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NAGAEV METHOD VIA KELLER-LIVERANI THEOREM

Loïc HERVÉ ¹ and Françoise PÈNE ²

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Abstract. *Nagaev's method, via the perturbation operator theorem of Keller and Liverani, has been exploited in recent papers to establish local limit and Berry-Essen type theorems for unbounded functionals of strongly ergodic Markov chains. The main difficulty of this approach is to prove Taylor expansions for the dominating eigenvalue of the Fourier kernels. This paper outlines this method and extends it by proving a multi-dimensional local limit theorem, a first-order Edgeworth expansion, and a multi-dimensional Berry-Esseen type theorem in the sense of Prohorov metric. When applied to uniformly or geometrically ergodic chains and to iterative Lipschitz models, the above cited limit theorems hold under moment conditions similar, or close, to those of the i.i.d. case.*

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¹I.R.M.A.R., UMR-CNRS 6625. Institut National des Sciences Appliquées de Rennes, 20, Avenue des Buttes de Couësmes CS 14 315, 35043 Rennes Cedex, France. Loic.Herve@insa-rennes.fr

²UMR-CNRS 6205, Université de Bretagne Occidentale, UFR des sciences et techniques, département de Mathématiques, 29285 Brest cedex, France. francoise.pene@univ-brest.fr

1. Introduction, setting, and notation. Let $(X_n)_n$ be a Markov chain with values in (E, \mathcal{E}) , with transition probability Q and with stationary distribution π . Let ξ be a π -centered random variable with values in \mathbb{R}^d (with $d \geq 1$). We are interested in probabilistic limit theorems for $(\xi(X_n))_n$ namely :

- central limit theorem (c.l.t.),
- rate of convergence in the central limit theorem : Berry Esseen type theorem,
- multi-dimensional local limit theorem,
- First-order Edgeworth expansion (when $d = 1$).

We want to establish these results under moment conditions on ξ as close as possible to those of the i.i.d. case (as usual i.i.d. is the short-hand for “independent and identically distributed”). Let us recall some facts about the case when $(Y_n)_n$ is a sequence of i.i.d. \mathbb{R}^d -valued random variables (r.v.) with null expectation. If $Y_1 \in \mathbb{L}^2$, we have the central limit theorem and, under some additional non-lattice type assumption, we have the local limit theorem. If $Y_1 \in \mathbb{L}^3$ and $d = 1$, we have the uniform Berry-Esseen theorem, and the first-order edgeworth expansion. If $Y_1 \in \mathbb{L}^m$ with $m = \max(3, \lfloor d/2 \rfloor + 1)$, then $(Y_n)_n$ satisfies a multi-dimensional Berry-Esseen type theorem (in the sense of Prohorov).

To get analogous results for Markov chain, we shall use and adapt Nagaev’s method. This method has been introduced in [?] [?] to establish limit theorems for sequences $(\xi(X_n))_n$, when ξ is a real-valued function and $(X_n)_{n \geq 0}$ is a uniformly ergodic Markov chain. This method is based on Fourier techniques and the usual perturbation operator theory applied to the Fourier kernels $Q(t)(x, dy) = e^{it\xi(y)}Q(x, dy)$ ($t \in \mathbb{R}$). The idea is that $\mathbb{E} \left[e^{it \sum_{k=1}^n \xi(X_k)} \right]$ is close enough to an expression of the form $\lambda(t)^n$ and the calculations are then similar to those of the i.i.d. case (let us recall that, if $(Y_n)_n$ is a sequence of i.i.d. random variables, then we have $\mathbb{E} \left[e^{it \sum_{k=1}^n Y_k} \right] = \left(\mathbb{E} [e^{itY_1}] \right)^n$). The use of Nagaev’s method always requires a strong ergodic assumption. This assumption is presented in section 2.

Ever since Nagaev’s works, this method has been widely extended to more general Markov chains. This is fully described in [?] [?] where references are given. Roughly speaking, this method yields the usual distributional limit theorems if Q is strongly ergodic w.r.t. some Banach function space \mathcal{B} and (Q, ξ) satisfies some operator-moment conditions on \mathcal{B} . This method is specially efficient when \mathcal{B} is a Banach algebra and ξ is in \mathcal{B} . Unfortunately, on the one hand, since Banach algebras are often composed of bounded functions, the condition $\xi \in \mathcal{B}$ implies that ξ must be bounded. On the other hand, usual models as geometrically ergodic chains or iterative Lipschitz models (typically $E = \mathbb{R}^p$) are strongly ergodic w.r.t. to some weighted supremum normed space or weighted Lipschitz-type space which are not Banach algebras, and the above mentioned operator-moment conditions then hold under very restrictive assumptions involving both Q and ξ . For instance, in these models, the usual Nagaev’s method cannot be efficiently applied to the sequence $(X_n)_n$ (i.e. $\xi(x) = x$) ; an explicit and typical counter-example will be presented in Section 3.

In recent works [?] [?] [?], a new procedure, based on the perturbation theorem of Keller-Liverani [?] (see also [?] p. 177), allows to get round the previous difficulty and to greatly improve Nagaev’s method when applied to unbounded functionals ξ ; the main results of [?] are applied to the Fourier kernels in Section 4. Our work outlines this new approach, and presents the applications, namely : a multi-dimensional local limit theorem, a one-dimensional

Berry Esseen Theorem, a first-order Edgeworth expansion, a multi-dimensional Berry-Esseen type theorem in the sense of Prohorov metric.

When the usual perturbation theorem is replaced with that of Keller-Liverani, the main difficulty consists in proving Taylor expansions for the dominating eigenvalue $\lambda(t)$ of the Fourier kernel $Q(t)$. This point is crucial here. Such expansions may be obtained as follows :

(A) To get Taylor expansion at $t = 0$, one can combine Nagaev's method with more probabilistic arguments such as martingale techniques [?]. In this paper, this method is just outlined : the local limit theorem obtained in [?] is extended to the multi-dimensional case in section 5, and the one-dimensional uniform Berry-Esseen theorem of [?] is just recalled for completeness in section 6.

(B) To establish the others limit theorems, we shall use a stronger property : the regularity of the eigen-elements of $Q(\cdot)$ on a neighbourhood of $t = 0$. We shall see in section 7 that this can be done by considering the action of $Q(t)$ on a "chain" of suitable function spaces instead of a single one as in the classical approach. This method, already introduced to investigate the clt for iterative Lipschitz models [?], is here specified and extended to general strongly ergodic chains. It will provide the one-dimensional Edgeworth expansion (section 8) and the multi-dimensional Berry-Esseen type theorem (section 9).

When $(X_n)_n$ is Harris recurrent, the so-called regenerative (or splitting) method provides limit theorems for $(\xi(X_n))_n$ including the uniform Berry-Esseen theorem [?] and Edgeworth expansions [?]. Here the Harris recurrence is not assumed, but we shall appeal to abstract operator-type assumptions. Despite the forbidding appearance of these assumptions, we want to point out that, when applied to the classical examples of strongly ergodic Markov chains, they will reduce to simple moment conditions on ξ and standard nonlattice conditions.

We apply our method to the three following examples :

1. Uniformly ergodic Markov chain, see [?].

As a first motivation of the abstract sections 4-9, the above cited limit theorems will be summarized in Section 2.3 in the simple case of the uniformly ergodic Markov chains. This simple model provides a good understanding of the improvements obtained in this work. Indeed, for results requiring $Y_1 \in \mathbb{L}^m$ in the i.i.d. case, whereas the usual Nagaev's method needs $\sup_{x \in E} \int |\xi(y)|^m Q(x, dy) < +\infty$ [?, ?, ?], the present method appeals to the moment conditions $\xi \in \mathbb{L}^m(\pi)$ or $\xi \in \mathbb{L}^{m+\varepsilon}(\pi)$.

Of course this example is quite restrictive, and another motivation of this work is to present applications to the two following markovian models of more practical use.

2. v -geometrically ergodic Markov chains, see e.g [?, ?].

Applications of our abstract results to v -geometrically ergodic Markov chains are given in section 10. When $Y_1 \in \mathbb{L}^m$ is needed in the i.i.d. case, the usual Nagaev's method requires for these models the condition $\sup_{x \in E} v(x)^{-1} \int |\xi(y)|^m v(y) Q(x, dy) < +\infty$ (see e.g [?]) which, in practice, often amounts to assuming that ξ is bounded [?]. Our method only requires that $|\xi|^m \leq C v$ or $|\xi|^{m+\varepsilon} \leq C v$, which extends the well-known condition $|\xi|^2 \leq C v$ used for proving the clt [?].

Since v -geometrically ergodic Markov chains (as well as uniformly ergodic Markov chains) are Harris recurrent and strongly mixing, other methods may be used to study the above mentioned limit theorems. For these two models, our method is as efficient

as the hitherto known ones, even better in some cases. More precise comparisons will be made afterwards.

3. The iterated random Lipschitz models, see e.g [?, ?].

Except when Harris recurrence and strong mixing hypotheses are assumed, not many works have been devoted to the refinements of clt for iterative models. Applications to this example are detailed in Section 11 : by considering the general weighted-Lipschitz functionals ξ of [?], the limit theorems will be stated under some usual moment and mean contractivity conditions which extend those of [?] [?] used to prove the c.l.t.. For instance, let us consider the typical example when $(X_n)_n$ is the usual linear autoregressive model in \mathbb{R}^d , namely $X_n = AX_{n-1} + \theta_n$ where A is a contractive $d \times d$ -matrix and $X_0, \theta_1, \theta_2, \dots$ are \mathbb{R}^d -valued independent r.v., and when the functional is $\xi(x) = x$. For this example, whereas the usual Nagaev's method requires exponential moment conditions for these statements [?], our adaptation gives the local limit theorem and the Berry-Esseen theorem under the expected moment hypotheses $\theta_1 \in \mathbb{L}^2$ and $\theta_1 \in \mathbb{L}^3$ respectively. The first-order edgeworth expansion and the multi-dimensional Berry Esseen theorem are established under stronger moment conditions : $\theta_1 \in \mathbb{L}^{2m+\varepsilon} < +\infty$ when only $Y_1 \in \mathbb{L}^m$ is needed in the i.i.d. case. All these results are new to our knowledge.

A slight correction of [?]. Also, one of our motivations is to correct slightly theorem 2.2.4 of [?] about the Knudsen gas model. This model involves a uniformly ergodic Markov chain $(X_n)_n$, and this result gave a rate of convergence in $n^{-1/2}$ (in the sense of Prohorov) in the multi-dimensional clt for $(\xi(X_n))_n$ under the hypothesis $\xi \in \mathbb{L}^3(\pi) \cap \mathbb{L}^{\lfloor d/2 \rfloor + 1}(\pi)$. The proof of this statement is not correct as it is written in [?]. Indeed, under this hypothesis, $t \mapsto Q(t)$ does not define a regular family of operators when acting on the single space of bounded functions (this holds under the more restrictive condition mentioned in the above example 1). Hence proposition 2.4.2 of [?] is not correct. By considering the action of $Q(t)$ on a "chain" of $\mathbb{L}^p(\pi)$ -spaces, the above procedure (B) will enable us to solve this problem ; in fact this will be done for any uniformly ergodic chain. This will provide the following slight correction (see Section 2.2 for details) : *Theorem 2.2.4 in [?] is valid under the condition $\xi \in \mathbb{L}^{3+\varepsilon}(\pi) \cap \mathbb{L}^{\lfloor d/2 \rfloor + 1 + \varepsilon}(\pi)$ for some $\varepsilon > 0$.*

Extensions. Mention that the procedures (A) and (B) may be also used to investigate renewal theorems [?] [?]. Anyway, our method, developed here for functionals of Markov chains, may be employed in other contexts where Fourier operators occur. First, by an easy adaptation of the hypotheses, the present limit theorems may be extended to the general setting of Markov random walks, whose our work is a special case. Second, these theorems may be also stated for the Birkhoff sums stemming from dynamical systems, by adapting the hypotheses to the so-called Perron-Frobenius operator (to pass from Markov chains to dynamical systems, see e.g [?] Chap. XI). Besides, [?] also used Nagaev's method together with Keller-Liverani theorem to prove convergences in distribution to stable laws in the stadium billiard.

Now let us present the general settings useful for the abstract study of Sections 4-9.

Probabilistic setting. $(X_n)_{n \geq 0}$ is a Markov chain with general state space (E, \mathcal{E}) , transition probability Q , stationary distribution π , initial distribution μ , and $\xi = (\xi_1, \dots, \xi_d)$ is a \mathbb{R}^d -

valued π -integrable function on E such that $\pi(\xi) = 0$ (i.e. each ξ_i is π -integrable and $\pi(\xi_i) = 0$). The associated random walk in \mathbb{R}^d is denoted by

$$S_n = \sum_{k=1}^n \xi(X_k).$$

We denote by $|\cdot|_2$ and $\langle \cdot, \cdot \rangle$ the euclidean norm and the canonical scalar product on \mathbb{R}^d . For any $t \in \mathbb{R}^d$ and $x \in E$, we define the Fourier kernels of (Q, ξ) as

$$Q(t)(x, dy) = e^{i\langle t, \xi(y) \rangle} Q(x, dy).$$

$\mathcal{N}(0, \Gamma)$ denotes the centered normal distribution associated to a covariance matrix Γ , and “ $\xrightarrow{\mathcal{D}}$ ” means “convergence in distribution”. Although $(X_n)_{n \geq 0}$ is not a priori the canonical version, we shall slightly abuse notation and write $\mathbb{P}_\mu, \mathbb{E}_\mu$ to refer to the initial distribution. For any μ -integrable function f , we shall often write $\mu(f)$ for $\int f d\mu$. For $x \in E$, δ_x will stand for the Dirac mass : $\delta_x(f) = f(x)$.

Functional setting. The Banach spaces on which Q will act will be composed of complex-valued π -integrable functions on E (or, of equivalence classes of such functions) and will be moreover supposed to be stable under complex conjugation. Such a space will be called a *function space*. The simplest examples are the space \mathcal{B}^∞ of bounded measurable functions on E , equipped with the supremum norm, and the usual Lebesgue space $\mathbb{L}^p(\pi)$, $1 \leq p \leq +\infty$.

Unless otherwise indicated, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a function space. If X is a Banach space, we denote by $\mathcal{L}(\mathcal{B}, X)$ the space of the bounded linear operators from \mathcal{B} to X , and by $\|\cdot\|_{\mathcal{B}, X}$ the associated operator norm, with the usual simplified notations $\mathcal{L}(\mathcal{B}) = \mathcal{L}(\mathcal{B}, \mathcal{B})$, $\mathcal{B}' = \mathcal{L}(\mathcal{B}, \mathbb{C})$, for which the associated norms are simply denoted by $\|\cdot\|_{\mathcal{B}}$. If $T \in \mathcal{L}(\mathcal{B})$, $r(T)$ denotes its spectral radius, and $r_{ess}(T)$ its essential spectral radius (see e.g [?] Chap. XIV). For any normed spaces A and B , the notation “ $A \hookrightarrow B$ ” means that the inclusion $A \subset B$ is a continuous embedding.

Throughout this paper, we consider function spaces \mathcal{B} containing 1_E and such that $\mathcal{B} \hookrightarrow \mathbb{L}^1(\pi)$ (which implies that π is in \mathcal{B}'), and Π denotes the rank-one projection defined on \mathcal{B} by :

$$\Pi f = \pi(f)1_E \quad (f \in \mathcal{B}).$$

2. Strong ergodicity assumption. We shall appeal to the following strong ergodicity condition on \mathcal{B} :

Condition (K1) : $\mathcal{B} \hookrightarrow \mathbb{L}^1(\pi)$, $1_E \in \mathcal{B}$, $Q \in \mathcal{L}(\mathcal{B})$, and $\lim_n \|Q^n - \Pi\|_{\mathcal{B}} = 0$.

When condition (K1) holds, we shall say that Q (or merely $(X_n)_n$) is *strongly ergodic* w.r.t. \mathcal{B} . One could also say “geometrically ergodic w.r.t. \mathcal{B} “. Indeed, one can easily see that the last property in (K1) is equivalent to :

$$\exists \kappa_0 < 1, \exists C > 0, \forall n \geq 1, \|Q^n - \Pi\|_{\mathcal{B}} \leq C \kappa_0^n.$$

It is worth noticing that the condition $\mathcal{B} \hookrightarrow \mathbb{L}^1(\pi)$ in (K1) implies $\pi \in \mathcal{B}'$. Besides, we shall repeatedly use the following obvious fact. If Q is strongly ergodic w.r.t. \mathcal{B} , and if $f \in \mathcal{B}$ is such that $\pi(f) = 0$, then the series $\sum_{k \geq 0} Q^k f$ is absolutely convergent in \mathcal{B} .

2.1. Examples.

(a) According to the terminology of [?], we say that $(X_n)_n$ is *uniformly ergodic* if we have (K1) on \mathcal{B}^∞ . This also corresponds to aperiodic Markov chains satisfying the so-called Doe-

blin condition. Nagaev's method was introduced with this example.

(b) Let $v : E \rightarrow [1, +\infty[$. The v -geometrically ergodic Markov chains [?] constitute a natural extension of Example (a). A Markov chain is said to be v -geometrically ergodic if its transition operator Q satisfies (K1) on the weighted supremum normed space $(\mathcal{B}_v, \|\cdot\|_v)$ composed of measurable complex-valued functions f on E such that $\|f\|_v = \sup_{x \in E} \frac{|f(x)|}{v(x)} < +\infty$.

(c) Under some usual moment and mean contractivity conditions, the *iterated random Lipschitz models* [?] satisfy (K1) on weighted Lipschitz-type spaces, see [?] (Sect. 5).

(d) Let $q(\cdot)$ be a semi-norm on \mathcal{B} , dominated by $\|\cdot\|_{\mathcal{B}}$. Suppose that Q continuously acts on both $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ and $(\mathcal{B}, q(\cdot))$, and satisfies the so-called Doeblin-Fortet inequalities : there exist $r < 1$ and $C > 0$ such that $\|Q^n f\|_{\mathcal{B}} \leq C r^n \|f\|_{\mathcal{B}} + C q(f)$ for all $n \geq 1$ and $f \in \mathcal{B}$. Finally, denoting by S the unit ball of $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$, assume $Q(S)$ is relatively compact in $(\mathcal{B}, q(\cdot))$. Then Q is quasi-compact on \mathcal{B} [?] [?], so that Q satisfies (K1) provided that it verifies some further irreducibility and aperiodicity conditions.

All the limit theorems investigated in this work are stated in Section 2.3 for Example (a). They will be applied to the examples (b) and (c) in Sections 10-11.

2.2. A central limit theorem in the stationary case. As a preliminary to the next limit theorems, we state here a well-known c.l.t. for $(\xi(X_n))_n$, which is a standard consequence of a theorem due to Gordin [?]. We shall then deduce a corollary based on Condition (K1). In this section, we shall only consider the stationary case. Also observe that, concerning distributional questions on $(\xi(X_n))_n$, one may without loss of generality assume that $(X_n)_{n \geq 0}$ is the canonical Markov chain associated to Q .

So we consider here the usual probability space $(E^{\mathbb{N}}, \mathcal{E}^{\otimes \mathbb{N}}, \mathbb{P}_{\pi})$ for the canonical Markov chain, still denoted by $(X_n)_{n \geq 0}$, with transition probability Q and initial stationary distribution π . Let θ be the shift operator on $E^{\mathbb{N}}$. As usual we shall say that $(X_n)_{n \geq 0}$ is ergodic if the dynamical system $(E^{\mathbb{N}}, \mathcal{E}^{\otimes \mathbb{N}}, \mathbb{P}_{\pi}, \theta)$ is ergodic.

Theorem (Gordin). *Assume that $(X_n)_{n \geq 0}$ is ergodic, and*

$$\forall i = 1, \dots, d, \quad \xi_i \in \mathbb{L}^2(\pi) \quad \text{and} \quad \check{\xi}_i := \sum_{n \geq 0} Q^n \xi_i \quad \text{converges in } \mathbb{L}^2(\pi).$$

Then $\frac{S_n}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \Gamma)$, where Γ is the covariance matrix defined by $\langle \Gamma t, t \rangle = \pi(\check{\xi}_t^2) - \pi((Q\check{\xi}_t)^2)$, where we set $\check{\xi}_t = \sum_{i=1}^d t_i \check{\xi}_i$.

Corollary 2.1. *Let us suppose that $(X_n)_n$ is ergodic, that (K1) holds on a function space $\mathcal{B} \hookrightarrow \mathbb{L}^2(\pi)$, and $\xi_i \in \mathcal{B}$ ($i = 1, \dots, d$). Then the c.l.t. of the previous theorem holds.*

Proof of Corollary. Since we have (K1) on \mathcal{B} , $\xi_i \in \mathcal{B}$ and $\pi(\xi_i) = 0$, the series $\check{\xi}_i = \sum_{n=0}^{+\infty} Q^n \xi_i$ converges in \mathcal{B} , thus in $\mathbb{L}^2(\pi)$. \square

In order to make easier the use of this corollary, let us recall the following sufficient condition for $(X_n)_n$ to be ergodic. This statement, again in relation with Condition (K1), is established in [?] (Th. IX.2) with the help of standard arguments based on the monotone class theorem.

Proposition 2.2. *Let us suppose that (K1) holds on a function space \mathcal{B} satisfying the following conditions : \mathcal{B} generates the σ -algebra \mathcal{E} , $\delta_x \in \mathcal{B}'$ for all $x \in E$, and $\mathcal{B} \cap \mathcal{B}^{\infty}$ is stable*

under product. Then $(X_n)_n$ is ergodic.

Of course, other methods exist to investigate the c.l.t. for Markov chains. Corollary 2.1 will be here our basic distributional statement : it will be easily applicable to our examples, and its interest is also to define the asymptotic covariance matrix Γ which will occur in all the others limit theorems. Anyway, the above definition of Γ provides the following classical characterisation of the case when Γ is degenerate.

Proposition 2.3. *Under the hypotheses of Corollary 2.1, Γ is non invertible if and only if*

$$\exists t \in \mathbb{R}^d, t \neq 0, \exists g \in \mathcal{B}, \quad \langle t, \xi(X_1) \rangle = g(X_0) - g(X_1) \quad \mathbb{P}_\pi - a.s..$$

Let us notice that this equivalence is still true for $\mathcal{B} = \mathbb{L}^2(\pi)$ if we know that :

$$\forall t \in \mathbb{R}^d, \sup_{n \geq 1} |n \langle \Gamma t, t \rangle - \mathbb{E}_\pi[\langle t, S_n \rangle^2]| < +\infty.$$

Proof. If $\langle t, \xi(X_1) \rangle = g(X_0) - g(X_1) \quad \mathbb{P}_\pi$ -a.s., then $\left(\frac{\langle t, S_n \rangle}{\sqrt{n}}\right)_n$ converges in distribution to the Dirac mass at 0 (which proves Γ is non invertible). Indeed, by stationarity, we have $\langle t, \xi(X_n) \rangle = g(X_{n-1}) - g(X_n) \quad \mathbb{P}_\pi$ -a.s. for all $n \geq 1$, so $\langle t, S_n \rangle = g(X_0) - g(X_n)$. Since we have $g \in \mathcal{B} \hookrightarrow \mathbb{L}^2(\pi)$ by hypothesis, this implies that $\lim_n \mathbb{E}_\pi[\left(\frac{\langle t, S_n \rangle}{\sqrt{n}}\right)^2] = 0$ and hence the desired statement. Conversely, let us suppose that Γ is not invertible. Then there exists $t \in \mathbb{R}^d, t \neq 0$, such that $\langle \Gamma t, t \rangle = 0$. From the definition of Γ given in the above theorem and from the obvious equality $\mathbb{E}_\pi[(\check{\xi}_t(X_1) - Q\check{\xi}_t(X_0))^2] = \pi(\check{\xi}_t^2) - \pi((Q\check{\xi}_t)^2)$, it follows that

$$\mathbb{E}_\pi[(\check{\xi}_t(X_1) - Q\check{\xi}_t(X_0))^2] = 0.$$

Thus $\check{\xi}_t(X_1) - Q\check{\xi}_t(X_0) = 0 \quad \mathbb{P}_\pi - a.s..$ Set $\xi_t(\cdot) = \langle t, \xi(\cdot) \rangle$. By definition of $\check{\xi}_t$, we have $\check{\xi}_t = \xi_t + Q\check{\xi}_t$, so

$$\xi_t(X_1) + Q\check{\xi}_t(X_1) - Q\check{\xi}_t(X_0) = 0 \quad \mathbb{P}_\pi - a.s..$$

This yields $\xi_t(X_1) = g(X_0) - g(X_1) \quad \mathbb{P}_\pi - a.s.$ with $g = Q\check{\xi}_t$. \square

As usual, a set $A \in \mathcal{E}$ is said to be π -full if $\pi(A) = 1$, and Q -absorbing if $Q(a, A) = 1$ for all $a \in A$. The previous proposition can be specified as follows.

Proposition 2.4. *Let $t \in \mathbb{R}^d, t \neq 0$, and let g be a measurable function on E such that :*

$$\langle t, \xi(X_1) \rangle = g(X_0) - g(X_1) \quad \mathbb{P}_\pi - a.s.,$$

then there exists a π -full Q -absorbing set $A \in \mathcal{E}$ such that we have :

$$\forall x \in A, \langle t, \xi(y) \rangle = g(x) - g(y) \quad Q(x, dy) - a.s..$$

Proof. For $x \in E$, set $A_x = \{y \in E : \langle t, \xi(y) \rangle = g(x) - g(y)\}$. By hypothesis we have $\int Q(x, A_x) d\pi(x) = 1$, and since $Q(x, A_x) \leq 1$, this gives $Q(x, A_x) = 1 \quad \pi$ -a.s.. Thus there exists a π -full set $A_0 \in \mathcal{E}$ such that $Q(x, A_x) = 1$ for $x \in A_0$. From $\pi(A_0) = 1$ and invariance of π , we also have $\pi(Q1_{A_0}) = 1$, and since $Q1_{A_0} \leq Q1_E = 1_E$, this implies that $Q(x, A_0) = 1 \quad \pi$ -a.s.. Again there exists a π -full set $A_1 \in \mathcal{E}$ such that $Q(x, A_0) = 1$ for $x \in A_1$. Repeating this procedure, one then obtains a family $\{A_n, n \geq 1\}$ of π -full sets satisfying by construction the condition : $\forall n \geq 1, \forall x \in A_n, Q(x, A_{n-1}) = 1$. Now the set $A := \bigcap_{n \geq 0} A_n$ is π -full and, for any $a \in A$, we have $Q(a, A_{n-1}) = 1$ for all $n \geq 1$, thus $Q(a, A) = 1$. This proves A is Q -absorbing, and the desired equality follows from the inclusion $A \subset A_0$. \square

2.3. The uniformly ergodic Markov chains. We suppose here that $(X_n)_{n \geq 0}$ is uniformly ergodic, as defined in Example (a) of Section 2.1, and we state below in this special case all the limit theorems investigated in this work.

Let us suppose that $\pi(|\xi|_2^2) < +\infty$. Then $(\frac{S_n}{\sqrt{n}})_n$ converges in distribution to a centered normal distribution $\mathcal{N}(0, \Gamma)$ for any initial distribution. This is a classical result [?] (see also [?]) that can be easily deduced, in the stationary case, from the statements of Section 2.2. Indeed, by definition we have (K1) on \mathcal{B}^∞ , so $(X_n)_n$ is ergodic by Proposition 2.2, and since Q also satisfies (K1) on $\mathbb{L}^2(\pi)$ [?], Corollary 2.1 applied with $\mathcal{B} = \mathbb{L}^2(\pi)$ gives the desired c.l.t..

