional traveling wave for the thermo-diffusive model. More specifically, the existence of such waves is proved, for example in [3] for a one-step chemistry, in [35] for a complex chemistry network of irreversible exothermic reactions and in [12] for a complex chemistry network of reversible exothermic reactions and for detailed transport. As far as the existence proofs are concerned, various tools have been used such as the Leray-Schauder degree in [3], [12] in order to prove the existence in bounded domain, the solution being then extended to the whole real line, or such as the topological degree directly defined for unbounded domains [35]. These tools are defined for elliptic operators, all variables experiencing diffusion. For condensed phase reaction fronts, a mass diffusion can then be first added, in order to still use these tools, the mass diffusion coefficient then approaching zero, as it was proposed in [24] for a one-step chemistry and in [19] for a complex chemistry network of irreversible exothermic reactions. We here make use of the technique developed in [19]: the existence proof is conducted for an elliptic system of equation where mass diffusion

can be related to a gaseous turbulent agitation. The topological degree defined in [36] for elliptic operators in unbounded domains can be used and we pass to the limit when the mass diffusion coefficient tends to zero which is equivalent to let the turbulent kinetic energy decrease to zero.

The key point of the paper is the ability to treat the coupled fluid-kinetic system. The model for the liquid phase is a kinetic model: the spray is described by its distribution function, satisfying the Williams equation, which is either of Boltzmann type [37], [38] or of Fokker-Planck type if we consider an averaged distribution function with an underlying gaseous turbulent agitation [10]. The corresponding phase space is thus two dimensional and the second order ODE deduced from the thermo-diffusive model in order to describe a traveling wave is coupled to a PDE, which makes the problem essentially more difficult.

In order to tackle this difficulty, we use a Eulerian discretization of the distribution function in the size phase space: the multi-fluid model derived from the kinetic model in [22]. For problems without dynamic effect (the droplet are assumed to be motionless or thermalized by the gas velocity fluctuations around a zero mean velocity) this model can be considered to be an upwind finite volume discretization in the droplet size of the distribution function. The corresponding numerical analysis is performed in [21], [18], showing that the method is first order under some compatibility conditions. This numerical analysis allows us to get back to the kinetic model, passing to the limit when the number of sections tends to infinity, as long as we are able to prove the existence of traveling wave for the finite dimensional spray elliptic system coupled to the elliptic thermo-diffusive model for the gas. We finally conclude with the existence of plane flames propagating in two-phase medium, for a Lewis number Le, which is the rate between the heat diffusion coefficient and the mass diffusion coefficient of the gaseous fuel, greater than 1.

The paper is organized as follow: the fluid-kinetic model and the discretization by the multi-fluid method are presented in Section 2. Some fundamentals on the topological degree are recalled in Section 3. It shows that, in order to use this tool, we have to add a diffusion term and some other terms controlled by a small added parameter. For a Lewis number such that $\mathrm{Le} \geqslant 1 + p$, where $p \geqslant 0$ only depends of the fuel vapor saturated mass fraction function and of the temperature of the fresh gas, some estimates on this system are presented in Section 4, which will be useful in order to prove existence of solutions in Section 5 and to pass to the limit when the small parameter and the diffusion coefficient tend to zero in Section 6. Finally, the numerical analysis conducted in [21] allows us to conclude on the existence of a traveling wave for the original system.

2 Polydispersed spray flame model

In this section, we introduce a reaction-diffusion-source model for the gaseous phase coupled to a kinetic description of the polydisperse spray. We first consider a global model and then concentrate on traveling waves.

2.1 Polydispersed spray flame

We investigate a polydispersed spray flame at constant pressure and density. The spray of fuel droplets is evaporating in the gaseous phase, with no dynamical effect: the gas is at rest and droplets are motionless. In order to describe such a problem, we use a simplified model which takes into account the main physical characteristics of the configuration: a thermo-diffusive model, with a one step chemistry, coupled with a

kinetic model of the liquid phase, describing the polydispersion of the spray. More precisely, in the gaseous phase, only one exothermic reaction $F \to P$ takes place between the fuel F and a product P, in the presence of an inert gas I, with constant mass fraction. The spray is supposed to be dilute (the liquid volume fraction is much smaller than one) in such a way that the coupling of the spray with the gaseous phase only occurs through added source terms in conservation equations.

The mass conservation implies that $\rho = \rho_{gas} + \rho_{liq}$ is a constant, where ρ_{gas} is the gas mass density and ρ_{liq} is the liquid mass density per unit of total volume. We note $Y_{gas} = \rho_{gas}/\rho$ and $Y_{liq} = \rho_{liq}/\rho$.

The gas is multi-component and $Y_{gas} = Y_F + Y_P + Y_I$ where Y_F/Y_{gas} (respectively Y_P/Y_{gas} or Y_I/Y_{gas}) is the mass fraction of fuel (respectively of product P or of inert gas I) in the gaseous phase. We then have $Y_{liq} + Y_F + Y_P = 1 - \omega$ if we set $Y_I = \omega$.

The thermo-diffusive model for this reaction-diffusion problem can then be written:

$$\frac{\partial T}{\partial t} - \kappa\, \Delta T = q\, k(T)\, Y_F\,, \tag{1}$$

$$\frac{\partial Y_F}{\partial t} - D\, \Delta Y_F = -k(T)\, Y_F + S^m, \tag{2}$$

where t is the time, x the space variable, $x \in \mathbb{R}^d$, T the gas temperature, κ the thermal diffusivity, D the diffusivity of the fuel. The function k is close to the Arrhenius coefficient: in order to avoid the cold boundary problem [40] [13], we introduce an ignition temperature T_c. So $k(T)$ is a regular function of T; it is zero for $T \in]-\infty, T_c]$, increasing for $T \in [T_c, T_c + \eta]$ and equal to the Arrhenius coefficient for $T \in [T_c + \eta, +\infty[$, with η a small positive real number. We assume that the vaporization has a negligible influence on the gas temperature, so that a source term S^m only appears in the fuel conservation equation. The expression of the source term S^m relies on a kinetic description of the spray.

Remark 1 Here, we use the equation on Y_F and not the equation on Y_P:

$$\frac{\partial}{\partial t} Y_P - D\, \Delta Y_P = k(T)\, Y_F + D\, \Delta Y_{liq}$$

because the variable Y_F is better adapted to our problem, even if it is not necessarily monotone along trajectories.

The dispersed phase is described by its distribution function $f(t, x, S)$ where S is the surface of the droplets. Because the droplets are motionless, the Williams' equation describing the evolution of f [37] reduces to the transport equation:

$$\frac{\partial}{\partial t} f + R_S\, \frac{\partial}{\partial S} f = 0, \tag{3}$$

where $R_S(T, Y_F)$ is the rate of change of droplet surface, due to evaporation. We assume that the d^2 law holds so that R_S is independent of the size of the droplets.

A model for the vaporization rate is given in [1]. In our case of zero Reynolds number, and if we preclude condensation, it reads:

$$R_S(T, Y_F) = \varphi\, \ln(1 + B_M(T, Y_F))^+, \qquad B_M(T, Y_F) = \frac{Y_{Fs}(T) - Y_F}{1 - Y_{Fs}(T)}, \tag{4}$$

where φ is a positive constant and, for a real u, u^+ is equal to u if u is positive and to zero otherwise. The function Y_{Fs} represents the fuel vapor saturated mass fraction. It is supposed to be equal to the fuel vapor satured molar fraction x_{Fs} and then only depends on the temperature, the pressure being constant. It is coherent with the system (1), (2), where the various molar masses of the species have been taken equal and the details of the chemical mechanism and composition have been neglected. The function $x_{Fs}(T)$ is drawn, for heptane, in Fig. 1.

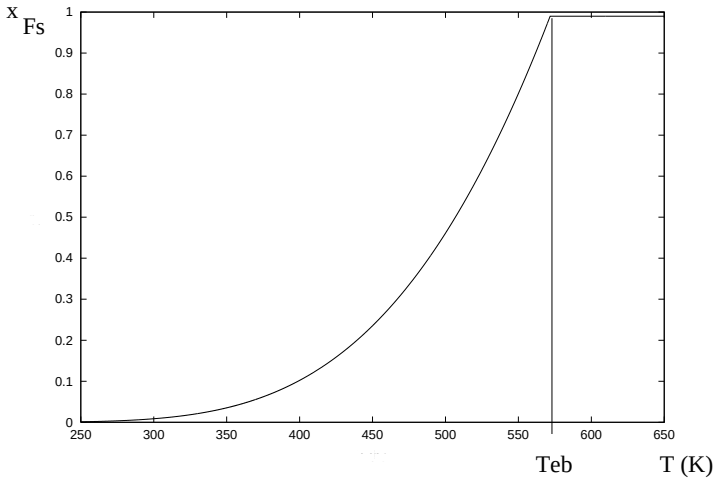


Figure 1: Fuel vapor satured molar fraction for heptane, as a function of the temperature, at the atmospheric pressure.

The function $Y_{Fs}(T)$ is then assumed regular (at least C^2) and such that:

$$\begin{array}{lll} \text{if} \quad 0 \leqslant T < T_{eb} - \nu & \text{then} \quad Y'_{Fs}(T) > 0, \quad Y''_{Fs}(T) > 0, \\ \text{if} \quad T_{eb} - \nu \leqslant T < T_{eb} & \text{then} \quad Y'_{Fs}(T) > 0, \quad Y''_{Fs}(T) \leqslant 0, \\ \text{if} \quad T \geqslant T_{eb} & \text{then} \quad Y_{Fs}(T) = 1 - \delta, \end{array}$$

where T_{eb} is the boiling temperature, ν is a small positive real number and δ is a number such that $0 < \delta \leqslant \omega < 1$. Those assumptions correspond, for example, to the shape of function x_{Fs} for heptane which is drawn in Fig. 1 and used for practical applications [20], [26].

Remark 2 The vaporization rate is bounded because the upper bound of $Y_{Fs}(T)$ is assumed to be strictly smaller than one. It is coherent with physics of droplet vaporization: the vaporization can not be instantaneous for large temperatures because of the presence of a diffusion layer.

The source term of fuel for the gaseous phase is then given by:

$$S^m(t,x,T,Y_F) = \frac{\rho_l}{4\sqrt{\pi}}\, R_S(T,Y_F) \int_0^{+\infty} S^{1/2}\, f(t,x,S)\, dS. \tag{5}$$

System (1), (2) and equation (3) are then coupled together through both $S^m(t,x)$ which is a non local term and through $R_S(T,Y_F)$ which depends on the local gas state.

The complete system of partial differential equations is then:

$$(\mathcal{PDE}) \quad \begin{cases} \dfrac{\partial T}{\partial t} - \kappa \Delta T = q\, k(T)\, Y_F, \\ \dfrac{\partial Y_F}{\partial t} - D\, \Delta Y_F = -k(T)\, Y_F + S^m(t,x,T,Y_F), \\ \dfrac{\partial f}{\partial t} + R_S(T,Y_F)\, \dfrac{\partial f}{\partial S} = 0, \end{cases}$$

with the source term:

$$S^m(t,x,T,Y_F) = \frac{\rho_l}{4\sqrt{\pi}}\, R_S(T,Y_F) \int_0^{+\infty} S^{1/2}\, f(t,x,S)\, dS.$$

The purpose of the present paper is then to investigate the existence of traveling wave solutions for the system $(\mathcal{PDE})$. The originality of this study is to use mathematical tools such as the topological degree for elliptic operator in unbounded domains in order to study a system of mixed type hyperbolic-parabolic which, even if simplified, still preserves the details of the vaporization process and its coupling with the propagation of a flame.

2.2 Solutions as mono-dimensional traveling waves

We investigate mono-dimensional plane polydispersed spray flames as traveling waves for the system $(\mathcal{PDE})$. We then consider $T(t,x)$, $Y_F(t,x)$ and $f(t,x,.)$ as functions of $z = x - ct$, with given boundary conditions at infinity. More precisely, we assume that at $+\infty$, a cold gas at temperature T_0 and a polydisperse spray coexist. The gas is only composed of fuel and inert gas, with mass fractions: $Y_F^{+\infty} = Y_{Fs}(T_0)$ and $Y_I = \omega$, respectively. The distribution function of the spray is given by a function f_0, assumed regular (at least C^2 on $\mathbb{R}^+$) and of compact support: $f_0(S) = 0$ for $S \geqslant S_{max}$. It is consistent with distribution which can be experimentally obtained [26] [20]. An example of such a function is given on Fig. 2. We also note $Y_{liq}^{+\infty}$ the total liquid mass density at $+\infty$ and we have:

$$Y_{liq}^{+\infty} = \frac{\rho_l}{6\sqrt{\pi}} \int_0^{\infty} S^{3/2} f_0(S) dS, \qquad Y_F^{+\infty} + Y_{liq}^{+\infty} = 1 - \omega.$$

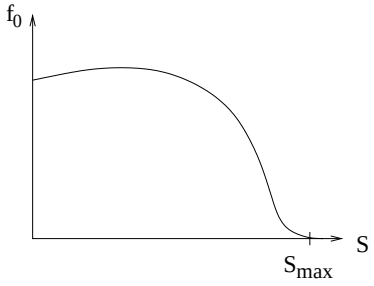
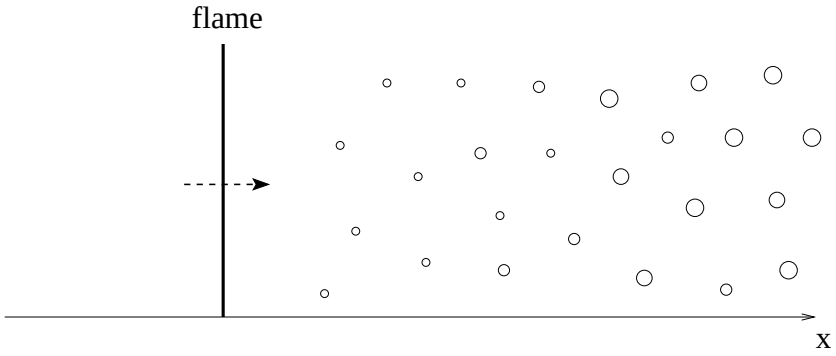


Figure 2: Distribution function of droplets at $+\infty$.

At $-\infty$, the liquid phase is completely vaporized and all the fuel is consumed. So, there is only a gaseous phase, at hot temperature $T_* > T_0$, composed of the product and the inert gas: $Y_F(-\infty) = 0$ and $Y_I = \omega$. A sketch of the configuration is given on Fig. 3.



at $-\infty$

- gaseous phase :
  $\rightarrow$ burned gas : $Y_F = 0$
  $\rightarrow$ inert gas : $Y_I = \delta$
  $\rightarrow$ temperature : T_*
- no liquid phase :
  $Y_{liq} = 0$

at $+\infty$

- gaseous phase :
  $\rightarrow$ fuel : $Y_F = Y_F^\infty$
  $\rightarrow$ inert gas : $Y_I = \delta$
  $\rightarrow$ temperature : T_0
- liquid phase : $Y_{liq} = Y_{liq}^\infty$
  fuel droplets
  $\rightarrow$ **polydisperse spray**

Figure 3: Sketch of the plane polydispersed spray flame.

We then define θ, α and ψ, the reduced temperature, mass fraction of fuel and distribution function such that:

$$T(t,x) = T_0 + (T_* - T_0)\,\theta(x - ct), \tag{6}$$

$$Y_F(t,x) = (1-\omega)\,\alpha(x - ct), \tag{7}$$

$$f(t,x,S) = (1-\omega)\,\psi(x - ct, S), \tag{8}$$

The distribution function at $+\infty$ in the new variables is then denoted ψ_0 (with $f_0 = (1-\omega)\psi_0$). Let us also introduce the limit of α at $+\infty$:

$$\alpha^\infty = \frac{Y_F^{+\infty}}{1-\omega}.$$

We then have:

$$\alpha^\infty + \frac{\rho_l}{6\sqrt{\pi}} \int_0^\infty S^{3/2}\psi_0(S)dS = 1.$$

The reduced ignition temperature is θ_c and the reduced boiling temperature θ_{eb}, with:

$$\theta_c = \frac{T_c - T_0}{T_* - T_0}, \qquad \theta_{eb} = \frac{T_{eb} - T_0}{T_* - T_0}.$$

For each function h of T, Y_F or f, we note $\tilde{h}$ the same function of θ, α or ψ. The function $\tilde{B}_M(\theta,\alpha)$ then reads:

$$\tilde{B}_M(\theta,\alpha) = \frac{\tilde{Y}_{Fs}(\theta) - (1-\omega)\alpha}{1 - \tilde{Y}_{Fs}(\theta)}.$$

The reduced chemical rate $\tilde{k}(\theta) = k(T)$ vanishes for $\theta \leqslant \theta_c$. We assume that $\theta_c \in]0,1[$ and $\theta_{eb} > 1$. We could eventually take $\theta_c = 0$ as in [25] or [5], but it is not the subject of this paper.

We obtain the system $(\mathcal{S}_0)$:

$$(\mathcal{S}_0) \quad \begin{cases} \kappa\,\theta'' + c\,\theta' + \tilde{k}(\theta)\,\alpha = 0, \\ D\,\alpha'' + c\,\alpha' - \tilde{k}(\theta)\,\alpha + S^m(z,\theta,\alpha) = 0, \\ c\,\partial_z\psi + \tilde{R}_S(\theta,\alpha)\,\partial_S\psi = 0, \qquad S \geqslant 0, \end{cases}$$

with the source term:

$$S^m(z,\theta,\alpha) = \frac{\rho_l}{4\sqrt{\pi}}\,\tilde{R}_S(\theta,\alpha)\int_0^{+\infty} S^{1/2}\,\psi(z,S)\,dS,$$

and with the boundary conditions at infinity:

$$(\mathcal{L}_0) \quad \begin{cases} \theta(+\infty) = 0, & \alpha(+\infty) = \alpha^\infty, & \psi(+\infty,S) = \psi_0(S), \\ \theta(-\infty) = 1, & \alpha(-\infty) = 0, & \psi(-\infty,S) = 0. \end{cases}$$

Remark 3 An integration of the equations (1), (2) and (3) with $z \in \mathbb{R}$ and $S \in \mathbb{R}^+$ shows that $T_* - T_0 = q(Y_F^{+\infty} + Y_{liq}^{+\infty})$. It's the reason why q disappears in the first equation of system $(\mathcal{S}_0)$.