In the sequel, whenever we consider uniformly ergodic Markov chains, we shall assume that the σ -algebra \mathcal{E} is countably generated (in order to apply [?] in Section 4.2).

For the sake of simplicity, we assume in this section that $(X_n)_n$ is stationary ($\mu = \pi$). It will be understood below that the previous covariance matrix Γ is invertible, and the next nonlattice condition will mean that there is no $a \in \mathbb{R}^d$, no closed subgroup H in \mathbb{R}^d , $H \neq \mathbb{R}^d$, no π -full Q -absorbing set $A \in \mathcal{E}$, and finally no bounded measurable function $\theta : E \rightarrow \mathbb{R}^d$ such that :

$$\forall x \in A, \quad \xi(y) + \theta(y) - \theta(x) \in a + H \quad Q(x, dy) - a.s..$$

The following statements will be afterwards specified and established as corollaries of the abstract results of Sections 4-9 :

- *If $\pi(|\xi|_2^2) < +\infty$ and ξ is nonlattice, then $(\xi(X_n))_n$ satisfies a multidimensional local limit theorem (Corollary 5.5).*
- *($d = 1$) If $\pi(|\xi|^3) < +\infty$, then $(\xi(X_n))_n$ satisfies a one-dimensional uniform Berry-Esseen theorem (Corollary 6.3).*
- *($d = 1$) If $\pi(|\xi|^\alpha) < +\infty$ with some $\alpha > 3$ and ξ is nonlattice, then $(\xi(X_n))_n$ satisfies a one-dimensional first-order Edgeworth expansion (Corollary 8.2).*
- *If $\pi(|\xi|_2^\alpha) < +\infty$ for some $\alpha > \max(3, \lfloor d/2 \rfloor + 1)$, then $(\xi(X_n))_n$ satisfies a multidimensional Berry-Esseen theorem in the sense of Prohorov (Corollary 9.2).*

These results, which are similar or very close to those of the i.i.d. case, are new to our knowledge. For details and comparison with prior results, see each above cited corollary. Let us just mention here that the first Nagaev's work [?], which investigated the real-valued functionals of uniformly ergodic Markov chains, established a clt, and a local limit theorem in the countable case, without requiring to the operator-moment conditions mentioned in introduction. More exactly, Nagaev first applied the perturbation method for bounded functionals, and used then some intricate truncation techniques in order to extend his results to unbounded functionals under the condition $\pi(|\xi|^2) < +\infty$. Afterwards this truncation method has not been used any more. In particular, the Edgeworth expansions in Nagaev's paper [?] were stated under strong operator-moment conditions of the type $\sup_{x \in E} \int_E |\xi(y)|^3 Q(x, dy) < +\infty$.

The Knudsen gas model. As pointed out in introduction, Corollary 9.2 just above summarized specifies the slightly uncorrect result of [?] concerning the Knudsen gas model studied by Boatto and Golse in [?], for which we just briefly recall here the link with the uniform ergodicity hypothesis, see [?] for details. Let (E, \mathcal{F}, π) be a probability space, let T be a π -preserving transformation. The Knudsen gas model can be investigated with the help of

the Markov chain $(X_n)_n$ on (E, \mathcal{F}, π) whose transition operator Q is given by :

$$Qf = \delta \pi(f) + (1 - \delta) f \circ T \quad (\text{for some } \delta \in (0, 1)).$$

Clearly $(X_n)_n$ is uniformly ergodic, so that the above statements apply to the Knudsen gas model. In particular, if $\pi(|\xi|_2^\alpha) < +\infty$ for some $\alpha > \max(3, \lfloor d/2 \rfloor + 1)$, then we have as a consequence of Corollary 9.2

$$\mathcal{P} \left(\pi_* \left(\frac{S_n}{\sqrt{n}} \right), \mathcal{N}(0, \Gamma) \right) = O(n^{-1/2}),$$

where \mathcal{P} denotes the Prohorov metric [?, ?], and $\pi_* \left(\frac{S_n}{\sqrt{n}} \right)$ is the law of $\frac{S_n}{\sqrt{n}}$ under P_π . Moreover, according to corollary 2.3.2 of [?] and to the remark following Corollary 9.2, this result is valid for non-invertible Γ as well as for invertible one.

3. The usual Nagaev's method and a typical counter-example. The characteristic function of S_n is linked to the Fourier kernels $Q(t)(x, dy) = e^{i\langle t, \xi(y) \rangle} Q(x, dy)$ of (Q, ξ) by the following formula (see e.g [?] p. 23)

$$(\mathbf{CF}) \quad \forall n \geq 1, \forall t \in \mathbb{R}^d, \mathbb{E}_\mu[e^{it \cdot S_n}] = \mu(Q(t)^n \mathbf{1}_E),$$

and Nagaev's method consists in applying to $Q(t)$ the standard perturbation operator theory [?]. For this to make sense, one must assume that Q satisfies (K1) (of Section 2) on a function space \mathcal{B} , that $Q(t) \in \mathcal{L}(\mathcal{B})$ for t near 0, and at least that $\|Q(t) - Q\|_{\mathcal{B}} \rightarrow 0$ when $t \rightarrow 0$. Then one may express $Q(t)^n$, hence $\mathbb{E}_\mu[e^{it \cdot S_n}]$, in function of $\lambda(t)^n$, where $\lambda(t)$ denotes the dominating eigenvalue of $Q(t)$, and roughly speaking, the usual Fourier techniques then supply limit theorems for $(\xi(X_n))_n$, provided that $\lambda(t)$ satisfies the required Taylor expansions.

This method is notably efficient when (K1) holds on a Banach algebra \mathcal{B} and $\xi \in \mathcal{B}$. Indeed, $Q(\cdot)$ is then analytic (see e.g [?] p. 58). Therefore, the Fourier techniques provide for $(\xi(X_n))_n$ a uniform Berry-Esseen theorem, local limit and renewal theorems under some non-arithmeticity hypothesis, and large deviations theorems by the use of the Laplace kernels, see e.g [?]. Nevertheless it is worth noticing that Banach algebras are generally composed of bounded functions, so that the condition $\xi \in \mathcal{B}$ implies ξ is bounded.

For instance, concerning the local limit theorem, if we want to repeat the proof of the i.i.d. case presented in [?], one needs a second-order Taylor expansion for $\lambda(t)$. To that end (in case $d = 1$ for simplicity), the kernels $\xi(y)Q(x, dy)$ and $\xi(y)^2 Q(x, dy)$ must act continuously on \mathcal{B} , see [?, ?, ?, ?] : this is what we called the operator-moment conditions in introduction. Of course, if ξ is unbounded, this requires very restrictive assumptions on Q . This problem occurs for geometrically ergodic chains or for iterative models, where typically $E = \mathbb{R}^d$, when we consider unbounded functionals as $\xi(x) = x$. In this case, as it is shown in the simple and typical example below, neither the previous conditions on (Q, ξ) , nor even the simple necessary condition $\|Q(t) - Q\|_{\mathcal{B}} \rightarrow 0$, hold in general.

Counter-example. Let $(X_n)_{n \geq 0}$ be the real-valued autoregressive chain defined by

$$X_n = aX_{n-1} + \theta_n \quad (n \in \mathbb{N}^*),$$

where $a \in (-1, 1)$, $a \neq 0$, X_0 is a real r.v. and $(\theta_n)_{n \geq 1}$ is a sequence of i.i.d.r.v., independent of X_0 . Assume θ_1 has a positive density p with finite variance. It is well-known that $(X_n)_{n \geq 0}$ is a Markov chain whose transition probability is : $(Qf)(x) = \int_{\mathbb{R}} f(ax + y) p(y) dy$.

Set $v(x) = 1 + x^2$ ($x \in \mathbb{R}$). Using the so-called drift condition [?] (§ 15.5.2), one can prove that $(X_n)_{n \geq 0}$ is v -geometrically ergodic (see def. in § 2.1). Now let us consider the functional

$\xi(x) = x$. We have for any $x \in \mathbb{R}$

$$Q(\xi^2 v)(x) \geq \int_{\mathbb{R}} (ax + y)^4 p(y) dy.$$

If $\int_{\mathbb{R}} y^4 p(y) dy = +\infty$, then $Q(\xi^2 v)$ is not defined. If $\int_{\mathbb{R}} y^4 p(y) dy < +\infty$, then $Q(\xi^2 v)$ is a polynomial function of degree 4, so that

$$\sup_{x \in E} \frac{|Q(\xi^2 v)(x)|}{1 + x^2} = +\infty,$$

that is, $Q(\xi^2 v) \notin \mathcal{B}_v$. Similarly we have $Q(|\xi|v) \notin \mathcal{B}_v$. Thus neither $\xi(y)Q(x, dy)$, nor $\xi(y)^2 Q(x, dy)$, continuously act on \mathcal{B}_v . Actually, let us show that even the continuity condition $\|Q(t) - Q\|_{\mathcal{B}_v} \rightarrow 0$ is not valid. To see that, it suffices to establish that, if $g(x) = x^2$, then $\|Q(t)g - Qg\|_v = \sup_{x \in \mathbb{R}} (1 + x^2)^{-1} |Q(t)g(x) - Qg(x)|$ does not converge to 0 when $t \rightarrow 0$. Set

$p_1(y) = yp(y)$ and $p_2(y) = y^2 p(y)$, and denote by $\hat{\phi}(t) = \int_{\mathbb{R}} \phi(y) e^{ity} dy$ the Fourier transform of any integrable function ϕ on \mathbb{R} . Then

$$Q(t)g(x) = \int_{\mathbb{R}} e^{it(ax+y)} (y + ax)^2 p(y) dy = e^{iatx} [\hat{p}_2(t) + 2ax\hat{p}_1(t) + a^2 x^2 \hat{p}(t)].$$

So $Q(t)g(x) - Qg(x) = \left(e^{iatx} \hat{p}_2(t) - \hat{p}_2(0) + 2ax [e^{iatx} \hat{p}_1(t) - \hat{p}_1(0)] \right) + a^2 x^2 [e^{iatx} \hat{p}(t) - 1]$.

Using the inequality $|e^{iu} - 1| \leq |u|$, we easily see that there exists a constant $C > 0$ such that

$$\sup_{x \in \mathbb{R}} (1 + x^2)^{-1} \left| e^{iatx} \hat{p}_2(t) - \hat{p}_2(0) + 2ax [e^{iatx} \hat{p}_1(t) - \hat{p}_1(0)] \right| \leq C \left(|t| + |\hat{p}_2(t) - \hat{p}_2(0)| + |\hat{p}_1(t) - \hat{p}_1(0)| \right).$$

By continuity of \hat{p}_1 and \hat{p}_2 , the last term converges to 0 as $t \rightarrow 0$. Now set

$$\psi(x, t) = (1 + x^2)^{-1} a^2 x^2 |e^{iatx} \hat{p}(t) - 1|.$$

We have $\sup_{x \in \mathbb{R}} \psi(x, t) \geq \psi\left(\frac{\pi}{at}, t\right) = \frac{a^2 \pi^2}{\pi^2 + a^2 t^2} |\hat{p}(t) - 1|$. Since this last term converges to $2a^2 \neq 0$ as $t \rightarrow 0$, this clearly implies the desired statement.

4. The starting point of Nagaev's method via Keller-Liverani theorem. Here we state general hypotheses under which the perturbation theorem of Keller-Liverani applies to the Fourier kernels $Q(t)$. Furthermore, as an example, we prove that these hypotheses hold for the uniformly ergodic chains. Finally we shall present two lemmas that specify the connection between the characteristic function of S_n and the dominating eigenvalue $\lambda(t)$ of $Q(t)$.

4.1. Keller-Liverani theorem. When applied with the Fourier kernels $Q(t)$ and the auxiliary semi-norm $\pi(|\cdot|)$, the hypotheses of [?] are ³ :

There exists a neighbourhood \mathcal{O} of 0 in \mathbb{R}^d such that $Q(t) \in \mathcal{L}(\mathcal{B})$ for each $t \in \mathcal{O}$, and :

(K2) $\sup\{\pi(|Q(t)f - Qf|), f \in \mathcal{B}, \|f\|_{\mathcal{B}} \leq 1\}$ converges to 0 when $t \rightarrow 0$ ($t \in \mathcal{O}$).

(K3) $\exists \kappa_1 < 1, \exists C > 0, \forall n \geq 1, \forall f \in \mathcal{B}, \forall t \in \mathcal{O}, \|Q(t)^n f\|_{\mathcal{B}} \leq C \kappa_1^n \|f\|_{\mathcal{B}} + C\pi(|f|)$.

(K4) For $t \in \mathcal{O}$, we have $r_{ess}(Q(t)) \leq \kappa_1$.

We call (K) the whole conditions (K1) (K2) (K3) (K4) (See § 2 for (K1)), we denote by κ any real number such that $\max\{\kappa_0, \kappa_1\} < \kappa < 1$, and we define the following set

³The auxiliary norm in [?] can be easily replaced by a semi-norm.

$$\mathcal{D}_\kappa = \left\{ z : z \in \mathbb{C}, |z| \geq \kappa, |z - 1| \geq \frac{1 - \kappa}{2} \right\}.$$

The next statement, which extends (K1) to $Q(t)$ for t close to 0, is a consequence of [?] (see also [?] p. 177).

Theorem (K-L). *Let us assume that Condition (K) holds. Then, for all $t \in \mathcal{O}$ (with possibly \mathcal{O} reduced), $Q(t)$ admits a dominating eigenvalue $\lambda(t) \in \mathbb{C}$, with a corresponding rank-one eigenprojection $\Pi(t)$, satisfying the following properties :*

$$\lim_{t \rightarrow 0} \lambda(t) = 1, \quad \sup_{t \in \mathcal{O}} \|Q(t)^n - \lambda(t)^n \Pi(t)\|_{\mathcal{B}} = O(\kappa^n),$$

$\sup\{\pi(|\Pi(t)f - \Pi f|), f \in \mathcal{B}, \|f\|_{\mathcal{B}} \leq 1\}$ converges to 0 when $t \rightarrow 0$, and

$$\sup \left\{ \|(z - Q(t))^{-1}\|_{\mathcal{B}}, t \in \mathcal{O}, z \in \mathcal{D}_\kappa \right\} < +\infty.$$

Despite their not very probabilistic appearance, the conditions (K2), (K3), (K4) are suited to many examples of strongly ergodic Markov chains : for instance, we shall prove that they hold, for any measurable functional ξ , in the case of uniformly and geometrically ergodic chains (Prop. 4.1, Lem. 10.1), and that they are valid under simple mean contractivity and moment conditions for iterative Lipschitz models (see § 11). Concerning these conditions, we shall use repeatedly the following simple remarks.

Some comments on the conditions (K2)-(K4).

(a) The continuity Condition (K2) is quite less restrictive than the condition $\|Q(t) - Q\|_{\mathcal{B}} \rightarrow 0$ required in the usual perturbation theorem. A sufficient condition for (K2) is that

(K2') $\sup\{\pi(|e^{i\langle t, \xi \rangle} - 1| |f|), f \in \mathcal{B}, \|f\|_{\mathcal{B}} \leq 1\}$ converges to 0 when $t \rightarrow 0$.

More precisely, (K2') implies that the continuity condition in [?] holds near each point $t_0 \in \mathcal{O}$. Indeed we have, for $f \in \mathcal{B}$,

$$\pi(|Q(t_0 + h)f - Q(t_0)f|) \leq \pi(Q|e^{i\langle h, \xi \rangle} - 1| |f|) = \pi(|e^{i\langle h, \xi \rangle} - 1| |f|).$$

For instance, (K2') holds if $\mathcal{B} \hookrightarrow \mathbb{L}^p(\pi)$ for some $p > 1$ (by Hölder's inequality and Lebesgue's theorem).

(b) Suppose that (K3) holds, and that, for each $t \in \mathcal{O}$, $Q(t)(S)$ is relatively compact in $(\mathcal{B}, \pi(|\cdot|))$, where S is the unit ball of $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$. Then (K4) automatically holds [?] [?]. This remark will be helpful to prove (K4) for iterative models (Section 11).

(c) If \mathcal{B} is a Banach lattice (i.e. $|f| \leq |g| \Rightarrow \|f\|_{\mathcal{B}} \leq \|g\|_{\mathcal{B}}$ for any $f, g \in \mathcal{B}$), then (K1) implies (K3) (indeed, $|Q(t)^n f| \leq Q^n |f|$, thus $\|Q(t)^n f\|_{\mathcal{B}} \leq \|Q^n |f|\|_{\mathcal{B}}$). However the previous compactness condition does not hold for the usual infinite dimensional Banach lattices, so that in this case, (K4) cannot be deduced from (K3) as above indicated.

4.2. Example of the uniformly ergodic Markov chains. Suppose $(X_n)_{n \geq 0}$ is a uniformly ergodic Markov chain. So its transition probability Q satisfies Condition (K1) of Section 2 on \mathcal{B}^∞ and on $\mathbb{L}^p(\pi)$ for each $1 < p < +\infty$ [?].

Proposition 4.1. *For any \mathbb{R}^d -valued measurable function ξ , (Q, ξ) satisfies (K2'), and (K3)-(K4) with $\mathcal{O} = \mathbb{R}^d$, on the function spaces \mathcal{B}^∞ and $\mathbb{L}^p(\pi)$ for each $1 < p < +\infty$. In particular Condition (K) holds on each of these spaces.*

Proof. Let $\|\cdot\|_p = \pi(|\cdot|^p)^{\frac{1}{p}}$ denote the norm on $\mathbb{L}^p(\pi)$ and let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Conditions (K2') and (K3) with $\mathcal{O} = \mathbb{R}^d$, follow from the above remarks (a) and (c). Property (K4) w.r.t. \mathcal{B}^∞ is proved in [?] with the help of [?]. To get (K4) w.r.t. $\mathbb{L}^p(\pi)$, $1 < p < +\infty$, we are going to repeat some arguments derived from [?]. These arguments are based on Doeblin's condition (which can be easily deduced from (K1) on \mathcal{B}^∞), namely : there exist $\ell \geq 1$, $\eta > 0$, and $\rho < 1$ such that

$$(\pi(A) \leq \eta) \Rightarrow (\forall x \in E, Q^\ell(x, A) \leq \rho^\ell).$$

Lemma 4.2. *There exist a nonnegative bounded measurable function α on $E \times E$ and a positive kernel $S(x, dy)$, $x \in E$, such that $Q^\ell(x, dy) = \alpha(x, y)d\pi(y) + S(x, dy)$ and $\|S\|_p \leq \rho^{\frac{\ell}{q}}$.*

Proof. Let us summarize the beginning of the proof in [?] (Lemma III.4) : using the differentiation of measures, there exist a nonnegative measurable function α' on $E \times E$ and a positive kernel $S'(x, dy)$ such that, for all $x \in E$, we have $Q^\ell(x, dy) = \alpha'(x, y)d\pi(y) + S'(x, dy)$, with $\pi(C_x) = 0$ and $S'(x, E \setminus C_x) = 0$ for some $C_x \in \mathcal{E}$. Set $\alpha = \alpha' 1_{\{\alpha' \leq \eta^{-1}\}}$, and for $x \in E$, let $L_x = \{y \in E : \alpha'(x, y) > \eta^{-1}\} \setminus C_x$. Then $Q^\ell(x, dy) = \alpha(x, y)d\pi(y) + S(x, dy)$ with $S(x, A) = Q^\ell(x, A \cap (C_x \cup L_x))$. We have

$$\forall x \in E, \quad 1 \geq Q^\ell(x, L_x) \geq \int_{L_x} \alpha'(x, y) d\pi(y) \geq \eta^{-1} \pi(L_x),$$

thus $\pi(L_x \cup C_x) = \pi(L_x) \leq \eta$, so that $Q^\ell(x, L_x \cup C_x) \leq \rho^\ell$.

Now let $f \in \mathbb{L}^p(\pi)$. We have $Sf(x) = \int_{C_x \cup L_x} f(y) Q^\ell(x, dy)$, and from Hölder's inequality w.r.t. the probability measure $Q^\ell(x, dy)$, we have

$$\|Sf\|_p^p = \int_E \left| \int_E f(y) 1_{C_x \cup L_x}(y) Q^\ell(x, dy) \right|^p d\pi(x) \leq \int_E Q^\ell |f|^p(x) Q^\ell(x, C_x \cup L_x)^{\frac{p}{q}} d\pi(x),$$

hence $\|Sf\|_p^p \leq (\rho^\ell)^{\frac{p}{q}} \pi(Q^\ell |f|^p) = (\rho^\ell)^{\frac{p}{q}} \pi(|f|^p)$ which is the stated estimate on $\|S\|_p$. \square

Now let us prove $r_{ess}(Q(t)) \leq \rho^{\frac{1}{q}}$. Since $|Q(t)^\ell f| \leq Q^\ell |f|$, there exists a complex-valued measurable function χ_t on $E \times E$ such that $Q(t)^\ell(x, dy) = \chi_t(x, y) Q^\ell(x, dy)$ with $|\chi_t| \leq 1$. So, by Lemma 4.2,

$$Q(t)^\ell(x, dy) = \chi_t(x, y) \alpha(x, y)d\pi(y) + \chi_t(x, y) S(x, dy) := \alpha_t(x, y)d\pi(y) + S_t(x, dy),$$

and, since $\alpha_t(\cdot, \cdot)$ is bounded, the associated kernel operator is compact on $\mathbb{L}^p(\pi)$ [?]. Recall that, if T is a bounded operator on a Banach space \mathcal{B} , then $r_{ess}(T) = \lim_n (\inf \|T^n - V\|_{\mathcal{B}})^{\frac{1}{n}}$ where the infimum is considered over the ideal of compact operators V on \mathcal{B} . This yields $r_{ess}(Q(t)^\ell) = r_{ess}(S_t) \leq r(S_t) \leq \|S_t\|_p \leq \|S\|_p \leq \rho^{\frac{\ell}{q}}$ (Lem. 4.2). Hence $r_{ess}(Q(t)) \leq \rho^{\frac{1}{q}}$. \square

4.3. Link between $\lambda(t)$ and the characteristic function of S_n . Recall that π [resp. μ] denotes the stationary distribution [resp. the initial distribution] of the chain. For convenience, let us repeat the basic formula (CF), already formulated in Section 3, which links the characteristic function of S_n with the Fourier kernels of (Q, ξ) :

$$(CF) \quad \forall n \geq 1, \forall t \in \mathbb{R}^d, \mathbb{E}_\mu[e^{i\langle t, S_n \rangle}] = \mu(Q(t)^n 1_E).$$

In this section, we assume that Condition (K) of section 4.1 holds on some function space \mathcal{B} .

We shall appeal to Theorem (K-L), in particular to the dominating eigenvalue $\lambda(t)$ of $Q(t)$, $t \in \mathcal{O}$, and to the real number κ for which we just remember that $\kappa < 1$.

Lemma 4.3. *If (K) (of section 4.1) holds and $\mu = \pi$, then there exists a function $\ell : \mathcal{O} \rightarrow \mathbb{C}$ such that*

$$\lim_{t \rightarrow 0} \ell(t) = 1 \quad \text{and} \quad \sup_{t \in \mathcal{O}} |\mathbb{E}_\pi[e^{i(t, S_n)}] - \lambda(t)^n \ell(t)| = O(\kappa^n).$$

Proof. Set $\ell(t) = \pi(\Pi(t)1_E)$ for $t \in \mathcal{O}$, where $\Pi(t)$ is the rank-one eigenprojection associated to $\lambda(t)$. Since $\pi \in \mathcal{B}'$, Lemma 4.3 directly follows from Theorem (K-L) and Formula (CF). \square

If the conditions (K2) and (K3) are fulfilled with, instead of $\pi(| \cdot |)$, any semi-norm $q(\cdot)$ on \mathcal{B} satisfying $q(\cdot) \leq c \| \cdot \|$ ($c > 0$), then the conclusions of Theorem (K-L) remain valid with in this case : $\sup\{q(|\Pi(t)f - \Pi f|), f \in \mathcal{B}, \|f\|_{\mathcal{B}} \leq 1\} \rightarrow 0$ when $t \rightarrow 0$, see [?]. The property $\lim_{t \rightarrow 0} \pi(\Pi(t)1_E) = 1$ required for Lemma 4.3 holds if we have furthermore $\pi(| \cdot |) \leq d q(\cdot)$ ($d > 0$), and Lemma 4.3 is then still valid. In fact, contrary to the condition (\widetilde{K}) of Section 5.2 for which the use of a general semi-norm may be interesting in practice, Condition (K) will be always fulfilled in our examples with the semi-norm $\pi(| \cdot |)$.

To extend Lemma 4.3 to the non-stationary case, we consider another function space $(\widetilde{\mathcal{B}}, \| \cdot \|_{\widetilde{\mathcal{B}}})$ such that $\mathcal{B} \subset \widetilde{\mathcal{B}}$, and the following condition :

Condition (\widetilde{K}) : we have (K) (of section 4.1) on \mathcal{B} , (K1) (of section 2) on $\widetilde{\mathcal{B}}$, and the following continuity condition : $\lim_{t \rightarrow 0} \|Q(t) - Q\|_{\mathcal{B}, \widetilde{\mathcal{B}}} = 0$.

Lemma 4.4. *If (\widetilde{K}) holds and $\mu \in \mathcal{B}' \cap \widetilde{\mathcal{B}}'$, the conclusions of Lemma 4.3 hold under \mathbb{P}_μ .*

Proof. Set $\ell(t) = \mu(\Pi(t)1_E)$ for $t \in \mathcal{O}$. Lemma 4.4 will follow from Theorem (K-L) and Formula (CF), provided that we prove $\lim_{t \rightarrow 0} \mu(\Pi(t)1_E) = 1$. For that, observe that $1 = \mu(\Pi 1_E)$, so $|\mu(\Pi(t)1_E) - 1| \leq \|\mu\|_{\widetilde{\mathcal{B}}} \|\Pi(t) - \Pi\|_{\mathcal{B}, \widetilde{\mathcal{B}}} \|1_E\|_{\mathcal{B}}$. Besides, as in the standard perturbation theory [?], $\Pi(t)$ can be expressed as the line integral of $(z - Q(t))^{-1}$ over a suitable oriented circle centered at $\lambda = 1$, see [?]. Then the formula

$$(z - Q(t))^{-1} - (z - Q)^{-1} = (z - Q)^{-1} [Q(t) - Q] (z - Q(t))^{-1}$$

and (\widetilde{K}) easily give $\|\Pi(t) - \Pi\|_{\mathcal{B}, \widetilde{\mathcal{B}}} \rightarrow 0$ as $t \rightarrow 0$, hence the desired convergence. \square

To illustrate the continuity condition in (\widetilde{K}) , let us prove the following lemma which holds without any moment condition on ξ . Suppose $(X_n)_{n \geq 0}$ is a uniformly ergodic Markov chain.

Proposition 4.5. *Let $1 \leq p' < p$. Then $\lim_{t \rightarrow 0} \|Q(t) - Q\|_{\mathbb{L}^p(\pi), \mathbb{L}^{p'}(\pi)} = 0$.*

Proof. Let us denote $\| \cdot \|_p$ for $\| \cdot \|_{\mathbb{L}^p(\pi)}$. Using the inequality $|e^{ia} - 1| \leq 2 \min\{1, |a|\}$ ($a \in \mathbb{R}$) and the Hölder inequality, one gets for $t_0, h \in \mathbb{R}$ and $f \in \mathbb{L}^p(\pi)$,

$$\begin{aligned} \|Q(t_0 + h)f - Q(t_0)f\|_{p'} &\leq \left\| Q(|e^{ih\xi} - 1| |f|) \right\|_{p'} \\ &\leq 2 \|\min\{1, |h\xi|\} |f|\|_{p'} \\ &\leq 2 \|\min\{1, |h\xi|\}\|_{\frac{pp'}{p-p'}} \|f\|_p, \end{aligned}$$

with $\|\min\{1, |h\xi|\}\|_{\frac{pp'}{p-p'}} \rightarrow 0$ when $h \rightarrow 0$ by Lebesgue's theorem. \square

5. A multi-dimensional local limit theorem. The two previous lemmas constitute the necessary preliminary to employ the usual Fourier techniques. However, it is worth noticing that, except $\lim_{t \rightarrow 0} \lambda(t) = 1$, the perturbation theorem of Keller-Liverani cannot yield anyway the Taylor expansions needed for $\lambda(t)$ in Fourier techniques. An abstract operator-type hypothesis $\mathcal{C}(m)$ will be presented in Section 7 in order to ensure the existence of m continuous derivatives for $\lambda(\cdot)$. But we want before to recall another method, based on weaker and more simple probabilistic c.l.t.-type assumptions, which provides second or third-order Taylor expansions of $\lambda(t)$ near $t = 0$. As in the i.i.d. case, these expansions are sufficient to establish a multi-dimensional local limit theorem, this is the goal of Section 5.1, and a one-dimensional uniform Berry-Esseen theorem which will be presented in Section 6. ⁴

Theorem 5.1 below has been established for real-valued functionals in [?] under slightly different hypotheses. Here we present an easy extension to the multidimensional case. The application to the uniformly ergodic chains is new (Section 5.3).

5.1. A general statement. To state the local limit theorem, one needs to introduce the two following conditions. The first one is the central limit assumption stated under \mathbb{P}_π for which one may appeal to Corollary 2.1 for instance. The second one is a spectral non-arithmeticity condition. Recall that, by hypothesis, we have $\pi(\xi) = 0$, so that $\mathbb{E}_\pi[S_n] = 0$.

Condition (CLT) : Under \mathbb{P}_π , $\frac{S_n}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \Gamma)$, with a non-singular matrix Γ .

Condition (S) : For all $t \in \mathbb{R}^d$, $Q(t) \in \mathcal{L}(\mathcal{B})$, and for each compact set K_0 in $\mathbb{R}^d \setminus \{0\}$, there exist $\rho < 1$ and $c \geq 0$ such that we have, for all $n \geq 1$ and $t \in K_0$, $\|Q(t)^n\|_{\mathcal{B}} \leq c\rho^n$.

Condition (S) constitutes the tailor-made hypothesis in order to operate in Nagaev's method the proofs of the i.i.d. limit theorems involving Fourier techniques and the so-called non-lattice assumption. It will be reduced to more practical hypotheses in Section 5.2.

We want to prove that, given some fixed positive function f on E , we have

$$(LLT) \quad \lim_n \left(\sqrt{\det \Gamma} (2\pi n)^{\frac{d}{2}} \mathbb{E}_\mu[f(X_n)g(S_n - a)] - e^{-\frac{1}{2n}\langle \Gamma^{-1}a, a \rangle} \pi(f) \int_{\mathbb{R}} g(x)dx \right) = 0,$$

for all compactly supported continuous function $g : \mathbb{R}^d \rightarrow \mathbb{R}$ and for all $a \in \mathbb{R}^d$.

The conditions on f and μ will be specified below.

Theorem 5.1. *Assume that Condition (CLT) holds, that (Q, ξ) satisfies Conditions (K) (of section 4.1) and (S) w.r.t. some function space \mathcal{B} , and suppose $f \in \mathcal{B}$, $f \geq 0$. Then we have (LLT) under \mathbb{P}_π . If (K) is replaced with (\tilde{K}) , then (LLT) holds under \mathbb{P}_μ if $\mu \in \mathcal{B}' \cap \tilde{\mathcal{B}}'$.*

Proof. In order to use Lemmas 4.3-4 and to write out the Fourier techniques of the i.i.d. case [?], one needs to establish a second-order Taylor expansion for $\lambda(t)$.

Lemma 5.2. *If assumptions (K) (of section 4.1) and (CLT) are fulfilled, then we have*

⁴These two limit theorems could also be deduced from respectively Conditions $\mathcal{C}(2)$ and $\mathcal{C}(3)$ of Section 7, but in practice, these two conditions are slightly more restrictive than those of Sections 5-6. For instance, compare $\mathcal{C}(2)$ and $\mathcal{C}(3)$ for uniformly ergodic chains (see Prop. 7.3) with the conditions of Coro. 5.5 and 6.3.

$\lambda(u) = 1 - \frac{1}{2}\langle \Gamma u, u \rangle + o(\|u\|^2)$ near $u = 0$.

Proof (sketch). For $d = 1$, the proof of Lemma 5.2 is presented in [?], let us just recall the main ideas. By hypothesis, $\frac{S_n}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$ under \mathbb{P}_π , with $\sigma^2 > 0$. It follows from Lévy's theorem and Lemma 4.3 that $\lim_n \lambda(\frac{t}{\sqrt{n}})^n = e^{-\frac{\sigma^2}{2}t^2}$, with uniform convergence on any compact set in \mathbb{R} . Then the fact that $\log \lambda(\frac{t}{\sqrt{n}})^n = n \log \lambda(\frac{t}{\sqrt{n}})$ and $\log \lambda(\frac{t}{\sqrt{n}}) \sim \lambda(\frac{t}{\sqrt{n}}) - 1$ gives for $t \neq 0$:

$$\left(\frac{\sqrt{n}}{t}\right)^2 \left(\lambda\left(\frac{t}{\sqrt{n}}\right) - 1\right) + \frac{\sigma^2}{2} = o(1) \quad \text{when } n \rightarrow +\infty.$$

Setting $u = \frac{t}{\sqrt{n}}$, it is then not hard to deduce the stated Taylor expansion (see [?] Lem. 4.2). These arguments can be readily repeated for $d \geq 2$. (To get $\log \lambda(\frac{t}{\sqrt{n}})^n = n \log \lambda(\frac{t}{\sqrt{n}})$ in $d \geq 2$, proceed as in [?] with $\psi(x) = \lambda(x\frac{t}{\sqrt{n}})$, $x \in [0, 1]$.) \square

If $f = 1_E$, then (LLT) follows from Lemma 4.3 or 4.4, according that $\mu = \pi$ or not, by writing out the i.i.d. Fourier techniques of [?]. In particular, Condition (S) plays the same part that the nonlattice condition of [?]. If $f \in \mathcal{B}$, $f \geq 0$, one can proceed in the same way by using the following equality, whose (CF) is a special case (see e.g [?] p. 23),

$$(CF') \quad \mathbb{E}_\mu[f(X_n) e^{itS_n}] = \mu(Q(t)^n f),$$

and some obvious extensions of Lemmas 4.3-4. See [?] for details in $d = 1$. \square

Neglecting for the moment the non-arithmeticity condition (S), we want now to complete the comparison with the usual Nagaev's method. Let us suppose that $(X_n)_n$ satisfies the strong ergodicity condition (K1) of Section 2. As already mentioned in Section 3, in order to establish (LLT), one may suppose that the kernel $\xi(y)^2 Q(x, dy)$ has a continuous action on \mathcal{B} : one then obtains directly a second-order Taylor expansion for $\lambda(\cdot)$, from which one can deduce both the clt and (LLT), see e.g [?]. Here the previous condition is replaced by the conditions (K2) (K3) (K4) of Theorem (K-L) which are fulfilled under very weak hypotheses in our examples. From Theorem 5.1, at least in the stationary case, (CLT) then implies (LLT) : of course, the conditions for (CLT) are weaker (and often much weaker) than the above second-order operator-moment condition. In other words, concerning the local limit theorem, Theorem 5.1 improves all the prior statements obtained with the help of the usual Nagaev's method, see e.g [?, ?, ?, ?]. This will be first illustrated in Section 5.3 with the uniformly ergodic Markov chains, see also Sections 10-11.

5.2. Study of Condition (S). When Nagaev's method is applied with the standard perturbation theory, it is well-known that the spectral condition (S) can be reduced to a more practical condition on (Q, ξ) , see e.g [?] [?] [?]. This reduction is based on some spectral arguments, and on simple properties of strict convexity which require some additional conditions on the function space \mathcal{B} , see (b1) (b2) below. Here the spectral arguments will be derived from Keller-Liverani theorem. To this end, conditions (K2)-(K4) must be extended to the whole family $\{Q(t), t \in \mathbb{R}^d\}$ as stated below. Moreover the use of a general semi-norm on \mathcal{B} (instead of $\pi(|\cdot|)$) in Keller-Liverani theorem will be here helpful in practice.

More precisely, let us assume that there exists a semi-norm $q(\cdot)$ on $(\mathcal{B}, \|\cdot\|)$ such that we have $q(\cdot) \leq c \|\cdot\|$ for some $c > 0$, and

$(\widehat{K2}) \quad \forall t \in \mathbb{R}^d, Q(t) \in \mathcal{L}(\mathcal{B})$ and $\limsup_{h \rightarrow 0} \{q(Q(t+h)f - Q(t)f), f \in \mathcal{B}, \|f\|_{\mathcal{B}} \leq 1\} = 0$

and, for all compact set K_0 in \mathbb{R}^d , there exists $\kappa \in (0, 1)$ such that :

$(\widehat{K3}) \quad \exists C > 0, \forall n \geq 1, \forall f \in \mathcal{B}, \forall t \in K_0, \|Q(t)^n f\|_{\mathcal{B}} \leq C \kappa^n \|f\|_{\mathcal{B}} + C q(|f|)$

$(\widehat{K4}) \quad \forall t \in K_0, r_{ess}(Q(t)) \leq \kappa.$

We call (\widehat{K}) the whole conditions $(K1)$ (of section 2), $(\widehat{K2}), (\widehat{K3}), (\widehat{K4})$.

Notice that, if $q(\cdot) = \pi(|\cdot|)$, then $(K2')$ implies $(\widehat{K2})$, see Rk. (a) in § 4.1. The uniformly ergodic chains satisfy with $q(\cdot) = \pi(|\cdot|)$ all the previous conditions on \mathcal{B}^∞ and $\mathbb{L}^p(\pi)$ for $p > 1$, see Prop. 4.1. This will be still valid, for geometrically ergodic chains w.r.t the weighted supremum-normed spaces of Section 10, and for the iterative models (by using another semi-norm) w.r.t. the weighted Lipschitz-type spaces of Section 11.

As usual, the relation " $f \leq g$ " in \mathcal{B} means, either " $f \leq g$ on E " if \mathcal{B} is composed of functions, or " $f \leq g$ π -a.s. on E " if \mathcal{B} is composed of equivalence classes of functions (as the $\mathbb{L}^p(\pi)$'s). Finally, for $e' \in \mathcal{B}'$, we say that $e' \geq 0$ if we have $e'(g) \geq 0$ for all $g \in \mathcal{B}, g \geq 0$.

In the next non-arithmeticity condition, we shall suppose that \mathcal{B} is a function space satisfying, either the following general condition (b1), in which f, g, f_n ($n \geq 0$) denote elements of \mathcal{B} ,

Condition (b1) :

$f \geq 0, f \neq 0 \Rightarrow \exists e' \in \mathcal{B}', e' \geq 0, e'(f) \neq 0$

$[\lim_n f_n = f \text{ in } \mathcal{B} \text{ and } g \leq f_n \text{ for all } n \geq 0] \Rightarrow g \leq f,$

or the following stronger *Condition (b2) :* $\forall x \in E, \delta_x \in \mathcal{B}'$.

We set : $\mathcal{B}_0 = \mathcal{B} \cap \mathbb{L}^\infty(\pi)$ under *Condition (b1)*, and $\mathcal{B}_0 = \mathcal{B} \cap \mathcal{B}^\infty$ under *Condition (b2)*.

Let us recall that $A \in \mathcal{E}$ is π -full if $\pi(A) = 1$, and Q -absorbing if $Q(a, A) = 1$ for all $a \in A$.

A non-arithmetic condition on ξ . Under the condition (b1) or (b2), we shall say that (Q, ξ) , or merely ξ , is arithmetic w.r.t. \mathcal{B} (and non-arithmetic w.r.t. \mathcal{B} in the opposite case) if there exist $t \in \mathbb{R}^d, t \neq 0, \lambda \in \mathbb{C}, |\lambda| = 1$, a π -full Q -absorbing set $A \in \mathcal{E}$, and a function w in \mathcal{B}_0 such that $|w|$ is nonzero constant on A , satisfying :

$$(*) \quad \forall x \in A, e^{i\langle t, \xi(y) \rangle} w(y) = \lambda w(x) \quad Q(x, dy) - a.s..$$

The two next propositions are expected results, and for convenience they will be proved in Section 12.

Proposition 5.3. Assume that Hypothesis (\widehat{K}) holds, and that we have (b1) or (b2). Then *Condition (S)* holds w.r.t. \mathcal{B} if and only if ξ is non-arithmetic w.r.t. \mathcal{B} .

In the usual Nagaev's method, this statement is for instance established in [?] (Prop. V.2) under *Condition (b2)*. Let us notice that (b2), which clearly implies (b1), doesn't hold on $\mathbb{L}^p(\pi)$, while (b1) is fulfilled on this space (take $e' = \pi$). The proof of Proposition 5.3 is an extension of that in [?]. See Section 12.1.

We now present a lattice-type criterion for (S) which is a natural extension of the i.i.d. case, and a well-known condition in the general context of Markov random walks.

A nonlattice condition on ξ . We shall say that (Q, ξ) , or merely ξ , is lattice (and non-lattice in the opposite case) if there exist $a \in \mathbb{R}^d$, a closed subgroup H in \mathbb{R}^d , $H \neq \mathbb{R}^d$, a π -full Q -absorbing set $A \in \mathcal{E}$, and a bounded measurable function $\theta : E \rightarrow \mathbb{R}^d$ such that

$$(**) \quad \forall x \in A, \quad \xi(y) + \theta(y) - \theta(x) \in a + H \quad Q(x, dy) - a.s..$$

Proposition 5.4. *In addition to the hypotheses of Proposition 5.3, assume that \mathcal{B}_0 is stable under product, that Condition (CLT) of Section 5.1 is fulfilled, and that $\pi(|\cdot|) \leq c q(\cdot)$. If ξ is nonlattice, then (S) holds on \mathcal{B} . The converse is true when, for any real-valued measurable function ψ on E , we have $e^{i\psi} \in \mathcal{B}$.*

This expected result, based on the study of the set $G = \{t \in \mathbb{R}^d : r(Q(t)) = 1\}$, will be established in Section 12.2. For the uniformly ergodic chains, Condition (S) w.r.t. \mathcal{B}^∞ , or $\mathbb{L}^p(\pi)$ for $p > 1$, is equivalent to the previous non-lattice hypothesis, provided $\pi(|\xi|_2^2) < +\infty$ in order to get (CLT) (see § 2.3). This will be extended to v -geometrically ergodic chains w.r.t. the spaces used in Section 10, if we have $|\xi|_2^2 \leq C v$ for some $C > 0$. For the iterative models, the above non-arithmetic condition on ξ will be a necessary and sufficient condition for (S) to be valid on the spaces used in Section 11, and the non-lattice assumption will be a sufficient condition for (S), but not (a priori) a necessary condition.

Condition (S) and invertibility of Γ . Observe that, if the conclusion of Proposition 2.4 holds for some real-valued measurable function g on E , then we clearly have (*) with $w(\cdot) = e^{ig(\cdot)}$ and $\lambda = 1$. Moreover, (**) is satisfied with $a = 0$, $\theta(x) = \{g(x)\} \frac{t}{|t|_2^2}$, and $H = (\mathbb{R} \cdot t)^\perp + \mathbb{Z} \frac{t}{|t|_2^2}$, where $\{\cdot\}$ stands for the fractionary part. Finally Condition (S) with respect to any \mathcal{B} containing w is then false because, in this case, the above mentioned equality (*) easily implies that $r(Q(t)) \geq 1$, see Lemma 12.2.

Consequently, given any function spaces \mathcal{B} and \mathcal{B}_2 , one can deduce the following facts from the previous remarks and the results of Section 2.2. If the hypotheses of Corollary 2.1 hold w.r.t. \mathcal{B}_2 and if $e^{i\psi(\cdot)} \in \mathcal{B}$ for all $\psi \in \mathcal{B}_2$, then we have the following implications, in which Γ denotes the covariance matrix of Section 2.2.

Condition (S) w.r.t. $\mathcal{B} \Rightarrow \Gamma$ is invertible

Non-arithmeticity w.r.t. $\mathcal{B} \Rightarrow \Gamma$ is invertible

(Q, ξ) is non-lattice $\Rightarrow \Gamma$ is invertible.

The above condition on \mathcal{B} is unnecessary for the last implication.

5.3. (LLT) for the uniformly ergodic Markov chains. Let us suppose that $(X_n)_{n \geq 0}$ is uniformly ergodic. Recall that, if $\pi(|\xi|_2^2) < +\infty$, then $(n^{-\frac{1}{2}} S_n)_n$ converges in distribution to a normal distribution $\mathcal{N}(0, \Gamma)$ (see § 2.3). The next statement is new to our knowledge.

Corollary 5.5. *Let us assume that $\pi(|\xi|_2^2) < +\infty$, that ξ is nonlattice, and that $d\mu = \phi d\pi$ with $\phi \in \mathbb{L}^r(\pi)$ for some $r > 1$. Then we have (LLT) for each function $f \geq 0$ in $\mathbb{L}^p(\pi)$ provided $p > \frac{r}{r-1}$.*

Proof. Let $r' = \frac{r}{r-1}$ and $p > r'$. From Proposition 4.1, we have (K), and even (\widehat{K}) with $q(\cdot) = \pi(|\cdot|)$, on $\mathcal{B} = \mathbb{L}^p(\pi)$. Moreover, Proposition 4.5 yields (\widetilde{K}) with $\widetilde{\mathcal{B}} = \mathbb{L}^{r'}(\pi)$. Besides,

since ξ is nonlattice, (S) holds on $\mathbb{L}^p(\pi)$ by Proposition 5.4. Finally $\mathcal{B}' \cap \tilde{\mathcal{B}}' = (\mathbb{L}^{r'}(\pi))'$ may be identified with $\mathbb{L}^r(\pi)$. Then Corollary 5.5 follows from Theorem 5.1. \square

6. A one-dimensional uniform Berry-Esseen theorem. Here we assume $d = 1$ (i.e. ξ is real-valued), we denote by \mathcal{N} the distribution function of $\mathcal{N}(0, 1)$, we suppose that Hypothesis (CLT) of section 5.1 holds with $\Gamma = \sigma^2 > 0$, and we set for any $u \in \mathbb{R}$:

$$\Delta_n(u) = \left| \mathbb{P}_\mu\left(\frac{S_n}{\sigma\sqrt{n}} \leq u\right) - \mathcal{N}(u) \right| \quad \text{and} \quad \Delta_n = \sup_{u \in \mathbb{R}} \Delta_n(u).$$

Theorem 6.1 and Proposition 6.2 below have been already presented in [?], we state them again for completeness. As in the previous section, the application to the uniformly ergodic chains, presented in Section 6.3, is new and it will give a good understanding of the present results in comparison with the usual Nagaev's method (and even with the application of Bolthausen's theorem to this special example). Other comparisons with prior works are presented in [?], they will be partially remembered in Sections 10-11.

6.1 A general statement. Let us reinforce Condition (CLT) by the following one :

$$\text{Condition (CLT')} : \exists C > 0, \forall t \in [-\sqrt{n}, \sqrt{n}], \quad |\mathbb{E}_\pi[e^{it\frac{S_n}{\sigma\sqrt{n}}}] - e^{-\frac{t^2}{2}}| \leq C \frac{|t|}{\sqrt{n}}.$$

Theorem 6.1 [?]. *Assume that Condition (\tilde{K}) of Section 4.3 and (CLT') are fulfilled, with the continuity condition of (\tilde{K}) strengthened by $\|Q(t) - Q\|_{\mathcal{B}, \tilde{\mathcal{B}}} = O(|t|)$. Then we have $\Delta_n = O(n^{-\frac{1}{2}})$ for any $\mu \in \mathcal{B}' \cap \tilde{\mathcal{B}}'$.*

Proof (sketch). See [?] for details. Let us assume for convenience that $\mu = \pi$. In order to copy the Fourier techniques used for the i.i.d. Berry-Essen theorem (see [?] [?]), we have to improve Lemmas 4.3 and 5.2 as follows :

- (a) $\sup_{t \in \mathcal{O}} \frac{1}{|t|} |\mathbb{E}_\pi[e^{i(t, S_n)}] - \lambda(t)^n \ell(t)| = O(\kappa^n)$, with $\sup_{t \in \mathcal{O}} \frac{|\ell(t) - 1|}{|t|} < +\infty$.
(b) $\lambda(u) = 1 - \sigma^2 \frac{u^2}{2} + O(u^3)$ near $u = 0$.

Assertion (a) can be easily derived from $\|Q(t) - Q\|_{\mathcal{B}, \tilde{\mathcal{B}}} = O(|t|)$ by proceeding as in Lemma 4.4. To get (b), one may repeat the short proof of Lemma 5.2 by starting here from the property $\lambda(\frac{t}{\sqrt{n}})^n - e^{-\frac{\sigma^2}{2}t^2} = O(\frac{|t|}{\sqrt{n}})$ which follows from (CLT') and (a). One then obtains $(\frac{\sqrt{n}}{t})^2(\lambda(\frac{t}{\sqrt{n}}) - 1) + \frac{\sigma^2}{2} = O(\frac{|t|}{\sqrt{n}})$, and setting $u = \frac{t}{\sqrt{n}}$, this leads to the expansion (b) (see Lem. IV.2 in [?]). \square

6.2. A sufficient condition for (CLT'). Actually, one of the difficulties in the previous theorem is to show Hypothesis (CLT'). By the use of martingale techniques derived from [?], we showed in [?] the next statement.

Proposition 6.2 [?]. *We have (CLT') when the two following conditions hold :*

- (G1) $\check{\xi} = \sum_{n=0}^{+\infty} Q^n \xi$ absolutely converges in $\mathbb{L}^3(\pi)$.
(G2) $\sum_{p=0}^{+\infty} Q^p \psi$ absolutely converges in $\mathbb{L}^{\frac{3}{2}}$, where $\psi = Q(\check{\xi}^2) - (Q\check{\xi})^2 - (\pi(\check{\xi}^2) - \pi((Q\check{\xi})^2)) 1_E$.

Let us notice that $\check{\xi}$ is the solution of the Poisson equation $\check{\xi} - Q\check{\xi} = \xi$, already introduced

in Gordin's theorem (see § 2.2). Also observe that the above function ψ can be expressed as $\psi = Q(\check{\xi}^2) - (Q\check{\xi})^2 - \sigma^2 1_E$, where σ^2 is the asymptotic variance of Gordin's theorem.

About the practical verification of (G1) (G2).

In practice, one often proceeds as follows to verify the two above conditions. Since $\pi(\xi) = 0$, Condition (G1) holds if Q is strongly ergodic w.r.t. some $\mathcal{B} \hookrightarrow \mathbb{L}^3(\pi)$ and if $\xi \in \mathcal{B}$. If moreover Q is strongly ergodic w.r.t. some $\mathcal{B}_2 \hookrightarrow \mathbb{L}^{\frac{3}{2}}(\pi)$ containing all the functions g^2 , $g \in \mathcal{B}$, then Condition (G2) holds. Indeed, under these hypotheses, $\check{\xi} \in \mathcal{B}$, thus $\psi \in \mathcal{B}_2$, and, since $\pi(\psi) = 0$, the series $\sum_{p=0}^{+\infty} Q^p \psi$ absolutely converges in \mathcal{B}_2 , thus in $L^{\frac{3}{2}}(\pi)$.

6.3. Application to uniformly ergodic Markov chains. Let us suppose that $(X_n)_{n \geq 0}$ is uniformly ergodic. It is proved in Nagaev's work [?], and in some of its extensions (see e.g [?]), that $\Delta_n = O(n^{-\frac{1}{2}})$ under the strong moment condition $\sup_{x \in E} \int_E |\xi(y)|^3 Q(x, dy) < +\infty$.

In the stationary case (ie. $\mu = \pi$), since $(X_n)_{n \geq 0}$ is strongly mixing (see [?]), Bolthausen's theorem [?] implies the previous estimate if $\pi(|\xi|^p) < +\infty$ for some $p > 3$. The next statement, which is new to our knowledge, only requires the expected third-order moment condition.

Corollary 6.3. *If $\pi(|\xi|^3) < +\infty$ and $\mu = \phi d\pi$, with some $\phi \in \mathbb{L}^3(\pi)$, then $\Delta_n = O(n^{-\frac{1}{2}})$.*

Proof. Set $\mathbb{L}^p = \mathbb{L}^p(\pi)$. Since Q is strongly ergodic on \mathbb{L}^3 and $\mathbb{L}^{\frac{3}{2}}$ [?], and since $\xi \in \mathbb{L}^3$ by hypothesis, this implies (G1) (G2), thus (CLT') (see § 6.2). Besides, (Q, ξ) verifies Condition (K) w.r.t. \mathbb{L}^3 and $\mathbb{L}^{\frac{3}{2}}$ (Prop. 4.1). Moreover we have $\|Q(t) - Q\|_{\mathbb{L}^3, \mathbb{L}^{\frac{3}{2}}} = O(|t|)$. Indeed, let $f \in \mathbb{L}^3$. Using $|e^{ia} - 1| \leq |a|$, one gets

$$\pi(|Q(t)f - Qf|^{\frac{3}{2}}) \leq \pi(|Q(e^{it\xi} - 1)|f|)^{\frac{3}{2}} \leq |t|^{\frac{3}{2}} \pi(Q(|\xi|^{\frac{3}{2}}|f|^{\frac{3}{2}})) = |t|^{\frac{3}{2}} \pi(|\xi|^{\frac{3}{2}}|f|^{\frac{3}{2}}),$$

and Schwarz inequality yields $\|Q(t)f - Qf\|_{\frac{3}{2}} \leq |t| (\pi(|\xi|^{\frac{3}{2}}|f|^{\frac{3}{2}}))^{\frac{2}{3}} \leq |t| \|\xi\|_3 \|f\|_3$. We have proved that the hypotheses of Theorem 6.1 are fulfilled with $\mathcal{B} = \mathbb{L}^3$ and $\tilde{\mathcal{B}} = \mathbb{L}^{\frac{3}{2}}$. Finally, $\mu \in (\mathbb{L}^3)' \cap (\mathbb{L}^{\frac{3}{2}})' = (\mathbb{L}^{\frac{3}{2}})'$ if and only if μ has the stated form. \square

7. Regularity of the eigen-elements of $Q(t)$. The goal of this section is to present an abstract operator-type Hypothesis, called $\mathcal{C}(m)$ ensuring that the dominating eigenvalue $\lambda(t)$ and the others eigen-elements of $Q(t)$ have m continuous derivatives on some neighbourhood \mathcal{O} of 0. The usual Nagaev's method already exploited this idea by considering the action of $Q(t)$ on a single function space, but as illustrated in Section 3, the resulting operator-moment conditions may be very restrictive in practice. The use of a "chain" of spaces developed here enables to greatly weaken these assumptions.

As a first example we shall see in Section 7.3 that, for the uniformly ergodic chains, Hypothesis $\mathcal{C}(m)$ reduces to $\pi(|\xi|_2^\alpha) < +\infty$ for some $\alpha > m$. This condition is slightly stronger than the assumption $\pi(|\xi|_2^m) < +\infty$ of the i.i.d. case ensuring that the common characteristic function has m continuous derivatives. But it is much weaker than the condition $\sup_{x \in E} (Q|\xi|_2^m)(x) < +\infty$ required in the usual Nagaev's method when applied with the single space \mathcal{B}^∞ . Other simple reductions of $\mathcal{C}(m)$ will be obtained in Sections 10-11 for geometrically ergodic chains and iterative models.