Compared to the usual system of equations for mono-dimensional homogeneous gaseous flames [19], the system is more complex because of the S dependence of the function ψ. But the equation on ψ being hyperbolic, an analytical solution for this function can be found, introducing the characteristics X:

$$\psi(z,S) = \psi_0(X(z,S)), \qquad X(z,S) = S + \frac{1}{c}\int_z^\infty \tilde{R}_S(\theta(y),\alpha(y))dy.$$

The system $(\mathcal{S}_0)$ can then be rewritten:

$$(\mathcal{S}_0') \quad \begin{cases} \kappa\,\theta'' + c\,\theta' + \tilde{k}(\theta)\,\alpha = 0, \\ D\,\alpha'' + c\,\alpha' - \tilde{k}(\theta)\,\alpha + S^m(z,\theta,\alpha) = 0, \end{cases}$$

with the source term and the characteristics:

$$S^m(z,\theta,\alpha) = \frac{\rho_l}{4\sqrt{\pi}}\,\tilde{R}_S(\theta,\alpha)\int_0^{\infty} S^{1/2}\,\psi_0(X(z,S))\,dS,$$
$$X(z,S) = S + \frac{1}{c}\int_z^\infty \tilde{R}_S(\theta(y),\alpha(y))dy.$$

and with the boundary conditions at infinity:

$$(\mathcal{L}_0') \quad \begin{cases} \theta(+\infty) = 0, & \alpha(+\infty) = \alpha^\infty, \\ \theta(-\infty) = 1, & \alpha(-\infty) = 0. \end{cases}$$

However, the source term S^m is then non local: it depends on all the vaporization in $[z, +\infty[$. In order to avoid this problem and apply techniques developed in [19], the system $(\mathcal{S}_0)$ is approximated by a finite dimensional one, which corresponds to a discretization of ψ as a function of S.

2.3 Use of a Eulerian multi-fluid model for the spray

Several type of methods have been developed in order to discretize the distribution function, tacking into account the polydispersion of the spray. But a Eulerian approach is needed here, in order to reintroduce local source terms. A fully Eulerian approach has been described and derived from the kinetic level of description in [22]: multi-fluid models. Moreover, the numerical analysis conducted in [21] shows that, in our mono-dimensional case without neither dynamic nor thermal effect, a multi-fluid vaporizing model provides an approximation of order 1 of the kinetic equation. It can be seen as a discretization of the continuous size phase space, using a finite volume formulation, the size intervals being called the sections in reference to [14]. We can call such a method a multi-fluid method in that the dispersed phase is described as a set of continuous media: "fluids", each "fluid" corresponding to a statistical average in a section: the cell defined by two fixed droplet sizes. The evolution of these coupled sections or "fluids" is governed by conservation equations for mass, momentum and enthalpy. In all previous references, only evaporation is considered and not condensation, thus resulting in a predefined upwind scheme; it is consistent with the model used for $\tilde{R}_S$.

Let us apply this multi-fluid method to our problem. We know that the function $\psi(z, S)$ is equal to zero if $S \geqslant S_{max}$ and we take a uniform partition of $[0, S_{max}]$: $S_i = i\,\Delta S$ with $i \in \{0, \ldots, N\}$ and $\Delta S = S_{max}/N$. The sections are then the intervals $[S_{i-1}, S_i]$. Let $\hat{\mu}_i$ denote the reduced mass density of the i^{th} section:

$$\hat{\mu}_i(z) = \frac{\rho_l}{6\sqrt{\pi}} \int_{S_{i-1}}^{S_i} S^{3/2}\, \psi(z, S)\, dS. \tag{9}$$

As in [21], the $\hat{\mu}_i$ are approximated by the μ_i such that:

$$\forall i \in \{1, \ldots, N\}, \qquad c\,\mu_i'(z) - \tilde{R}_S(\theta, \alpha)\,[(M_i + F_i)\,\mu_i(z) - F_{i+1}\,\mu_{i+1}(z)] = 0. \tag{10}$$

The term with M_i is the mass flux between the section i and the gas and the term with F_i is the mass flux between the section i and the section $i-1$ ($F_{N+1} = 0$). If we choose to approximate ψ by a function of S constant in each section, then the constant coefficients M_i and F_i are [21]:

$$\forall i \in \{1, \ldots, N\}, \qquad M_i = \frac{3\int_{S_{i-1}}^{S_i} \sigma^{1/2} d\sigma}{2\int_{S_{i-1}}^{S_i} \sigma^{3/2} d\sigma}, \qquad F_i = \frac{S_{i-1}^{3/2}}{\int_{S_{i-1}}^{S_i} \sigma^{3/2} d\sigma}. \tag{11}$$

The boundary conditions are given by:

$$\mu_i(+\infty) = \mu_i^{\infty} = \frac{\rho_l}{6\sqrt{\pi}} \int_{S_{i-1}}^{S_i} S^{3/2}\, \psi_0(S)\, dS, \qquad \mu_i(-\infty) = 0. \tag{12}$$

It is useless to consider a last section with 0 mass density at $+\infty$, because the mass density in this section would then be identically equal to 0. We then assume that the upper bound S_{max} has been chosen as the first point beyond which f_0 vanishes. It implies that μ_N^{∞} is positive.

The techniques developed in [19] are then applied to the system $(\mathcal{S}_N)$:

$$(\mathcal{S}_N) \quad \begin{cases} \kappa\,\theta'' + c\,\theta' + \tilde{k}(\theta)\,\alpha & = 0, \\ D\,\alpha'' + c\,\alpha' - \tilde{k}(\theta)\,\alpha + \tilde{R}_S(\theta,\alpha)\,\sum M_i\,\mu_i & = 0, \\ c\,\mu_i' \qquad\qquad -\tilde{R}_S(\theta,\alpha)\,[(M_i+F_i)\,\mu_i - F_{i+1}\,\mu_{i+1}] & = 0, \end{cases}$$

with the boundary conditions:

$$(\mathcal{L}_N) \quad \begin{cases} \theta(+\infty) = 0, \quad \alpha(+\infty) = \alpha^\infty, \quad \mu_i(+\infty) = \mu_i^\infty, \\ \theta(-\infty) = 1, \quad \alpha(-\infty) = 0, \qquad \mu_i(-\infty) = 0 \end{cases}$$

The scheme is of order 1 in the surface step ΔS, as shown in [21] and we will be able to recover the fluid-kinetic formulation $(\mathcal{S}_0)$, from the system $(\mathcal{S}_N)$, by taking the limit $N \to \infty$ (see section 6). We want to use the topological degree theory in order to prove the existence of solutions c, θ, α and μ_i for the system $(\mathcal{S}_N)$. However, the topological degree is defined for elliptic operator [36], and the operator corresponding to the system $(\mathcal{S}_N)$ is mixed hyperbolic/elliptic because there is no droplet diffusion. We then introduce a small diffusion coefficient ζ, and, as in [19], we will study the limit when this diffusion coefficient tends to zero. The system with diffusion then reads:

$$(\mathcal{S}_\zeta) \quad \begin{cases} \kappa\,\theta'' + c\,\theta' + \tilde{k}(\theta)\,\alpha & = 0, \\ D\,\alpha'' + c\,\alpha' - \tilde{k}(\theta)\,\alpha + \tilde{R}_S(\theta,\alpha)\,\sum M_i\,\mu_i & = 0, \\ \zeta\,\mu_i'' + c\,\mu_i' \qquad\qquad -\tilde{R}_S(\theta,\alpha)\,[(M_i+F_i)\,\mu_i - F_{i+1}\,\mu_{i+1}] & = 0, \end{cases}$$

with the boundary conditions $(\mathcal{L}_N)$. We remark in the next subsection that the added diffusion can be interpreted as due to a small turbulent mixing.

2.4 Introducing turbulence and diffusion

When the droplets have their own dynamics, the distribution function f depends also on the droplet velocity u_l: $f(t,x,S,u_l)$. The transport equation then reads [37], [38]:

$$\frac{\partial f}{\partial t} + u_l \cdot \nabla_x f + \frac{\partial\, R_S\, f}{\partial S} + \nabla_{u_l}(F\,f) = 0, \tag{13}$$

where F is the drag acceleration. If we consider a Stokes drag, it reads:

$$F = \frac{\alpha}{S}(u - u_l),$$

where u is the gas velocity and α is a constant. The function R_S is assumed to be independent of u, thus neglecting the convective correction term [39], [22].

For turbulent flows, the gas velocity can be divided between its average value $\bar{u}$ and a fluctuation u' which is of Gaussian type: $u = \bar{u} + u'$. As shown in [27] and [7], an equation on the average $\bar{f}$ of f reads:

$$\frac{\partial \bar{f}}{\partial t} + u_l \cdot \nabla_x \bar{f} + \frac{\partial\, R_S\, \bar{f}}{\partial S} + \nabla_{u_l}(\bar{F}\,\bar{f}) - \nabla_{u_l} \cdot \left(D_x\,\nabla_x \bar{f} + D_{u_l}\,\nabla_{u_l} \bar{f}\right) = 0, \tag{14}$$

where the averaged drag force reads:

$$F = \frac{\alpha}{S}(\bar{u} - u_l).$$

The random fluctuations in the gas velocity then generate, on average, a diffusion process in the phase space.

As in [22], we make an assumption on the shape of $\bar{f}$: in this article, the probability density function of the spray is supposed to be such that at a given size, there is only one characteristic velocity, with no dispersion around it. But the considered flames were laminar. Here, we have to consider a dispersion around the characteristic velocity, because of the turbulence. So, we assume [28]:

$$\bar{f}(t, x, S, u_l) = n(t, x, S)\, \varphi_{\sigma(t,x,S)}(u_l - \bar{u}_l(t, x, S)),$$

where φ_σ is a Gaussian of dispersion σ:

$$\varphi_\sigma(v) = \frac{1}{(4\pi\sigma^2)^{3/2}} \exp\left(-\frac{3\,v^2}{4\,\sigma^2}\right).$$

From the equation (14), we obtain a semi-kinetic model used for the multi-fluid method [28]:

$$\frac{\partial n}{\partial t} + \nabla_x \cdot (n\bar{u}_l) + \frac{\partial R_S\, n}{\partial S} = 0 \tag{15}$$

$$\frac{\partial n\,\bar{u}_l}{\partial t} + \nabla_x \cdot \left(n\,\bar{u}_l \otimes \bar{u}_l + n\,\frac{2}{3}\sigma^2\right) + \frac{\partial R_S\, n\,\bar{u}_l}{\partial S} = -\frac{\alpha}{S}\, n\,(\bar{u} - \bar{u}_l) - D_x \nabla_x n \tag{16}$$

$$\begin{aligned}\frac{\partial}{\partial t}\left[n\,\left(\frac{\bar{u}_l^2}{2} + \sigma^2\right)\right] + \nabla_x \cdot \left(n\,\frac{2}{3}\sigma^2\,\bar{u}_l\right) + \frac{\partial}{\partial S}\left[R_S\, n\,\left(\frac{\bar{u}_l^2}{2} + \sigma^2\right)\right] = -\frac{\alpha}{S}\,(\bar{u} - \bar{u}_l)\cdot\bar{u}_l \\ - 2n\,\frac{\alpha}{S}\,\sigma^2 - D_x \nabla_x \cdot (n\,\bar{u}_l) + 3n\,D_{u_l}\end{aligned} \tag{17}$$

In the case $\sigma^2 = 0$ and $D_x = D_{u_l} = 0$, we recover the equations used in the laminar case [22].

Remark 4 We can also remark that D_{u_l} introduces a source term in the σ^2 equation:

$$n\frac{\partial \sigma^2}{\partial t} + n\,\bar{u}_l \cdot \nabla_x \sigma^2 + n\,\frac{2}{3}\sigma^2\,\nabla_x \cdot \bar{u}_l + R_S\, n\frac{\partial \sigma^2}{\partial S} = -2n\,\frac{\alpha}{S}\,\sigma^2 - D_x\, n\,\nabla_x \cdot \bar{u}_l + 3n\,D_{u_l}.$$

So, take $\sigma^2 = 0$ and $D_x \neq 0$ or $D_{u_l} \neq 0$ is incoherent.

In the case of very small droplets, D_x takes the following asymptotic expression:

$$D_x = \frac{2}{3}\,\kappa\,\frac{\alpha}{S},$$

where κ is the turbulent kinetic energy and the 3 comes from the space dimension. The source term in the equation of momentum conservation then reads:

$$-\frac{\alpha}{S}\left(n\,\bar{u}_l - n\,\bar{u} + \frac{2}{3}\,\kappa\,\nabla_x n\right).$$

So, taking the formal singular limit of small S, we obtain:

$$n\,\bar{u}_l \approx n\,\bar{u} - \frac{2}{3}\,\kappa\,\nabla_x n,$$

and the mass conservation equation can be rewritten:

$$\frac{\partial n}{\partial t} + \nabla_x \cdot (n\bar{u}) - \frac{2}{3}\,\kappa\,\Delta n + \frac{\partial R_S\, n}{\partial S} = 0. \tag{18}$$

We then have a spatial diffusion coupled to the vaporization, the constant κ being the turbulent kinetic energy of the gas.

If the averaged velocity of the droplets is equal to zero, the multi-fluid model corresponding to the kinetic model (18) reads

$$\frac{\partial n}{\partial t} - \zeta\,\Delta n + \frac{\partial R_S\, n}{\partial S} = 0, \tag{19}$$

with $\zeta = \frac{2}{3}\kappa$. This equation is linear in n. So, multiplying (19) by $\frac{\rho_l}{6\sqrt{\pi}} S^{3/2}$ and integrating it between S_{i-1} and S_i, we obtain a multi-fluid model which is exactly the equation on μ_i in the system $(\mathcal{S}_\zeta)$ Thus, the diffusion process on the mass variables has a physical interpretation and it allows us to apply the topological degree theory to this elliptic system.

3 Topological degree

In order to prove the existence of traveling waves, the topological degree is a useful tool. It is defined as follow (see, e. g. [6]): let E_1 and E_2 be two Banach spaces. Suppose we are given a class ϕ of operators acting from E_1 to E_2 and a class H of homotopies, i.e., maps $A_\tau(u) : E_1 \times [0,1] \to E_2$, $\tau \in [0,1]$, $u \in E_1$ such that for any $\tau \in [0,1]$ fixed, $A_\tau(u) \in \phi$. Assume moreover that for any bounded open set $D \subset E_1$ and any operator $A \in \phi$ such that $A(u) \neq 0$ for $u \in \partial D$ (∂D denotes the boundary of the set D), there is an integer $\gamma(A, D)$ satisfying the following conditions:

1. *Homotopy invariance*. Let $A_\tau(u) \in H$ and $A_\tau(u) \neq 0$ for $u \in \partial D$ and $\tau \in [0,1]$. Then, we have:

$$\gamma(A_0, D) = \gamma(A_1, D).$$

2. *Additivity*. Let $D \subset E_1$ be an arbitrary open set in E_1, and D_1, $D_2 \subset D$ be open sets such that $D_1 \cap D_2 = \emptyset$. Suppose that $A \in \phi$ does not vanish for $u \in \bar{D} \setminus (D_1 \cup D_2)$. Then, we have:

$$\gamma(A, D) = \gamma(A, D_1) + \gamma(A, D_2).$$

3. *Normalization*. There exists a bounded linear operator $J : E_1 \to E_2$ with a bounded inverse defined on E_2 such that for any bounded open set $D \subset E_1$ with $0 \in D$, $\gamma(J, D) = 1$.

The integer $\gamma(A, D)$ is called the topological degree.

Since we consider a problem in unbounded domains, the Leray-Schauder theory [23] cannot be directly applied. However the topological degree theory for elliptic operators can be extended to unbounded domains [8], [9], [34], [33], [36]. One of the approaches is based on the theory of Fredholm operators [11], [16], [32]. Here, we use the topological degree constructed in [36] for elliptic, Fredholm and proper operators in unbounded domain and weighted Hölder spaces. This theory directly applies when the domain is the whole real line.

The topological degree is useful in order to investigate the existence of solution because, for an operator $A \in \phi$ without zero at the boundaries of a bounded open set D, if the topological degree of (A, D) is not equal to zero, then the problem $Aw = 0$ admits a solution. Moreover, the topological degree can be directly calculated for simple operators: for example, if $A \in \phi$ admits only one zero w and if the linearized operator $A'(w)$ has no zero eigenvalue, then, for a bounded open set D which contains w, the topological degree of (A, D) is 1 or -1. The invariance by homotopy allows us to evaluate the topological degree for more complex operators.

The aim of this section is not to describe the construction of the topological degree or to present the whole theory. For the sake of legibility, we only present the application of the theory to traveling wave systems.

3.1 General reaction-diffusion operator

We introduce weighted Hölder spaces $C^{j+\delta}_{\mu}(\mathbb{R}) = \{u : u\mu \in C^{j+\delta}(\mathbb{R})\}$. The topological degree is constructed [36] in the weighted Hölder spaces, so that the operators are proper and, generically, it cannot be constructed in spaces without weight [2]. As a weight, we take a polynomial function $\mu(x) = 1 + x^2$. We note that functions exponentially decreasing at infinity belong to these spaces. The following type of reaction-diffusion operators:

$$Aw = aw'' + c(w)w' + F(w),$$

operate from $E_1 = C^{2+\delta}_{\mu}(\mathbb{R})$ onto $E_2 = C^{\delta}_{\mu}(\mathbb{R})$, where a is a diagonal matrix, with positive, constant terms on the diagonal and F vanishes at w^+ and w^-, with $w^+ \neq w^-$. The function w belongs to E, a set of functions u such that $u - (1 - \Phi)w^+ - \Phi w^- \in E_1$, where $\Phi(z)$ is a C^∞ function which is identically equal to 1 for $z \in]-\infty, 0]$ and identically equal to 0 for $z \in [1, +\infty[$. The functional $c(w)$, which replaces the wave velocity, is defined by:

$$c(w) = \ln \int_{\mathbb{R}} |w(x) - w^+|^2 \, \sigma(z) dz, \tag{20}$$

where σ is an increasing function on $\mathbb{R}$, of limit 0 at $-\infty$ and 1 at $+\infty$ and such that $\int_{-\infty}^{0} \sigma(x)dx < \infty$. This functional is such that, if we note $w_h(z) = w(z + h)$, $c(w_h)$ is a decreasing bijection from $\mathbb{R}$ to $\mathbb{R}$, as a function of h. It is also such that, for a solution w of $Aw = 0$: $< c'(w), w' > \neq 0$. It was introduced in order to solve some problems due to the invariance by translation and zero eigenvalue problem [36]. Suppose that we have the constant c and not the functional. If there exists a solution $w(z)$ of the problem $Aw = 0$ with $w \in E$, then the function $w_h(z) = w(z + h)$ is also a solution, for all $h \in \mathbb{R}$. The weighted norm $\|\mu(z)(w - (1 - \Phi)w^+ - \Phi w^-)\|$ tends to infinity as h tends to $\pm\infty$. Therefore, a bounded domain D in the set E does not contain any solution or the branch of solutions intersects its boundary. So, the degree cannot be used. Moreover, if w is a solution of $Aw = 0$, then zero is an eigenvalue of the linearized operator $A'(w)$ (with the eigenvector w') and it is not easy to evaluate the topological degree of A. The functionalization of the parameter c allows us to solve these problems. It also moves the zero eigenvalue of the linearized problem about a traveling wave to the left half-plane, and it singles out one particular element of the family of solutions [35]. Moreover it eliminates the unknown constant from the problem.