Roughly speaking one can say that Hypothesis $\mathcal{C}(m)$ below (together with possibly the non-arithmeticity condition) allows to extend to strongly ergodic Markov chains the classical

i.i.d. limit theorems established with Fourier techniques. This will be illustrated in Sections 8-9 by proving a one-dimensional Edgeworth expansion and a multi-dimensional Berry-Esseen type theorem in the sense of Prohorov metric. This has been also exploited in [?] to prove a multi-dimensional renewal theorem.

7.1. Regularity of $(z - Q(\cdot))^{-1}$. Before dealing with the regularity of the eigen-elements of $Q(t)$, we investigate that of the function $t \mapsto (z - Q(t))^{-1}$, where $(z - Q(t))^{-1}$ is seen as an operator from a Banach space \mathcal{B} to a Banach space $\tilde{\mathcal{B}}$. The arguments below were partially developed in [?] in the special case of iterative models, and in [?] which presents a statement similar to Proposition 7.1 below, but only concerning the Taylor expansions of $(z - Q(\cdot))^{-1}$ at $t = 0$.

Let \mathcal{O} be an open subset of \mathbb{R}^d , X a vector normed space. Then, for $m \in \mathbb{N}$, we shall say that $U \in \mathcal{C}^m(\mathcal{O}, X)$ if U is a function from \mathcal{O} to X which admits m continuous derivatives. For convenience, $\mathcal{C}^\ell(\mathcal{O}, \mathcal{B}_1, \mathcal{B}_2)$ will stand for $\mathcal{C}^\ell(\mathcal{O}, \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2))$.

Let \mathcal{B} and $\tilde{\mathcal{B}}$ be two function spaces such that $\mathcal{B} \hookrightarrow \tilde{\mathcal{B}}$. Let m be a positive integer.

Hypothesis $\mathcal{C}(m)$. *There exist an interval I , a family of function spaces $(\mathcal{B}_\theta, \theta \in I)$ and two functions $T_0 : I \rightarrow \mathbb{R}$ and $T_1 : I \rightarrow \mathbb{R}$ such that, for all $\theta \in I$, there exists a neighbourhood \mathcal{V}_θ of 0 in \mathbb{R}^d such that we have for $j = 1, \dots, m$:*

$$(0) [T_0(\theta) \in I \Rightarrow \mathcal{B}_\theta \hookrightarrow \mathcal{B}_{T_0(\theta)}] \quad \text{and} \quad [T_1(\theta) \in I \Rightarrow \mathcal{B}_\theta \hookrightarrow \mathcal{B}_{T_1(\theta)}]$$

$$(1) \text{ If } T_0(\theta) \in I, \text{ then } Q(\cdot) \in \mathcal{C}^0(\mathcal{V}_\theta, \mathcal{B}_\theta, \mathcal{B}_{T_0(\theta)})$$

$$(2) \text{ If } \theta_j := T_1(T_0 T_1)^{j-1}(\theta) \in I, \text{ then } Q(\cdot) \in \mathcal{C}^j(\mathcal{V}_\theta, \mathcal{B}_\theta, \mathcal{B}_{\theta_j}), \text{ and each partial derivative } \frac{\partial^j Q}{\partial t_{p_1} \dots \partial t_{p_j}}(t) \text{ is defined by means of the kernel}$$

$$Q_{(p_1, \dots, p_j)}(t)(x, dy) = i^j \left(\prod_{s=1}^j \xi_{p_s}(y) \right) e^{i\langle t, \xi(y) \rangle} Q(x, dy)$$

$$(3) \quad Q(\cdot) \text{ satisfies the hypothesis (K) of section 4.1 on } \mathcal{B}_\theta$$

$$(4) \quad \text{There exists } a \in \bigcap_{k=0}^m [T_0^{-1}(T_0 T_1)^{-k}(I) \cap (T_1 T_0)^{-k}(I)] \text{ such that we have } \mathcal{B} = \mathcal{B}_a \text{ and } \tilde{\mathcal{B}} = \mathcal{B}_{(T_0 T_1)^m T_0(a)}.$$

To fix ideas, let us introduce a more restrictive but simpler hypothesis :

Hypothesis $\mathcal{C}'(m)$. *There exist $A > m$ and a family of function spaces $(\mathcal{B}_\theta, \theta \in [0, A])$ such that $\mathcal{B}_0 = \mathcal{B}$, $\mathcal{B}_A = \tilde{\mathcal{B}}$ and, for all $\theta, \theta' \in [0, A]$ with $0 \leq \theta < \theta' \leq A$, we have :*

$$(a) \quad \mathcal{B}_\theta \hookrightarrow \mathcal{B}_{\theta'},$$

$$(b) \quad \text{there exists a neighbourhood } \mathcal{V} = \mathcal{V}_{\theta, \theta'} \text{ of 0 in } \mathbb{R}^d \text{ such that, for any } j \in \{0, \dots, m\} \text{ with } j < \theta' - \theta, \text{ we have } Q \in \mathcal{C}^j(\mathcal{V}, \mathcal{B}_\theta, \mathcal{B}_{\theta'}),$$

$$(c) \quad Q(\cdot) \text{ satisfies the hypothesis (K) of section 4.1 on } \mathcal{B}_\theta.$$

It is easy to see that Hypothesis $\mathcal{C}'(m)$ implies Hypothesis $\mathcal{C}(m)$ (by taking $a = 0$, $T_0(x) = x + \varepsilon$ and $T_1(x) = x + 1 + \varepsilon$ for some well chosen $\varepsilon > 0$). Actually Hypothesis $\mathcal{C}'(m)$ will be satisfied in all our examples, but Hypothesis $\mathcal{C}(m)$ is more general and, despite its apparent complexity, might be more natural to establish than hypothesis $\mathcal{C}'(m)$ (see the end of section 7.3).

Let us come back to hypothesis $\mathcal{C}(m)$. The condition on a in (4) means that $a, T_0a, T_1T_0a, T_0T_1T_0a, \dots, (T_0T_1)^m T_0(a)$ belong to I , and from (0), it follows that the corresponding family of \mathcal{B}_θ 's is increasing with respect to the continuous embedding. In particular, $\theta := T_0(a)$ and $\theta_m := T_1(T_0T_1)^{m-1}(\theta)$ are in I , therefore we have $Q(\cdot) \in \mathcal{C}^m(\mathcal{V}_\theta, \mathcal{B}_\theta, \mathcal{B}_{\theta_m})$ by (2). It then follows that $Q(\cdot) \in \mathcal{C}^m(\mathcal{V}_\theta, \mathcal{B}, \tilde{\mathcal{B}})$. In practice, we may have $Q(\cdot) \in \mathcal{C}^m(\mathcal{V}_\theta, \mathcal{B}, \mathcal{B}_{T_1^m(a)})$, but the introduction of T_0 will enable us to get $(z - Q(\cdot))^{-1} \in \mathcal{C}^m(\mathcal{O}, \mathcal{B}, \tilde{\mathcal{B}})$ for some neighbourhood \mathcal{O} of $t = 0$ and for suitable $z \in \mathbb{C}$.

Notation. Recall that we set $\mathcal{D}_\kappa = \{z \in \mathbb{C} : |z| \geq \kappa, |z - 1| \geq (1 - \kappa)/2\}$ for any $\kappa \in (0, 1)$. Under Hypothesis $\mathcal{C}(m)$, we have (K) on \mathcal{B} , so from Theorem (K-L) of Section 4.1, if t belongs to some neighbourhood \mathcal{U}_a of 0 in \mathbb{R}^d and if $z \in \mathcal{D}_{\kappa_a}$ for some $\kappa_a \in (0, 1)$, then $(z - Q(t))^{-1}$ is a bounded operator on $\tilde{\mathcal{B}}$, and we shall set $R_z(t) = (z - Q(t))^{-1}$. It is worth noticing that we also have $R_z(t) \in \mathcal{L}(\mathcal{B}, \tilde{\mathcal{B}})$ for all $t \in \mathcal{U}_a$ and $z \in \mathcal{D}_{\kappa_a}$. In the case $d \geq 2$, for $t = (t_1, \dots, t_d)$, $R_z^{(\ell)}(t)$ will stand for any partial derivative of the form $\frac{\partial^\ell R_z}{\partial t_{i_1} \dots \partial t_{i_\ell}}(t)$.

Proposition 7.1. *Under Hypothesis $\mathcal{C}(m)$, there exist a neighbourhood $\mathcal{O} \subset \mathcal{U}_a$ of 0 in \mathbb{R}^d and $\kappa \in (\kappa_a, 1)$ such that $R_z(\cdot) \in \mathcal{C}^m(\mathcal{O}, \mathcal{B}, \tilde{\mathcal{B}})$ for all $z \in \mathcal{D}_\kappa$, and*

$$\sup\{\|R_z^{(\ell)}(t)\|_{\mathcal{B}, \tilde{\mathcal{B}}}, z \in \mathcal{D}_\kappa, t \in \mathcal{O}\} < +\infty, \quad \ell = 0, \dots, m.$$

The proof of Proposition 7.1 is presented in Appendix A. It is based on general and elementary derivation arguments. Anyway, Proposition 7.1 is valid for any family $\{Q(t), t \in \mathcal{O}\}$ of operators satisfying the five conditions in $\mathcal{C}(m)$, except the particular form of the derivatives in (2). Actually this special form will not be used in the proof of Proposition 7.1, but of course, it will occur in the effective verification of Hypothesis $\mathcal{C}(m)$ in our examples.

Remarks.

(a) For the sake of simplicity, in our examples, we shall verify $\mathcal{C}(m)$ in the case $d = 1$, and we shall simply denote by $Q^{(k)}$ the k -th derivative of $Q(\cdot)$ occurring in $\mathcal{C}(m)$, which is defined for $k = 0, \dots, m$ by the kernel

$$Q_k(t)(x, dy) = i^k \xi(y)^k e^{it\xi(y)} Q(x, dy) \quad (t \in \mathbb{R}, x \in E).$$

(b) We know that $\mathcal{C}(m)$ yields $Q(\cdot) \in \mathcal{C}^m(\mathcal{V}_\theta, \mathcal{B}, \tilde{\mathcal{B}})$, thus $\frac{\partial^m Q}{\partial t^m}(0) \in \mathcal{L}(\mathcal{B}, \tilde{\mathcal{B}})$ ($k = 1, \dots, d$). From $1_E \in \mathcal{B}$ and $\pi \in \tilde{\mathcal{B}}'$ (by (K1) on $\mathcal{B}, \tilde{\mathcal{B}}$) and from (2), it follows that $\pi(\frac{\partial^m Q}{\partial t^m}(0)1_E) = i^m \pi(Q\xi_k^m) = i^m \pi(\xi_k^m)$ is defined. So, in substance, Hypothesis $\mathcal{C}(m)$ implies $\pi(|\xi|_2^m) < +\infty$ (this is actually true if m is even). However, in our examples, we shall need some slightly more restrictive moment conditions to be able to prove $\mathcal{C}(m)$.

(c) According to the proof of proposition 7.1, we do not need (1), (2) and (3) of Hypothesis $\mathcal{C}(m)$ to be true for any θ but only for $\theta = a, T_0a, T_1T_0a, T_0T_1T_0a, \dots, (T_0T_1)^m T_0(a)$, hence only for a finite number of θ . This remark will be of no relevance for checking $\mathcal{C}(m)$, but it will be important to define the set \mathcal{O} and the real number κ in the proof of Proposition 7.1.

7.2. Regularity of the eigen-elements of $Q(\cdot)$. Suppose that Hypothesis $\mathcal{C}(m)$ holds for some $m \in \mathbb{N}^*$, and as above let us use the notation of Theorem (K-L) of Section 4.1 for $Q(t)$ acting on \mathcal{B} : if $t \in \mathcal{U}_a$, $\lambda(t)$ is the dominating eigenvalue of $Q(t)$, $\Pi(t)$ is the

associated rank-one eigenprojection, and we define in $\mathcal{L}(\mathcal{B})$: $N(t) = Q(t) - \lambda(t)\Pi(t)$. Since $\Pi(t)Q(t) = Q(t)\Pi(t) = \lambda(t)\Pi(t)$, we have $N(t)^n = Q(t)^n - \lambda(t)^n\Pi(t)$ for all $n \geq 1$. It follows from Theorem (K-L) that $Q(t)^n = \lambda(t)^n\Pi(t) + N(t)^n$, with $\|N(t)^n\|_{\mathcal{B}} \leq C\kappa_a^n$.

The operators $Q(t)$, $R_z(t)$, $\Pi(t)$ and $N(t)^n$ are viewed as elements of $\mathcal{L}(\mathcal{B})$ when we appeal to the spectral theory, and as elements of $\mathcal{L}(\mathcal{B}, \tilde{\mathcal{B}})$ for stating our results of derivation.

Corollary 7.2. *Under Hypothesis $\mathcal{C}(m)$, there exists a neighbourhood \mathcal{V} of 0 in \mathbb{R}^d such that :*

(i) $\Pi(\cdot) \in \mathcal{C}^m(\mathcal{V}, \mathcal{B}, \tilde{\mathcal{B}})$;

(ii) for all $n \geq 1$, $N_n(\cdot) := N(\cdot)^n \in \mathcal{C}^m(\mathcal{V}, \mathcal{B}, \tilde{\mathcal{B}})$, and

$$\exists C > 0, \forall n \geq 1, \forall \ell = 0, \dots, m : \sup_{t \in \mathcal{V}} \|N_n^{(\ell)}(t)\|_{\mathcal{B}, \tilde{\mathcal{B}}} \leq C\kappa^n;$$

(iii) $\lambda(\cdot) \in \mathcal{C}^m(\mathcal{V}, \mathbb{C})$.

Proof of Corollary 7.2. Let $t \in \mathcal{O}$, with \mathcal{O} introduced in Proposition 7.1.

(i) As in the standard perturbation theory, the eigenprojection $\Pi(t)$ is defined in [?] by

$$\Pi(t) = \frac{1}{2i\pi} \int_{\Gamma_1} R_z(t) dz,$$

where this line integral is considered on the oriented circle Γ_1 centered at $z = 1$, with radius $(1 - \kappa)/2$ (thus $\Gamma_1 \subset \mathcal{D}_\kappa$). Then, by Proposition 7.1, $\Pi(\cdot) \in \mathcal{C}^m(\mathcal{O}, \mathcal{B}_a, \tilde{\mathcal{B}})$.

(ii) In the same way, one can write

$$N(t)^n = \frac{1}{2i\pi} \int_{\Gamma_0} z^n R_z(t) dz,$$

where Γ_0 is here the oriented circle, centered at $z = 0$, with radius κ (thus $\Gamma_0 \subset \mathcal{D}_\kappa$). By Proposition 7.1, we have $N_n(\cdot) \in \mathcal{C}^m(\mathcal{O}, \mathcal{B}, \tilde{\mathcal{B}})$ with $N_n^{(\ell)}(t) = \frac{1}{2i\pi} \int_{\Gamma_0} z^n R_z^{(\ell)}(t) dz$ for $\ell = 1, \dots, m$. Hence the stated inequalities.

(iii) Since $\lim_{t \rightarrow 0} \pi(\Pi(t)1_E) = \pi(\Pi 1_E) = 1$ (see Th. (K-L)), there exists a neighbourhood \mathcal{V} of 0 contained in \mathcal{O} such that $\pi(\Pi(t)1_E) \neq 0$ for any $t \in \mathcal{V}$. From $Q(t) = \lambda(t)\Pi(t) + N(t)$, it follows that

$$\lambda(t) = \frac{\pi(Q(t)1_E - N(t)1_E)}{\pi(\Pi(t)1_E)}.$$

Besides, from the remark following the statement of $\mathcal{C}(m)$, we have $Q(\cdot) \in \mathcal{C}^m(\mathcal{V}, \mathcal{B}, \tilde{\mathcal{B}})$ (with possibly \mathcal{V} reduced). Now, since $1_E \in \mathcal{B}$ and $N(\cdot)$, $\Pi(\cdot)$ are in $\mathcal{C}^m(\mathcal{V}, \mathcal{B}, \tilde{\mathcal{B}})$, the functions $Q(\cdot)1_E$, $N(\cdot)1_E$, $\Pi(\cdot)1_E$ are in $\mathcal{C}^m(\mathcal{V}, \tilde{\mathcal{B}})$. Finally we have $\pi \in \tilde{\mathcal{B}}'$ (by (K1) on $\tilde{\mathcal{B}}$). This gives (iii). \square

7.3. Hypothesis $\mathcal{C}(m)$ for the uniformly ergodic chains. Let us suppose that $(X_n)_{n \geq 0}$ is uniformly ergodic. Let $m \in \mathbb{N}^*$, and let us investigate Hypothesis $\mathcal{C}(m)$ by using a family $\{\mathcal{B}_\theta = \mathbb{L}^\theta(\pi), r \leq \theta \leq s\}$ for some suitable $1 < r < s$.

Proposition 7.3. *If $\pi(|\xi|_2^\alpha) < +\infty$ with $\alpha > m$, then $\mathcal{C}(m)$ holds with $\mathcal{B} = \mathbb{L}^s(\pi)$ and $\tilde{\mathcal{B}} = L^r(\pi)$ for any $s > \frac{\alpha}{\alpha - m}$ and $1 < r < \frac{\alpha s}{\alpha + ms}$.*

We give the proof for $d = 1$. The extension to $d \geq 2$ is obvious by the use of partial derivatives.

Proof. Let us notice that the condition on s implies that $\frac{\alpha s}{\alpha + ms} > 1$, so one may choose r as stated, and we have $r < s$. Let $\varepsilon > 0$ be such that $r = \frac{\alpha s}{\alpha + ms + \varepsilon(m+1)s}$. Let us prove $\mathcal{C}(m)$ with

$\mathcal{B}_\theta = \mathbb{L}^\theta(\pi)$, $I = [r; s]$, $a = s$, and finally $T_0(\theta) = \frac{\alpha\theta}{\alpha+\varepsilon\theta}$ and $T_1(\theta) = \frac{\alpha\theta}{\alpha+\theta}$. Since $T_0T_1 = T_1T_0$, one gets $T_0^kT_1^j(\theta) = \frac{\alpha\theta}{\alpha+(j+\varepsilon k)\theta}$, in particular $(T_0T_1)^mT_0(s) = r$, so the space $\tilde{\mathcal{B}}$ introduced in $\mathcal{C}(m)$ is $\tilde{\mathcal{B}} = L^r(\pi)$. Since $T_0(\theta) < \theta$ and $T_1(\theta) < \theta$, we have (0), and Proposition 4.5 gives (1) of $\mathcal{C}(m)$. To prove (2), let $j \in \{1, \dots, m\}$, and let $\theta \in I$ such that $\theta_j := T_1(T_0T_1)^{j-1}(\theta) \in I$. We have $\theta_j < T_1^j(\theta)$, thus $\mathbb{L}^{T_1^j(\theta)}(\pi) \hookrightarrow \mathbb{L}^{\theta_j}(\pi)$, so the regularity property in (2) follows from the following Lemma where $Q_k(t)$ stands for the kernel defined in remark (a) of Section 7.1.

Lemma 7.4. *Let $1 \leq j \leq m$. Then $Q(\cdot) \in \mathcal{C}^j(\mathbb{R}, \mathbb{L}^\theta(\pi), \mathbb{L}^{T_1^j(\theta)}(\pi))$ with $Q^{(k)} = Q_k$ ($k = 0, \dots, j$).*

Proof. We denote $\|\cdot\|_p$ for $\|\cdot\|_{L^p(\pi)}$, and $\|\cdot\|_{p,q}$ for $\|\cdot\|_{L^p(\pi), L^q(\pi)}$. Let us first show that $Q_k(\cdot) \in \mathcal{C}^0(\mathbb{R}, \mathcal{B}_\theta, \mathcal{B}_{T_1^j(\theta)})$ for any $k = 0, \dots, j$. The case $k = 0$ follows from Proposition 4.5. For $1 \leq k \leq j$, we have for $t_0, h \in \mathbb{R}$ and $f \in \mathbb{L}^\theta$,

$$\begin{aligned} \|Q_k(t_0+h)f - Q_k(t_0)f\|_{T_1^j(\theta)} &= \|Q_k(t_0+h)f - Q_k(t_0)f\|_{\frac{\alpha\theta}{\alpha+j\theta}} \\ &\leq 2 \left\| \xi^k \min\{1, |h\xi|\} f \right\|_{\frac{\alpha\theta}{\alpha+j\theta}} \\ &\leq 2 \left\| \xi^k \min\{1, |h\xi|\} \right\|_{\frac{\alpha}{j}} \|f\|_\theta \end{aligned}$$

with $\left\| \xi^k \min\{1, |h\xi|\} \right\|_{\frac{\alpha}{j}} \rightarrow 0$ when $h \rightarrow 0$ by Lebesgue's theorem. Now let us prove $Q'_k = Q_{k+1}$ in $\mathcal{L}(\mathcal{B}_\theta, \mathcal{B}_{T_1^j(\theta)})$ for $k = 0, \dots, j-1$. Using $|e^{ia} - 1 - ia| \leq 2|a| \min\{1, |a|\}$, one gets for $t_0, h \in \mathbb{R}$ and $f \in \mathcal{B}_\theta$:

$$\begin{aligned} \|Q_k(t_0+h)f - Q_k(t_0)f - hQ_{k+1}(t_0)f\|_{\frac{\alpha\theta}{\alpha+j\theta}} &\leq \left\| Q(|\xi|^k(e^{ih\xi} - 1 - ih\xi)f) \right\|_{\frac{\alpha\theta}{\alpha+j\theta}} \\ &\leq 2|h| \left\| |\xi|^{(k+1)} \min\{1, |h|\xi|\} |f| \right\|_{\frac{\alpha\theta}{\alpha+j\theta}}, \end{aligned}$$

and the previous computations yield $\|Q_k(t_0+h) - Q_k(t_0) - hQ_{k+1}(t_0)\|_{\theta, T_1^j(\theta)} = o(|h|)$. \square

We know that Q satisfies (K) on $L^p(\pi)$ for every $p \in]1; +\infty[$ (Prop. 4.1). Hence we have (3) of $\mathcal{C}(m)$, and (4) is obvious from the definition of T_0 , T_1 and r . \square

For this example, one can also use proposition 4.5 and lemma 7.4 to prove that Hypothesis $\mathcal{C}'(m)$ is satisfied by taking $A > m$ such that $r = \frac{\alpha s}{\alpha + As}$ and by setting $\mathcal{B}_\theta := L^{\frac{\alpha s}{\alpha + \theta s}}(\pi)$.

8. A one-dimensional first-order Edgeworth expansion. In this section we assume that $d = 1$ (i.e. ξ is a real-valued measurable function on E).

Let us recall that, when $(X_n)_n$ is Harris recurrent, the regenerative method provides Edgeworth expansions under some ‘‘block’’ moment conditions [?] [?]. Here we do not assume Harris recurrence. We present an alternative statement. To that effect, we shall appeal to Hypothesis $\mathcal{C}(3)$ of section 7.1 which ensures (Corollary 7.2) that the dominating eigenvalue $\lambda(t)$ of $Q(t)$ is three times continuously differentiable : then one shall be able to repeat the arguments of the i.i.d. first-order Edgeworth expansion of [?] (Th. 1 p. 506). By assuming $\mathcal{C}(m)$ for any fixed $m \geq 3$ and writing out the i.i.d. arguments of [?] (Th. 2 p. 508), the present method may also be employed to prove higher-order Edgeworth expansions.

We denote by η the density function of $\mathcal{N}(0,1)$ and by \mathcal{N} its distribution function. The next theorem extends the first-order Edgeworth expansion of the i.i.d. case, with an additional asymptotic bias, namely $b_\mu = \lim_n \mathbb{E}_\mu[S_n]$ which depends on the initial distribution μ . As for i.i.d.r.v., this bias is zero in the stationary case (i.e. $b_\pi = 0$).

Theorem 8.1. *Suppose that Hypothesis C(3) of section 7.1 holds with some function spaces $\mathcal{B}, \tilde{\mathcal{B}}$, that the non-arithmeticity condition (S) of Section 5.1 holds on \mathcal{B} , and finally that the initial distribution μ is in $\tilde{\mathcal{B}}'$. Then the real numbers*

$$\sigma^2 = \lim_n \frac{1}{n} \mathbb{E}_\mu[S_n^2] = \lim_n \frac{1}{n} \mathbb{E}_\pi[S_n^2], \quad m_3 = \lim_n \frac{1}{n} \mathbb{E}_\pi[S_n^3], \quad b_\mu = \lim_n \mathbb{E}_\mu[S_n],$$

are well-defined, and if $\sigma > 0$, the following expansion holds uniformly in $u \in \mathbb{R}$

$$(E) \quad \mathbb{P}_\mu \left(\frac{S_n}{\sigma\sqrt{n}} \leq u \right) = \mathcal{N}(u) + \frac{m_3}{6\sigma^3\sqrt{n}}(1-u^2)\eta(u) - \frac{b_\mu}{\sigma\sqrt{n}}\eta(u) + o\left(\frac{1}{\sqrt{n}}\right).$$

It will be seen in the proof of Lemmas 8.3-4 below that

$$\left| \sigma^2 - \frac{1}{n} \mathbb{E}_\mu[S_n^2] \right| = O\left(\frac{1}{n}\right) \quad \text{and} \quad \left| m_3 - \frac{1}{n} \mathbb{E}_\pi[S_n^3] \right| = O\left(\frac{1}{n}\right).$$

Case of uniformly ergodic Markov chains.

In this special case, the expansion (E) was established in [?] for any initial distribution, under some hypothesis on the absolute continuous component of $Q(x, dy)$ w.r.t. π and under the following restrictive operator-moment condition : there exists $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $g(u) \rightarrow +\infty$ when $|u| \rightarrow +\infty$ and $\sup_{x \in E} \int_E |\xi(y)|^3 g(|\xi(y)|) Q(x, dy) < +\infty$. In [?], this result is slightly improved, more precisely (E) is established under the weaker (but still restrictive) moment condition $\sup_{x \in E} \int_E |\xi(y)|^3 Q(x, dy) < +\infty$ and under some refinements of the non-lattice condition given in [?].

In the stationary case (i.e. under \mathbb{P}_π), the general asymptotic expansions established in [?] apply to this example. They yield (E) when $\pi(|\xi|^4) < +\infty$ and the r.v. $\xi(X_0)$ satisfies the so-called Cramér's condition. With the help of Theorem 8.1, one obtains here the following improvement.

Corollary 8.2. *Let us suppose that $(X_n)_{n \geq 0}$ is uniformly ergodic. If $\pi(|\xi|^\alpha) < +\infty$ with some $\alpha > 3$ and if ξ satisfies the nonlattice condition of Proposition 5.4, then we have (E) for any initial distribution of the form $d\mu = \phi d\pi$, where $\phi \in \mathbb{L}^{r'}(\pi)$ for some $r' > \frac{\alpha}{\alpha-3}$.*

Proof. Let r' be fixed as above and let r be such that $\frac{1}{r} + \frac{1}{r'} = 1$. Then $1 < r < \frac{\alpha}{3}$, and since $\frac{\alpha s}{\alpha+3s} \nearrow \frac{\alpha}{3}$ when $s \rightarrow +\infty$, on can choose s such that $s > \frac{\alpha}{\alpha-3}$ and $\frac{\alpha s}{\alpha+3s} > r$. By Proposition 7.3, we have C(3) with $\mathcal{B} = \mathbb{L}^s(\pi)$ and $\tilde{\mathcal{B}} = \mathbb{L}^r(\pi)$. Finally we have (S) on $\mathbb{L}^s(\pi)$ by Proposition 5.4. \square

Proof of Theorem 8.1. We shall appeal repeatedly to the notations and the conclusions of Theorem (K-L) (cf. § 4.1) and of corollary 7.2 (case $m = 3$). The existence of σ^2 , m_3 and b_μ follows from the two next lemmas.