Then, if the condition

$$(\mathcal{C}): \qquad \forall \lambda \geqslant 0, \qquad \forall \xi \in \mathbb{R}, \qquad det(-a\,\xi^2 + c(w)\,i\,\xi + F'(w^{\pm}) - \lambda\, I) \neq 0 \tag{21}$$

is realized, we can define the topological degree for the operator A.

3.2 Introduction of a perturbed system

The topological degree can not be defined for the operator corresponding to the system $(\mathcal{S}_\zeta)$, $(\mathcal{L}_N)$. Indeed, this operator is not regular because the function $\tilde{R}_S(\theta,\alpha) = \varphi \ln(1+\tilde{B}_M(\theta,\alpha))^+$ is not C^1. Moreover the condition $(\mathcal{C})$ is not valid, neither at $-\infty$, nor at $+\infty$. The system $(\mathcal{S}_\zeta)$ is then perturbed, so that the topological degree can be defined for the operator corresponding to the new system. First, we replace $\tilde{R}_S(\theta,\alpha)$ by $R_r(\theta,\alpha)$ such that:

$$R_r(\theta,\alpha) = \varphi\left(1-\Phi\left(\frac{\tilde{B}_M(\theta,\alpha)}{r}\right)\right)\ln(1+\tilde{B}_M(\theta,\alpha)),$$

where r is a small positive parameter with at least $r \leqslant 1$ and $\Phi(z)$ is a C^∞ function such that:

$$\begin{cases} \text{for } z \leqslant 0 & \Phi(z) = 1, \\ \text{for } 0 < z < 1 & \Phi'(z) < 0, \\ \text{for } z \geqslant 1 & \Phi(z) = 0. \end{cases} \tag{22}$$

The function R_r tends to $\tilde{R}_S$ when r tends to zero and we then denote $R_0 = \tilde{R}_S$. Second, we add a small positive parameter ϵ and introduce the perturbed system:

$$(\mathcal{S}_{\zeta,\epsilon,r}) \quad \begin{cases} \kappa\,\theta'' + c\,\theta' + \tilde{k}(\theta)\,\alpha + \epsilon\,\tilde{k}(\theta)\,(1-\theta-\alpha) = 0, \\ D\,\alpha'' + c\,\alpha' - \tilde{k}(\theta)\,\alpha + R_r(\theta,\alpha)\,\sum M_i\,\mu_i = 0, \\ \zeta\,\mu_i'' + c\,\mu_i' - R_r(\theta,\alpha)\,[(M_i+F_i)\,\mu_i - F_{i+1}\,\mu_{i+1}] - \epsilon\,\tilde{k}(\theta)\,\mu_i = 0, \end{cases}$$

with the boundary conditions $(\mathcal{L}_N)$. After a change of variable which will be defined later, and the functionalization (20) of c, we can define and use the topological degree for the corresponding operator.

Contrary to what was made in preceding studies, as in [35], the added terms are defined from functions already existing in the initial system $(\mathcal{S}_0)$, as in [19]. It allows us to preserve the monotonicity of variables, as it will be shown in the next section. Moreover, it makes simpler the homotopies which transform the system to one for which the topological degree can be evaluated.

As shown in Fig. 4, the sketch of the existence proof is as follows: we use the topological degree in order to prove the existence of solutions for $(\mathcal{S}_{\zeta,\epsilon,r})$ (Section 5) and we take the limits $r \to 0$, $\epsilon \to 0$ and $\zeta \to 0$ in order to prove existence of solutions for $(\mathcal{S}_N)$ (Section 6). These steps require a priori estimates, independent of the homotopy parameter τ for the first step and independent of r, ϵ and ζ for the second step. These estimates have a common part. Consequently, we begin with general estimates (Section 4), independent of all parameters τ, r, ϵ and ζ.

$$\big(\mathcal{S}_0\big) \quad + \big(\mathcal{L}_0\big)$$

$$\kappa\,\theta'' + c\,\theta' + \tilde{k}(\theta)\,\alpha = 0,$$
$$D\,\alpha'' + c\,\alpha' - \tilde{k}(\theta)\,\alpha + S^m = 0,$$
$$c\,\partial_z\psi + \tilde{R}_S(\theta,\alpha)\,\partial_S\psi = 0, \qquad S \geqslant 0.$$

$N \to \infty$ $\uparrow$ $\downarrow$ discretization: Multi−fluid model

$$\big(\mathcal{S}_N\big) \quad + \big(\mathcal{L}_N\big)$$

$$\kappa\,\theta'' + c\,\theta' + \tilde{k}(\theta)\,\alpha = 0,$$
$$D\alpha'' + c\alpha' - \tilde{k}(\theta)\,\alpha + \tilde{R}_S(\theta,\alpha)\sum M_i\mu_i = 0,$$
$$c\mu_i' - \tilde{R}_S(\theta,\alpha)\,[(M_i+F_i)\mu_i - F_{i+1}\mu_{i+1}] = 0.$$

$r \to 0$, $\epsilon \to 0$, $\zeta \to 0$ $\uparrow$ $\downarrow$ addition of a diffusion (turbulent mixing) and small parameters

$$\big(\mathcal{S}_{\zeta,\epsilon,r}\big) \quad + \big(\mathcal{L}_N\big)$$

$$\kappa\,\theta'' + c\,\theta' + \tilde{k}\,\alpha + \epsilon\tilde{k}\,(1-\theta-\alpha) = 0,$$
$$D\,\alpha'' + c\,\alpha' - \tilde{k}\,\alpha + R_r\sum M_i\mu_i = 0,$$
$$\zeta\,\mu_i'' + c\,\mu_i' - R_r\,[(M_i+F_i)\mu_i - F_{i+1}\mu_{i+1}] - \epsilon\tilde{k}\,\mu_i = 0,$$

$\uparrow$ $\downarrow$ change of variables
$$w = (\zeta\theta, \zeta\alpha, \zeta\mu_1, \ldots, \zeta\mu_N)$$
functionnalization of c

$\big(\mathcal{S}_{rd}\big)$ a topological degree can be defined and used

Figure 4: Sketch of the proof.

4 A priori estimates

Here, we give a priori estimates independent of all the parameters r, ϵ and ζ and also of the homotopy parameter τ which will be introduced in the Section 5. We then consider the following system:

$$(\mathcal{S}_{\zeta,\epsilon,r,\tau}) \quad \begin{cases} \kappa\,\theta'' + c\,\theta' + \tilde{k}\,\alpha + \epsilon\,\tilde{k}\,(1-\theta-\alpha) = 0, \\ D\,\alpha'' + c\,\alpha' - \tilde{k}\,\alpha + (1-\tau) R_r \sum M_i\,\mu_i = 0, \\ \zeta\,\mu_i'' + c\,\mu_i' - (1-\tau) R_r\,[(M_i+F_i)\,\mu_i - F_{i+1}\,\mu_{i+1}] - \epsilon\,\tilde{k}\,\mu_i = 0, \end{cases}$$

with the boundary conditions $(\mathcal{L}_N)$, with $0 \leqslant \tau \leqslant 1$, with a small non negative parameter ϵ, with a small positive parameter ζ and with a Lewis number $\mathrm{Le} = \kappa/D$ larger than $1+p$, where p is a non negative real and will be defined later, in order to prevent the condensation phenomena. Estimates given in this Section are general enough, so that they are used to have a priori estimates in the weighted Hölder space $C^{2+\delta}_{\mu}(\mathbb{R})$ independent of τ in order to prove existence of solution of system $(\mathcal{S}_{\zeta,\epsilon,r})$ in Section 5 and to have a priori estimates in the Hölder space $C^{2+\delta}(\mathbb{R})$ independent of r, ϵ and ζ (or of ϵ and ζ for $r=0$) in order to pass to the limits $r \to 0$, $\epsilon \to 0$ and $\zeta \to 0$ in Section 6.

4.1 Preliminary lemmas

To prove monotonicity of functions or to determine a priori estimates, we use properties of differential equations given by the two following lemmas:

Lemma 4.1 *Let I be the interval $]-\infty, b]$ or $]-\infty, b[$. Let u be a function in $C^{1+\delta}(\mathbb{R})$ and a a positive constant. If $u'(z) + a\,u(z)$ is non negative for $z \in I$ then $u(z)$ is non negative for $z \in I$. If, moreover, $u'(z) + a\,u(z)$ is positive for $z \in]-\infty, x_0[$ with $x_0 \in I$, then $u(z)$ is positive for $z \in I$.*

Proof. – We introduce the function $\Gamma(z) = u(z)\exp(a\,z)$ defined for $z \in I$. This function is C^1 with zero limit at $-\infty$ and $\Gamma'(z) = [u'(z) + a\,u(z)]\exp(a\,z)$. Studying the variations of Γ, we prove the lemma. □

Lemma 4.2 *Let a be a positive number, $g(z)$ a non negative continuous function defined for $z \in \mathbb{R}$ and $F(z,v)$ a continuous function defined for $(z,v) \in \mathbb{R}^2$. The function F is assumed to be of the same sign as v and such that:*

$$\forall v \geqslant 0 \quad \forall z \in \mathbb{R} \qquad F(z,v) \leqslant f(z)\,v,$$

where $f(z)$ is a non negative continuous function. Let u be a function in $C^{2+\delta}(\mathbb{R})$ with non negative limits $u^{\pm}$ at $\pm\infty$ and solution of the equation:

$$\forall z \in \mathbb{R} \qquad u''(z) + a\,u'(z) - F(z,u(z)) + g(z) = 0. \tag{23}$$

We make one of these assumptions:

1. *g is the null function and u^+ or u^- is not equal to zero.*
2. *g is not the null function.*

Then u is positive.

Proof. – First, we show that u is non negative. If it is not true, then u reaches its negative minimum in a real z_m. So, in this point, we have:

$$u''(z_m) = F(z_m, u(z_m)) - g(z_m), \qquad u''(z_m) \geqslant 0, \qquad u(z_m) < 0.$$

Then $u''(z_m)$ is equal to zero. Let us suppose that z_m is the biggest real for which u reaches its minimum. Then, for a small positive real b and for $z \in]z_m, z_m + b]$, $u(z)$ is negative and $u'(z)$ is positive. So, because of the equation (23), $u''(z)$ is negative and then $u'(z)$ is a decreasing function for $z \in]z_m, z_m + b]$. But it is inconsistent with the inequality $u'(z_m) = 0 < u'(z_m + b)$ and u is necessarily non negative.

We can apply the maximum principle to u because this function satisfies the inequality:

$$u''(z) + a\, u'(z) - f(z)\, u(z) + g(z) = F(z, u(z)) - f(z)\, u(z) \leqslant 0.$$

If u vanishes in a point, then u is identically equal to 0 and g is the null function. It is inconsistent with the assumptions and the lemma is proved. □

4.2 Monotonicity

First, we give monotonicity properties of solutions of $(\mathcal{S}_{\zeta,\epsilon,r,\tau})$. Let introduce the variable $S_{\theta\alpha}$, the partial sums S_i and their limits S_i^∞ at $+\infty$:

$$S_{\theta\alpha} = \frac{\kappa}{D\,(1-\epsilon)}(1-\theta) - \alpha, \qquad S_i = \sum_{j=i}^{N} \mu_j, \qquad S_i^\infty = \sum_{j=i}^{N} \mu_j^\infty.$$

We also introduce the number p defined by:

$$\begin{aligned} p &= \max\left\{ \frac{\theta \tilde{Y}''_{Fs}(\theta)}{\tilde{Y}'_{Fs}(\theta)}, \quad 0 \leqslant \theta \leqslant \theta_{eb} - \frac{\nu}{T_* - T_0} \right\} && \text{if } \ \theta_{eb} > \frac{\nu}{T_* - T_0} \\ p &= 0 && \text{if } \ \theta_{eb} \leqslant \frac{\nu}{T_* - T_0}. \end{aligned}$$

First, we give result of monotonicity for some variables:

Proposition 4.3 *If, for $\tau \in [0,1]$, $r \in [0,1]$, $\epsilon \geqslant 0$ and $0 < \zeta < D \leqslant \kappa$, such that and $\epsilon(1-\tau) \neq 0$, there exists a scalar c and functions θ, α and μ_i in $C^{2+\delta}(\mathbb{R})$ solution of $(\mathcal{S}_{\zeta,\epsilon,r,\tau})$, then:*

$$c > 0, \qquad 0 < \alpha < 1, \qquad \forall i \quad \mu_i > 0, \qquad \forall i \quad S_i' > 0, \qquad \theta' < 0, \qquad S'_{\theta\alpha} > 0.$$

Moreover, if $D \leqslant \dfrac{\kappa}{1+p}$ then $\tilde{B}_M(\theta, \alpha) > 0$.

Proof. – Let us first prove that c is positive. Because θ_c is positive, there exists a real z_c such that for $z \geqslant z_c$, $\theta(z)$ is smaller than θ_c. Then the equation on θ' can be solved for $z \geqslant z_c$: $\theta'(z) = \theta'(z_c)\exp(-c(z-z_c)/k)$. But this function tends to zero when z tends to $+\infty$ and because of the uniqueness for the corresponding Cauchy problem, we have $\theta'(z_c) \neq 0$. Then c is positive. Because of the invariance by translation, we can assume without lost of generality that $\theta(0) = \theta_c$ and:

$$\forall z \geqslant 0, \qquad \theta(z) = \theta_c \exp\left(-\frac{c}{\kappa} z\right). \tag{24}$$

We now prove that the μ_i are positive. The function $R_r(\theta(z), \alpha(z))$ of z is continuous and tends to $\phi \ln(1/\delta) > 0$ as z tends to $-\infty$. We can apply Lemma 4.2, first for μ_N, then for μ_{N-1}, down to μ_1. It proves that the μ_i are positive.

Then, we prove that S_i is an increasing function. This function satisfies the following equation:

$$\zeta\, S_i'' + c\, S_i' = (1-\tau) R_r(\theta, \alpha) \sum_{j=i}^{N} M_j\, \mu_j + (1-\tau) R_r(\theta, \alpha)\, F_i\, \mu_i + \epsilon \tilde{k}(\theta)\, S_i.$$

Consequently, the quantity $\zeta S_i'' + c S_i'$ is nonnegative and also positive for sufficiently small z, where $R_r(\theta(z), \alpha(z))$ and $\tilde{k}(\theta(z))$ are positive. Making use of Lemma 4.1, we obtain that S_i' is positive.

Applying Lemma 4.2 to α, we show that α is positive. We now prove that $1 - \alpha - \zeta/D\, S_1$ is non negative. This function is denoted u. The equations on α and μ_i of $(\mathcal{S}_{\zeta,\epsilon,r,\tau})$ are integrated between z and $+\infty$ and added. We obtain, for all $z \in \mathbb{R}$:

$$D\, u'(z) + c\, u(z) = \int_z^{+\infty} \tilde{k}(\theta(x))\, [\alpha(x) + \epsilon\, S_1(x)]\, dx + c\left(1 - \frac{\zeta}{D}\right) S_1(z) \leqslant 0.$$

Lemma 4.1 then shows that u is non negative. So, we have the inequality:

$$\alpha \leqslant 1 - \frac{\zeta}{D} S_1 < 1.$$

Let us then show that θ is a decreasing function, smaller than 1: $\theta' \leqslant 0$ and $\theta \leqslant 1$. If there exists a real z_o such that $\theta'(z_o) > 0$ and $\theta(z_o) \leqslant 1$, then we have $\theta''(z_o) < 0$ because of the first equation of $(\mathcal{S}_{\zeta,\epsilon,r,\tau})$. So θ' is decreasing around z_o. It is then easy to see that $\theta'(z)$ is decreasing and positive and $\theta(z)$ is increasing and smaller than 1 for $z \leqslant z_o$. It is inconsistent with the limit of θ at $-\infty$. So, if $\theta(z)$ is smaller or equal to 1 then it is decreasing. We also have to get rid of the case:

$$\begin{aligned} &\theta(z) > 1 && \text{for } z < z_o, \\ &\theta(z) \leqslant 1, \quad \theta'(z) \leqslant 0 && \text{for } z \geqslant z_o. \end{aligned}$$

In such a case, we denote ψ the function $\theta + \alpha + \sum \mu_i - 1$. This function $\psi(z)$ is positive for $z \leqslant z_o$ and its sign is not constant because of the equality $\int_{\mathbb{R}} \tilde{k}(\theta(x))\psi(x)dx = 0$, obtained through the integration on $\mathbb{R}$ and the sum of the equations of $(\mathcal{S}_{\zeta,\epsilon,r,\tau})$. So, we can introduce the first point a where ψ vanishes. The system $(\mathcal{S}_{\zeta,\epsilon,r,\tau})$ is integrated between $-\infty$ and a and we sum all equations, thus giving:

$$D\, \psi'(a) + c\, \psi(a) = \epsilon \int_{-\infty}^{a} \tilde{k}(\theta(x))\psi(x)\, dx - (\kappa - D)\theta'(a) + (D - \zeta) S_1'(a) > 0.$$

Since $\psi(a) = 0$, we obtain $\psi'(a) > 0$. But $\psi(z)$ is positive for $z < a$ and vanishes for $z = a$. The assumption made was then false and θ is decreasing.

We then prove that θ is strictly decreasing. Indeed if $\theta'(z)$ vanishes, then $\theta''(z)$ is equal to zero. Because of the first equation of $(\mathcal{S}_{\zeta,\epsilon,r,\tau})$ and because of the preceding results, we necessarily have $\theta(z) \leqslant \theta_c$ and $z \leqslant 0$. It is inconsistent with the equation (24). So, θ' is negative.

Multiplying the first equation of $(\mathcal{S}_{\zeta,\epsilon,r,\tau})$ by $1/(1-\epsilon)$ and adding the second equation, we obtain the equation for $S_{\theta\alpha}$:

$$D\, S''_{\theta\alpha} + c\, S'_{\theta\alpha} = -\frac{c}{1-\epsilon}\theta' \left(\frac{\kappa}{D} - 1\right) + \frac{\epsilon}{1-\epsilon}\tilde{k}(\theta)(1-\theta) + (1-\tau)R_r(\theta,\alpha)\sum M_i\mu_i.$$

So, the quantity $D\, S''_{\theta\alpha}(z) + c\, S'_{\theta\alpha}(z)$ is non negative and positive for $z < 0$. Applying Lemma 4.2, we obtain $S'_{\theta\alpha} > 0$.