Lemma 8.3. *We have $\lambda'(0) = 0$ and $\mu(\Pi'(0)1_E) = i \sum_{k \geq 1} \mu(Q^k \xi) = i \lim_n \mathbb{E}_\mu[S_n]$.*

Proof. By deriving the equality $Q(\cdot)\Pi(\cdot)1_E = \lambda(\cdot)\Pi(\cdot)1_E$, one gets

$$Q'(0)1_E + Q\Pi'(0)1_E = \lambda'(0)1_E + \Pi'(0)1_E \quad \text{in } \tilde{\mathcal{B}}.$$

Thus $\pi(Q'(0)1_E) + \pi(\Pi'(0)1_E) = \lambda'(0) + \pi(\Pi'(0)1_E)$. This gives $\lambda'(0) = i\pi(Q\xi) = i\pi(\xi) = 0$, and $iQ(\xi) + Q\Pi'(0)1_E = \Pi'(0)1_E$ in $\tilde{\mathcal{B}}$. Therefore we have $\Pi'(0)1_E - \pi(\Pi'(0)1_E) = i\sum_{k \geq 1} Q^k \xi$. This serie is absolutely convergent in $\tilde{\mathcal{B}}$ since $\pi(Q\xi) = 0$, $Q\xi = -iQ'(0)1_E \in \tilde{\mathcal{B}}$ and Q is strongly ergodic on $\tilde{\mathcal{B}}$. Moreover, we have $\pi(\Pi'(0)1_E) = 0$. Indeed, by deriving $\Pi(t)^2 = \Pi(t)$, we get $2\pi(\Pi'(0)1_E) = \pi(\Pi(0)\Pi'(0)1_E + \Pi'(0)\Pi(0)1_E) = \pi(\Pi'(0)1_E)$. Since $\mu \in \tilde{\mathcal{B}}'$, this yields the first equality of the second assertion. The second one is obvious. \square

Lemma 8.4. *We have $\lim_n \frac{1}{n} \mathbb{E}_\mu[S_n^2] = -\lambda''(0)$ and $\lim_n \frac{1}{n} \mathbb{E}_\pi[S_n^3] = i\lambda^{(3)}(0)$.*

Proof of Lemma 8.4. For convenience, let us assume that $\mu = \pi$ and prove the two equalities of Lemma 8.4 at once (see Rk. below). Since $\mathbb{E}_\pi[|\xi(X_k)|^3] = \pi(|\xi|^3) < +\infty$, we have $\mathbb{E}_\pi[|S_n^3|] < +\infty$, so

$$\mathbb{E}_\pi[e^{itS_n}] = 1 - \mathbb{E}_\pi[S_n^2] \frac{t^2}{2} - i \mathbb{E}_\pi[S_n^3] \frac{t^3}{6} + o_n(t^3).$$

Besides, Formula (CF) (cf. § 3) and the equality $Q(t)^n = \lambda(t)^n \Pi(t) + N(t)^n$ (see § 7.2) give

$$\mathbb{E}_\pi[e^{itS_n}] = \lambda(t)^n \pi(\Pi(t)1_E) + \pi(N(t)^n 1_E),$$

and, since $\lambda'(0) = 0$ and $\pi(\Pi'(0)1_E) = 0$ (Lemma 8.3), it follows from Hypothesis $\mathcal{C}(3)$ and Corollary 7.2 that

$$\lambda(t)^n = 1 + n \frac{\lambda''(0)}{2} t^2 + n \frac{\lambda^{(3)}(0)}{6} t^3 + o_n(t^3), \quad \pi(\Pi(t)1_E) = 1 + ct^2 + dt^3 + o(t^3),$$

with some $c, d \in \mathbb{C}$, and since $N(0)1_E = 0$, we have $\pi(N(t)^n 1_E) = e_n t + f_n t^2 + g_n t^3 + o_n(t^3)$ for all $n \geq 1$, with some $e_n, f_n, g_n \in \mathbb{C}$. Moreover, from Assertion (ii) in Corollary 7.2, it follows that the sequences $(e_n)_n$, $(f_n)_n$ and $(g_n)_n$ are bounded. From the previous expansions, one can write another third order Taylor expansion for $\mathbb{E}_\pi[e^{itS_n}]$, from which we easily deduce the following equalities (and so Lemma 8.4) :

$$n\lambda''(0) + 2c + 2f_n = -\mathbb{E}_\pi[S_n^2] \quad \text{and} \quad n\lambda^{(3)}(0) + 6d + 6g_n = -i \mathbb{E}_\pi[S_n^3]. \quad \square$$

Remark. By using the above arguments with second-order Taylor expansions, it can be easily proved that the first equality of Lemma 8.4 is valid under Hypothesis $\mathcal{C}(2)$ for any $\mu \in \tilde{\mathcal{B}}'$. To prove $\mathbb{E}_\mu[S_n^2] < +\infty$ under Hypothesis $\mathcal{C}(2)$ and for $\mu \in \tilde{\mathcal{B}}'$, notice that $Q''(0)1_E = -Q(\xi^2) \in \tilde{\mathcal{B}}$, so $Q^k(\xi^2) \in \tilde{\mathcal{B}}$ for $k \geq 1$, and $\mathbb{E}_\mu[\xi(X_k)^2] = \mu(Q^k \xi^2) < +\infty$.

The proof of the Edgeworth expansion (E) is close to that of the i.i.d. case [?] (XVI.4). For convenience, one may assume, without any loss of generality, that $\sigma = 1$ (of course this reduction also leads to alter the constants m_3 and b_μ). Set

$$G_n(u) = \mathcal{N}(u) + \frac{m_3}{6\sqrt{n}}(1-u^2)\eta(u) - \frac{b_\mu}{\sqrt{n}}\eta(u) \quad (u \in \mathbb{R}).$$

Then G_n has a bounded derivative g_n on \mathbb{R} whose Fourier transform γ_n is given by

$$\gamma_n(t) = \gamma_{0,n}(t) + \gamma_{\mu,n}(t), \quad \text{where} \quad \gamma_{0,n}(t) = e^{-\frac{1}{2}t^2} \left(1 + \frac{m_3}{6\sqrt{n}}(it)^3 \right) \quad \text{and} \quad \gamma_{\mu,n}(t) = e^{-\frac{1}{2}t^2} \left(i \frac{b_\mu}{\sqrt{n}} t \right).$$

Let us notice that the part $\gamma_{0,n}(t)$ has the same form as in the i.i.d. context.

Let $\phi_n(t) = \mathbb{E}_\mu[e^{itS_n}]$. The first question is to prove the so-called Berry-Esseen inequality

$$\sup_{u \in \mathbb{R}} \left| \mathbb{P}_\mu \left(\frac{S_n}{\sqrt{n}} \leq u \right) - G_n(u) \right| \leq \frac{1}{\pi} \int_{-T}^T \left| \frac{\phi_n(\frac{t}{\sqrt{n}}) - \gamma_n(t)}{t} \right| dt + \frac{24m}{\pi T},$$

where $m = \sup\{|G'_n(u)|, n \geq 1, u \in \mathbb{R}\}$. To do this, let us observe that all the hypotheses of Lemma 2 in Section XVI.3 of [?], which provides this inequality, are satisfied, except $\gamma'_n(0) = 0$ because of the additional term $\gamma_{\mu,n}(t)$ in $\gamma_n(t)$. However it can be easily seen that the above cited lemma of [?] still holds under the condition that $\frac{\gamma_n(t)-1}{t}$ is continuous at the origin. Indeed the argument in [?] (p. 511) deriving from the Riemann-Lebesgue theorem then remains valid. Obviously the previous condition on γ_n is fulfilled since $\frac{\gamma_{\mu,n}(t)}{t} = i \frac{b_\mu}{\sqrt{n}} e^{-\frac{1}{2}t^2}$. Thus we have the desired Berry-Esseen inequality and we can now proceed as in [?] : let $\varepsilon > 0$, let $T = a\sqrt{n}$ with a such that $\frac{24m}{\pi a} < \varepsilon$. So $\frac{24m}{\pi T} \leq \frac{\varepsilon}{\sqrt{n}}$. Let $0 < \delta < a$ such that $[-\delta, \delta]$ is contained in the interval \mathcal{O} of Theorem (K-L) applied on \mathcal{B} , and write

$$\int_{-a\sqrt{n}}^{a\sqrt{n}} \left| \frac{\phi_n(\frac{t}{\sqrt{n}}) - \gamma_n(t)}{t} \right| dt = \int_{\delta\sqrt{n} \leq |t| \leq a\sqrt{n}} + \int_{|t| \leq \delta\sqrt{n}} := A_n + B_n.$$

The property (E) then follows from the two next lemmas. \square

Lemma 8.5. *There exists $N_0 \in \mathbb{N}^*$ such that $A_n \leq \frac{\varepsilon}{\sqrt{n}}$ for all $n \geq N_0$.*

Proof. From Formula (CF) (cf. § 3), Condition (S) (cf § 5.1) on \mathcal{B} applied with $K_0 = [-a, -\delta] \cup [\delta, a]$, and from $\mu \in \tilde{\mathcal{B}}' \subset \mathcal{B}'$, there exist $\rho < 1$ and $c' \geq 0$ such that we have, for $n \geq 1$ and $u \in K : |\phi_n(u)| = |\mu(Q(u)^n 1_E)| \leq c' \rho^n$. So

$$\int_{\delta\sqrt{n} \leq |t| \leq a\sqrt{n}} \frac{|\phi_n(\frac{t}{\sqrt{n}})|}{|t|} dt = \int_{\delta \leq |u| \leq a} \frac{|\phi_n(u)|}{|u|} du \leq \frac{2a}{\delta} c' \rho^n.$$

Moreover, for n sufficiently large, we have $\int_{\delta\sqrt{n} \leq |t| \leq a\sqrt{n}} \frac{|\gamma_n(t)|}{|t|} dt \leq \int_{|t| \geq \delta\sqrt{n}} |\gamma_n(t)| dt$. We easily deduce Lemma 8.5 from the two last estimates. \square

Lemma 8.6. *There exists $N'_0 \in \mathbb{N}^*$ such that $B_n \leq \frac{\varepsilon}{\sqrt{n}}$ for all $n \geq N'_0$.*

Proof. Using $\gamma_n(t) = \gamma_{0,n}(t) + \gamma_{\mu,n}(t)$ and the equality $\phi_n(t) = \lambda(t)^n \mu(\Pi(t)1_E) + \mu(N(t)^n 1_E)$ which follows from (CF) and from Theorem (K-L), one can write for any t such that $|t| \leq \delta\sqrt{n}$

$$\begin{aligned} \phi_n(\frac{t}{\sqrt{n}}) - \gamma_n(t) &= \left(\lambda(\frac{t}{\sqrt{n}})^n - \gamma_{0,n}(t) \right) + \lambda(\frac{t}{\sqrt{n}})^n \left(\mu(\Pi(\frac{t}{\sqrt{n}})1_E) - 1 - i b_\mu \frac{t}{\sqrt{n}} \right) \\ &\quad + i b_\mu \frac{t}{\sqrt{n}} \left(\lambda(\frac{t}{\sqrt{n}})^n - e^{-\frac{1}{2}t^2} \right) + \mu(N(\frac{t}{\sqrt{n}})^n 1_E) \\ &:= i_n(t) + j_n(t) + k_n(t) + \ell_n(t). \end{aligned}$$

Therefore : $B_n \leq \int_{|t| \leq \delta\sqrt{n}} \left(|i_n(t)| + |j_n(t)| + |k_n(t)| + |\ell_n(t)| \right) \frac{dt}{|t|} := I_n + J_n + K_n + L_n$.

Then Lemma 8.6 follows from the assertions (i)-(l) below for which, as in the i.i.d. case, we shall repeatedly appeal to the following remark : using the Taylor expansion $\lambda(t) = 1 - \frac{t^2}{2} + o(t^2)$ near 0 (use Lemma 8.3-4 and $\sigma^2 = 1$), one can choose the real number δ such that $|\lambda(u)| \leq 1 - \frac{u^2}{4} \leq e^{-\frac{u^2}{4}}$ when $|u| \leq \delta$, hence we have $|\lambda(\frac{t}{\sqrt{n}})|^n \leq e^{-\frac{t^2}{4}}$ for any $|t| \leq \delta\sqrt{n}$.

(i) $\exists N_1 \in \mathbb{N}^*, \forall n \geq N_1, I_n \leq \frac{\varepsilon}{\sqrt{n}}$. This can be proved exactly as in the i.i.d. case [?] since

we have $\lambda(t) = 1 - \frac{t^2}{2} - i \frac{m_3}{6} t^3 + o(t^3)$ (Lemmas 8.3-4).

(j) $\exists N_2 \in \mathbb{N}^*, \forall n \geq N_2, J_n \leq \frac{\varepsilon}{\sqrt{n}}$. Indeed, since $u \mapsto \mu(\Pi(u)1_E)$ has two continuous derivatives on $[-\delta, \delta]$ (Coro. 7.2) and $\mu(\Pi'(0)1_E) = ib_\mu$ (Lemma 8.3), there exists $C > 0$ such that :

$$J_n \leq \int_{|t| \leq \delta\sqrt{n}} e^{-\frac{t^2}{4}} \frac{Ct^2}{n} \frac{dt}{|t|} \leq \frac{C}{n} \int_{-\infty}^{+\infty} e^{-\frac{t^2}{4}} |t| dt.$$

(k) $\exists N_3 \in \mathbb{N}^*, \forall n \geq N_3, K_n \leq \frac{\varepsilon}{\sqrt{n}}$. Indeed we have $K_n \leq \frac{|b_\mu|}{\sqrt{n}} \int_{|t| \leq \delta\sqrt{n}} \left| \lambda\left(\frac{t}{\sqrt{n}}\right)^n - e^{-\frac{1}{2}t^2} \right| dt$, and from the already mentioned second order Taylor expansion of $\lambda(t)$ and Lebesgue's theorem, it follows that this last integral converges to 0 when $n \rightarrow +\infty$.

(l) $\exists N_4 \in \mathbb{N}^*, \forall n \geq N_4, L_n \leq \frac{\varepsilon}{\sqrt{n}}$. Indeed, the function $\chi_n : u \mapsto \mu(N(u)^n 1_E)$ is continuously derivable on $[-\delta, \delta]$ and there exists $C' > 0$ such that we have for all $n \geq 1$ and $u \in [-\delta, \delta] : |\chi'_n(u)| \leq C' \kappa^n$ (Corollary 7.2(ii)). Since $N(0)1_E = 0$, one then obtains $|\mu(N(u)^n 1_E)| \leq C' \kappa^n |u|$ for $|u| \leq \delta$, so $L_n \leq \frac{C'}{\sqrt{n}} \kappa^n 2\delta\sqrt{n} = 2C'\delta \kappa^n = o(\frac{1}{\sqrt{n}})$. \square

9. A multi-dimensional Berry-Esseen theorem. We want to estimate the rate of convergence in the central limit theorem for a \mathbb{R}^d -valued function $\xi = (\xi_1, \dots, \xi_d)$. A natural way to do this is in the sense of the Prohorov metric. Let us recall the definition of this metric and some well-known facts about it. We denote by $\mathcal{B}(\mathbb{R}^d)$ the Borel σ -algebra of \mathbb{R}^d and by $\mathcal{M}_1(\mathbb{R}^d)$ the set of probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.

The Prohorov metric [?, ?]. For all P, Q in $\mathcal{M}_1(\mathbb{R}^d)$, we define :

$$\mathcal{P}(P, Q) := \inf \left\{ \varepsilon > 0 : \forall B \in \mathcal{B}(\mathbb{R}^d), (P(B) - Q(B^\varepsilon)) \leq \varepsilon \right\},$$

where B^ε is the open ε -neighbourhood of B .

The Ky Fan metric for random variables. If X and Y are two \mathbb{R}^d -valued random variables defined on the same probability space $(E_0, \mathcal{T}_0, P_0)$, we define :

$$\mathcal{K}(X, Y) := \inf \{ \varepsilon > 0 : P_0(|X - Y|_2 > \varepsilon) < \varepsilon \}.$$

Let us recall that $\lim_{n \rightarrow +\infty} \mathcal{K}(X_n, Y) = 0$ means that $(X_n)_n$ converges in probability to Y .

Proposition ([?] Corollary 11.6.4). For all P, Q in $\mathcal{M}_1(\mathbb{R}^d)$, the quantity $\mathcal{P}(P, Q)$ is the infimum of $\mathcal{K}(X, Y)$ over the couples (X, Y) of \mathbb{R}^d -valued random variables defined on the same probability space, whose distributions are respectively P and Q .

For any $n \geq 1$, $\mu_*\left(\frac{S_n}{\sqrt{n}}\right)$ stands for the law of $\frac{S_n}{\sqrt{n}}$ under \mathbb{P}_μ , and we denote by $S_n^{\otimes 2}$ the random variable with values in the set of matrices of order d given by :

$$(S_n^{\otimes 2})_{i,j} = \sum_{k,\ell=1}^n \xi_i(X_k) \xi_j(X_\ell).$$

Theorem 9.1. Let us fix $m := \max(3, \lfloor d/2 \rfloor + 1)$. Suppose that Hypothesis $\mathcal{C}(m)$ (of section 7.1) holds on some function spaces $\mathcal{B}, \tilde{\mathcal{B}}$ and that μ is in $\tilde{\mathcal{B}}'$. Then the following limits exist and are equal :

$$\Gamma := \lim_{n \rightarrow +\infty} \frac{1}{n} \mathbb{E}_\pi[S_n^{\otimes 2}] = \lim_{n \rightarrow +\infty} \frac{1}{n} \mathbb{E}_\mu[S_n^{\otimes 2}].$$

If Γ is invertible, then $\left(\frac{S_n}{\sqrt{n}}\right)_n$ converges in distribution under \mathbb{P}_μ to the gaussian distribution $\mathcal{N}(0, \Gamma)$, and we have

$$\mathcal{P}\left(\mu_*\left(\frac{S_n}{\sqrt{n}}\right), \mathcal{N}(0, \Gamma)\right) = O(n^{-1/2}).$$

In the i.i.d. case, the conclusion of Theorem 9.1 holds if the random variables admit a moment of order m . For the uniformly ergodic Markov chains, one gets the following close statement which is a consequence of theorem 9.1 and of proposition 7.3 (proceed as for Corollary 8.2).

Corollary 9.2. *Let us suppose that $(X_n)_{n \geq 0}$ is uniformly ergodic, that $\pi(|\xi|_2^\alpha) < +\infty$ for some $\alpha > m := \max(3, \lfloor d/2 \rfloor + 1)$, and that the initial distribution satisfies $d\mu = \phi d\pi$ with $\phi \in \mathbb{L}^{r'}(\pi)$ for some $r' > \frac{\alpha}{\alpha - m}$. Then the conclusion of theorem 9.1 is true.*

Concerning still the special case of uniform ergodicity, notice that [?] provides a multidimensional uniform Berry-Esseen type estimate when $\pi(|\xi|_2^4) < +\infty$. However, the hypothesis $\mu = \pi$ (i.e. $(X_n)_n$ is stationary) and Cramér's condition for $\xi(X_0)$ are required in [?], while the (Prohorov) estimate in Corollary 9.2, and more generally in Theorem 9.1, is valid in the non-stationary case and without any lattice-type condition.

Let us mention that Theorem 9.1 remains true when Γ is non invertible if, for every $\beta \in \mathbb{R}^d$ such that $\langle \beta, \Gamma \beta \rangle = 0$, we are able to prove that $\sup_n \|\langle \beta, S_n \rangle\|_\infty < +\infty$. In this case, up to a linear change of coordinates and to a possible change of d , we are led to the invertible case (see section 2.4.2 of [?]). This remark applies to the Knudsen gas model (see § 2.3).

When $d = 1$, Theorem 9.1 gives the uniform Berry-Esseen result under Condition $\mathcal{C}(3)$ if the asymptotic variance σ^2 is nonzero. This is an easy consequence of the definition of \mathcal{P} by taking $B = (-\infty, x]$ and $B = (x, +\infty)$. However, as already mentioned, $\mathcal{C}(3)$ is in practice a little more restrictive than the conditions of Section 6 ; for instance, compare the expected condition $\pi(|\xi|^3) < +\infty$ of Corollary 6.3 with that of Corollary 9.2 (case $d = 1$).

The proof of theorem 9.1 is based on corollary 7.2, on lemmas 8.3 and 8.4 and on the following result due to Yurinskii [?] :

Proposition 9.3. *Let Q be some non degenerate d -dimensional normal distribution. There exists a real number $c_0 > 0$ such that, for any real number $T > 0$ and for any Borel probability measure P admitting moments of order $\lfloor \frac{d}{2} \rfloor + 1$, we have :*

$$\mathcal{P}(P, Q) \leq c_0 \left[\frac{1}{T} + \left(\int_{|t|_2 < T} \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor + 1} \sum_{\{i_1, \dots, i_k\} \in \{1, \dots, d\}^k} \left| \frac{\partial^k}{\partial t_{i_1} \dots \partial t_{i_k}} (P(e^{i(t, \cdot)}) - Q(e^{i(t, \cdot)})) \right|^2 dt \right)^{\frac{1}{2}} \right].$$

Proof of theorem 9.1. The proof uses corollary 7.2 which is applied here under Hypothesis $\mathcal{C}(m)$ with m defined in Theorem 9.1. In particular we have $m \geq 3$, and we shall use repeatedly the fact that $1_E \in \mathcal{B}$ and $\pi, \mu \in \tilde{\mathcal{B}}'$ (the fact that $\pi \in \tilde{\mathcal{B}}'$ follows from Condition (K1) on $\tilde{\mathcal{B}}$). Since the proof has common points with the proof given in section 2.4.1 of [?], we do not give all the details. We shall refer to [?] for some technical points.

The existence of the asymptotic covariance matrix Γ as defined in Theorem 9.1 follows from the next lemma in which ∇ and $Hess$ denote the gradient and the Hessian matrix.

Lemma 9.4. *We have $\nabla\lambda(0) = 0$ and $\lim_n \frac{1}{n} \mathbb{E}_\mu[S_n^{\otimes 2}] = -\text{Hess}\lambda(0)$.*

Proof. These properties have been proved in the case $d = 1$ (Lemmas 8.3-4). We deduce from them the multidimensional version by considering, for any $\alpha \in \mathbb{R}^d$, the function $t \mapsto Q(t\alpha)$ defined on \mathbb{R} . \square

Without any loss of generality, up to a linear change of variables, we may suppose that the covariance matrix Γ is the identity matrix.

Let $\beta > 0$ be such that the closed ball $\{u \in \mathbb{R}^d : |u|_2 \leq \beta\}$ is contained in the set \mathcal{O} of Corollary 7.2. In the following, the couple (t, n) ($t \in \mathbb{R}^d$, $n \geq 2$) will always satisfy the condition $|t|_2 < \beta\sqrt{n}$. For such a couple, we have : $\frac{t}{\sqrt{n}} \in \mathcal{O}$. As already stated by convention in Section 7, for any function F defined on an open set of \mathbb{R}^d , $F^{(k)}$ will merely denote any partial derivative of order k of $F(\cdot)$.

Set $\Xi_n(t) := \mathbb{E}_\mu[e^{i\langle t, \frac{S_n}{\sqrt{n}} \rangle}] - e^{-\frac{|t|_2^2}{2}}$. According to Proposition 9.3, it is enough to prove that we have for $k = 0, \dots, \lfloor \frac{d}{2} \rfloor + 1$

$$(I) \quad \left(\int_{|t|_2 \leq \beta\sqrt{n}} |\Xi_n^{(k)}(t)|^2 dt \right)^{\frac{1}{2}} = O\left(\frac{1}{\sqrt{n}}\right).$$

From the decomposition $\mathbb{E}_\mu[e^{i\langle u, S_n \rangle}] = \lambda(u)^n \mu(\Pi(u)1_E) + \mu(N(u)^n 1_E)$ which is valid for $u \in \mathcal{O}$, it follows that

$$\begin{aligned} \Xi_n^{(k)} &= \left(\lambda\left(\frac{\cdot}{\sqrt{n}}\right)^n - e^{-\frac{|\cdot|_2^2}{2}} \right)^{(k)} + \left\{ \lambda\left(\frac{\cdot}{\sqrt{n}}\right)^n \left(\mu(\Pi\left(\frac{\cdot}{\sqrt{n}}\right)1_E) - 1 \right) \right\}^{(k)} + \left(\mu(N\left(\frac{\cdot}{\sqrt{n}}\right)^n 1_E) \right)^{(k)} \\ &:= A_n^{(k)} + B_n^{(k)} + C_n^{(k)} \end{aligned}$$

where the functions A_n , B_n and C_n , defined on the set $\{t : |t|_2 < \beta\sqrt{n}\}$, are implicitly given by the above equality. In the sequel, we merely use the notation $F_n(t) = O(G_n(t))$ to express that $|F_n(t)| \leq C|G_n(t)|$ for some $C \in \mathbb{R}_+$ independent of (t, n) such that $|t|_2 \leq \beta\sqrt{n}$.

Setting $N_n(\cdot) = N(\cdot)^n$, Proposition 7.2(ii) yields

$$|C_n^{(k)}(t)| = n^{-\frac{k}{2}} \left| \mu(N_n^{(k)}\left(\frac{t}{\sqrt{n}}\right)1_E) \right| = O(n^{-\frac{k}{2}} \kappa^n) \quad \text{with some } \kappa < 1.$$

So $\int_{|t|_2 \leq \beta\sqrt{n}} |C_n^{(k)}(t)|^2 dt = O(n^{\frac{d}{2}-k} \kappa^{2n}) = O\left(\frac{1}{\sqrt{n}}\right)$. Now (I) will be clearly valid provided that we have, for some square Lebesgue-integrable function $\chi(\cdot)$ on \mathbb{R}^d :

$$(II) \quad |A_n^{(k)}(t)| + |B_n^{(k)}(t)| = O\left(\frac{1}{\sqrt{n}} \chi(t)\right).$$

To prove this estimate for the term $A_n^{(k)}$, one can proceed as in the i.i.d. case. Indeed, according to the previous lemma, the function $\lambda(\cdot)$ then satisfies the same properties and plays exactly the same role, as the common characteristic function of the i.i.d. case (see section 3 of [?] and lemma 8 of [?]).

To study $B_n^{(k)}(t)$, set $\lambda_n(t) = \lambda\left(\frac{t}{\sqrt{n}}\right)^n$ for any (t, n) such that $|t|_2 \leq \beta\sqrt{n}$ and, for $|u|_2 \leq \beta$, set $\alpha(u) = \mu(\Pi(u)1_E) - 1$. With these notations, we have $B_n(t) = \lambda_n(t) \alpha\left(\frac{t}{\sqrt{n}}\right)$, and any partial derivative $B_n^{(k)}(t)$ is a finite sum of terms of the form

$$B_{n,p,q}^{(k)}(t) := \lambda_n^{(p)}(t) n^{-\frac{q}{2}} \alpha^{(q)}\left(\frac{t}{\sqrt{n}}\right) \quad \text{with } p+q=k.$$

Lemma 9.5. For $p = 0, \dots, m$, we have $|\lambda_n^{(p)}(t)| = O\left((1 + |t|_2^p) e^{-\frac{|t|_2^2}{4}}\right)$.