Let note $y(z) = \tilde{Y}_{Fs}(\theta(z)) - (1-\omega)\alpha(z)$. We want to prove that this function is positive if $D \leqslant \dfrac{\kappa}{1+p}$. It satisfies the equation:

$$D\, y''(z) + c\, y'(z) - F(z, y(z)) + g(z) = 0,$$

with

$$\begin{aligned} F(z,v) = &\left(1 + \frac{D\,(1-\epsilon)}{\kappa\,(1-\omega)}\,\tilde{Y}'_{Fs}(\theta(z))\right)\tilde{k}(\theta(z))\,v \\ &+ (1-\tau)(1-\omega)\varphi \ln\left(1 + \frac{v}{1-\tilde{Y}_{Fs}(\theta(z))}\right)\left(1 - \Phi\left(\frac{v}{r(1-\tilde{Y}_{Fs}(\theta(z)))}\right)\right)\sum_{i=1}^{N} M_i\,\mu_i(z), \end{aligned}$$

and

$$\begin{aligned} g(z) = &-\theta'(z)\left[D\,\theta'(z)\,\tilde{Y}''_{Fs}(\theta(z)) + c\left(1 - \frac{D}{\kappa}\right)\tilde{Y}'_{Fs}(\theta(z))\right] \\ &+ \tilde{k}(\theta(z))\left[\tilde{Y}_{Fs}(\theta(z))\left(1 + \frac{(1-\epsilon)\,D}{(1-\omega)\,\kappa}\,\tilde{Y}'_{Fs}(\theta(z))\right) + \epsilon\,\frac{D}{\kappa}\,(1-\theta(z))\,\tilde{Y}'_{Fs}(\theta(z))\right]. \end{aligned}$$

An integration of the θ equation between z and $+\infty$ shows that $\kappa\,\theta' + c\,\theta \geqslant 0$. We can then give a lower bound for g, independent of c:

$$g(z) \geqslant \theta'(z)^2\left[\frac{\kappa - D}{\theta(z)}\tilde{Y}'_{Fs}(\theta(z)) - D\tilde{Y}''_{Fs}(\theta(z))\right] + \tilde{k}(\theta(z))\tilde{Y}_{Fs}(\theta(z)).$$

For $D \leqslant \dfrac{\kappa}{1+p}$, the g function is non negative and is not the zero function since $g \geqslant \tilde{k}(\theta)\tilde{Y}_{Fs}(\theta)$. We then can apply Lemma 4.2 which show that y is positive. Consequently, the proposition is proved. □

4.3 General estimates

From this subsection, α^∞ is supposed to be greater than θ_c and we introduce a real, $\beta \in]0,1[$, such that

$$\theta_c < \beta\alpha^\infty < \alpha^\infty. \tag{25}$$

This assumption is not very restrictive because θ_c can be chosen as small as we want and $\alpha^\infty > 0$. Estimates independent of r, ζ, ϵ and τ, and also inequalities, useful for Section 6, are given in the following theorem:

Theorem 4.4 *For $0 \leqslant r \leqslant 1$, $0 < \zeta < D \leqslant \dfrac{\kappa}{1+p}$, $0 \leqslant \epsilon \leqslant 1-\beta$ and $0 \leqslant \tau \leqslant 1$ such that $\epsilon(1-\tau) \neq 0$, there exists constants $M > 0$, $c_a^- > 0$ and c^+ independent of r, ζ, ϵ and τ and $M_\zeta > 0$ independent of r, ϵ and τ such that, if there exists a scalar c and functions θ, α and μ_i in $C^{2+\delta}(\mathbb{R})$ solution of $(\mathcal{S}_{\zeta,\epsilon,r,\tau})$ then:*

$$\|\theta\|_{C^{2+\delta}(\mathbb{R})} \leqslant M_\zeta, \qquad \|\alpha\|_{C^{2+\delta}(\mathbb{R})} \leqslant M_\zeta \qquad \|\mu_i\|_{C^{2+\delta}(\mathbb{R})} \leqslant M_\zeta \qquad c_a^- \leqslant c \leqslant c^+,$$

and, for $r = 0$:

$$\|\theta\|_{C^{2+\delta}(\mathbb{R})} \leqslant M, \qquad \|\alpha\|_{C^{2+\delta}(\mathbb{R})} \leqslant M \qquad \|\mu_i\|_{C^{2+\delta}(\mathbb{R})} \leqslant M.$$

If we choose $\theta(0) = \theta_c$, we also have the inequalities:

$$(1-\epsilon)(\alpha^\infty - \alpha) \leqslant \theta, \tag{26}$$

$$\forall i \in \{1,\dots,N\}, \qquad \sum_{j=i}^{N} \prod_{k=j+1}^{N} \left(1 + \frac{F_k}{M_k}\right)(\mu_j^\infty - \mu_j) \geqslant 0, \tag{27}$$

$$\forall z \in \mathbb{R}^+, \quad \forall i \in \{1,\dots,N\}, \qquad 0 \leqslant S_i^\infty - S_i \leqslant \left(1 + \frac{F_i}{M_i}\right)(\theta(z) - \alpha^\infty + \alpha(z)), \tag{28}$$

$$\forall z \in \mathbb{R}^+, \qquad \theta(z) + \alpha(z) + S_1(z) \geqslant 1. \tag{29}$$

Proof. – We already know, because of Proposition (4.3), that c is positive, θ and $\sum \mu_i$ are monotone and α and all the μ_i are positive and bounded.

We now give an estimate for θ', distinguishing two cases: $c \geqslant 1$ and $c < 1$. If c is bigger than 1, then $\theta'(z)$ reaches its minimum at $z = z_m$ and for all $z \in \mathbb{R}$, we have:

$$|\theta'(z)| \leqslant |\theta'(z_m)| = \left| -\frac{1-\epsilon}{c}\tilde{k}(\theta(z_m))\alpha(z_m) - \frac{\epsilon}{c}\tilde{k}(\theta(z_m))[1-\theta(z_m)] \right| \leqslant \tilde{k}(1).$$

Thus, this involves that $|\theta'|$ is smaller than $\tilde{k}(1)$.
If c is smaller than 1, suppose that $\theta'(z) > \tilde{k}(1)$, for a $z \in \mathbb{R}$. We define z_o as the biggest real, smaller than z and such that $|\theta'(z_o)| = \tilde{k}(1)$. So, for $x \in [z_o, z]$, $|\theta'(x)|$ is bigger than $\tilde{k}(\theta(x))$. The first equation of $(\mathcal{S}_{\zeta,\epsilon,r,\tau})$ is then integrated between z_o and z:

$$|\theta'(z)| = \left| \theta'(z_o) - \frac{c}{\kappa}[\theta(z) - \theta(z_o)] - \frac{1-\epsilon}{\kappa}\int_{z_o}^{z} \tilde{k}(\theta(y))\alpha(y)dy - \frac{\epsilon}{\kappa}\int_{z_o}^{z} \tilde{k}(\theta(y))(1-\theta(y))dy \right|. \tag{30}$$

The term $|\theta'(z_o)|$ is upperly bounded by $\tilde{k}(1)$, c by 1, $|\theta(z) - \theta(z_o)|$ by 1, $|\alpha(y)|$ by 1 and $\tilde{k}(\theta(y))$ by $|\theta'(y)|$. Then, in all the cases, $|\theta'|$ is upperly bounded by $\tilde{k}(1) + \frac{2}{\kappa}$.

As in the proof of Proposition 4.3, we choose $\theta(0) = \theta_c$. Then c can be evaluated from $\theta'(0)$:

$$c = \frac{-\theta'(0)}{\theta_c}.$$

With what precedes, we have the upper bound for the wave velocity:

$$c \leqslant c^+ = \frac{\kappa\tilde{k}(1) + 2}{\theta_c}. \tag{31}$$

In order to have a lower bound for c, we first show the inequality $\theta + (1-\epsilon)\alpha \geqslant (1-\epsilon)\alpha^\infty$. The first equation of $(\mathcal{S}_{\zeta,\epsilon,r,\tau})$ and the second multiplied by $1-\epsilon$ are integrated between z and $+\infty$ and added. With $u = \theta + (1-\epsilon)(\alpha - \alpha^\infty)$, we obtain:

$$D\,u' + c\,u \geqslant -(\kappa - D)\theta' \geqslant 0.$$

Because of Lemma 4.1, u is necessarily non negative and $\theta + (1-\epsilon)\alpha \geqslant (1-\epsilon)\alpha^\infty$. So:

$$(1-\epsilon)\alpha \geqslant [(1-\epsilon)\alpha^\infty - \theta]^+.$$

The first equation of $(\mathcal{S}_{\zeta,\epsilon,r,\tau})$ is then multiplied by θ' and integrated on $\mathbb{R}$. With the preceding inequality, we obtain:

$$\begin{aligned} c\int_{\mathbb{R}} (\theta'(x))^2 dx &\geqslant -\int_{\mathbb{R}} \tilde{k}(\theta(x))[(1-\epsilon)\alpha^\infty - \theta(x)]^+\theta'(x)dx - \epsilon\int_{\mathbb{R}} \tilde{k}(\theta(x))(1-\theta(x))\theta'(x)dx, \\ &\geqslant \int_0^{(1-\epsilon)\alpha^\infty} \tilde{k}(y)[(1-\epsilon)\alpha^\infty - y]dy + \epsilon\int_0^1 \tilde{k}(y)(1-y)dy. \end{aligned}$$

With the assumption (25) and the estimates of θ', we find the lower bound for c:

$$c \geqslant c_a^- = \frac{\kappa}{2 + \kappa\tilde{k}(1)} \int_{\theta_c}^{\beta\alpha^\infty} \tilde{k}(y)[\beta\alpha^\infty - y]dy > 0. \tag{32}$$

In order to have estimates in $C^{2+\delta}(\mathbb{R})$, we can give here estimates in $C^3(\mathbb{R})$, because the solutions in $C^{2+\delta}(\mathbb{R})$ of $(\mathcal{S}_{r,\epsilon,\zeta,\tau})$ are necessarily regular. With the lower bound of c and bounds of θ, θ', α and μ_i, it is then easy to prove that α', μ_i', α'', θ'' and θ''' are bounded independently of r, ζ, ϵ and τ. We also prove that μ_i'' has bounds independent of r, ϵ and τ, but dependent of ζ. For $r > 0$, we have to derive the equations on α and μ_i and to give a bound, independent of r, ϵ and τ of the $R_r(\theta(z), \alpha(z))$ derivative in order to have estimates of α''' and μ_i'''. This function can be rewritten:

$$R_r(\theta(z), \alpha(z)) = \varphi\psi_r(\tilde{B}_M(\theta(z), \alpha(z))), \qquad \psi_r(x) = \left(1 - \Phi\left(\frac{x}{r}\right)\right)\ln(1+x).$$

But $\tilde{B}_M(\theta(z), \alpha(z))$ is positive and smaller than $1 + \frac{1}{\delta}$ and its derivative is bounded independently of all parameters. Moreover, $\psi_r'(x)$ is bounded independently of r for $x \in [0, 1 + \frac{1}{\delta}]$:

$$\begin{aligned} 0 \leqslant \psi_r'(x) &= \frac{1 - \Phi\left(\frac{x}{r}\right)}{1+x} - \frac{x}{r}\Phi'\left(\frac{x}{r}\right)\frac{\ln(1+x)}{x} \\ &\leqslant 1 + \max\left\{y\,\Phi'(y),\ y \in [0,1]\right\} \times \max\left\{\frac{\ln(1+x)}{x},\ x \in \left]0, 1 + \frac{1}{\delta}\right]\right\}. \end{aligned}$$

It allows us to prove the first estimates of the theorem.

For $r = 0$, because $\tilde{B}_M > 0$, the function $R_0(\theta(z), \alpha(z))$ is regular for $z \in \mathbb{R}$ and it is easy to prove that α''', μ_i'', μ_i''' are bounded independently of ζ, ϵ and τ (for μ_i'', μ_i''', we use the first and second derivative of the μ_i equation and we evaluate the maximum and the minimum of these functions), thus showing the second estimates of the theorem.

The inequality (27) is proved thanks to Lemma 4.1. Indeed, let us denote

$$u = \sum_{j=i}^{N} \prod_{k=j+1}^{N} \left(1 + \frac{F_k}{M_k}\right) (\mu_j^\infty - \mu_j).$$

Multiplying the equations on the μ_i of $(\mathcal{S}_{\zeta,\epsilon,r,\tau})$ for the index i to N by $\displaystyle\prod_{k=j+1}^{N} \left(1 + \frac{F_k}{M_k}\right)$, and integrating there sum between z and $+\infty$, the equation on u appears:

$$\zeta\, u'(z) + c(u(+\infty) - u(z)) = (M_i + F_i) \prod_{k=1}^{N} \left(1 + \frac{F_k}{M_k}\right) (1-\tau) \int_z^{+\infty} R_r(\theta(y), \alpha(y))\, \mu_i(y)\, dy \geqslant 0$$

and the inequality is shown.

Let us then show inequality (28). Let us denote $v(z)$ the function defined by:

$$v(z) = \frac{M_i}{F_i + M_i}(S_i^\infty - S_i(z)) + \alpha^\infty - \alpha(z) - \theta(z).$$

We then multiply by $\dfrac{M_i}{F_i + M_i}$ the equations on μ_i in $(\mathcal{S}_{\zeta,\epsilon,r,\tau})$ for the index i to N, integrate these equations as well as first and second equations of $(\mathcal{S}_{\zeta,\epsilon,r,\tau})$ between z and $+\infty$, and sum up all these equations. We finally get the following equation for v:

$$\begin{aligned} D\, v'(z) + c\, v(z) + \epsilon \int_z^{+\infty} \tilde{k}(\theta(y))\, v(y)\, dy &= (\kappa - D)\theta'(z) + (\zeta - D)\frac{M_i}{F_i + M_i} S_i'(z) \\ &- \left[\frac{F_i}{F_i + M_i} S_i^\infty + \sum_{k=1}^{i-1} \mu_k^\infty\right] \epsilon \int_z^{+\infty} \tilde{k}(\theta(y))\, dy \\ &- (1-\tau) \int_z^{+\infty} R_r(\theta(y), \alpha(y)) \left[\sum_{k=1}^{i-1} M_k\, \mu_k(y) + \sum_{k=i+1}^{N} \frac{F_i}{F_i + M_i} M_k \mu_k(y)\right] dy. \end{aligned}$$

Because θ and S_i are decreasing and because of the inequality $\zeta \leqslant D \leqslant \kappa$, the function defined by the left hand side of the preceding equation is then non positive. If $v(0)$ is positive, then $v'(0)$ is negative, and necessarily, $v(z)$ is positive and decreasing for $z \leqslant 0$. It is impossible because v has a non positive limit at $-\infty$. So, $v(0)$ is non positive, and for $z \geqslant 0$, we have $D\, v'(z) + c\, v(z) \leqslant 0$. As for Lemma 4.1, introducing the function $\Gamma(z) = u(z) \exp(a\, z)$, we prove that $v(z) \leqslant 0$ for $z \geqslant 0$ and inequality (28) is proved.

Let us now turn to inequality (29). We note $w = 1 - \theta - \alpha - S_1$. Integrating the equations of $(\mathcal{S}_{\zeta,\epsilon,r,\tau})$ between z and $+\infty$, and adding, the equation on w yields:

$$D\, w'(z) + c\, w(z) + \epsilon \int_z^{+\infty} \tilde{k}(\theta(y))\, w(y)\, dy = (\kappa - D)\theta'(z) - (D - \zeta) S_1'(z) \leqslant 0.$$

As for v, we then prove that $w(z) \leqslant 0$ for $z \geqslant 0$. The theorem is then proved. □

4.4 Behavior of solutions at infinity

We have suggested in the section 3, that a change of variables will be done in order to be able to define the topological degree. This change of variable is defined in the subsection 5.3. But for that, we have to show that the functions θ, α and μ_i, solutions of $(\mathcal{S}_{\zeta,\epsilon,r,\tau})$ are exponentially decreasing; this behavior is characterized in the following Proposition:

Proposition 4.5 *If the scalar c and the functions θ, α and μ_i in $C^{2+\delta}(\mathbb{R})$ are solution of $(\mathcal{S}_{\zeta,\epsilon,r,\tau})$, then, for z greater than $z_c = \theta^{-1}(\theta_c)$, we have:*

$$\theta(z) \leqslant \theta_c \exp\left(-\frac{c_a^-}{\kappa}(z-z_c)\right), \qquad |\alpha^\infty - \alpha(z)| \leqslant \max\left\{\frac{1}{1-\epsilon}, \mathcal{N}\right\} \theta_c \exp\left(-\frac{c_a^-}{\kappa}(z-z_c)\right),$$

$$\forall i \in \{1,\dots,N\}, \qquad 0 \leqslant S_i^\infty - S_i(z) \leqslant (1+\mathcal{N})\left(1+\frac{F_i}{M_i}\right)\theta_c \exp\left(-\frac{c_a^-}{\kappa}(z-z_c)\right),$$

$$\forall i \in \{1,\dots,N-1\}, \qquad |\mu_i^\infty - \mu_i(z)| \leqslant (2+\frac{F_i}{M_i}+\frac{F_{i+1}}{M_{i+1}})(1+\mathcal{N})\theta_c \exp\left(-\frac{c_a^-}{\kappa}(z-z_c)\right),$$

with $\mathcal{N} = \sup_{x\in[0,1]} \tilde{Y}_{Fs}(x)$.

Proof. – We have already remarked that, for $z \geqslant z_c$, $\theta(z) = \theta_c \exp(-c(z-z_c)/\kappa)$ and the first inequality is proved, thanks to Theorem 4.4.

Theorem 4.4 also gives the result: $(1-\epsilon)(\alpha^\infty - \alpha) \leqslant \theta$. Because of Proposition 4.3, we have: $\alpha - \alpha^\infty \leqslant \tilde{Y}_{Fs}(\theta) - \tilde{Y}_{Fs}(0) \leqslant \mathcal{N}\theta$. The second inequality is then proved.