Assume this lemma for the moment. Since we have, by Proposition 7.2(i), $\alpha\left(\frac{t}{\sqrt{n}}\right) = O\left(\frac{|t|_2}{\sqrt{n}}\right)$ and $\alpha^{(q)}\left(\frac{t}{\sqrt{n}}\right) = O(1)$ for $1 \leq q \leq m$, this lemma gives for $q = 0$

$$B_{n,k,0}^{(k)}(t) = O\left((1 + |t|_2^k) e^{-\frac{|t|_2^2}{4}}\right) O\left(\frac{|t|_2}{\sqrt{n}}\right) = O\left(\frac{1}{\sqrt{n}} (1 + |t|_2^{k+1}) e^{-\frac{|t|_2^2}{4}}\right),$$

and for $q \geq 1$: $B_{n,p,q}^{(k)}(t) = O\left((1 + |t|_2^p) e^{-\frac{|t|_2^2}{4}}\right) O(n^{-\frac{q}{2}}) = O\left(\frac{1}{\sqrt{n}} (1 + |t|_2^{k+1}) e^{-\frac{|t|_2^2}{4}}\right)$.

So all the $B_{n,p,q}^{(k)}(t)$'s are $O\left(\frac{1}{\sqrt{n}} \chi(t)\right)$ with $\chi(t) = (1 + |t|_2^{k+1}) e^{-\frac{|t|_2^2}{4}}$, and this gives the estimate (II) for $B_n^{(k)}(t)$, and finally the proof of (I) is complete.

Proof of Lemma 9.5. Recall Γ is by hypothesis the identity matrix, so $\lambda(u) = 1 - \frac{|u|_2^2}{2} + o(|u|_2^2)$ as u goes to 0 (use Lemma 9.4). Hence, for $|u|_2 \leq \beta$ with β possibly reduced,

$$|\lambda(u)| \leq \left| \lambda(u) - 1 + \frac{|u|_2^2}{2} \right| + \left| 1 - \frac{|u|_2^2}{2} \right| \leq \frac{|u|_2^2}{4} + \left(1 - \frac{|u|_2^2}{2}\right) \leq 1 - \frac{|u|_2^2}{4} \leq e^{-\frac{|u|_2^2}{4}},$$

so $|\lambda\left(\frac{t}{\sqrt{n}}\right)| \leq e^{-\frac{|t|_2^2}{4n}}$ and $|\lambda\left(\frac{t}{\sqrt{n}}\right)^n| \leq (e^{-\frac{|t|_2^2}{4n}})^n = e^{-\frac{|t|_2^2}{4}}$. This gives the estimate of the lemma for $p = 0$. Now, in the case $p \geq 1$, one can prove by a straightforward induction that $\lambda_n^{(p)}(t)$ is a finite sum of terms of the form

$$\gamma(t, n) := n(n-1) \cdots (n-j+1) n^{-\frac{p}{2}} \lambda^{(s_1)}\left(\frac{t}{\sqrt{n}}\right) \cdots \lambda^{(s_j)}\left(\frac{t}{\sqrt{n}}\right) \lambda\left(\frac{t}{\sqrt{n}}\right)^{n-j},$$

with $j \in \{1, \dots, p\}$, $s_i \geq 1$, and $s_1 + \cdots + s_j = p$ (for convenience, j, s_1, \dots, s_j have been neglected in the above notation $\gamma(t, n)$). So we must prove that, given such fixed j, s_1, \dots, s_j , we have $\gamma(t, n) = O\left((1 + |t|_2^p) e^{-\frac{|t|_2^2}{4}}\right)$. To that effect, let us observe that $\lambda^{(1)}\left(\frac{t}{\sqrt{n}}\right) = O\left(\frac{|t|_2}{\sqrt{n}}\right)$ since $\lambda^{(1)}(0) = 0$, and that $\lambda^{(s)}\left(\frac{t}{\sqrt{n}}\right) = O(1)$ for any $s = 2, \dots, m$. This leads to define $a = \text{Card}\{i : s_i = 1\}$. Then we have

$$\begin{aligned} \gamma(t, n) &= O\left(n^{j-\frac{p}{2}} \frac{|t|_2^a}{n^{\frac{a}{2}}} (e^{-\frac{|t|_2^2}{4n}})^{n-j}\right) = O\left(e^{\frac{j}{4}} \left|\frac{t}{\sqrt{n}}\right|_2^2 n^{j-\frac{p}{2}-\frac{a}{2}} |t|_2^a e^{-\frac{|t|_2^2}{4}}\right) \\ &= O\left(n^{\frac{1}{2}(2j-p-a)} (1 + |t|_2^p) e^{-\frac{|t|_2^2}{4}}\right). \end{aligned}$$

For the last estimate, we used the fact that $\left|\frac{t}{\sqrt{n}}\right|_2 \leq \beta$ and $a \leq p$. Finally observe that we have $p = s_1 + \cdots + s_j \geq a + 2(j-a)$ by definition of the number a , thus $2j-p-a \leq 0$, so that the desired estimate on $\gamma(t, n)$ follows from the previous one. \square

10. Application to geometrically ergodic Markov chains. For the moment, the abstract results of the previous sections have been only applied to the simple but restrictive example of uniformly ergodic Markov chains. This section and the next one present applica-

tions to more general and practicable markovian models, namely the so-called geometrically ergodic Markov chains and the random iterative models. The interest of these models for statistical applications and for stochastic algorithms are fully described in [?] [?], and of course, the rate of convergence in the c.l.t. and the Edgeworth expansions are of great importance in practice, see e.g [?] [?]. For these models, all the previously studied limit theorems will be stated under general and simple moment conditions.

Throughout this section, we suppose that the σ -field \mathcal{E} is countably generated, that $(X_n)_{n \geq 0}$ is aperiodic and ψ -irreducible w.r.t. a certain positive σ -finite measure ψ on E .

Moreover, given an unbounded function $v : E \rightarrow [1, +\infty[$, we assume $(X_n)_{n \geq 0}$ is v -geometrically ergodic, that is $\pi(v) < +\infty$ and there exist real numbers $\kappa_0 < 1$ and $C \geq 0$ such that we have, for all $n \geq 1$ and $x \in E$,

$$\sup \left\{ |Q^n f(x) - \pi(f)|, f : E \rightarrow \mathbb{C} \text{ measurable}, |f| \leq v \right\} \leq C \kappa_0^n v(x).$$

If w is an unbounded function defined on E and taking values in $[1, +\infty[$, we denote by $(\mathcal{B}_w, \|\cdot\|_w)$ the weighted supremum-normed space of measurable complex-valued functions f on E such that

$$\|f\|_w = \sup_{x \in E} \frac{|f(x)|}{w(x)} < +\infty.$$

Let us observe that $\mu \in \mathcal{B}'_w$ if $\mu(w) < +\infty$. In particular we have $\pi \in \mathcal{B}'_v$ by hypothesis. Clearly, v -geometrical ergodicity means that Q is strongly ergodic w.r.t. \mathcal{B}_v .

For $0 < \theta \leq 1$, we set $\mathcal{B}_\theta = \mathcal{B}_{v^\theta}$ and $\|\cdot\|_\theta = \|\cdot\|_{v^\theta}$. In particular $\mathcal{B}_1 = \mathcal{B}_v$ and $\|\cdot\|_1 = \|\cdot\|_v$. The next lemma will be repeatedly used in the proofs below (here ξ is only supposed to be measurable).

Lemma 10.1. *For all $0 < \theta \leq 1$, Condition (\widehat{K}) of section 5.2 holds on \mathcal{B}_θ with the semi-norm $q(\cdot) = \pi(|\cdot|)$. In particular Condition (K) of Section 4.1 holds on \mathcal{B}_θ .*

Proof. The property (K1) (cf. section 2) on \mathcal{B}_θ - i.e. $(X_n)_{n \geq 0}$ is v^θ -geometrically ergodic - follows from the well-known link between geometric ergodicity and the so-called drift criterion [?]. More precisely, under the aperiodicity and ψ -irreducibility hypotheses, the w -geometric ergodicity for some $w : E \rightarrow [1, +\infty[$ is equivalent to the following condition : there exist $r < 1$, $M \geq 0$ and a small set $C \in \mathcal{E}$ such that $Qw_0 \leq rw_0 + M1_C$, where w_0 is a function equivalent to w in the sense that $c^{-1}w \leq w_0 \leq cw$ for some $c \in \mathbb{R}_+^*$. From that and since the function $t \mapsto t^\theta$ is concave on \mathbb{R}_+ , v -geometric ergodicity implies, by virtue of Jensen's inequality, that

$$Q(v_0^\theta) \leq (rv_0 + M1_C)^\theta \leq r^\theta v_0^\theta + M^\theta 1_C,$$

where v_0 stands for some function equivalent to v . Thus $(X_n)_{n \geq 0}$ is v^θ -geometrically ergodic. Besides, since $\pi(|e^{i(h, \xi)} - 1| |f|) \leq \|f\|_\theta \pi(|e^{i(h, \xi)} - 1| v^\theta)$ for $f \in \mathcal{B}_\theta$, Condition (K2') (cf. § 4.1), thus $(\widehat{K2})$, is fulfilled by Lebesgue's theorem. Besides we have $(\widehat{K3})$ by Remark (c) in § 4.1. The property $(\widehat{K4})$ w.r.t. \mathcal{B}_θ is a consequence of [?], see [?]. \square

If $|\xi|_2^2 \leq Cv$ for some $C > 0$, then $(\frac{S_n}{\sqrt{n}})_n$ converges to a normal distribution $\mathcal{N}(0, \Gamma)$ for any initial distribution. This is a classical result [?] which can be also deduced, in the stationary case, from the statements of Section 2.2. Indeed, the condition $|\xi|_2^2 \leq Cv$ implies that the

coordinate functions ξ_i of ξ belong to the space $\mathcal{B}_{\frac{1}{2}}$. Since $\pi(v) < +\infty$, we have $\mathcal{B}_{\frac{1}{2}} \hookrightarrow \mathcal{L}^2(\pi)$, and the previous lemma shows that Q is strongly ergodic on $\mathcal{B}_{\frac{1}{2}}$. So the desired c.l.t. follows from Proposition 2.2 and Corollary 2.1 both applied with $\mathcal{B} = \mathcal{B}_{\frac{1}{2}}$.

Recall that, without additional assumptions, this central limit theorem does not hold under the weaker condition $\pi(|\xi|_2^2) < +\infty$ (see [?]). In the same way, the limit theorems below will hold under moment conditions of the type $|\xi|_2^\alpha \leq Cv$ with some suitable exponent $\alpha \geq 2$, and some positive constant C . So the real number α will measure the order in these moment conditions, and as for the uniformly ergodic Markov chains (see the summary given in § 2.3), the values of α will be similar or very close to the optimal orders of the i.i.d. case.

The hypotheses of Assertions (a)-(d) below will imply that the above cited c.l.t. holds, and it will be then understood that Γ is non-singular (this means $\sigma^2 > 0$ in case $d = 1$). The nonlattice condition below is that of Proposition 5.4. Finally we suppose that the initial distribution μ is such that $\mu(v) < +\infty$.

Corollary 10.2.

- (a) *If $|\xi|_2^2 \leq Cv$ and ξ is nonlattice, then we have (LLT) of Theorem 5.1 with $\mathcal{B} = \mathcal{B}_{\frac{1}{2}}$.*
- (b) *(Case $d = 1$) If $|\xi|^3 \leq Cv$, then the uniform Berry-Essen estimate holds : $\Delta_n = O(n^{-\frac{1}{2}})$.*
- (c) *(Case $d = 1$) If $|\xi|^\alpha \leq Cv$ with some $\alpha > 3$ and ξ is nonlattice, then the first-order Edgeworth expansion (E) of Theorem 8.1 holds.*
- (d) *If $|\xi|_2^\alpha \leq Cv$ with some $\alpha > \max(3, \lfloor d/2 \rfloor + 1)$, then the (Prohorov) Berry-Essen estimate holds : $\mathcal{P}\left(\mu_*\left(\frac{S_n}{\sqrt{n}}\right), \mathcal{N}(0, \Gamma)\right) = O(n^{-1/2})$.*

Using the usual Nagaev method, (a) was established in [?] for bounded functional ξ . Assertion (a) extends the result of [?] stated under a kernel condition on Q . From Bolthausen's theorem [?], the one-dimensional uniform Berry-Essen theorem holds under \mathcal{IP}_π (stationary case) if $\pi(|\xi|^p) < +\infty$ for some $p > 3$. Assertion (b), already presented in [?], extends this result to the non-stationary case under an alternative third-order moment condition. Assertion (c) was established in [?] for bounded functional ξ , and (d) is new to our knowledge.

Proof of Corollary 10.2. (a) Condition (CLT) of section 5.1 is fulfilled by hypothesis, and we have (K) on $\mathcal{B} = \mathcal{B}_{\frac{1}{2}}$ (Lem. 10.1). Besides, since $g := |\xi|_2 \in \mathcal{B}_{\frac{1}{2}}$, one gets for all $f \in \mathcal{B}_{\frac{1}{2}}$

$$|Q(t)f - Qf| \leq Q(|e^{i(t, \xi)} - 1| f)| \leq |t|_2 \|g\|_{\frac{1}{2}} \|f\|_{\frac{1}{2}} Qv.$$

Since $\frac{Qv}{v}$ is bounded, this proves the continuity condition of (\widetilde{K}) (cf. section 4.3) with $\widetilde{\mathcal{B}} = \mathcal{B}_v$. Finally, from Lemma 10.1, the non-arithmeticity condition (S) of section 5.1 (w.r.t. any space \mathcal{B}_θ) is equivalent to the nonlattice condition of Proposition 5.4. So Theorem 5.1 applies.

(b) We have (K) (of section 4.1) w.r.t. $\mathcal{B} = \mathcal{B}_{\frac{1}{3}}$ (Lem. 10.1). Since $\mathcal{B}_{\frac{1}{3}} \subset \mathcal{L}^3(\pi)$ and $\xi \in \mathcal{B}_{\frac{1}{3}}$, Condition (G1) of § 6.2 is satisfied, and we have also (G2) of § 6.2 because (K1) holds on $\mathcal{B}_{\frac{2}{3}} \hookrightarrow \mathcal{L}^{\frac{3}{2}}(\pi)$ and this space contains all the functions g^2 , $g \in \mathcal{B}_{\frac{1}{3}}$. Finally, as above, one gets $|Q(t)f - Qf| \leq |t| \|\xi\|_{\frac{1}{3}} \|f\|_{\frac{1}{3}} Qv^{\frac{2}{3}}$ for all $f \in \mathcal{B}_{\frac{1}{3}}$. Since $\frac{Qv^{\frac{2}{3}}}{v^{\frac{2}{3}}}$ is bounded, the continuity condition in Theorem 6.1 holds with $\mathcal{B} = \mathcal{B}_{\frac{1}{3}}$ and $\widetilde{\mathcal{B}} = \mathcal{B}_{\frac{2}{3}}$. So (b) follows from Theorem 6.1. Assertions (c) and (d) are direct consequences of Theorems 8.1 and 9.1 (respectively) and of

the next proposition. \square

Proposition 10.3. *If $|\xi|_2^\alpha \leq C v$ with $\alpha > m$ ($m \in \mathbb{N}^*$), then $\mathcal{C}(m)$ holds with $\mathcal{B} = \mathcal{B}_\varepsilon$, $\tilde{\mathcal{B}} = \mathcal{B}_v$, for any $\varepsilon > 0$ such that $\varepsilon + \frac{m+(2m+1)\varepsilon}{\alpha} \leq 1$.*

Proof. For convenience, let us assume that $d = 1$. The extension to $d \geq 2$ is obvious by the use of partial derivatives. We take $I = [\varepsilon, 1]$ and $\mathcal{B}_\theta = \mathcal{B}_{v^\theta}$, $T_0(\theta) = \theta + \frac{\varepsilon}{\alpha}$, $T_1(\theta) = \theta + \frac{1+\varepsilon}{\alpha}$, $a = \varepsilon$. With these notations, the proof of $\mathcal{C}(m)$ is a direct consequence of the two following lemmas.

Lemma 10.4. *For any $k = 0, \dots, m$ and $\gamma, \gamma' > 0$ such that $\gamma + \frac{k}{\alpha} < \gamma' \leq 1$, we have $Q_k \in \mathcal{C}^0(\mathbb{R}, \mathcal{B}_\gamma, \mathcal{B}_{\gamma'})$.*

Proof. Let $0 < \delta \leq 1$ such that $\gamma + \frac{k+\delta}{\alpha} \leq \gamma'$. Using the inequality $|e^{ia} - 1| \leq 2|a|^\delta$ ($a \in \mathbb{R}$), one gets for $t, t_0 \in \mathbb{R}$ and $f \in \mathcal{B}_\gamma$:

$$|Q_k(t)f - Q_k(t_0)f| \leq Q(|\xi|^k |e^{i(t-t_0)\xi} - 1| |f|) \leq C \frac{k+\delta}{\alpha} |t - t_0|^\delta \|f\|_\gamma Q(v^{\frac{k+\delta}{\alpha} + \gamma}),$$

hence $\|Q_k(t)f - Q_k(t_0)f\|_{\gamma'} \leq C \frac{k+\delta}{\alpha} |t - t_0|^\delta \|f\|_\gamma \|Q(v^{\gamma'})\|_{\gamma'}$. \square

Lemma 10.5. *For any $k = 0, \dots, m-1$ and $\gamma, \gamma' > 0$ such that $\gamma + \frac{k+1}{\alpha} < \gamma' \leq 1$, we have $Q_k \in \mathcal{C}^1(\mathbb{R}, \gamma, \gamma')$ with $Q'_k = Q_{k+1}$.*

Proof. Let $0 < \delta \leq 1$ such that $\gamma + \frac{k+1+\delta}{\alpha} \leq \gamma'$. Using $|e^{ia} - 1 - ia| \leq 2|a|^{1+\delta}$ and proceeding as above, one gets $\|Q_k(t)f - Q_k(t_0)f - (t - t_0)Q_{k+1}(t_0)f\|_{\gamma'} \leq C \frac{k+1+\delta}{\alpha} |t - t_0|^{1+\delta} \|f\|_\gamma \|Q(v^{\gamma'})\|_{\gamma'}$ for $t_0, t \in \mathbb{R}$ and $f \in \mathcal{B}_\gamma$. Since $Q_{k+1} \in \mathcal{C}^0(\mathbb{R}, \gamma, \gamma')$, this yields the desired statement. \square

Remark. The above proof of (b) shows that Assertion (b) holds also under the alternative following hypotheses : $(X_n)_{n \geq 0}$ is $v^{\frac{2}{3}}$ -geometrically ergodic, $\mu(v^{\frac{2}{3}}) < +\infty$, $|\xi|^3 \leq C v$, and finally $\pi(v) < +\infty$ in order to have $\mathcal{B}_{\frac{1}{3}} \subset \mathcal{L}^3(\pi)$.

11. Applications to iterative Lipschitz models. Here (E, d) is a non-compact metric space in which every closed ball is compact. We endow it with its Borel σ -field \mathcal{E} .

11.1. Iterative Lipschitz models. Let (G, \mathcal{G}) be a measurable space, let $(\theta_n)_{n \geq 1}$ be a sequence of i.i.d.r.v. taking values in G . Let X_0 be a E -valued r.v. independent of $(\theta_n)_n$, and finally let $F : E \times G \rightarrow E$ be a measurable function. We set

$$X_n = F(X_{n-1}, \theta_n), \quad n \geq 1.$$

For $\theta \in G$, $x \in E$, we set $F_\theta x = F(x, \theta)$ and we suppose that $F_\theta : E \rightarrow E$ is Lipschitz continuous. Then $(X_n)_{n \geq 1}$ is called an iterative Lipschitz model [?] [?]. It is a Markov chain and its transition probability is :

$$Qf(x) = \mathbb{E}[f(F(x, \theta_1))].$$

Let x_0 be a fixed point in E . As in [?], we shall appeal to the following r.v. :

$$\mathcal{C} = \sup \left\{ \frac{d(F_{\theta_1} x, F_{\theta_1} y)}{d(x, y)}, x, y \in E, x \neq y \right\} \quad \text{and} \quad \mathcal{M} = 1 + \mathcal{C} + d(F(x_0, \theta_1), x_0).$$

As a preliminary, let us present a sufficient condition for the existence and the unicity of an invariant distribution. The following proposition is proved in [?] [Th. I].

Proposition 11.1. *Let $\alpha \in (0, 1]$, $\theta \in \mathbb{R}_+$. Under the moment condition $\mathbb{E}[\mathcal{M}^{\alpha(\theta+1)}] < +\infty$ and the mean contractivity condition $\mathbb{E}[\mathcal{C}^\alpha \max\{\mathcal{C}, 1\}^{\alpha\theta}] < 1$, there exists a unique stationary distribution, π , and we have $\pi(d(\cdot, x_0)^{\alpha(\theta+1)}) < +\infty$.*

More precise statements can be found in the litterature (see e.g [?] [?]). However, the hypotheses occuring in Proposition 11.1 are convenient in our context and are similar to those introduced later.

Finally, we shall suppose that ξ satisfies the following condition, with given $S, s \geq 0$:

$$(\mathbf{L})_s \quad \forall (x, y) \in E \times E, \quad |\xi(x) - \xi(y)|_2 \leq S d(x, y) [1 + d(x, x_0) + d(y, x_0)]^s.$$

For convenience, Condition $(\mathbf{L})_s$ has been stated as a weighted-Lipschitz condition w.r.t. to the distance $d(\cdot, \cdot)$ on E . However, by replacing $d(\cdot, \cdot)$ with the distance $d(\cdot, \cdot)^a$ ($0 < a \leq 1$), Condition $(\mathbf{L})_s$ then corresponds to the general weighted-Hölder condition of [?].

Section 11.2 below will introduce weighted Hölder-type spaces and investigate all the hypotheses of the previous sections. Using these preliminary statements, we shall see in Section 11.3 that the limit theorems of the preceding sections then apply to $(\xi(X_n))_n$ under some mean contractivity and moment conditions. These conditions will focus on the random variables \mathcal{C} , \mathcal{M} and will depend on the real number s of Condition $(\mathbf{L})_s$.

To compare with the i.i.d. case, let us summarize the results obtained in Section 11.3 in the following special setting : $(X_n)_n$ is a \mathbb{R}^d -valued iterative Lipschitz sequence such that $\mathcal{C} < 1$ a.s.. For convenience we also assume that $(X_n)_n$ is stationary, with stationary distribution π , and we consider the random walk associated to $\xi(x) = x - \mathbb{E}_\pi[X_0]$, that is :

$$S_n = X_1 + \dots + X_n - n\mathbb{E}_\pi[X_0].$$

Finally suppose that $\mathbb{E}[\mathcal{M}^2] < +\infty$. Then the sequence $(\frac{S_n}{\sqrt{n}})_n$ converges to $\mathcal{N}(0, \Gamma)$ [?], and we shall assume that Γ is invertible. Corollaries of § 11.3 will then provide the following results :

- (i) *Local limit theorem* : ξ nonlattice \Rightarrow (LLT) of section 5.1 with $f = 1_E$,
- (ii) *(d = 1) Uniform Berry-Essen type theorem* : $\mathbb{E}[\mathcal{M}^3] < +\infty \Rightarrow \Delta_n = O(n^{-\frac{1}{2}})$,
- (iii) *(d = 1) First-order Edgeworth expansion* : $\mathbb{E}[\mathcal{M}^{6+\varepsilon_0}] < +\infty$, ξ nonlattice \Rightarrow (E) of section 8,
- (iv) *Multi-dimensional Berry-Esseen theorem (with Prohorov metric)* : $\mathbb{E}[\mathcal{M}^{2m+\varepsilon_0}] < +\infty$ with $m = \max(3, \lfloor d/2 \rfloor + 1) \Rightarrow$ the conclusion of theorem 9.1 holds.

More generally, the previous assertions will apply to $(\xi(X_n))_n$ whenever ξ is a Lipschitz continuous function on E (i.e. $s = 0$ in $(\mathbf{L})_s$).

Example. The autoregressive models.

A simple and typical example is the autoregressive chain defined in \mathbb{R}^d by

$$X_n = A_n X_{n-1} + \theta_n \quad (n \in \mathbb{N}^*),$$

where $(A_n, \theta_n)_{n \geq 1}$ is a i.i.d. sequence of r.v. taking values in $\mathcal{M}_d(\mathbb{R}) \times \mathbb{R}^d$, independent of X_0 . ($\mathcal{M}_d(\mathbb{R})$ denotes the set of real $d \times d$ -matrices.) Assume that we have $|A_1| < 1$ a.s., where $|\cdot|$ denotes here both some norm on \mathbb{R}^d and the associated matrix norm. Taking the

distance $d(x, y) = |x - y|$ on \mathbb{R}^d , we have $\mathcal{C} = |A_1|$ and $\mathcal{M} \leq 2 + |\theta_1|$. So the above moment conditions in (i)-(iv) only concern $|\theta_1|$.

The special value $A_n = 0$ corresponds to the i.i.d. case ($S_n = \theta_1 + \dots + \theta_n - n\mathbb{E}[\theta_1]$), and Statements (i) (ii) give again the corresponding classical i.i.d. limit theorems. On the other hand, the moment condition in (iii) is more restrictive than that of the i.i.d. case : indeed the i.i.d. first-order Edgeworth expansion is valid under the condition $\mathbb{E}[|\theta_1|^3] < +\infty$ [?]. The moment condition in (iv) is also stronger than the expected one. (See Rk. in § 11.3.)

Finally let us recall that, in the case when $A_n = A$ is a fixed nonrandom contractive matrix, and θ_1 has a density w.r.t. the Lebesgue measure on \mathbb{R}^d , with a moment of order $r > 0$, then $(X_n)_n$ is v -geometrically ergodic with $v(x) = (1 + |x|)^r$, see [?] (§ 15.5.2). So, in this case, one can apply the statements of Corollary 10.2 which, when they are applied as above with $\xi(x) = x - \mathbb{E}_\pi[X_0]$, yield the first-order Edgeworth expansion if ξ is nonlattice and $r \geq 3 + \varepsilon$ for some $\varepsilon > 0$, and the multi-dimensional Berry-Esseen theorem if $r \geq \max(3, \lfloor d/2 \rfloor + 1) + \varepsilon$. These moment conditions are the expected ones up to $\varepsilon > 0$.

11.2. Preliminary results. The function spaces used here are weighted Hölder-type spaces defined as follows.

For $x \in E$, we set $p(x) = 1 + d(x, x_0)$. Given $\alpha \in (0, 1]$ and $\gamma > 0$, we denote by $\mathcal{B}_{\alpha, \gamma}$ the space of \mathbb{C} -valued functions on E satisfying the following condition

$$m_{\alpha, \gamma}(f) = \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)^\alpha p(x)^{\alpha\gamma} p(y)^{\alpha\gamma}}, x, y \in E, x \neq y \right\} < +\infty.$$

Set $|f|_{\alpha, \gamma} = \sup_{x \in E} \frac{|f(x)|}{p(x)^{\alpha(\gamma+1)}}$ and $\|f\|_{\alpha, \gamma} = m_{\alpha, \gamma}(f) + |f|_{\alpha, \gamma}$.

Then $(\mathcal{B}_{\alpha, \gamma}, \|\cdot\|_{\alpha, \gamma})$ is a Banach space. The next result which concerns Condition (K1) on $\mathcal{B}_{\alpha, \gamma}$ is established in [?] [Th. 5.5].

Proposition 11.2. *Let $\gamma > 0$. If $\mathbb{E}[\mathcal{M}^{\alpha(\gamma+1)} + \mathcal{C}^\alpha \mathcal{M}^{2\alpha\gamma}] < +\infty$, $\mathbb{E}[\mathcal{C}^\alpha \max\{\mathcal{C}, 1\}^{2\alpha\gamma}] < 1$, then Q is strongly ergodic on $\mathcal{B}_{\alpha, \gamma}$.*

Now we give a sufficient condition for the central limit theorem in the stationary case. Similar statements are presented in [?], and in [?] when ξ is Lipschitz continuous (i.e. $s = 0$ in $(\mathbf{L})_s$).

Proposition 11.3. *If $\mathbb{E}[\mathcal{M}^{2s+2} + \mathcal{C}^{\frac{1}{2}} \mathcal{M}^{2s+1}] < +\infty$ and $\mathbb{E}[\mathcal{C}^{\frac{1}{2}} \max\{\mathcal{C}, 1\}^{2s+\frac{3}{2}}] < 1$, then, under \mathbb{P}_π , $(\frac{S_n}{\sqrt{n}})_n$ converges to a normal distribution $\mathcal{N}(0, \Gamma)$.*

Proof. We apply proposition 11.1 with $\alpha = \frac{1}{2}$ and $\theta = 4s + 3$. This yields the existence and the unicity of π , and $\pi(d(\cdot, x_0)^{2s+2}) < +\infty$. Set here $\mathcal{B} = \mathcal{B}_{\frac{1}{2}, 2s+1}$. For $f \in \mathcal{B}$, we have $|f| \leq |f|_{\frac{1}{2}, 2s+1} p(x)^{s+1}$. Thus $\mathcal{B} \hookrightarrow \mathbb{L}^2(\pi)$. Besides, from $(\mathbf{L})_s$, it can be easily seen that the coordinate functions of ξ belong to \mathcal{B} , and by Proposition 11.2, Q is strongly ergodic on \mathcal{B} . We conclude by applying Proposition 2.2 and Corollary 2.1 with $\mathcal{B} = \mathcal{B}_{\frac{1}{2}, 2s+1}$. \square

The possibility of considering $\alpha < 1$ as above is important. To see that, consider for instance the case $s = 0$ (i.e. ξ is Lipschitz continuous on E). Then $\xi \in \mathcal{B}_{1, \gamma}$ for any $\gamma > 0$, and we could also consider $\mathcal{B} = \mathcal{B}_{1, \gamma}$ in the previous proof, but it is worth noticing that the condition

$\mathcal{B}_{1,\gamma} \hookrightarrow \mathbb{L}^2(\pi)$ would then require the moment condition $\pi(d(\cdot, x_0)^{2(1+\gamma)}) < +\infty$ which is stronger than $\pi(d(\cdot, x_0)^2) < +\infty$ used above. Anyway, we shall often appeal below to the conditions $\gamma \geq 1 + s$, namely $\gamma \geq 1$ in the case $s = 0$, and $\mathbb{E}[\mathcal{M}^{\alpha(\gamma+1)}] < +\infty$. If $\gamma = 1$ for instance, the previous condition is $\mathbb{E}[\mathcal{M}] < +\infty$ if $\alpha = \frac{1}{2}$, while it is $\mathbb{E}[\mathcal{M}^2] < +\infty$ if $\alpha = 1$.

Now we investigate the action of the Fourier kernels $Q(t)$ on the space $\mathcal{B}_{\alpha,\gamma}$. The proofs of Propositions 11.4-8 below present no theoretical problem. However the presence of Lipschitz coefficients in the definition of $\mathcal{B}_{\alpha,\gamma}$ makes the computations quite more technical than those seen for uniformly ergodic, or geometrically ergodic, chains. For convenience, these proofs are presented in Appendix B. The arguments will be derived from [?]. However, the next four statements improve the corresponding ones in [?].

Proposition 11.4. *Condition (K) of section 4.1 holds w.r.t. the function space $\mathcal{B}_{\alpha,\gamma}$ if $\gamma \geq 1 + s$ and $\mathbb{E}[\mathcal{M}^{\alpha(\gamma+1)} + \mathcal{C}^\alpha \mathcal{M}^{2\alpha\gamma}] < +\infty$, $\mathbb{E}[\mathcal{C}^\alpha \max\{\mathcal{C}, 1\}^{2\alpha\gamma}] < 1$.*

Proposition 11.5. *We have $\|Q(t) - Q\|_{\mathcal{B}_{\alpha,\gamma}, \mathcal{B}_{\alpha,\gamma'}} \rightarrow 0$ when $t \rightarrow 0$ if $\gamma' > \gamma \geq 1 + s$ and $\mathbb{E}[\mathcal{M}^{\alpha(\gamma'+1)} + \mathcal{C}^\alpha \mathcal{M}^{\alpha(\gamma'+\gamma)}] < +\infty$.*

Proposition 11.6. *We have $\|Q(t) - Q\|_{\mathcal{B}_{\alpha,\gamma}, \mathcal{B}_{\alpha,\gamma'}} = O(|t|)$ if $\gamma > 0$, $\gamma' \geq \gamma + \frac{s+1}{\alpha}$, and $\mathbb{E}[\mathcal{M}^{\alpha(\gamma'+1)} + \mathcal{C}^\alpha \mathcal{M}^{\alpha(\gamma'+\gamma)} + \mathcal{C}^\alpha \mathcal{M}^{\alpha\gamma'}] < +\infty$.*

Proposition 11.7. *We have $\mathcal{C}(m)$ of section 7.1 ($m \in \mathbb{N}^*$) with $\mathcal{B} = \mathcal{B}_{\alpha,\eta}$ and $\tilde{\mathcal{B}} = \mathcal{B}_{\alpha,\eta'}$ for any η, η' such that $\eta \geq 1 + s$, $\eta' > \eta + \frac{m(s+1)}{\alpha}$, and $\mathbb{E}[\mathcal{M}^{\alpha(\eta'+1)} + \mathcal{C}^\alpha \mathcal{M}^{2\alpha\eta'}] < +\infty$, $\mathbb{E}[\mathcal{C}^\alpha \max\{\mathcal{C}, 1\}^{2\alpha\eta'}] < 1$.*

Concerning the spectral condition (S) of section 5.1, we now study the possibility of applying the results of Section 5.2. Observe that this cannot be done with the help of Proposition 11.4 because it only yields (K3) (K4) near $t = 0$. By considering another auxiliary semi-norm on $\mathcal{B}_{\alpha,\gamma}$, we shall prove in Appendix B.5 the following result for which the hypotheses are somewhat more restrictive than that of Proposition 11.4.

Proposition 11.8. *Assume $\mathbb{E}[\mathcal{M}^{\alpha(\gamma'+1)} + \mathcal{C}^\alpha \mathcal{M}^{\alpha(\gamma'+\gamma)}] < +\infty$ and $\mathbb{E}[\mathcal{C}^\alpha \max\{\mathcal{C}, 1\}^{2\alpha\gamma}] < 1$ for some $\gamma' > \gamma > 1 + s$. Then we have (S) on $\mathcal{B}_{\alpha,\gamma}$ iff ξ is non-arithmetic w.r.t. $\mathcal{B}_{\alpha,\gamma}$. If moreover (CLT) holds, then the two previous equivalent conditions hold if ξ is nonlattice.*

11.3. Limit theorems for $(\xi(X_n))_n$. *The hypotheses of Corollaries 11.9-11 below will imply those of Proposition 11.3. Consequently the clt stated in this proposition will hold automatically, and it will be understood that Γ is non-singular.*

Concerning the conditions imposed in this section on the initial distribution μ , it is worth noticing that, if $\mu(d(\cdot, x_0)^{\alpha(1+\gamma)}) < +\infty$, then $\mu \in \mathcal{B}'_{\alpha,\gamma}$. Also mention that the next conditions imposed on μ will be always satisfied for $\mu = \pi$ or $\mu = \delta_x$ ($x \in E$) (for π it comes from proposition 11.1).

Local limit theorem. To present a simple application of Theorem 5.1, let us simply investigate Statement (LLT) of section 5.1 with $f = 1_E$. We want to prove that, for any compactly

supported continuous function $g : \mathbb{R}^d \rightarrow \mathbb{R}$, we have

$$(LLT') \quad \lim_n \left(\sqrt{\det \Gamma} (2\pi n)^{\frac{d}{2}} \mathbb{E}_\mu [g(S_n - a)] - e^{-\frac{1}{2n} \langle \Gamma^{-1} a, a \rangle} \int_{\mathbb{R}} g(x) dx \right) = 0.$$

Corollary 11.9. *Suppose that $\mathbb{E}[\mathcal{M}^{2s+2} + \mathcal{C}^{\frac{1}{2}} \mathcal{M}^{2s+1+\delta}] < +\infty$ for some $\delta > 0$, that $\mathbb{E}[\mathcal{C}^{\frac{1}{2}} \max\{\mathcal{C}, 1\}^{2s+\frac{3}{2}}] < 1$, that ξ satisfies $(\mathbf{L})_s$ and is nonlattice, and finally that we have $\mu(d(\cdot, x_0)^{\frac{2+s+\delta}{2}}) < +\infty$. Then we have (LLT').*

Proof. By using the above preliminary statements, we can prove that the hypotheses of Theorem 5.1 are fulfilled. Indeed, the mean contractivity and moment hypotheses imply those of Proposition 11.3, so we have (CLT). Besides, let $\alpha = \frac{1}{2}$, $\varepsilon > 0$, and set $\gamma = 1 + s + \varepsilon$ and $\gamma' = \gamma + \varepsilon$. Then, by choosing $0 < \varepsilon \leq \frac{\delta}{2}$ sufficiently small, we have (K) on $\mathcal{B} = \mathcal{B}_{\frac{1}{2}, \gamma}$ and (K1) on $\tilde{\mathcal{B}} = \mathcal{B}_{\frac{1}{2}, \gamma'}$ (Prop. 11.4, 11.2). Moreover the continuity condition in (\tilde{K}) of section 4.3 holds (Prop. 11.5), and finally, since ξ is nonlattice, (S) holds on $\mathcal{B}_{\alpha, \gamma}$ (Prop. 11.8). Furthermore we have $\mathcal{B}' \cap \tilde{\mathcal{B}}' = \tilde{\mathcal{B}}'$, and $\mu(d(\cdot, x_0)^{\frac{2+s+\delta}{2}}) < +\infty$ implies $\mu \in \tilde{\mathcal{B}}'$. \square

According to the previous proof, the property (LLT) may be also investigated with positive function $f \in \mathcal{B}_{\frac{1}{2}, \gamma}$ (for some suitable $\gamma > 1 + s$), and the sufficient nonlattice condition can be replaced by the more precise non-arithmeticity condition (w.r.t. $\mathcal{B}_{\frac{1}{2}, \gamma}$) of Proposition 5.3. Finally, if $s = 0$ (i.e. ξ is Lipschitz continuous on E), as for example $\xi(x) = \|x\|$, and if we have $\mathcal{C} < 1$ a.s., then (LLT') is valid under the expected moment condition $\mathbb{E}[\mathcal{M}^2] < +\infty$.

Uniform Berry-Essen type theorem. Here we assume $d = 1$.

Corollary 11.10. *Suppose $\mathbb{E}[\mathcal{M}^{3(s+1)} + \mathcal{C}^{\frac{1}{2}} \mathcal{M}^{4s+3}] < +\infty$ and $\mathbb{E}[\mathcal{C}^{\frac{1}{2}} \max\{\mathcal{C}, 1\}^{4s+3}] < 1$, that ξ satisfies $(\mathbf{L})_s$, and $\mu(d(\cdot, x_0)^{2(s+1)}) < +\infty$. Then $\Delta_n = O(n^{-\frac{1}{2}})$.*

Proof. To apply Theorem 6.1, we have to prove (CLT') of section 6.1 and to find some spaces \mathcal{B} and $\tilde{\mathcal{B}}$ on which (\tilde{K}) holds with the additional condition $\|Q(t) - Q\|_{\mathcal{B}, \tilde{\mathcal{B}}} = O(|t|)$. To investigate (CLT'), we shall use the procedure based on conditions (G1)-(G2) (see § 6.2). In particular this procedure requires that $\xi \in \mathcal{B} \leftrightarrow \mathbb{L}^3(\pi)$. Since $(\mathbf{L})_s$ implies $\xi \in \mathcal{B}_{\frac{1}{2}, 2s+1}$, let us consider $\mathcal{B} = \mathcal{B}_{\frac{1}{2}, 2s+1}$. We have (K) on \mathcal{B} (Prop. 11.4). Now set $\tilde{\mathcal{B}} = \mathcal{B}_{\frac{1}{2}, 4s+3}$. Then Q is strongly ergodic on $\tilde{\mathcal{B}}$ (Prop. 11.2), and the above estimates on $\|Q(t) - Q\|_{\mathcal{B}, \tilde{\mathcal{B}}}$ is valid (Prop. 11.6). Moreover, for $f \in \mathcal{B}$, we have $|f| \leq |f|_{\frac{1}{2}, 2s+1} p^{s+1}$, and since $\pi(d(\cdot, x_0)^{3(s+1)}) < +\infty$ (use Prop. 11.1 with $\alpha = \frac{1}{2}$, $\theta = 6s + 5$), one gets $\mathcal{B} \leftrightarrow \mathbb{L}^3(\pi)$. This gives (G1). Besides, it can be easily seen that $\tilde{\mathcal{B}}$ contains all the functions g^2 , $g \in \mathcal{B}$, and since each $f \in \tilde{\mathcal{B}}$ satisfies $|f| \leq |f|_{\frac{1}{2}, 4s+3} p^{2(s+1)}$, one obtains $\tilde{\mathcal{B}} \leftrightarrow \mathbb{L}^{\frac{3}{2}}(\pi)$. This proves (G2). Finally the moment condition on μ implies $\mu \in \tilde{\mathcal{B}}' = \mathcal{B}' \cap \tilde{\mathcal{B}}'$. \square

The first-order Edgeworth expansion. We assume $d = 1$, and for convenience, we investigate the property (E) of Theorem 8.1 under the hypothesis that \mathcal{C} is strictly contractive a.s..

Corollary 11.11. *Let us suppose that $\mathcal{C} < 1$ a.s. and that $\mathbb{E}[\mathcal{M}^{6(s+1)+\varepsilon_0}] < +\infty$ for some $\varepsilon_0 > 0$, that ξ satisfies $(\mathbf{L})_s$ and is nonlattice, and $\mu(d(\cdot, x_0)^{3(s+1)+\varepsilon_0}) < +\infty$. Then we have (E).*

Proof. To check the hypotheses of Theorem 8.1, let us first observe that the hypothesis $\mathcal{C} < 1$ a.s. implies $\mathbb{E}[\mathcal{C}^\alpha \max\{\mathcal{C}, 1\}^\beta] < 1$ for any $\alpha \in (0, 1]$ and $\beta \geq 0$. We have $\pi(d(\cdot, x_0)^{6(s+1)}) < +\infty$ (Prop. 11.1). From $|\xi(x)| \leq p(x)^{s+1}$, it follows that $\pi(|\xi|^3) < +\infty$. Let us prove that $\mathcal{C}(3)$ holds w.r.t. $\mathcal{B} = \mathcal{B}_{\alpha, \eta}$ and $\tilde{\mathcal{B}} = \mathcal{B}_{\alpha, \eta'}$ for suitable α, η, η' . Let $\delta > 0$, $\eta = s + 1 + \delta$, and $\eta' = \eta + \frac{3(s+1)}{\alpha} + \delta$. We have $\alpha(\eta' + 1) \leq 2\alpha\eta'$, and $2\alpha\eta' = 6(s+1) + 2\alpha(\eta + \delta)$. Now let us choose $0 < \alpha \leq 1$ such that $2\alpha\eta' \leq 6(s+1) + \varepsilon_0$. Then Proposition 11.7 yields the desired property. To study Condition (S), observe that the hypotheses of Proposition 11.8 are fulfilled with $\gamma = \eta$ and $\gamma' = \eta + \delta$ (to check the moment condition, use $\gamma' \leq \eta'$). Therefore, since ξ is nonlattice, we have (S) on $\mathcal{B}_{\alpha, \eta}$. Finally, we have $\alpha\eta' = 3(s+1) + \alpha(\eta + \delta) \leq 3(s+1) + \frac{\varepsilon_0}{2}$, and we have $\alpha \leq \frac{\varepsilon_0}{2(\eta + \delta)} \leq \frac{\varepsilon_0}{2}$, thus $\alpha(\eta' + 1) \leq 3(s+1) + \varepsilon_0$. So the moment hypothesis on μ implies that $\mu \in \tilde{\mathcal{B}}'$. \square

Other similar statements may be derived by proceeding as above. For instance, let us consider $0 < \alpha \leq 1$ (fixed here), $\eta = s + 1 + \delta$, and $\eta' = \eta + \frac{3(s+1)}{\alpha} + \delta$ with some small $\delta > 0$, and suppose that we have $\mathbb{E}[\mathcal{M}^{\alpha(\eta'+1)} + \mathcal{C}^\alpha \mathcal{M}^{2\alpha\eta'}] < +\infty$, $\mathbb{E}[\mathcal{C}^\alpha \max\{\mathcal{C}, 1\}^{2\alpha\eta'}] < 1$, and $\mu(d(\cdot, x_0)^{\alpha(\eta'+1)}) < +\infty$. Then we have (E) if ξ is non-arithmetic w.r.t. $\mathcal{B}_{\alpha, \eta}$.

The multi-dimensional Berry-Essen theorem with Prohorov distance. Again we give a statement in the particular case when $\mathcal{C} < 1$. From theorem 9.1 and 11.7, we get the following.

Corollary 11.12. *Suppose $\mathcal{C} < 1$ a.s and $\mathbb{E}[\mathcal{M}^{2m(s+1)+\varepsilon_0}] < +\infty$ for some $\varepsilon_0 > 0$ and with $m := \max(3, \lfloor d/2 \rfloor + 1)$, that ξ satisfies $(\mathbf{L})_s$ and $\mu(d(\cdot, x_0)^{m(s+1)+\frac{\varepsilon_0}{2}+1}) < +\infty$. Then the conclusion of theorem 9.1 holds.*

Proof. Let $\alpha \in (0, 1)$ be such that $(1+s) < \frac{\varepsilon_0}{2\alpha}$. Let us take $\eta = 1+s$ and $\eta' = \frac{m(1+s)}{\alpha} + \frac{\varepsilon_0}{2\alpha} > \eta + \frac{m(1+s)}{\alpha}$. We have : $\mathbb{E}[\mathcal{M}^{2\alpha\eta'}] < +\infty$. Since $\mathcal{C} < 1$ and since $\alpha(\eta' + 1) \leq 2\alpha\eta'$, the hypothesis of proposition 11.7 is satisfied. Hence we have $\mathcal{C}(m)$ with $\mathcal{B} = \mathcal{B}_{\alpha, \eta}$ and $\tilde{\mathcal{B}} = \mathcal{B}_{\alpha, \eta'}$. Moreover, the moment hypothesis on μ insures that μ is in $\tilde{\mathcal{B}}'$. \square

Remark. The moment conditions on \mathcal{M} in Corollaries 11.11-12 is twice the expected one. This is due to the weight $p(x)^{\alpha\gamma} p(y)^{\alpha\gamma}$ in the definition of $\mathcal{B}_{\alpha, \gamma}$ which yields the term $\mathcal{M}^{2\alpha\gamma}$ when one investigates $m_{\alpha, \gamma}(Q(t)f)$, see Appendix B. This drawback, which only occurs here when $\mathcal{C}(m)$ is used, could be possibly removed with a more appropriate choice of spaces.

Extension. Mention that all the previous statements remain valid when, in the hypotheses, the r.v. \mathcal{C} is replaced with the following one :

$$\mathcal{C}^{(n_0)} = \sup \left\{ \frac{d(F_{\theta_1} \cdots F_{\theta_{n_0}} x, F_{\theta_1} \cdots F_{\theta_{n_0}} y)}{d(x, y)}, x, y \in E, x \neq y \right\} \quad (n_0 \in \mathbb{N}^*).$$

The proofs of the preliminary statements in Section 11.2 are then similar.

12. More on the non-arithmeticity condition. This section presents some complements concerning the spectral condition (S) of Section 5.1, in particular we prove Propositions 5.3-4.

12.1. Proof of Proposition 5.3. We suppose that the conditions of Proposition 5.3 hold. Recall that Condition (S) on \mathcal{B} states that, for each compact set K_0 in $\mathbb{R}^d \setminus \{0\}$, there exist $\rho < 1$ and $c \geq 0$ such that we have, for all $n \geq 1$ and $t \in K_0$, $\|Q(t)^n\|_{\mathcal{B}} \leq c\rho^n$.

We have to prove that (S) is not true if and only if there exist $t \in \mathbb{R}^d$, $t \neq 0$, $\lambda \in \mathbb{C}$, $|\lambda| = 1$, a π -full Q -absorbing set $A \in \mathcal{E}$, and a function w in \mathcal{B}_0 such that $|w|$ is nonzero constant on A , satisfying :

$$(*) \quad \forall x \in A, \quad e^{i\langle t, \xi(y) \rangle} w(y) = \lambda w(x) \quad Q(x, dy) - a.s.,$$

where \mathcal{B}_0 is defined as $\mathcal{B}_0 = \mathcal{B} \cap \mathcal{L}^\infty(\pi)$ or $\mathcal{B}_0 = \mathcal{B} \cap \mathcal{B}^\infty$ according that we have (b1) or (b2).

Lemma 12.1. *Let $t \in \mathbb{R}^d$ such that $r(Q(t)) \geq 1$. Then*

(i) $r(Q(t)) = 1$ and $Q(t)$ is quasi-compact.

(ii) We have (*) with λ , A and w as above stated.

Proof. (i) By $(\widehat{K4})$, we have $r_{ess}(Q(t)) < 1 \leq r(Q(t))$, thus $Q(t)$ is quasi-compact on \mathcal{B} . Now let λ be any eigenvalue of modulus $r(Q(t))$, and let $f \neq 0$ be an associated eigenfunction in \mathcal{B} . We have $|\lambda|^n |f| = |Q(t)^n f| \leq Q^n |f|$. By (b1), there exists $e' \in \mathcal{B}'$, $e' \geq 0$, such that $e'(|f|) \neq 0$, and (K1) of Section 2 gives $\sup_n e'(Q^n |f|) < +\infty$. From $|\lambda|^n e'(|f|) \leq e'(Q^n |f|)$, it then follows that $|\lambda| \leq 1$, and finally $r(Q(t)) = 1$. Hence (i).

By (i), there exists $\lambda \in \mathbb{C}$, $|\lambda| = 1$ and $w \in \mathcal{B}$, $w \neq 0$, such that $Q(t)w = \lambda w$. From $Q(t)^n w = \lambda^n w$, one gets $|w| \leq Q^n |w|$, and since $Q^n |w| \rightarrow \pi(|w|)$ in \mathcal{B} by (K1), it follows from (b1) that $|w| \leq \pi(|w|)$, either on E , or π -a.s. on E , according to the nature of \mathcal{B} . From now, if \mathcal{B} is composed of equivalence classes (w.r.t. π), w is replaced with any measurable function of its class, and for convenience, this function will be still denoted by w . Since $v = \pi(|w|) - w \geq 0$ and $\pi(v) = 0$, we have $|w| = \pi(|w|)$ π -a.s. Let us define the set

$$A_0 = \{z \in E : |w(z)| = \pi(|w|)\}.$$

Then we have $\pi(A_0) = 1$ (i.e. A_0 is π -full).

Proof of (ii) under Condition (b2). It is proved in [?] (Prop. V.2) that, under Condition (b2), we have (*) with $w \in \mathcal{B} \cap \mathcal{B}^\infty$ and $A = A_0$. Let us briefly resume the main arguments. From (b2), we have here $|w| \leq \pi(|w|)$ on E . Thus $w \in \mathcal{B}^\infty$. Besides, the equality $Q(t)w(x) = \int_E e^{i\langle t, \xi(y) \rangle} w(y) Q(x, dy) = \lambda w(x)$ is valid for all $x \in E$. Let $x \in A_0$. Then this equality, together with the previous inequality, give (*). Finally (*) shows that A_0 is Q -absorbing.

Proof of (ii) under Condition (b1). Here one may have $Q(t)w = \lambda w$ a.s., so the previous arguments must be slightly modified as follows. First, by proceeding as in the proof of Proposition 2.4, one can easily get a π -full Q -absorbing set $B \subset A_0$. Besides the following set is clearly π -full :

$$C = \{z \in E : \forall n \geq 1, Q(t)^n w(z) = \lambda^n w(z)\}.$$

So the set $A = B \cap C$ is also π -full. Let $x \in A$. We have

$$Q(t)w(x) = \int_E e^{i\langle t, \xi(y) \rangle} w(y) Q(x, dy) = \lambda w(x).$$

Since $Q(x, B) = 1$ (B is Q -absorbing), one can replace E by B in the previous integral, and since $|\lambda^{-1} w(x)^{-1} e^{i\langle t, \xi(y) \rangle} w(y)| = 1$ for all $y \in B$, we then obtain the equality (*). It remains to prove that A is Q -absorbing. To that effect, we must just prove that $Q(x, C) = 1$ for any $x \in A$. Set $D_x = \{y \in E : e^{i\langle t, \xi(y) \rangle} w(y) = \lambda w(x)\}$. We know that $Q(x, D_x) = 1$, and from $\lambda^{n+1} w(x) = \int_{D_x} e^{i\langle t, \xi(y) \rangle} Q(t)^n w(y) Q(x, dy)$ ($n \geq 1$), we deduce that

$$\lambda^n = \int_{D_x} w(y)^{-1} Q(t)^n w(y) Q(x, dy).$$

Since $Q(x, B) = 1$, this equality holds also with B instead of D_x . Besides, for any $y \in B$, we

have $|Q(t)^n w(y)| \leq Q^n |w|(y) = \int_B |w(z)| Q^n(y, dz) = \pi(|w|)$, so that $|w(y)^{-1} Q(t)^n w(y)| \leq 1$. So, for some $D_{x,n} \in \mathcal{E}$ such that $Q(x, D_{x,n}) = 1$, we have $Q(t)^n w(y) = \lambda^n w(y)$ for each $y \in D_{x,n}$. From $\bigcap_{n \geq 1} D_{x,n} \subset C$, one gets $Q(x, C) = 1$ as claimed. \square

Lemma 12.2. *Let $t \in \mathbb{R}^d$. If the equality (*) holds with λ , A and w as stated at the beginning of this section, then we have $r(Q(t)) \geq 1$.*

Proof. By integrating (*), one gets $Q(t)w = \lambda w$ on A , and since A is Q -absorbing, this gives $Q(t)^n w = \lambda^n w$ on A for all $n \geq 1$. Suppose $r(Q(t)) < 1$. Then $\lim_n Q(t)^n w = 0$ in \mathcal{B} , and since $\mathcal{B} \hookrightarrow \mathbb{L}^1(\pi)$, we have $\lim_n \pi(|Q(t)^n w|) = 0$, but this is impossible because $|Q(t)^n w| = |w|$ on A , and by hypothesis $|w|$ is a nonzero constant on A and $\pi(A) = 1$. \square

The previous lemmas show that, for any fixed $t \in \mathbb{R}^d$, we have $r(Q(t)) \geq 1$ iff the equality (*) holds for some λ , A and w as stated at the beginning of this section. Consequently, in order to prove the equivalence of Proposition 5.3, it remains to establish the following lemma whose proof is based on the use of Keller-Liverani theorem.