The third inequality is easy to prove from (28) and the last inequality is due to $\mu_i = S_i - S_{i+1}$. □

5 Existence result for the system $(\mathcal{S}_{\zeta,\epsilon,r})$

In this section, the parameters ζ, ϵ and r are supposed fixed such that:

$$0 < \zeta < D \leqslant \frac{\kappa}{1+p}, \qquad 0 < \epsilon \leqslant 1-\beta, \qquad 0 < r \leqslant 1.$$

Moreover, we denote Φ, a C^∞ function with the property (22), and E, the space:

$$E = \left\{ v, \quad \Phi - v \in C_\mu^{2+\delta}(\mathbb{R}) \right\}.$$

We then want to prove the existence of a solution of the system $(\mathcal{S}_{\zeta,\epsilon,r})$. For that, we evaluate the topological degree of the corresponding operator of this system, using two continuous transformations, leading us to a simpler system. The first transformation is defined by the operator A_τ:

$$A_\tau \begin{pmatrix} \theta \\ \alpha \\ \mu_1 \\ \vdots \\ \mu_N \end{pmatrix} = \begin{pmatrix} \kappa\,\theta'' \\ D\,\alpha'' \\ \zeta\,\mu_1'' \\ \vdots \\ \zeta\,\mu_N'' \end{pmatrix} + c \begin{pmatrix} \theta' \\ \alpha' \\ \mu_1' \\ \vdots \\ \mu_N' \end{pmatrix} + \begin{pmatrix} \tilde{k}(\theta)\,\alpha + \epsilon\,\tilde{k}(\theta)(1-\theta-\alpha) \\ -\tilde{k}(\theta)\,\alpha + (1-\tau)R_r(\theta,\alpha)\sum M_i\,\mu_i \\ -(1-\tau)R_r(\theta,\alpha)[(M_1+F_1)\,\mu_1 - F_2\,\mu_2] - \epsilon\,\tilde{k}(\theta)\,\mu_1 \\ \vdots \\ -(1-\tau)R_r(\theta,\alpha)(M_N+F_N)\,\mu_N - \epsilon\,\tilde{k}(\theta)\,\mu_N \end{pmatrix}. \quad (33)$$

The second transformation is defined by the operator B_τ:

$$B_\tau \begin{pmatrix} \theta \\ \alpha \\ \mu_1 \\ \vdots \\ \mu_N \end{pmatrix} = \begin{pmatrix} \kappa\,\theta'' \\ D_\tau\,\alpha'' \\ \zeta_\tau\,\mu_1'' \\ \vdots \\ \zeta_\tau\,\mu_N'' \end{pmatrix} + c \begin{pmatrix} \theta' \\ \alpha' \\ \mu_1' \\ \vdots \\ \mu_N' \end{pmatrix} + \begin{pmatrix} \tilde{k}(\theta)\,\alpha + \epsilon_\tau\,\tilde{k}(\theta)(1-\theta-\alpha) \\ -\tilde{k}(\theta)\,\alpha \\ -\epsilon_\tau\,\tilde{k}(\theta)\,\mu_1 \\ \vdots \\ -\epsilon_\tau\,\tilde{k}(\theta)\,\mu_N \end{pmatrix}, \tag{34}$$

with $\epsilon_\tau = \tau + (1-\tau)\epsilon$, $D_\tau = \kappa\tau + (1-\tau)D$ and $\zeta_\tau = \kappa\tau + (1-\tau)\zeta$.

The system $(\mathcal{S}_{\zeta,\epsilon,r})$ is then $A_0 w = 0$. First, we see that these transformations yield to a simpler system $B_1 w = 0$, for which the existence and the uniqueness of solutions have already been proved. We then show how to define and use the topological degree. Finally, we give estimates in the weighted Hölder space which allow us to prove the existence of solutions for $(\mathcal{S}_{\zeta,\epsilon,r})$.

5.1 Existence and uniqueness for the transformed system

Let us begin with an investigation of system $B_1 w = 0$:

Proposition 5.1 *Up to a translation, there exists only one solution c, θ, α, μ_i of the system:*

$$\begin{cases} \kappa\,\theta'' + c\,\theta' + \tilde{k}(\theta)\,(1-\theta) = 0, \\ \kappa\,\alpha'' + c\,\alpha' - \tilde{k}(\theta)\,\alpha = 0, \\ \kappa\,\mu_i'' + c\,\mu_i' - \tilde{k}(\theta)\,\mu_i = 0, \end{cases} \tag{35}$$

such that θ, $(\alpha^\infty - \alpha)/\alpha^\infty$ and $(\mu_i^\infty - \mu_i)/\mu_i^\infty$ are in E.

Proof. – The proof comes from the existence and the uniqueness, up to a translation, of the solution $c \in \mathbb{R}$ and $\theta \in E$ of:

$$\kappa\,\theta'' + c\,\theta' + \tilde{k}(\theta)\,(1-\theta) = 0. \tag{36}$$

The unique solution of (35) is then c, θ, $\alpha = \alpha^\infty(1-\theta)$, $\mu_i = \mu_i^\infty(1-\theta)$. □

5.2 Estimates in the Hölder space

Estimates independent of τ are given by Theorem 4.4 for solutions of $A_\tau(\theta,\alpha,\mu) = 0$ and their behavior at $+\infty$ is given by Proposition 4.5. Let us give similar results for solutions of $B_\tau(\theta,\alpha,\mu) = 0$. First, we give monotonicity properties:

Proposition 5.2 *If, for $0 \leqslant \tau \leqslant 1$, there exists a scalar c and functions θ, α and μ_i in $C^{2+\delta}(\mathbb{R})$ solution of $B_\tau(\theta,\alpha,\mu) = 0$, then:*

$$c > 0, \qquad \alpha' > 0, \qquad \forall i \quad \mu_i' \geqslant 0, \qquad \theta' < 0.$$

Proof. – As for Proposition 4.3, we prove that $c > 0$. We then apply Lemma 4.2 in order to prove that $\alpha > 0$ and $\mu_i \geqslant 0$ and Lemma 4.1 in order to prove that θ is decreasing and α, μ_i are increasing. □

The a priori estimates and behavior at $+\infty$ are then given by the following theorem:

Theorem 5.3 *There exists $M > 0$, $c_b^- > 0$ and c^+ such that, for each $\tau \in [0,1]$, if there exists a scalar c and functions θ, α and μ_i in $C^{2+\delta}(\mathbb{R})$ solution of $B_\tau(\theta, \alpha, \mu) = 0$ then:*

$$\|\theta\|_{C^{2+\delta}(\mathbb{R})} \leqslant M, \qquad \|\alpha\|_{C^{2+\delta}(\mathbb{R})} \leqslant M \qquad \|\mu_i\|_{C^{2+\delta}(\mathbb{R})} \leqslant M \qquad c_b^- \leqslant c \leqslant c^+.$$

Moreover, if $z_c = \theta^{-1}(\theta_c)$, the behavior of the functions at $+\infty$ is given, for $z \geqslant z_c$, by:

$$\theta(z) \qquad = \theta_c \exp\left(-\frac{c}{\kappa}(z - z_c)\right) \qquad \leqslant \exp\left(-\frac{c_b^-}{\kappa}(z - z_c)\right) \tag{37}$$

$$\alpha^\infty - \alpha(z) \; = \alpha(z_c) \exp\left(-\frac{c}{D_\tau}(z - z_c)\right) \leqslant \exp\left(-\frac{c_b^-}{\kappa}(z - z_c)\right) \tag{38}$$

$$\mu_i^\infty - \mu_i(z) = \mu_i(z_c) \exp\left(-\frac{c}{\zeta_\tau}(z - z_c)\right) \leqslant \exp\left(-\frac{c_b^-}{\kappa}(z - z_c)\right) \tag{39}$$

Proof. – As for Theorem 4.4, we prove that $|\theta'|$ is upperly bounded by $\tilde{k}(1) + \frac{2}{\kappa}$ and then that c is upperly bounded by $c^+ = (\kappa\tilde{k}(1) + 2)/\theta_c$.

We have also $\theta \geqslant (1-\epsilon_\tau)(\alpha^\infty - \alpha)$ and we then obtain, integrating the first equation of $B_\tau(\theta, \alpha, \mu) = 0$ multiplied by θ':

$$\begin{aligned} c\int_{\mathbb{R}} (\theta'(x))^2 dx &\geqslant \int_0^{(1-\epsilon_\tau)\alpha^\infty} \tilde{k}(y)[(1-\epsilon_\tau)\alpha^\infty - y]dy + \epsilon_\tau \int_0^1 \tilde{k}(y)(1-y)dy \\ &\geqslant \epsilon \int_0^1 \tilde{k}(y)(1-y)dy. \end{aligned}$$

Then we give a lower bound for c:

$$c \geqslant c_b^- = \frac{\kappa\,\epsilon}{2 + \kappa\,\tilde{k}(1)} \int_0^1 \tilde{k}(y)(1-y)dy.$$

It is then easy to prove that α', μ_i', α'', μ_i'', θ'' and also θ''', α''', μ_i''' are bounded independently of τ.

The behavior of functions at $+\infty$ is easy to see and the theorem is proved. □

5.3 Definition of the topological degree

In this subsection, we define the topological degree for the considered operators A_τ and B_τ, after a change of variables. Let introduce a real λ such that

$$0 < \lambda < \frac{c^-}{\kappa}, \qquad c^- = \min\{c_a^-, c_b^-\} > 0,$$

where c_a^- and c_b^- are the lower bounds for the wave velocities introduced in Theorem 4.4 and in Theorem 5.3. Let ξ denote the function defined by $\xi(z) = \exp(\lambda(1 - \Phi(z))z)$, where Φ is a C^∞ function described by (22). The change of variables is given by:

$$t = \xi\theta, \qquad a = \xi\frac{\alpha^\infty - \alpha}{\alpha^\infty}, \qquad m_i = \xi\frac{\mu_i^\infty - \mu_i}{\mu_i^\infty}. \tag{40}$$

Then, we have:

$$\forall z \leqslant 0 \qquad t(z) = \theta(z) \qquad a(z) = \frac{\alpha^\infty - \alpha(z)}{\alpha^\infty} \qquad m_i(z) = \frac{\mu_i^\infty - \mu_i(z)}{\mu_i^\infty} \tag{41}$$

$$\forall z \geqslant 1 \qquad t(z) = \theta(z)e^{\lambda z} \qquad a(z) = \frac{\alpha^\infty - \alpha(z)}{\alpha^\infty}e^{\lambda z} \qquad m_i(z) = \frac{\mu_i^\infty - \mu_i(z)}{\mu_i^\infty}e^{\lambda z} \tag{42}$$

For solutions of $A_\tau(\theta, \alpha, \mu) = 0$ or of $B_\tau(\theta, \alpha, \mu) = 0$ in the weighted Hölder space, θ is necessarily decreasing and we can define $z_c = \theta^{-1}(\theta_c)$. Because of Proposition 4.5 for A_τ and Theorem 5.3 of B_τ, we know that there exists a constant $\mathcal{C}$, independent of τ, such that if $z \geqslant z_c$ and $z \geqslant 1$, then:

$$\begin{aligned}
t(z) &\leqslant \mathcal{C}\exp\left(\frac{c^-}{\kappa}z_c\right)\exp\left(\left(\lambda - \frac{c^-}{\kappa}\right)z\right),\\
|a(x)| &\leqslant \mathcal{C}\exp\left(\frac{c^-}{\kappa}z_c\right)\exp\left(\left(\lambda - \frac{c^-}{\kappa}\right)z\right),\\
|m_i(x)| &\leqslant \mathcal{C}\exp\left(\frac{c^-}{\kappa}z_c\right)\exp\left(\left(\lambda - \frac{c^-}{\kappa}\right)z\right).
\end{aligned} \tag{43}$$

After change of variables in $A_\tau(\theta, \alpha, \mu) = 0$, the obtained system is:

$$\begin{cases}
\kappa t'' + (c - 2\kappa l_1)t' + [(2l_1^2 - l_2)\kappa - l_1 c]t + F_1(t, a) = 0,\\
Da'' + (c - 2Dl_1)a' + [(2l_1^2 - l_2)D - l_1 c]a + F_2(t, a, m) = 0,\\
\zeta m_i'' + (c - 2\zeta l_1)m_i' + [(2l_1^2 - l_2)\zeta - l_1 c]m_i + F_3(t, a, m) = 0,
\end{cases} \tag{44}$$

with $l_1 = \xi'/\xi$, $l_2 = \xi''/\xi$ and

$$\begin{aligned}
F_1(t, a) \quad &= \tilde{k}\left(\frac{t}{\xi}\right)\left[(1 - \epsilon)\alpha^\infty(\xi - a) + \epsilon\,(\xi - t)\right],\\
F_2(t, a, m) &= \tilde{k}\left(\frac{t}{\xi}\right)(\xi - a) - (1 - \tau)\,R_r\left(\frac{t}{\xi}, \alpha^\infty\left(1 - \frac{a}{\xi}\right)\right)\sum M_i\frac{\mu_i^\infty}{\alpha^\infty}(\xi - \mathrm{m_i}),\\
F_3(t, a, m) &= (1 - \tau)\,R_r\left(\frac{t}{\xi}, \alpha^\infty\left(1 - \frac{a}{\xi}\right)\right)\left[(M_i + F_i)(\xi - m_i) - F_{i+1}\frac{\mu_{i+1}^\infty}{\mu_i^\infty}(\xi - m_{i+1})\right]\\
&\quad + \epsilon\,\tilde{k}\left(\frac{t}{\xi}\right)(\xi - m_i).
\end{aligned}$$

Let note $\tilde{A}_\tau$ the corresponding operator to (44), with the functional $c(t, a, m)$, defined by (20), replacing c. The problems $A_\tau(\theta, \alpha, m) = 0$ and $\tilde{A}_\tau(t, a, m) = 0$ are then equivalent because of (41) and (43). For the operator B_τ, we can also define the operator $\tilde{B}_\tau$ by the same change of variables and the problems $B_\tau(\theta, \alpha, m) = 0$ and $\tilde{B}_\tau(t, a, m) = 0$ are also equivalent.

For $z \geqslant 1$, $l_1(z) = \lambda$ and $l_2(z) = \lambda^2$, and because of the choice of λ, we know that $\lambda(\kappa\lambda - c) < 0$, $\lambda(D\lambda - c) < 0$ and $\lambda(\zeta\lambda - c) < 0$. For the operator $\tilde{A}_\tau$ and also for the operator $\tilde{B}_\tau$, the condition (21) is then valid and we can define the topological degree. These operators are homotopies.

5.4 Estimates in the weighted Hölder space

In order to find a priori estimates in the weighted Hölder space, we first prove that there is no 'infinite translation' of the solutions of $A_\tau(\theta, \alpha, m)$ and $B_\tau(\theta, \alpha, m)$, when τ varies. It means, for example, that the real z_c for which $\theta(z_c) = \theta_c$ has bounds, independent of τ. It is due to the choice of the functional $c(w)$. Then, the behavior of the solutions at $\pm\infty$ allows us to find the estimates.

Let note $\mathfrak{L} = 1 + 1/(\alpha^\infty)^2 + 1/\sum(\mu_i^\infty)^2$. Up to an eventual translation, we can choose σ such that

$$\int_{-\infty}^{0} \sigma(z)dz < \frac{\exp(c^-)}{\mathfrak{L}}. \tag{45}$$

Lemma 5.4 *For solution θ, α, m_i of $A_\tau(\theta, \alpha, m) = 0$ or of $B_\tau(\theta, \alpha, m) = 0$, the real $z_c = \theta^{-1}(\theta_c)$ is bounded independently of $\tau \in [0, 1]$.*

Proof. – The functional c is defined by:

$$c(t, a, m) = \ln\left(\int_{\mathbb{R}} [t^2(z) + a^2(z) + \sum_i m_i^2(z)]\sigma(z)dz\right).$$

In the variables θ, α and μ_i:

$$\begin{aligned}\tilde{c}(\theta, \alpha, \mu) &= c\left(\xi\theta, \xi\left(\frac{\alpha^\infty - \alpha}{\alpha^\infty}\right), \xi\left(\frac{\mu_1^\infty - \mu_1}{\mu_1^\infty}\right), \ldots, \xi\left(\frac{\mu_N^\infty - \mu_N}{\mu_N^\infty}\right)\right) \\ &= \ln\left(\int_{\mathbb{R}} \left[\theta^2(z) + \left(\frac{\alpha^\infty - \alpha}{\alpha^\infty}\right)^2 + \sum_i \left(\frac{\mu_i^\infty - \mu_i}{\mu_i^\infty}\right)^2\right] \xi^2(z)\sigma(z)dz\right).\end{aligned}$$

Let assume that $z_c > 0$. We then have:

$$\tilde{c}(\theta, \alpha, \mu) \geqslant \ln\int_0^{z_c} |\theta(z)|^2\xi^2(z)\sigma(z)dz \geqslant \ln(\theta_c^2 z_c\sigma(0)). \tag{46}$$

However, Theorem 4.4 or Theorem 5.3 say that $\tilde{c}(\theta, \alpha, \mu) \leqslant c^+$. We then have the upper bound for z_c independent of τ:

$$z_c \leqslant \frac{e^{c^+}}{\theta_c^2\sigma(0)}. \tag{47}$$

Let now assume that $z_c < 0$. Theorem 4.4 or Theorem 5.3 show that $\tilde{c}(\theta, \alpha, \mu) \geqslant c^-$. Then:

$$\begin{aligned}e^{c^-} \leqslant e^{\tilde{c}(\theta,\alpha,\mu)} = &\int_{-\infty}^{0} \left[\theta^2(z) + \left(\frac{\alpha^\infty - \alpha(z)}{\alpha^\infty}\right)^2 + \sum_i \left(\frac{\mu_i^\infty - \mu_i(z)}{\mu_i^\infty}\right)^2\right] \sigma(z)dz \\ &+ \int_0^{+\infty} \left[\theta^2(z) + \left(\frac{\alpha^\infty - \alpha(z)}{\alpha^\infty}\right)^2 + \sum_i \left(\frac{\mu_i^\infty - \mu_i(z)}{\mu_i^\infty}\right)^2\right] \xi^2(z)\sigma(z)dz.\end{aligned}$$

With the relations (43) and because $\Phi(z) \leqslant \exp(\lambda\, z)$ for $z \geqslant 0$, we obtain:

$$e^{c^-} \leqslant \mathfrak{L} \int_{-\infty}^{0} \sigma(z)dz + \mathfrak{L} \int_{0}^{+\infty} e^{2\lambda z} e^{-2c^- z_c/\kappa} dz.$$

So:

$$\frac{e^{c^-}}{\mathfrak{L}} \leqslant \int_{-\infty}^{0} \sigma(z)dz + \frac{1}{2(c^-/\kappa - \lambda)} e^{2c^- z_c/\kappa},$$

and because of (45), the lemma is proved. □

Before giving more estimates let us prove a lemma, which allows us to evaluate the behavior at $+\infty$ or at $-\infty$ of solution of some ordinary differential equation:

Lemma 5.5 *Let note I the interval $]-\infty, z_0]$, a and b two positive reals and y a function in $C^{2+\delta}(I)$ such that:*

- *y is non negative,*
- *$y(z)$ tends to 0 when z tends to $-\infty$,*
- *for all $z \in I$, $y''(z) + a\, y'(z) - b\, y(z) \geqslant 0$.*

Then, for $z \in I$:

$$y(z) \leqslant y(z_0) \exp\left(\frac{\sqrt{a^2+4b} - a}{2}(z - z_0)\right) \tag{48}$$

$$0 \leqslant y'(z) \leqslant \left[y'(z_0) + \frac{a + \sqrt{a^2+4b}}{2} y(z_0)\right] \exp\left(\frac{\sqrt{a^2+4b} - a}{2}(z - z_0)\right) \tag{49}$$

Proof. – Let introduce the function Λ_1 defined for $z \in I$ by:

$$\Lambda_1(z) = \left(y'(z) + \frac{a - \sqrt{a^2+4b}}{2} y(z)\right) \exp\left(\frac{a + \sqrt{a^2+4b}}{2} z\right).$$

A computation of derivative shows that this function is increasing. Moreover, it tends to 0 at $-\infty$. Then it is non negative and $y'(z) + (a - \sqrt{a^2+4b})/2\, y(z) \geqslant 0$. We also introduce the function Γ defined for $z \in I$ by:

$$\Gamma(z) = y(z) \exp\left(\frac{a - \sqrt{a^2+4b}}{2} z\right).$$

This function is also increasing because of the previous result, and simply writing $\Gamma(z) \leqslant \Gamma(z_0)$, we show the validity of the first inequality (48).

Let now introduce the function Λ_2 defined for $z \in I$ by:

$$\Lambda_2(z) = \left(y'(z) + \frac{a + \sqrt{a^2+4b}}{2} y(z)\right) \exp\left(\frac{a - \sqrt{a^2+4b}}{2} z\right).$$

This function is increasing and then $\Lambda_2(z) \leqslant \Lambda_2(z_0)$ for $z \leqslant z_0$. Because y is non negative, we obtain the second inequality (49) and the lemma is proved. □

We can then give estimates in the weighted Hölder space:

Theorem 5.6 *There exists a constant $M > 0$ independent of $\tau \in [0,1]$ such that, if t, a, m_i in E are solution of $\tilde{A}_\tau(t, a, m_{1,\tau}, \ldots, m_{N,\tau}) = 0$, then:*

$$\|\Phi - t\|_{C^{2+\delta}_\mu(\mathbb{R})} \leqslant M, \qquad \|\Phi - a\|_{C^{2+\delta}_\mu(\mathbb{R})} \leqslant M, \qquad \|\Phi - m_i\|_{C^{2+\delta}_\mu(\mathbb{R})} \leqslant M, \tag{50}$$

Proof. – Let note $\theta = t/\zeta \in E$, $\alpha = \alpha^\infty(1 - a/\zeta) \in E$, $\mu_i = \mu_i^\infty(1 - m_i/\zeta)$ which are solution of $A_\tau(\theta, \alpha, \mu_1, \ldots, \mu_N) = 0$.

Theorem 4.4, relations (41) and (43) and upper bound for $z_c = \theta^{-1}(\theta_c)$ in Lemma 5.4 give upper bound independent of τ for the norms of t, a and m_i in the Hölder space $C^{2+\delta}(\mathbb{R})$. Estimates independent of τ in the space $C^{2+\delta}_\mu(\mathbb{R}^+)$ are straightforward because of Proposition 4.5. We then only have to find estimates independent of τ in the space $C^{2+\delta}_\mu(\mathbb{R}^-)$ of the functions $1 - \theta$, α and μ_i, which are respectively equal to $1 - t$, $\alpha^\infty(1 - a)$ and $\mu_i^\infty(1 - m_i)$ on $\mathbb{R}^-$.

We know that $\theta(z_c) = \theta_c$, $\theta'(z_c) = -c\theta_c/\kappa$ and $\theta''(z_c) = c^2\theta_c/\kappa^2 > 0$. Let then note z_d the larger real in $]-\infty, z_c]$ such that $\theta''(z_d) = 0$ (θ'' have necessarily a zero in $]-\infty, z_c]$: in the other case, θ' would be increasing and upperly bounded by $\theta'(z_c) = -c\theta_c/\kappa$ in $]-\infty, z_c]$, but it is impossible). The function θ' is increasing on $[z_d, z_c]$ and then:

$$\forall z \in [z_d, z_c] \qquad \theta'(z) \leqslant -\frac{c\theta_c}{\kappa}. \tag{51}$$

An integration of this inequality between z_d and z_c give a lower bound for z_d:

$$z_d \geqslant z_c - \frac{\kappa}{c^- \theta_c}. \tag{52}$$

So, as z_c, z_d admits a lower bound independent of τ.

Moreover, for $z = z_d$, the equation on θ give:

$$-c\theta'(z_d) = (1-\epsilon)\tilde{k}(\theta(z_d))(1 - \alpha(z_d)) + \epsilon\tilde{k}(\theta(z_d))(1 - \theta(z_d)) \leqslant \tilde{k}(\theta(z_d)).$$

Applying (51) to $z = z_d$ and because $\tilde{k}(\theta(z))$ is a decreasing function, we obtain, for $z \leqslant z_d$:

$$\tilde{k}(\theta(z)) \geqslant \frac{c^2\theta_c}{\kappa}. \tag{53}$$

Let consider increasing variables $1 - \theta$, $S_{\theta\alpha} = (1-\theta)\kappa/[D(1-\epsilon)] - \alpha$ and $S_i = \sum_{k=i}^N \mu_k$. Because of (53), those variables are such that, for $z \leqslant z_d$:

$$-\kappa\,\theta''(z) - c\,\theta'(z) \geqslant \epsilon\frac{c^2\,\theta_c}{\kappa}(1 - \theta(z)), \tag{54}$$

$$D\,S''_{\theta\alpha}(z) + c\,S'_{\theta\alpha}(z) \geqslant \epsilon\frac{D^2\,c^2\,\theta_c}{\kappa^2}S_{\theta\alpha}(z), \tag{55}$$

$$\zeta\,S''_i(z) + c\,S'_i(z) \geqslant \epsilon\frac{c^2\,\theta_c}{\kappa}S_i(z). \tag{56}$$

We apply Lemma 5.5 to those variables. First for $1 - \theta$, we obtain, for $z \leqslant z_d$:

$$0 \leqslant 1 - \theta(z) \leqslant [1 - \theta(z_d)] \exp\left(\frac{c}{2\,\kappa}[\sqrt{1 + 4\epsilon\,\theta_c} - 1](z - z_d)\right)$$
$$0 \leqslant -\theta'(z) \leqslant \left[-\theta'(z_d) + \frac{c}{2\,\kappa}(1 + \sqrt{1 + 4\epsilon\,\theta_c})\theta(z_d)\right] \exp\left(\frac{c}{2\,\kappa}[\sqrt{1 + 4\epsilon\,\theta_c} - 1](z - z_d)\right)$$

We have also remarked that there exists a constant M, independent of τ, which upperly bound the norms of θ, α and μ_i in the Hölder space $C^{2+\delta}(\mathbb{R})$. The constants c^- and c^+ are also independent of τ and $0 < c^- \leqslant c \leqslant c^+$. Then:

$$0 \leqslant 1 - \theta(z) \leqslant \exp\left(\frac{c^-}{2\,\kappa}[\sqrt{1+4\epsilon\,\theta_c}-1](z-z_d)\right), \tag{57}$$

$$0 \leqslant -\theta'(z) \leqslant \left[M + \frac{c^+}{2\,\kappa}(1+\sqrt{1+4\epsilon\,\theta_c})\right]\exp\left(\frac{c^-}{2\,\kappa}[\sqrt{1+4\epsilon\,\theta_c}-1](z-z_d)\right). \tag{58}$$

In the same way, for $S'_{\theta\alpha}$, if $z \leqslant z_d$:

$$0 \leqslant S'_{\theta\alpha}(z) \leqslant \left[M + \frac{c^+}{2\,D}\left(1+\sqrt{1+4\epsilon\frac{D^2}{\kappa^2}\theta_c}\right)\right]\exp\left(\frac{c^-}{2\,D}\left[\sqrt{1+4\epsilon\frac{D^2}{\kappa^2}\theta_c}-1\right](z-z_d)\right).$$

As $1-\theta$, the functions α and α' then exponentially tend to zero at $-\infty$:

$$\begin{aligned}
0 \leqslant \alpha(z) \leqslant& \frac{\kappa}{(1-\epsilon)D}(1-\theta(z)) \leqslant \frac{\kappa}{(1-\epsilon)D}\exp\left(\frac{c^-}{2\,\kappa}[\sqrt{1+4\epsilon\,\theta_c}-1](z-z_d)\right),\\
\alpha'(z) \leqslant& -\frac{\kappa}{(1-\epsilon)D}\theta'(z)\\
\leqslant& \frac{\kappa}{(1-\epsilon)D}\left[M + \frac{c^+}{2\,\kappa}(1+\sqrt{1+4\epsilon\,\theta_c})\right]\exp\left(\frac{c^-}{2\,\kappa}[\sqrt{1+4\epsilon\,\theta_c}-1](z-z_d)\right),\\
-\alpha'(z) \leqslant& S'_{\theta\alpha}(z)\\
\leqslant& \left[M + \frac{c^+}{2\,D}\left(1+\sqrt{1+4\epsilon\frac{D^2}{\kappa^2}\theta_c}\right)\right]\exp\left(\frac{c^-}{2\,D}\left[\sqrt{1+4\epsilon\frac{D^2}{\kappa^2}\theta_c}-1\right](z-z_d)\right).
\end{aligned}$$

Finally, applying Lemma 5.5 to S_i, we have, for $z \leqslant z_d$:

$$0 \leqslant S_i(z) \leqslant \exp\left(\frac{c^-}{2\,\zeta}\left[\sqrt{1+4\epsilon\frac{\zeta}{\kappa}\theta_c}-1\right](z-z_d)\right) \tag{59}$$

$$0 \leqslant S'_i(z) \leqslant \left[M + \frac{c^+}{2\,\zeta}\left(1+\sqrt{1+4\epsilon\frac{\zeta}{\kappa}\theta_c}\right)\right]\exp\left(\frac{c^-}{2\,\zeta}\left[\sqrt{1+4\epsilon\frac{\zeta}{\kappa}\theta_c}-1\right](z-z_d)\right) \tag{60}$$

The functions μ_i and μ'_i then exponentially tend to zero at $-\infty$ because $0 \leqslant \mu_i \leqslant S_i$, $\mu'_i = S'_i - S'_{i+1} \leqslant S'_i$ and $-\mu'_i = S'_{i+1} - S'_i \leqslant S'_{i+1}$.

We then have an upper bound, independent of τ, for the norm of $1-\theta$, α and μ_i in the space $C^{2+\delta}_\mu(\mathbb{R}^-)$ and the theorem is proved. □

Theorem 5.7 *There exists a constant $M > 0$ such that, for each $\tau \in [0,1]$, if t, a, m_i in E are solution of $\tilde{B}_\tau(t, a, m_1, \ldots, m_N) = 0$, then:*

$$\|\Phi - t\|_{C^{2+\delta}_\mu(\mathbb{R})} \leqslant M, \qquad \|\Phi - a\|_{C^{2+\delta}_\mu(\mathbb{R})} \leqslant M, \qquad \|\Phi - m_i\|_{C^{2+\delta}_\mu(\mathbb{R})} \leqslant M, \tag{61}$$

Proof. – Let note $\theta = t/\zeta \in E$, $\alpha = \alpha^\infty(1 - a/\zeta) \in E$, $\mu_i = \mu_i^\infty(1 - m_i/\zeta)$ which are solution of $B_\tau(\theta, \alpha, \mu_1, \ldots, \mu_N) = 0$.

Theorem 5.3, relations (41) and (43) and the upper bound for $z_c = \theta^{-1}(\theta_c)$ in Lemma 5.4 give an upper bound independent of τ for the norms of t, a and m_i in the Hölder space $C^{2+\delta}(\mathbb{R})$. Estimates independent of τ in the space $C_\mu^{2+\delta}(\mathbb{R}^+)$ are straightforward because of Theorem 5.3. We then only have to find estimates independent of τ in the space $C_\mu^{2+\delta}(\mathbb{R}^-)$ of the functions $1 - \theta$, α and μ_i, which are respectively equal to $1 - t$, $\alpha^\infty(1 - a)$ and $\mu_i^\infty(1 - m_i)$ on $\mathbb{R}^-$.

We introduce, as in the proof of Theorem 5.6, the largest real z_d in $]-\infty, z_c]$ such that $\theta''(z_d) = 0$ and we have:

$$z_d \geqslant z_c - \frac{\kappa}{c^-\,\theta_c}, \qquad \tilde{k}(\theta(z)) \geqslant \frac{c^2\theta_c}{\kappa}.$$

The increasing variables $1 - \theta$, α and μ_i are then such that, for $z \leqslant z_d$:

$$-\kappa\,\theta''(z) - c\,\theta'(z) \geqslant \epsilon\frac{c^2\,\theta_c}{\kappa}(1 - \theta(z)), \tag{62}$$

$$D_\tau\,\alpha''(z) + c\,\alpha'(z) \geqslant \frac{c^2\,\theta_c}{\kappa}\alpha(z), \tag{63}$$

$$\zeta_\tau\,\mu_i''(z) + c\,\mu_i'(z) \geqslant \epsilon\frac{c^2\,\theta_c}{\kappa}\mu_i(z). \tag{64}$$

Applying Lemma 5.5 to those variables, we obtain:

$$\begin{aligned}
&0 \leqslant 1 - \theta(z) \leqslant \exp\left(\frac{c^-}{2\,\kappa}[\sqrt{1 + 4\epsilon\,\theta_c} - 1](z - z_d)\right),\\
&0 \leqslant -\theta'(z) \leqslant \left[M + \frac{c^+}{2\,\kappa}(1 + \sqrt{1 + 4\epsilon\,\theta_c})\right]\exp\left(\frac{c^-}{2\,\kappa}[\sqrt{1 + 4\epsilon\,\theta_c} - 1](z - z_d)\right),\\
&0 \leqslant \alpha(z) \leqslant \exp\left(\frac{c^-}{2\,\kappa}\left[\sqrt{1 + 4\epsilon\,\theta_c} - 1\right](z - z_d)\right),\\
&0 \leqslant \alpha'(z) \leqslant \left[M + \frac{c^+}{2\,D}(1 + \sqrt{1 + 4\epsilon\,\theta_c})\right]\exp\left(\frac{c^-}{2\,\kappa}[\sqrt{1 + 4\epsilon\,\theta_c} - 1](z - z_d)\right),\\
&0 \leqslant \mu_i(z) \leqslant \exp\left(\frac{c^-}{2\,\kappa}\left[\sqrt{1 + 4\epsilon\,\theta_c} - 1\right](z - z_d)\right),\\
&0 \leqslant \mu_i'(z) \leqslant \left[M + \frac{c^+}{2\,\zeta}(1 + \sqrt{1 + 4\epsilon\,\theta_c})\right]\exp\left(\frac{c^-}{2\,\kappa}[\sqrt{1 + 4\epsilon\,\theta_c} - 1](z - z_d)\right).
\end{aligned}$$

We then have an upper bound, independent of τ, for the norm of $1 - \theta$, α and μ_i in the space $C_\mu^{2+\delta}(\mathbb{R}^-)$ and the theorem is proved. □

5.5 Conclusion

For ζ, ϵ and r fixed such that $0 < \zeta < D \leqslant \dfrac{\kappa}{1+p}$, $0 < \epsilon \leqslant 1 - \beta$ and $0 < r \leqslant 1$, we then have defined two homotopies between the operator of $(\mathcal{S}_{\zeta,\epsilon,r})$ and a simpler operator vanishing only at one point. Moreover, we also have given a priori estimates in the weighted Hölder space, independent of τ. We can then conclude with the existence of solution of $(\mathcal{S}_{\zeta,\epsilon,r})$.

Theorem 5.8 *There exists a scalar $c > 0$ and functions θ, α, μ_i solution of $(\mathcal{S}_{\zeta,\epsilon,r})$, such that θ, $(\alpha^\infty - \alpha)/\alpha^\infty$ and $(\mu_i^\infty - \mu_i)/\mu_i^\infty$ are in E.*

Proof. – In fact, we prove that there exists solution t, a, m_i in E of system $\tilde{A}_0(t, a, m) = 0$, evaluating its topological degree.

Because of Theorem 5.7, an open ball $\mathcal{B}_1$ can be chosen in E^{N+2}, which contain all the zero of the homotopy $\tilde{B}_\tau$. The topological degrees of $(\tilde{B}_0 = \tilde{A}_1, \mathcal{B}_1)$ and $(\tilde{B}_1, \mathcal{B}_1)$ are then equal. Moreover, the problem $B_1 w = 0$ admits a unique solution w_0. Because of the functionalization of c (20), there is no eigenvalue for linearized operator at w_0. The topological degree of $(B_1, \mathcal{B}_1)$ is then 1 or -1. So, the topological degree of $(A_1, \mathcal{B}_1)$ is different from zero.

Because of Theorem 5.6, an open ball $\mathcal{B}_2$ can be chosen in E^{N+2}, which contain all the zero of the homotopy $\tilde{A}_\tau$. The topological degrees of $(\tilde{A}_0, \mathcal{B}_2)$ and $(\tilde{A}_1, \mathcal{B}_2)$ are then equal and not equal to zero. The theorem is then proved. □

6 Limits

We saw that, for each $\zeta \in]0, D[$, with $D \leqslant \dfrac{\kappa}{1+p}$, for each $\epsilon \in]0, 1-\beta]$ and for each $r \in]0, 1]$, there exists a solution of $(\mathcal{S}_{\zeta,\epsilon,r})$. We then show successively, the existence of solution for $(\mathcal{S}_{\zeta,\epsilon,0})$, for $(\mathcal{S}_\zeta)$, for $(\mathcal{S}_N)$ and then for $(\mathcal{S}_0)$.

6.1 Limit on the parameter r

Let now note θ_r, α_r and $\mu_{i,r}$ a solution of $(\mathcal{S}_{\zeta,\epsilon,r})$, for each $r \in]0, 1[$. Theorem 4.4 give estimates in $C^{2+\delta}(\mathbb{R})$ for θ_r, α_r and $\mu_{i,r}$, independent of r but eventually dependent of ζ. These a priori estimates allows us to pass to the limit when r tends to 0, with convergent subsequences, thus giving a solution of the limit system. But the used compactness property is only valid for function defined in bounded domains. The limit solution can be extended to the whole real line but the convergence can then not be uniform on $\mathbb{R}$. So the boundary conditions of each function have to be recovered, thanks to inequalities given in Proposition 4.3 and in Theorem 4.4. We then have the Theorem:

Theorem 6.1 *There exists a scalar c and functions θ, α and μ_i in $C^{2+\delta}(\mathbb{R})$ solution of the system $(\mathcal{S}_{\zeta,\epsilon,0})$.*

Proof. – Let note $r_n = 1/n$ for $n \geqslant 1$ and c_n θ_n, α_n, $\mu_{i,n}$ a solution of $(\mathcal{S}_{\zeta,\epsilon,r_n})$ such that $\theta_n(0) = \theta_c$. Since the sequence $(\theta_n)_{n\in\mathbb{N}}$ is bounded in $C^{2+\delta}(\mathbb{R})$, then for any bounded subset $[-q, q]$, with $q \in \mathbb{N}^*$, we can choose a subsequence $(\theta_{j,q})_{j\in\mathbb{N}}$ which converges to some limiting function θ in $C^2([-q, q])$. We can extend the limiting function θ to the whole real line and choose a subsequence such that the functions $\theta_{j_k} - \theta$ converge to zero in C^2 on every bounded real interval. Moreover, the function θ is in $C^{2+\delta}(\mathbb{R})$: indeed, it is sufficient to verify the inequality

$$\forall x \in \mathbb{R}, \quad \forall y \in \mathbb{R}/\{x\}, \qquad \frac{|\theta''(x) - \theta''(y)|}{|x-y|^\delta} \leqslant M.$$

It follows from the similar inequality for the functions θ_{j_k} with the constant M independent of n and from the C^2 convergence of the sequence $(\theta_{j_k})_{k\in\mathbb{N}}$ to θ in each bounded interval. We also can extract from $(\alpha_{j_k})_{k\in\mathbb{N}}$

and $(\mu_{i,j_k})_{k\in\mathbb{N}}$ subsequences still noted $(\alpha_{j_k})_{k\in\mathbb{N}}$ and $(\mu_{i,j_k})_{k\in\mathbb{N}}$ which converge to functions α and μ_i of $C^{2+\delta}(\mathbb{R})$, the convergence being C^2 on all bounded interval. Without loss of generality we can assume that the sequence $(c_{j_k})_{k\in\mathbb{N}}$, bounded by c_a^- and c^+, converge to $c>0$. Moreover, for each $z\in\mathbb{R}$, the sequence $R_{r_n}(\theta_n(z),\alpha_n(z))$ converges to $\tilde{R}_S(\theta(z),\alpha(z))$. Then, θ, α and μ_i are solution of the equations of $(\mathcal{S}_{\zeta,\epsilon,0})$. Because $\tilde{B}_M$ is non negative, they are regular functions (they belong to the Hölder space $C^{2+\delta}(\mathbb{R})$). We still have to find the limits of the functions at $\pm\infty$.

The function θ is decreasing and $S_i=\sum_{j=i}^N\mu_j$, for $i=1$ to N, are increasing functions. Moreover, those functions are bounded: then, they admit finite limits at $\pm\infty$. Since $\theta(0)=\theta_c$, we have for $z\geqslant 0$: $\theta(z)=\theta_c\exp(-c\,z/\kappa)$. Then $\theta(z)$ tends to 0 as z tends to $+\infty$ and $\theta(z)>\theta_c$ for $z<0$. Because of the differential equation for θ, we know that $\tilde{k}(\theta(z))\alpha(z)$ tends to 0 as z tends to $-\infty$. Then, necessarily, $\alpha(z)$ tends to 0 as z tends to $-\infty$.

Because of the differential equation for S_1, we know that $R_0(\theta(z),\alpha(z))\sum_{j=1}^N M_j\mu_j(z)$ tends to 0 as z tends to $\pm\infty$. At $-\infty$, $R_0(\theta(z),\alpha(z))$ has a positive limit. The functions $\mu_j(z)$ being positive, they then tend to 0 as z tends to $-\infty$. At $+\infty$, $R_0(\theta(z),\alpha(z))$ necessarily tends to 0 and then, $\alpha(z)$ tends to $\tilde{Y}_{Fs}(0)=\alpha^\infty$ as z tends to $+\infty$.

Integrating on $\mathbb{R}$ the sum of the equation of $(\mathcal{S}_{\zeta,\epsilon,0})$, we obtain: $\lim_{+\infty}S_1=\lim_{-\infty}\theta-\alpha^\infty\leqslant 1-\alpha^\infty$. Moreover, for $z\geqslant 0$, we know that $\theta(z)+\alpha(z)+S_1(z)\geqslant 1$. Passing to the limit $z\to+\infty$, we have $\lim_{+\infty}S_1\geqslant 1-\alpha^\infty$. Then $S_1(z)$ tends to $1-\alpha^\infty=S_1^\infty$ as z tends to $+\infty$ and $\theta(z)$ tends to 1 as z tends to $-\infty$.

Let note l_j the limit of $\mu_j(z)$ as $z\to+\infty$. Because of (27), we have:

$$\forall i\in\{1,\dots,N\},\qquad \sum_{j=i}^N\prod_{k=j+1}^N\left(1+\frac{F_k}{M_k}\right)(\mu_j^\infty-l_j)\geqslant 0.$$

Let note $\lambda_j=\prod_{k=j+1}^N\left(1+\frac{F_k}{M_k}\right)-\prod_{k=j+2}^N\left(1+\frac{F_k}{M_k}\right)$. The previous equation then reads:

$$\sum_{j=i}^N\sum_{k=j}^N\lambda_k(\mu_j^\infty-l_j)\geqslant 0.$$

Introducing the sums $\mathcal{S}_k=\sum_{j=1}^k(\mu_j^\infty-l_j)$, we have: $\sum_{k=i}^N\lambda_k\mathcal{S}_k\geqslant(\sum_{k=i}^N\lambda_k)\mathcal{S}_{i-1}$, with $\mathcal{S}_0=0$. We know that $\mathcal{S}_N=0$. Then it is easy to prove that $\mathcal{S}_i\leqslant 0$ for $i\in\{1,\dots,N\}$. But the first inequality is $\sum_{k=1}^N\lambda_kS_k\geqslant 0$. Then $\mathcal{S}_i=0$ for all i, the $\mu_j(z)$ tend to μ_j^∞ as $z\to+\infty$ and the theorem is proved. □

6.2 Limit on the parameter ϵ

Thanks to the estimates for $\tau=r=0$ of Theorem 4.4 and to inequalities of Proposition 4.3 and of Theorem 4.4, we can pass to the limit when ϵ tends to 0:

Theorem 6.2 *There exists a scalar c and functions θ, α and μ_i in $C^{2+\delta}(\mathbb{R})$ solution of the system $(\mathcal{S}_\zeta)$.*

Proof. – As for the proof of Theorem 6.1, because of estimates of Theorem 4.4, we can find a sequence ϵ_n, converging to zero, such that the solutions c_n, θ_n, α_n, $\mu_{i,n}$ of $(\mathcal{S}_{\zeta,\epsilon_n,0})$ converge to a scalar c and functions θ, α and μ_i in $C^{2+\delta}(\mathbb{R})$, the convergence being C^2 on all bounded interval.

The inequalities used in order to recover the limits of the functions at $\pm\infty$ can still be used here. The theorem is then proved. □

6.3 Limit on the diffusion coefficient

Let now note θ_ζ, α_ζ and $\mu_{i,\zeta}$ a solution of $(\mathcal{S}_\zeta)$, for each $\zeta \in [0, D]$. Estimates of Theorem 4.4 and inequalities of Proposition 4.3 and of Theorem 4.4 are also valid for $\tau = \epsilon = r = 0$. We can then pass to the limit when ζ tends to 0 in the system $(\mathcal{S}_\zeta)$ and also in the inequalities, thus giving other inequalities useful for the next step:

Theorem 6.3 *There exists a scalar c and functions θ, α and μ_i in $C^{2+\delta}(\mathbb{R})$ solution of the system $(\mathcal{S}_N)$. Moreover,*

$$0 < c^- \leqslant c \leqslant c^+, \qquad 0 \leqslant \alpha \leqslant 1, \qquad \tilde{B}_M(\theta, \alpha) \geqslant 0, \qquad \forall i \quad \mu_i \geqslant 0, \qquad \forall i \quad S_i' \geqslant 0, \qquad \theta' \leqslant 0.$$

Proof. – The proof is very similar to the proof of Theorem 6.2: because of estimates of Theorem 4.4, we can find a sequence ζ_n, converging to zero, such that the solutions c_n, θ_n, α_n, $\mu_{i,n}$ of $(\mathcal{S}_{\zeta_n})$ converge to a scalar c and functions in $C^{2+\delta}(\mathbb{R})$: θ, α and μ_i, the convergence being C^2 on all bounded interval.

The inequalities used in order to recover the limits of the functions at $\pm\infty$ can still be used here and the theorem is proved. □

6.4 Limit on the number of sections

We have here to pass to the limit when the number of sections goes to infinity. We then use the numerical analysis of the sectional approach given in [21] [18]. But in these articles, the distribution function for the kinetic problem was known. Here, we have to prove the existence of the distribution function for the limit problem and to recover the kinetic equation. In a first step, we then construct an approximation of this distribution. In a second step, we show the convergence of this approximation and of the solutions for the gaseous phase when the number of sections goes to infinity.

6.4.1 Construction of an approximation of the distribution function

In order to construct an approximation of the distribution function, we introduce the characteristics corresponding to gaseous variables solutions for the discretized system $(\mathcal{S}_N)$. A distribution function can then be defined as the transport of the distribution ψ_0 along these characteristics. We then check in a proposition, using the numerical analysis of the sectional method, that the source term for the gaseous phase in the discretized system is close to the one calculated from the approximated distribution function. This last result will be essential for the passage to the limit in the equations.

Let us note, $c^{(N)}$, $\theta^{(N)}$, $\alpha^{(N)}$ and $\mu_i^{(N)}$ in $C^{2+\delta}(\mathbb{R})$ solution of the system $(\mathcal{S}_N)$, with $\theta^{(N)}(0) = \theta_c$. We also renote $F_i^{(N)}$ and $M_i^{(N)}$ the constants in the equations on the $\mu_i^{(N)}$. The function $\tilde{R}_S(\theta^{(N)}(z), \alpha^{(N)}(z))$

is non negative and upperly bounded by $\varphi(1+\mathcal{N})\theta^{(N)}(z)/\delta$ and $\theta^{(N)}$ is exponentially decreasing on $\mathbb{R}^+$. Then, the function $\tilde{R}_S(\theta^{(N)}(z),\alpha^{(N)}(z))$ is integrable on each interval $[z,+\infty[$. Let then introduce the characteristics $X^{(N)}$ and the function $\psi^{(N)}$ defined by:

$$X^{(N)}(z,S) = S + \frac{1}{c^{(N)}}\int_z^{\infty} \tilde{R}_S(\theta^{(N)}(y),\alpha^{(N)}(y))\,dy \tag{65}$$

$$\psi^{(N)}(z,S) = \psi_0(X(z,S)). \tag{66}$$

The function $\psi^{(N)}$ is a solution of the system:

$$\begin{aligned} &c^{(N)}\partial_z\psi(z,S) + R_S(\theta^{(N)}(y),\alpha^{(N)}(y))\partial_S\psi^{(N)}(z,S) = 0 \qquad && z\in\mathbb{R} \quad S\in\mathbb{R}^+ \\ &\psi^{(N)}(+\infty,S) = \psi_0(S) && S\in\mathbb{R}^+. \end{aligned}$$

We want then to prove that the source term S^m is close to a moment of this function $\psi^{(N)}$:

Proposition 6.4 *There exists a constant $\mathcal{K}$ independent of N and such that, for all $z\in\mathbb{R}$:*

$$\left|\sum_{i=1}^{N} M_i^{(N)}\,\mu_i^{(N)}(z) - \frac{\rho_l}{4\sqrt{\pi}}\int_0^{+\infty} S^{1/2}\,\psi^{(N)}(z,S)\,dS\right| \leqslant \mathcal{K}\,\Delta S.$$

Proof. – First, because of equations (65) and (66), the function $\psi^{(N)}(z,S)$ is null for $z\leqslant \mathrm{z}_m$ such that:

$$\int_{z_m}^{+\infty} \tilde{R}_S(\theta^{(N)}(z),\alpha^{(N)}(z))dz = c^{(N)}\,S_{max}.$$

Such a z_m exists because the function $\tilde{R}_S(\theta^{(N)}(z),\alpha^{(N)}(z))$ tends to $-\varphi\ln\delta$ when z tends to $-\infty$.

Let introduce the mass densities in each section, calculated with the distribution function $\psi^{(N)}$:

$$\bar{\mu}_i^{(N)}(z) = \frac{\rho_l}{6\sqrt{\pi}}\int_0^{+\infty} S^{3/2}\,\psi^{(N)}(z,S)\,dS.$$

In order to prove the proposition, we introduce two terms:

$$\begin{aligned} \left|\sum_{i=1}^{N} M_i^{(N)}\mu_i^{(N)}(z) - \frac{\rho_l}{4\sqrt{\pi}}\int_0^{+\infty} S^{1/2}\psi^{(N)}(z,S)dS\right| &\leqslant \left|\sum_{i=1}^{N} M_i^{(N)}(\mu_i^{(N)}(z)-\bar{\mu}_i^{(N)}(z))\right| \\ &+ \left|\sum_{i=1}^{N} M_i^{(N)}\bar{\mu}_i^{(N)}(z) - \frac{\rho_l}{4\sqrt{\pi}}\int_0^{+\infty} S^{1/2}\psi^{(N)}(z,S)dS\right|. \end{aligned} \tag{67}$$

We begin by giving an upper bound of the first term. It is proved in [21], [18] that the used multi-fluid method is of order 1. More precisely, if

$$\gamma_i^{(N)} = c^{(N)}\partial_z\bar{\mu}_i^{(N)} - \tilde{R}_S(\theta^{(N)},\alpha^{(N)})[(M_i^{(N)}+F_i^{(N)})\,\bar{\mu}_i^{(N)} - F_{i+1}^{(N)}\,\bar{\mu}_{i+1}^{(N)}]$$

then $\gamma_i^{(N)}(z) = 0$ for $z\leqslant z_m$ and for $z > z_m$:

$$|\gamma_i^{(N)}(z)| \leqslant \mathcal{C}\tilde{R}_S(\theta^{(N)}(z),\alpha^{(N)}(z))\sqrt{S_i}\,\Delta S^2.$$

with

$$\mathcal{C} = \frac{\rho_l}{6\sqrt{\pi}} \left[6\|\psi_0'\|_{L^\infty(\mathbb{R}^+)} + \frac{3}{4} S_{max} \|\psi_0''\|_{L^\infty(\mathbb{R}^+)} \right].$$

Let now introduce, for each section, the difference between the calculated mass density and the 'real' mass density: $e_i^{(N)}(z) = \mu_i^{(N)}(z) - \bar{\mu}_i^{(N)}(z)$. Those functions are such that:

$$c^{(N)} \partial_z e_i^{(N)} - \tilde{R}_S(\theta^{(N)}, \alpha^{(N)})[(M_i^{(N)} + F_i^{(N)})\, e_i^{(N)} - F_{i+1}^{(N)}\, e_{i+1}^{(N)}] = -\gamma_i^{(N)},$$

and $e_i^{(N)}(z)$ tends to 0 when z tends to $\pm\infty$. Then, $|e_i^{(N)}|$ reaches its maximum at z_o such that $\partial_z e_i^{(N)}(z_o) = 0$. We have then the inequality:

$$\|e_i^{(N)}\|_{L^\infty(\mathbb{R})} \leqslant |e_i^{(N)}(z_o)| \leqslant \frac{F_{i+1}^{(N)}}{M_i^{(N)} + F_i^{(N)}} |e_{i+1}^{(N)}(z_o)| + \frac{1}{M_i^{(N)} + F_i^{(N)}} \left| \frac{\gamma_i^{(N)}(z_o)}{\tilde{R}_S(\theta^{(N)}(z_o), \alpha^{(N)}(z_o))} \right|.$$

Let note $\beta_i = \int_{S_{i-1}}^{S_i} \sigma^{3/2} d\sigma$. Then:

$$\|e_i^{(N)}\|_{L^\infty(\mathbb{R})} \leqslant \frac{\beta_i}{\beta_{i+1}} \|e_{i+1}^{(N)}\|_{L^\infty(\mathbb{R})} \mathcal{C} \frac{\beta_i}{S_i} \Delta S^2.$$

We obtain an upper bound for $\|e_i^{(N)}\|_{L^\infty(\mathbb{R})}$:

$$\|e_i^{(N)}\|_{L^\infty(\mathbb{R})} \leqslant \mathcal{C} \Delta S^2 \left(\sum_{k=i}^{N} \frac{1}{S_k} \right) \beta_i.$$

So, the first term of (67) can be upperly bounded:

$$\begin{aligned} \left| \sum_{i=1}^{N} M_i^{(N)} e_i^{(N)}(z) \right| &\leqslant \mathcal{C} \Delta S^2 \sum_{i=1}^{N} \left(\sum_{k=i}^{N} \frac{1}{S_k} \right) \frac{3}{2} \int_{S_{i-1}}^{S_i} \sigma^{1/2} d\sigma \leqslant \mathcal{C} \Delta S^2 \sum_{k=1}^{N} \sqrt{S_k} \\ &\leqslant \mathcal{C} \left[\frac{2}{3} S_{max}^{3/2} + \Delta S \sqrt{S_{max}} \right] \Delta S \end{aligned}$$

We now have to upperly bound the second term of (67).

$$\left| \sum_{i=1}^{N} M_i^{(N)} \bar{\mu}_i^{(N)}(z) - \frac{\rho_l}{4\sqrt{\pi}} \int_0^{+\infty} S^{1/2} \psi^{(N)}(z, S) dS \right| \leqslant \frac{\rho_l}{4\sqrt{\pi}} \|\psi_0\|_{L^\infty(\mathbb{R}^+)} \sum_{i=1}^{N} \int_{S_{i-1}}^{S_i} S^{1/2} \left| u_i(S) \right| dS$$

with

$$u_i(S) = \frac{S \int_{S_{i-1}}^{S_i} \sigma^{1/2} d\sigma}{\int_{S_{i-1}}^{S_i} \sigma^{3/2} d\sigma} - 1 = \frac{\int_{S_{i-1}}^{S_i} (S - \sigma) \sigma^{1/2} d\sigma}{\int_{S_{i-1}}^{S_i} \sigma^{3/2} d\sigma}.$$

So:

$$\sum_{i=1}^{N} \int_{S_{i-1}}^{S_i} S^{1/2} \left| u_i(S) \right| dS \leqslant \frac{2}{3} \int_0^{\Delta S} S^{1/2} dS + \sum_{i=2}^{N} \frac{\Delta S}{S_{i-1}} \int_{S_{i-1}}^{S_i} S^{1/2} dS$$

But:

$$\frac{3}{2}\sum_{i=2}^{N}\frac{1}{S_{i-1}}\int_{S_{i-1}}^{S_i} S^{1/2}dS = \sum_{i=1}^{N-1}\left[S_{i+1}^{1/2}\left(1+\frac{\Delta S}{S_i}\right)-S_i^{1/2}\right]$$
$$= \sqrt{S_{max}} - \sqrt{\Delta S} + \Delta S\sum_{i=1}^{N-1}\frac{1}{\sqrt{S_i}}\sqrt{1+\frac{\Delta S}{S_i}}.$$

We can then upperly bound the preceding term:

$$\sum_{i=1}^{N}\int_{S_{i-1}}^{S_i} S^{1/2}\left|u_i(S)\right|dS \leqslant \frac{2}{3}\Delta S\left(\sqrt{S_{max}}+\sqrt{2\Delta S}\sum_{i=1}^{N-1}\frac{1}{\sqrt{i}}\right)$$
$$\leqslant \frac{2}{3}\Delta S\left(\sqrt{S_{max}}+\sqrt{2\Delta S}\int_0^N\frac{1}{\sqrt{x}}dx\right)$$
$$\leqslant \frac{2}{3}(1+2\sqrt{2})\Delta S\sqrt{S_{max}}.$$

We can then upperly bound the second term of (67):

$$\left|\sum_{i=1}^{N} M_i^{(N)}\bar{\mu}_i^{(N)}(z) - \frac{\rho_l}{4\sqrt{\pi}}\int_0^{+\infty} S^{1/2}\psi^{(N)}(z,S)dS\right| \leqslant \frac{\rho_l}{6\sqrt{\pi}}(1+2\sqrt{2})\|\psi_0\|_{L^\infty(\mathbb{R}^+)}\sqrt{S_{max}}\,\Delta S,$$

and the proposition is proved. □

6.4.2 Passage to the limit

We can then pass to the limit, using the techniques similar to ones of previous subsections:

Theorem 6.5 *There exists a scalar c and functions θ, α and μ_i in $C^{2+\delta}(\mathbb{R})$ solution of the system $(\mathcal{S}_0)$. Moreover, $\tilde{B}_M > 0$.*

Proof. – The constants c^-, c^+ of Theorem 4.4 are independent of N. We then have the estimates, for all $N \in \mathbb{N}^*$:

$$0 < c^- \leqslant c^{(N)} \leqslant c^+, \qquad 0 \leqslant \alpha^{(N)} \leqslant 1, \qquad 0 \leqslant -\theta^{(N)\prime} \leqslant \tilde{k}(1)+\frac{2}{\kappa}, \qquad B_M^{(N)} \geqslant 0.$$

Because $\tilde{R}_S(\theta^{(N)},\alpha^{(N)})$ is bounded by the constant $\varphi(2+\delta)/\delta$ independent of N and z, it is easy to have upper bounds independent of N for $\|\alpha^{(N)\prime}\|_{L^\infty(\mathbb{R})}$ and then for $\|\theta^{(N)}\|_{C^{2+\delta}(\mathbb{R})}$ and $\|\alpha^{(N)}\|_{C^2(\mathbb{R})}$. As for Theorems 6.2 and 6.3, there exists a scalar c, functions $\theta \in C^{2+\delta}(\mathbb{R})$ and $\alpha \in C^2(\mathbb{R})$ and subsequences $c^{(N_k)}$, $\theta^{(N_k)}$ and $\alpha^{(N_k)}$ which converge to c, θ and α, the convergence being C^2 in all bounded interval for the functions.

Let prove that the function $\psi^{(N_k)}$ have also a limit. We know that, for $z \geqslant 0$:

$$|\tilde{R}_S(\theta^{(N_k)}(z),\alpha^{(N_k)}(z))| \leqslant \frac{\varphi}{\delta}(\mathcal{N}+1)\theta^{(N_k)}(z) = \theta_c\exp\left(-\frac{c^{(N_k)}}{\kappa}z\right) \leqslant \theta_c\exp\left(-\frac{c^-}{\kappa}z\right),$$

and, in the same way

$$|\tilde{R}_S(\theta(z),\alpha(z))| \leqslant \theta_c \exp\left(-\frac{c^-}{\kappa} z\right).$$

Then $\tilde{R}_S(\theta^{(N_k)},\alpha^{(N_k)})$ converge uniformly to $\tilde{R}_S(\theta,\alpha)$. So, the characteristic $X^{(N_k)}(z,S)$ converges to $X(z,S) = S + \frac{1}{c}\int_z^{+\infty} \tilde{R}_S(\theta(y),\alpha(y))dy$ and $\psi^{(N_k)}(z,S) = \psi_0(X^{(N_k)}(z,S))$ converges to $\psi(z,S) = \psi_0(X(z,S))$. Those convergences are uniform in $S \in \mathbb{R}^+$. So:

$$\lim_{k\to\infty} \frac{\rho_l}{4\sqrt{\pi}} \int_0^{+\infty} S^{1/2}\psi^{(N_k)}(z,S)dS = \frac{\rho_l}{4\sqrt{\pi}} \int_0^{+\infty} S^{1/2}\psi(z,S)dS$$

and because of Proposition 6.4, θ, α, ψ, c are solution of $(\mathcal{S}_0)$.

We only have to find the limits of the functions θ and α at $\pm\infty$. We saw that for $z \geqslant 0$, $\theta(z) = \theta_c \exp(-c\,z/\kappa)$ and $\lim_{z\to+\infty}\theta(z) = 0$. Because θ has a limit at $-\infty$ and because of the equation on θ, we have $\lim_{z\to-\infty}\alpha(z) = 0$. Moreover $\tilde{R}_S(\theta(z),\alpha(z))$ tends to 0 as z tends to $+\infty$. Then $\lim_{z\to+\infty}\alpha(z) = \tilde{Y}_{Fs}(0) = \alpha^\infty$. We also have

$$\begin{aligned}\sum \mu_i^{(N_k)} = \sum \bar{\mu}_i^{(N_k)} + O(N_k) &= \frac{\rho_l}{6\sqrt{\pi}} \int_0^{+\infty} S^{3/2}\psi^{(N_k)}(z,S)dS + O(N_k)\\ &\to \frac{\rho_l}{6\sqrt{\pi}} \int_0^{+\infty} S^{3/2}\psi(z,S)dS.\end{aligned}$$

Then

$$\theta + \alpha + \frac{\rho_l}{6\sqrt{\pi}} \int_0^{+\infty} S^{3/2}\,\psi(z,S)\,dS \geqslant 1.$$

As for $\psi^{(N_k)}(z,S)$, $\psi(z,S) = 0$ for sufficiently large z and then $\theta(z) + \alpha(z) \geqslant 1$. Then, necessarily, $\lim_{z\to-\infty}\theta(z) = 1$ and the theorem is proved. □

7 Conclusion

In this paper, we have proved the existence of traveling wave solution for the system of partial differential equations $(\mathcal{PDE})$ of mixed type hyperbolic-parabolic describing the propagation of a plane flame in a polydisperse fuel spray. Let us recall the system satisfied by the traveling wave solution profiles :

$$(\mathcal{S}_0) \quad \begin{cases} \kappa\,\theta'' + c\,\theta' + \tilde{k}(\theta)\,\alpha = 0,\\ D\,\alpha'' + c\,\alpha' - \tilde{k}(\theta)\,\alpha + S^m(z,\theta,\alpha) = 0,\\ c\,\partial_z\psi + \tilde{R}_S(\theta,\alpha)\,\partial_S\psi = 0, \qquad S \geqslant 0, \end{cases}$$

with the source term:

$$S^m(z,\theta,\alpha) = \frac{\rho_l}{4\sqrt{\pi}}\,\tilde{R}_S(\theta,\alpha) \int_0^{+\infty} S^{1/2}\,\psi(z,S)\,dS,$$

and with the boundary conditions at infinity:

$$(\mathcal{L}_0) \quad \begin{cases} \theta(+\infty) = 0, \quad \alpha(+\infty) = \alpha^\infty, \quad \psi(+\infty,S) = \psi_0(S),\\ \theta(-\infty) = 1, \quad \alpha(-\infty) = 0, \qquad \psi(-\infty,S) = 0. \end{cases}$$

The originality of the present contribution is fourfold. First, from the modeling point of view, we use a detailed model, coupling a thermo-diffusive model for the gaseous phase to a kinetic description of the vaporization of a polydisperse spray of liquid fuel droplets. More specifically, we obtain the existence of a solution under the condition that Lewis number is such that $\mathrm{Le} \geqslant 1 + p$, where the positive real p only depends on the fuel saturated vapor mass fraction $Y_{Fs}(T)$ and on the temperature of the cold gas T_0:

$$p = \max\left\{ \frac{(T - T_0)Y''_{Fs}(T)}{Y'_{Fs}(T)}, \quad T_0 \leqslant T \leqslant T_{eb} - \nu \right\} \quad \text{if } T_0 < T_{eb} - \nu$$
$$p = 0 \qquad \text{if } T_0 \geqslant T_{eb} - \nu.$$

This condition insures that no condensation phenomenon is taking place. It is coherent with the chosen model; however, it would be very interesting to further investigate if this condition can be weakened so that the existence of the traveling wave persists down to $\mathrm{Le} = 1$. Second, the use of Eulerian multi-fluid models in order to treat the droplet size phase space allows us to circumvent the difficulty of a non-local vaporization source term in the reactant parabolic equation. It is not only an interesting way of treating the vaporization process but, and this is the third aspect of the originality of the paper, it is compatible, with the addition of a droplet mass density diffusion process through some weak gaseous turbulent agitation and allows us to use the techniques of topological degree in unbounded domains for elliptic operators. In other terms, if we would first take the limit of an infinite number of sections in the discretization of the droplet size phase space, we could prove the existence of a traveling wave in the presence of a small turbulent agitation in the gaseous phase resulting in a diffusion process at the level of the droplets. Four, we use a new way of perturbing the system of elliptic equations in order to define an elliptic, Fredholm and proper operator in weighted Hölder spaces, in such a way that the monotonicity and homotopies are treated in a much simpler way. Finally, this paper is a first step forward to providing the qualitative analysis of two-phase flame propagation in using models bridging the gap between too simplified theoretical models and practical applications.

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References

[1] B. Abramzon and W.A. Sirignano. Droplet vaporization model for spray combustion calculations. *Int. J. Heat Mass Transfer*, 32:1605–1618, 1989.

[2] Cristelle Barillon and Vitaly A. Volpert. Topological degree for a class of elliptic operators in R^n. *Topol. Methods Nonlinear Anal.*, 14(2):275–293, 1999.

[3] Henri Berestycki, Basil Nicolaenko, and Bruno Scheurer. Traveling wave solutions to reaction-diffusion systems modeling combustion. In *Nonlinear partial differential equations (Durham, N.H., 1982)*, pages 189–208. Amer. Math. Soc., Providence, R.I., 1983.

[4] P. Berthonnaud and K. Domelevo. Existence et stabilité d'ondes planes pour un système de combustion diphasique monodimensionnel. prépublication 18, MIP UMR 5640, Toulouse, 2002.

[5] Alexis Bonnet. Propagation of flames in the limit of zero ignition temperature. *Arch. Rational Mech. Anal.*, 138(3):205–238, 1997.

[6] Felix E. Browder. Degree theory for nonlinear mappings. In *Nonlinear functional analysis and its applications, Part 1 (Berkeley, Calif., 1983)*, pages 203–226. Amer. Math. Soc., Providence, RI, 1986.

[7] J. F. Clouet and K. Domelevo. Solution of a kinetic stochastic equation modeling a spray in a turbulent gas flow. *Math. Models Methods Appl. Sci.*, 7(2):239–263, 1997.

[8] E. N. Dancer. Boundary-value problems for ordinary differential equations on infinite intervals. *Proc. London Math. Soc. (3)*, 30:76–94, 1975.

[9] E. N. Dancer. Boundary-value problems for ordinary differential equations of infinite intervals. II. *Quart. J. Math. Oxford Ser. (2)*, 28(109):101–115, 1977.

[10] K. Domelevo and L. Sainsaulieu. A numerical method for the computation of the dispersion of a cloud of particles by a turbulent gas flow field. *J. Comput. Phys.*, 133(2):256–278, 1997.

[11] Christiano Clemente Fenske. Extensio gradus ad quasdam applicationes Fredholmii. *Mitt. Math. Sem. Giessen*, (Heft 121):65–70, 1976.

[12] Vincent Giovangigli. Plane laminar flames with multicomponent transport and complex chemistry. *Math. Models Methods Appl. Sci.*, 9(3):337–378, 1999.

[13] Claude Godrèche and Paul Manneville, editors. *Hydrodynamics and nonlinear instabilities*. Cambridge University Press, Cambridge, 1998. With chapters by Bernard Castaing, P. Huerre and M. Rossi, Vincent Hakim, Stephan Fauve, and G. Joulin and P. Vidal.

[14] J.B. Greenberg, I. Silverman, and Y. Tambour. On the origin of spray sectional conservation equations. *Combustion and Flame*, 93:90–96, 1993.

[15] J.B. Greenberg, I. Silverman, and Y. Tambour. On droplet enhancement of the burning velocity of laminar premixed spray flames. *Combustion and Flame*, 113:271–273, 1998.

[16] C. A. S. Isnard. The topological degree on Banach manifolds. In *Global analysis and its applications (Lectures, Internat. Sem. Course, Internat. Centre Theoret. Phys., Trieste, 1972), Vol. II*, pages 291–313. Internat. Atomic Energy Agency, Vienna, 1974.

[17] A. Kalma and J. B. Greenberg. Special features of the combustion of a propagating flame in a polydisperse fuel spray cloud. *International Journal of Turbo and Jet Engines*, 14:201–216, 1997.

[18] F. Laurent and M. Massot. Numerical analysis of Eulerian multi-fluid models in the context of kinetic formulations for dilute evaporating sprays. in preparation, 2002.

[19] F. Laurent, M. Massot, and Vit. A. Volpert. Propagation de flammes gazeuses dans la limite d'une diffusion massique nulle. *C. R. Math. Acad. Sci. Paris*, 335(4):405–410, 2002.

[20] F. Laurent, V. Santoro, M. Noskov, A. Gomez, M.D. Smooke, and M. Massot. Accurate treatment of size distribution effects in polydispersed spray diffusion flames: multi-fluid modeling, computations and experiments. prépublication, soumis, MAPLY, UMR 5585 Lyon, 2002. (http://maply.univ-lyon1.fr/publis/publiv/2002/publis.html).

[21] Frédérique Laurent. Analyse numérique d'une méthode multi-fluide Eulérienne pour la description de sprays qui s'évaporent. *C. R. Math. Acad. Sci. Paris*, 334(5):417–422, 2002.

[22] Frédérique Laurent and Marc Massot. Multi-fluid modeling of laminar poly-dispersed spray flames: origin, assumptions and comparison of the sectional and sampling methods. *Combust. Theory and Modelling*, 5:537–572, 2001.

[23] S. Leray and J. Schauder. Topologie et équations fonctionnelles. *Ann. Sci. Ecole Norm. Sup.*, 113:45–78, 1934.

[24] Elisabeth Logak and Vincent Loubeau. Travelling wave solutions to a condensed phase combustion model. *Asymptotic Anal.*, 12(4):259–294, 1996.

[25] Martine Marion. Mathematical study of a model with no ignition temperature for laminar plane flames. In *Reacting flows: combustion and chemical reactors, Part 2 (Ithaca, N.Y., 1985)*, pages 239–252. Amer. Math. Soc., Providence, RI, 1986.

[26] M. Massot, M. Kumar, A. Gomez, and M. D. Smooke. Counterflow spray diffusion flames of heptane: computations and experiments. In *Proceedings of the 27th Symp. (International) on Combustion, The Comb. Institute*, pages 1975–1983, 1998.

[27] M. W. Reeks. On a kinetic equation for the transport of particles in turbulent flows. *Phys. Fluids*, 3:446–456, 1991.

[28] J. Réveillon and M. Massot. Lagrangian/Eulerian analysis of the subgrid dispersion of vaporizing polydispersed sprays in turbulent flows. CTR summer, Center for Turbulence Research, Stanford University, 2002.

[29] J. Réveillon, M. Massot, and L. Vervisch. Direct numerical simulation of turbulent sprays. In *Proceedings of the workshop: Trends in Numerical and Physical Modeling for Industrial Multiphase Flows*, Sept. 2001.

[30] J. Réveillon and L. Vervisch. Spray vaporization in nonpremixed turbulent combustion modeling: a single droplet model. *Combustion and Flame*, 121:75–90, 2000.

[31] I. Silverman, J. B. Greenberg, and Y. Tambour. Asymptotic analysis of a premixed polydisperse spray flame. *SIAM J. Appl. Math.*, 51(5):1284–1303, 1991.

[32] I. V. Skrypnik. Nonlinear elliptic equations of higher order. *Gamoqeneb. Math. Inst. Sem. Mohsen. Anotacie.*, (7):51–52, 1973.

[33] A. I. Vol′pert and V. A. Vol′pert. Application of the theory of the rotation of vector fields to the investigation of wave solutions of parabolic equations. *Trudy Moskov. Mat. Obshch.*, 52:58–109, 247, 1989.

[34] A. I. Vol′pert and V. A. Volpert. Construction of the Leray-Schauder degree for elliptic operators in unbounded domains. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 11(3):245–273, 1994.

[35] Aizik I. Volpert, Vitaly A. Volpert, and Vladimir A. Volpert. *Traveling wave solutions of parabolic systems*. American Mathematical Society, Providence, RI, 1994. Translated from the Russian manuscript by James F. Heyda.

[36] V. A. Volpert, A. I. Volpert, and J. F. Collet. Topological degree for elliptic operators in unbounded cylinders. *Adv. Differential Equations*, 4(6):777–812, 1999.

[37] F.A. Williams. Spray combustion and atomization. *Phys. Fluids*, 1:541–545, 1958.

[38] F.A. Williams. *Combustion Theory (Combustion Science and Engineering Series)*. ed F A Williams (Reading, MA: Addison-Wesley), 1985.

[39] M. C. Yuen and L. W. Chen. On drag of evaporating liquid droplets. *Combustion Science and Technology*, 14:147–154, 1973.

[40] Ya. B. Zel′dovich, G. I. Barenblatt, V. B. Librovich, and G. M. Makhviladze. *The mathematical theory of combustion and explosions*. Consultants Bureau [Plenum], New York, 1985. Translated from the Russian by Donald H. McNeill.

[41] Ya. B. Zel′dovich and D. A. Frank-Kamenetskii. A theory of thermal propagation of flame. *Acta Physicochim*, 9:341–350, 1938.