Lemma 12.3. *We have : (S) $\Leftrightarrow \forall t \in \mathbb{R}^d, t \neq 0, r(Q(t)) < 1$.*

Proof. The direct implication is obvious. For the converse, let us consider a compact set K_0 in $\mathbb{R}^d \setminus \{0\}$. Let us first prove that

$$r_{K_0} = \sup\{r(Q(t)), t \in K_0\} < 1.$$

For that, let us assume that $r_{K_0} = 1$. Then there exists a subsequence $(\tau_k)_k$ in K_0 such that we have $\lim_k r(Q(\tau_k)) = 1$. For $k \geq 1$, let λ_k be a spectral value of $Q(\tau_k)$ such that $|\lambda_k| = r(Q(\tau_k))$. By compactness, one may assume that the sequences $(\tau_k)_k$ and $(\lambda_k)_k$ converge. Let $\tau = \lim_k \tau_k$ and $\lambda = \lim_k \lambda_k$; observe that $\tau \in K_0$, thus $\tau \neq 0$, and $|\lambda| = 1$. Besides, by (\widehat{K}_2) $(\widehat{K}3)$ $(\widehat{K}4)$, the $Q(t)$'s satisfy the conditions of Keller-Liverani theorem near τ . From [?] (p. 145), it follows that λ is a spectral value of $Q(\tau)$, but this impossible since, by hypothesis, $r(Q(\tau)) < 1$. This shows the claimed statement.

Let $\rho \in (r_{K_0}, 1)$. By applying [?] to $Q(\cdot)$ near any point $t_0 \in K_0$, there exists a neighbourhood \mathcal{O}_{t_0} of t_0 such that $\sup\{\|(z - Q(t))^{-1}\|_{\mathcal{B}}, t \in \mathcal{O}_{t_0}, |z| = \rho\} < +\infty$. Since K_0 is compact, one gets $\sup\{\|(z - Q(t))^{-1}\|_{\mathcal{B}}, t \in K_0, |z| = \rho\} < +\infty$. Finally let Γ be the oriented circle defined by $\{|z| = \rho\}$. Then the inequality stated in (S) follows from the following usual spectral formula

$$Q(t)^n = \frac{1}{2i\pi} \int_{\Gamma} z^n (z - Q(t))^{-1} dz. \quad \square$$

12.2. Proof of Proposition 5.4. Suppose that the hypotheses of Proposition 5.4 are fulfilled, except (CLT) for the moment. Let us recall that these hypotheses contain those of Proposition 5.3. We set $G = \{t \in \mathbb{R}^d : r(Q(t)) = 1\}$.

We know that (S) is equivalent to $G = \{0\}$ (Lemmas 12.1-3). Consequently, the following proposition implies Proposition 5.4.

Proposition 12.4. *G is a closed subgroup of $(\mathbb{R}^d, +)$. Moreover, if Condition (CLT) of section 5.1 holds, then G is discrete, and we have then the following properties.*

(i) *If $G \neq \{0\}$, then there exist a point $a \in \mathbb{R}^d$, a closed subgroup H in \mathbb{R}^d of the form $H = (\text{vect } G)^\perp \oplus \Delta$, where Δ is a discrete subgroup of \mathbb{R}^d , a π -full Q -absorbing set $A \in \mathcal{E}$,*

and a bounded measurable function $\theta : E \rightarrow \mathbb{R}^d$ such that

$$(**) \quad \forall x \in A, \quad \xi(y) + \theta(y) - \theta(x) \in a + H \quad Q(x, dy) - a.s..$$

(ii) If (**) holds with a π -full Q -absorbing set $A \in \mathcal{E}$, a subgroup $H \neq \mathbb{R}^d$, and a measurable function $\theta : E \rightarrow \mathbb{R}^d$ such that $e^{i\langle t, \theta \rangle} \in \mathcal{B}$ for all $t \in \mathbb{R}^d$, then $G \neq \{0\}$.

Proof. Let $g_1, g_2 \in G$, and for $k = 1, 2$, using Lemma 12.1, let λ_k, A_k , and w_k be the elements associated with g_k in (*). Then $A = A_1 \cap A_2$ is a π -full Q -absorbing set, and $g_1 - g_2$ satisfies (*) with A , $\lambda = \lambda_1 \overline{\lambda_2}$, and with $w = w_1 \overline{w_2}$ in \mathcal{B}_0 since, by hypothesis, \mathcal{B}_0 is stable under product. Thus $g_1 - g_2 \in G$ by Lemmas 12.1-2. Besides $0 \in G$ since $Q1_E = 1_E$. So G is a subgroup of $(\mathbb{R}^d, +)$. To prove that G is closed, let us consider any sequence $(t_n)_n \in G^{\mathbb{N}}$ such that $\lim t_n = t$ in \mathbb{R}^d . By quasi-compactness (Lemma 12.1), each $Q(t_n)$ admits an eigenvalue, say λ_n , of modulus one. Now let λ be a limit point of the sequence $(\lambda_n)_n$. Then $|\lambda| = 1$, and from [?] (p. 145), it follows that λ is a spectral value of $Q(t)$, so $r(Q(t)) \geq 1$, and $t \in G$ by Lemma 12.1. To prove G is discrete under Hypothesis (CLT), let us first suppose for convenience that the semi-norm $q(\cdot)$ in (\widehat{K}) is $\pi(|\cdot|)$. Then we have (K), and Lemma 5.2 gives $\lambda(t) = 1 - \frac{1}{2}\langle \Gamma t, t \rangle + o(\|t\|^2)$ for t near 0, where $\lambda(t)$ denotes the dominating eigenvalue of $Q(t)$. Hence we have $r(Q(t)) = |\lambda(t)| < 1$ for t near 0, $t \neq 0$. This proves that 0 is an isolated point in G , hence G is discrete. If $q(\cdot) \neq \pi(|\cdot|)$, observe that Hypothesis (\widehat{K}) ensures that the conclusions of Theorem (K-L) are valid with here $q(\Pi(t)f - \Pi f) \rightarrow 0$ when $t \rightarrow 0$ for each $f \in \mathcal{B}$. However, since $\pi(|\cdot|) \leq c q(\cdot)$ by hypothesis, the last property also holds with $\pi(|\cdot|)$ instead of $q(\cdot)$, so Lemma 5.2 remains valid, and we can proceed as above.

Proof of (i). Set $G = \mathbb{Z}a_1 \oplus \dots \mathbb{Z}a_p$ with $p \leq d$, and let λ_k, A_k , and w_k be the elements associated with a_k in (*). Then $A = \bigcap_{k=1}^p A_k$ is a π -full Q -absorbing set, and if $x \in A$ and $g = n_1 a_1 + \dots + n_p a_p$ is any element of G , we deduce from (*) applied to each a_k , and by product that :

$$\forall x \in A, \quad e^{i\langle g, \xi(y) \rangle} \prod_{k=1}^p w_k(y)^{n_k} = \prod_{k=1}^p \lambda_k^{n_k} \prod_{k=1}^p w_k(x)^{n_k} \quad Q(x, dy) - a.s..$$

Since $|w_k|$ is a nonzero constant function on A , one may assume without loss of generality that $|w_k|_A = 1_A$, so that there exists a measurable function $\alpha_k : E \rightarrow [0, 2\pi[$ such that we have, for all $z \in A$: $w_k(z) = e^{i\alpha_k(z)}$. For $z \in A$, we set $V(z) = (\alpha_1(z), \dots, \alpha_p(z))$ in \mathbb{R}^p . Since the linear map $\chi : h \mapsto (\langle a_1, h \rangle, \dots, \langle a_p, h \rangle)$ is clearly bijective from $\text{vect}(G)$ into \mathbb{R}^p , one can define the element $\chi^{-1}(V(z))$ which satisfies $\langle a_k, \chi^{-1}(V(z)) \rangle = \alpha_k(z)$ for each $k = 1, \dots, p$. Finally let $\theta : E \rightarrow \mathbb{R}^d$ be a bounded measurable function such that $\theta(z) = \chi^{-1}(V(z))$ for all $z \in A$. Then we have $w_k(z) = e^{i\langle a_k, \theta(z) \rangle}$ for any $z \in A$ and $k = 1, \dots, p$. Consequently one gets $\prod_{k=1}^p w_k(z)^{n_k} = e^{i\langle g, \theta(z) \rangle}$ for $z \in A$, and the above equality becomes, by setting $\lambda_g = \prod_{k=1}^p \lambda_k^{n_k}$,

$$\forall x \in A, \quad e^{i\langle g, \xi(y) + \theta(y) - \theta(x) \rangle} = \lambda_g \quad Q(x, dy) - a.s..$$

For any $g \in G$, let us define $\beta_g \in \mathbb{R}$ such that $\lambda_g = e^{i\beta_g}$, and for $x \in \mathbb{R}^d$, set $T_g(x) = \langle g, x \rangle$. The previous property yields

$$\forall x \in A, \quad \xi(y) + \theta(y) - \theta(x) \in \bigcap_{g \in G} T_g^{-1}(\beta_g + 2\pi\mathbb{Z}) \quad Q(x, dy) - a.s..$$

Now let us define $H = \bigcap_{g \in G} T_g^{-1}(2\pi\mathbb{Z})$. Then H is a subgroup of \mathbb{R}^d , and the elements of $\bigcap_{g \in G} (T_g^{-1}(\beta_g + 2\pi\mathbb{Z}))$ are in the same class modulo H . That is :

$$\exists a \in \mathbb{R}^d, \cap_{g \in G} (T_g^{-1}(\beta_g + 2\pi\mathbb{Z})) \subset a + H.$$

This proves (**), and it remains to establish that H has the stated form. Actually, since H is closed, H is of the form $H = F \oplus \Delta$, where F and Δ are respectively a subspace and a discrete subgroup in \mathbb{R}^d . So we have to prove that $F = (\text{vect } G)^\perp$.

Let $x \in (\text{vect } G)^\perp$. Since $(\text{vect } G)^\perp = \cap_{g \in G} T_g^{-1}(\{0\}) \subset H$, we have $x = f + d$ for some $f \in F$, $d \in \Delta$, and for $\alpha \in \mathbb{R}$, the fact that $\alpha x \in (\text{vect } G)^\perp \subset H$ yields $\alpha x = f_\alpha + d_\alpha$ with some $f_\alpha \in F$ and $d_\alpha \in \Delta$. But we also have the unique decomposition $\alpha x = \alpha f + \alpha d$ in $F \oplus \text{vect } \Delta$. Hence we have $\alpha d = d_\alpha \in \Delta$, and since Δ is discrete and α can take any real value, we have necessary $d = 0$. That is, $x \in F$.

Conversely, let $f \in F$ and let $g \in G$. Since $F \subset H$, we have $\langle g, f \rangle \in 2\pi\mathbb{Z}$. Now let α be any fixed nonzero irrational number. Since $\alpha f \in F \subset H$, we have $\alpha \langle g, f \rangle = \langle g, \alpha f \rangle \in 2\pi\mathbb{Z}$. Hence $\langle g, f \rangle = 0$. This gives $f \in (\text{vect } G)^\perp$.

Proof of (ii). Let $t \in H^\perp$, $t \neq 0$. Then $\langle t, \xi(y) \rangle + \langle t, \theta(y) \rangle - \langle t, \theta(x) \rangle = \langle t, a \rangle \quad Q(x, dy)$ -a.s. for all $x \in A$. Setting $w(\cdot) = e^{i\langle t, \theta(\cdot) \rangle}$ and $\lambda = e^{i\langle t, a \rangle}$, this yields for all $x \in A$

$$e^{i\langle t, \xi(y) \rangle} w(y) = \lambda w(x) \quad Q(x, dy) - a.s..$$

Since $w \in \mathcal{B}$ by hypothesis, this gives (*), and Lemmas 12.1-2 implies that $t \in G$. □

Appendix A. Proof of Proposition 7.1.

The proof of Proposition 7.1 is presented in the case $d = 1$. The extension to $d \geq 2$ is obvious by the use of the partial derivatives. Let I be an interval in \mathbb{R} and let $(\mathcal{B}_\theta, \theta \in I)$ be a family of function spaces. Recall that we set $\mathcal{D}_\kappa = \{z \in \mathbb{C} : |z| \geq \kappa, |z - 1| \geq (1 - \kappa)/2\}$ for any $\kappa \in (0, 1)$. Given two function spaces \mathcal{B} and $\tilde{\mathcal{B}}$ of the previous family, let us introduce the following hypothesis.

Hypothesis $\mathcal{D}(m)$ ($m \in \mathbb{N}^*$). For all $\theta \in I$ there exists a neighbourhood \mathcal{V}_θ of 0 in \mathbb{R}^d such that, for all $j = 1, \dots, m$, we have :

$$(0) \quad [T_0(\theta) \in I \Rightarrow \mathcal{B}_\theta \hookrightarrow \mathcal{B}_{T_0(\theta)}] \quad \text{and} \quad [T_1(\theta) \in I \Rightarrow \mathcal{B}_\theta \hookrightarrow \mathcal{B}_{T_1(\theta)}]$$

$$(1) \quad T_0(\theta) \in I \text{ implies that } Q(\cdot) \in \mathcal{C}^0(\mathcal{V}_\theta, \mathcal{B}_\theta, \mathcal{B}_{T_0(\theta)})$$

$$(2) \quad \theta_j := T_1(T_0 T_1)^{j-1}(\theta) \in I \text{ implies that } Q(\cdot) \in \mathcal{C}^j(\mathcal{V}_\theta, \mathcal{B}_\theta, \mathcal{B}_{\theta_j})$$

(3') There exists a real number $\kappa_\theta \in (0, 1)$ such that, for all $\kappa \in [\kappa_\theta, 1)$, there exists a neighbourhood $\mathcal{V}_{\theta, \kappa} \subseteq \mathcal{V}_\theta$ of 0 in \mathbb{R}^d such that, $R_z(t) := (z - Q(t))^{-1} \in \mathcal{L}(\mathcal{B}_\theta)$ for all $z \in \mathcal{D}_\kappa$ and all $t \in \mathcal{V}_{\theta, \kappa}$, and we have $M_{\theta, \kappa} := \sup \left\{ \|R_z(t)\|_\theta, t \in \mathcal{V}_{\theta, \kappa}, z \in \mathcal{D}_\kappa \right\} < +\infty$

(4) There exists $a \in \bigcap_{k=0}^m [T_0^{-1}(T_0 T_1)^{-k}(I) \cap (T_1 T_0)^{-k}(I)]$ such that we have $\mathcal{B} = \mathcal{B}_a$ and $\tilde{\mathcal{B}} = \mathcal{B}_{(T_0 T_1)^m T_0(a)}$.

Conditions (0) (1) (2) and (4) are exactly those of $\mathcal{C}(m)$. According to Theorem (K-L) of section 4.1, (3) implies (3'). Hence $\mathcal{C}(m)$ implies $\mathcal{D}(m)$, and Proposition 7.1 will be a consequence of the next proposition. Let us notice that, from (4), we have

$$\Theta_a = \{a, T_0 a, T_1 T_0 a, T_0 T_1 T_0 a, \dots, (T_0 T_1)^m T_0(a)\} \subset I.$$

Let us define $\kappa = \max_{\theta \in \Theta_a} \kappa_\theta \in (0, 1)$, and $\mathcal{O} = \bigcap_{\theta \in \Theta_a} \mathcal{V}_{\theta, \kappa}$.

Proposition A. Under Hypothesis $\mathcal{D}(m)$, if κ and \mathcal{O} are as stated before, then we have $R_z(\cdot) \in \mathcal{C}^m(\mathcal{O}, \mathcal{B}, \tilde{\mathcal{B}})$ for all $z \in \mathcal{D}_\kappa$, and $\sup\{\|R_z^{(\ell)}(t)\|_{\mathcal{B}, \tilde{\mathcal{B}}}, z \in \mathcal{D}_\kappa, t \in \mathcal{O}\} < +\infty$ for each $\ell = 0, \dots, m$.

We shall write $\|\cdot\|_{\theta, \theta'}$ for $\|\cdot\|_{\mathcal{B}_\theta, \mathcal{B}_{\theta'}}$ and $\|\cdot\|_\theta$ for $\|\cdot\|_{\mathcal{B}_\theta}$. Moreover, $f = (f_z(t))_{z,t} \in \mathcal{C}^\ell(\theta, \theta')$ will mean that there exists $\tilde{\kappa} \in (0, 1)$ such that, for all $\kappa \in [\tilde{\kappa}, 1)$, there exists a neighbourhood \mathcal{U}_κ of 0 in \mathbb{R}^d and we have :

$$\forall z \in \mathcal{D}_\kappa, f_z(\cdot) \in \mathcal{C}^\ell(\mathcal{U}_\kappa, \mathcal{B}_\theta, \mathcal{B}_{\theta'}) \quad \text{and} \quad \sup_{z \in \mathcal{D}_\kappa, t \in \mathcal{U}_\kappa, j=0, \dots, \ell} \|f_z^{(j)}(t)\|_{\theta, \theta'} < +\infty.$$

By convention, we set $(UV)_z(t) := U_z(t)V_z(t)$, and when $U = (U_z(t))_{z,t} \in \mathcal{C}^\ell(\theta, \theta')$, we set $U^{(k)} = (U_z^{(k)}(t))_{z,t}$. We shall use repeatedly the next obvious results.

Lemma. Let $\theta_1, \theta_2, \theta_3$ and θ_4 be in I .

(i) If $V \in \mathcal{C}^k(\theta_2, \theta_3)$ and if $\mathcal{B}_{\theta_1} \hookrightarrow \mathcal{B}_{\theta_2}$ and if $\mathcal{B}_{\theta_3} \hookrightarrow \mathcal{B}_{\theta_4}$, then $V \in \mathcal{C}^k(\theta_2, \theta_4)$, $V \in \mathcal{C}^k(\theta_1, \theta_3)$, and $V \in \mathcal{C}^k(\theta_1, \theta_4)$.

(ii) If $V \in \mathcal{C}^0(\theta_1, \theta_2)$, $U \in \mathcal{C}^0(\theta_2, \theta_3)$, then $UV \in \mathcal{C}^0(\theta_1, \theta_3)$.

(iii) If $V \in \mathcal{C}^0(\theta_1, \theta_2)$, $U \in \mathcal{C}^1(\theta_2, \theta_4)$, and $V \in \mathcal{C}^1(\theta_1, \theta_3)$, $U \in \mathcal{C}^0(\theta_3, \theta_4)$, then we have $UV \in \mathcal{C}^1(\theta_1, \theta_4)$, and $(UV)' = U'V + UV'$.

Proof of Proposition A. Lemmas 0-1 below will be our basic statements.

Lemma 0. *If $\theta, T_0(\theta) \in I$, then $(R_z(t))_{z,t} \in \mathcal{C}^0(\theta, T_0(\theta))$.*

Proof. Let $\kappa \in [\max(\kappa_\theta, \kappa_{T_0(\theta)}), 1)$. Let $\mathcal{U}_{\theta, \kappa}^{(0)} = \mathcal{V}_{\theta, \kappa} \cap \mathcal{V}_{T_0(\theta), \kappa}$. From the usual operator formula $Id - W^{n+1} = (Id - W) \sum_{k=0}^n W^k$, one easily deduces the following equality, which is valid for any bounded linear operators S and T on a Banach space such that S and $S - T$ are invertible :

$$(*) \quad (S - T)^{-1} = \sum_{k=0}^n (S^{-1}T)^k S^{-1} + (S^{-1}T)^{n+1} (S - T)^{-1}.$$

With $n = 0$, $S = z - Q(t_0)$, $T = Q(t) - Q(t_0)$, thus $S - T = z - Q(t)$, Formula (*) yields

$$\forall z \in \mathcal{D}_\kappa, \forall t \in \mathcal{U}_{\theta, \kappa}^{(0)}, \quad R_z(t) - R_z(t_0) = R_z(t_0) (Q(t) - Q(t_0)) R_z(t).$$

Using the constants $M_{\theta, \kappa}$ and $M_{T_0(\theta), \kappa}$, Condition (1) in $\mathcal{D}(m)$ gives the desired property. \square

Lemma 1. *If $\theta, T_0(\theta), T_1 T_0(\theta), T_0 T_1 T_0(\theta) \in I$, then we have $(R_z(t))_{z,t} \in \mathcal{C}^1(\theta, T_0 T_1 T_0(\theta))$ and $R' = R Q' R$.*

Proof. Let us define $\theta_1 = T_0(\theta)$, $\theta_2 = T_1 T_0(\theta)$, $\theta_3 = T_0 T_1 T_0(\theta)$ and $\kappa_\theta^{(1)} = \max(\kappa_\theta, \kappa_{\theta_1}, \kappa_{\theta_2}, \kappa_{\theta_3})$. Let us consider a real number $\kappa \in [\kappa_\theta^{(1)}, 1)$. We define $\mathcal{U}_{\theta, \kappa}^{(1)} = \mathcal{U}_{\theta, \kappa}^{(0)} \cap \mathcal{U}_{\theta_2, \kappa}^{(0)}$. Let $t_0, t \in \mathcal{U}_{\theta, \kappa}^{(1)}$ and $z \in \mathcal{D}_\kappa$. Formula (*) with $n = 1$, $S = z - Q(t_0)$, $T = Q(t) - Q(t_0)$ gives

$$R_z(t) = R_z(t_0) + R_z(t_0) [Q(t) - Q(t_0)] R_z(t_0) + \vartheta_z(t),$$

with $\vartheta_z(t) := R_z(t_0) [Q(t) - Q(t_0)] R_z(t_0) [Q(t) - Q(t_0)] R_z(t)$. But we have :

$$\frac{\|\vartheta_z(t)\|_{\theta, \theta_3}}{|t - t_0|} \leq \|R_z(t)\|_{\theta_3} \|Q(t) - Q(t_0)\|_{\theta_2, \theta_3} \|R_z(t_0)\|_{\theta_2} \frac{\|Q(t) - Q(t_0)\|_{\theta_1, \theta_2}}{|t - t_0|} \|R_z(t)\|_{\theta, \theta_1}$$

which goes to 0 as t goes to t_0 , uniformly in $z \in \mathcal{D}_\kappa$ (according to condition (2) and with the use of $M_{\theta_3, \kappa}$, $M_{\theta_2, \kappa}$ and $M_{\theta, \kappa}$). In the same way, we have :

$$\begin{aligned} & \|R_z(t_0)(Q(t) - Q(t_0))R_z(t_0) - (t - t_0)R_z(t_0)Q'(t_0)R_z(t_0)\|_{\theta, \theta_3} \\ & \leq M_{\theta_2, \kappa} \|Q(t) - Q(t_0) - (t - t_0)Q'(t_0)\|_{\theta_1, \theta_2} M_{\theta, \kappa} = o(t - t_0). \end{aligned}$$

This shows that $R'_z(t_0) = R_z(t_0)Q'(t_0)R_z(t_0)$ in $\mathcal{L}(\mathcal{B}_\theta, \mathcal{B}_{\theta_3})$. Moreover, $(R_z(t))_{z,t} \in \mathcal{C}^0(\theta, \theta_1)$, $(Q'(t))_{z,t} \in \mathcal{C}^0(\theta_1, \theta_2)$, and $(R_z(t))_{z,t} \in \mathcal{C}^0(\theta_2, \theta_3)$, so $(R'_z(t))_{z,t} \in \mathcal{C}^0(\theta, \theta_3)$. \square

By Lemma 0, the following assertion holds :

(H_0) If $\theta \in I$ and if $T_0(\theta) \in I$, then $R = (R_z(t))_{z,t} \in \mathcal{C}^0(\theta, T_0(\theta))$.

For $\ell = 1, \dots, m$, let us denote by (H_ℓ) the following assertion :

(H_ℓ) If $\theta \in \bigcap_{k=0}^{\ell} [T_0^{-1}(T_0 T_1)^{-k}(I) \cap (T_1 T_0)^{-k}(I)]$, then $R = (R_z(t))_{z,t} \in \mathcal{C}^\ell(\theta, (T_0 T_1)^\ell T_0(\theta))$ and $R^{(\ell)} = \sum_{(i,j,k) \in \mathcal{E}_\ell} R^{(i)} Q^{(1+j)} R^{(k)}$, with

$$\mathcal{E}_\ell = \{(i, j, k) \in \mathbb{Z}^3 : i \geq 0, j \geq 0, k \geq 0, i + j + k = \ell - 1\}.$$

We want to prove (H_m) by induction. By Lemma 1, (H_1) holds.

Lemma 2. *Let $1 \leq \ell \leq m - 1$. If (H_0), (H_1), \dots , (H_ℓ) hold, then we have ($H_{\ell+1}$).*

Proof. Let $\theta \in \bigcap_{k=0}^{\ell+1} [T_0^{-1}(T_0T_1)^{-k}(I) \cap (T_1T_0)^{-k}(I)]$. From $\mathcal{B}_{(T_0T_1)^\ell T_0(\theta)} \hookrightarrow \mathcal{B}_{(T_0T_1)^{\ell+1}T_0(\theta)}$ and (H_ℓ) , we have $R = (R_z(t))_{z,t} \in \mathcal{C}^\ell(\theta, (T_0T_1)^{\ell+1}T_0(\theta))$ and : $R^{(\ell)} = \sum_{(i,j,k) \in \mathcal{E}_\ell} R^{(i)}Q^{(1+j)}R^{(k)}$.

Let $(i, j, k) \in \mathcal{E}_\ell$. We have to prove that $R^{(i)}Q^{(1+j)}R^{(k)} \in \mathcal{C}^1(\theta, (T_0T_1)^{\ell+1}T_0(\theta))$ and that :

$$\left(R^{(i)}Q^{(1+j)}R^{(k)}\right)' = R^{(i+1)}Q^{(1+j)}R^{(k)} + R^{(i)}Q^{(2+j)}R^{(k)} + R^{(i)}Q^{(1+j)}R^{(k+1)}.$$

Since $1 \leq k+1 \leq \ell$ and by induction hypothesis, we have :

$$R^{(k)} \in \mathcal{C}^1(\theta, (T_0T_1)^{(k+1)}T_0(\theta)) \quad \text{and} \quad R^{(k)} \in \mathcal{C}^0(\theta, (T_0T_1)^kT_0(\theta)).$$

Moreover, since $2+j \leq \ell+1 \leq m$ and according to $\mathcal{D}(m)$, we have :

$$Q^{(1+j)} \in \mathcal{C}^0\left((T_0T_1)^{(k+1)}T_0(\theta), T_1(T_0T_1)^{(k+j+1)}T_0(\theta)\right)$$

$$\text{and} \quad Q^{(1+j)} \in \mathcal{C}^1\left((T_0T_1)^kT_0(\theta), T_1(T_0T_1)^{k+j+1}T_0(\theta)\right).$$

According to the first lemma of this appendix, $Q^{(1+j)}R^{(k)} \in \mathcal{C}^1(\theta, T_1(T_0T_1)^{k+j+1}T_0(\theta))$ and $(Q^{(1+j)}R^{(k)})' = Q^{(2+j)}R^{(k)} + Q^{(1+j)}R^{(k+1)}$.

Analogously we have $Q^{(1+j)}R^{(k)} \in \mathcal{C}^0(\theta, T_1(T_0T_1)^{k+j}T_0(\theta))$, and, since $i+1 \leq \ell$, we have :

$$R^{(i)} \in \mathcal{C}^0\left(T_1(T_0T_1)^{k+j+1}T_0(\theta), (T_0T_1)^{k+j+i+2}T_0(\theta)\right)$$

$$\text{and} \quad R^{(i)} \in \mathcal{C}^1\left(T_1(T_0T_1)^{k+j}T_0(\theta), (T_0T_1)^{k+j+i+2}T_0(\theta)\right).$$

Since $k+j+i+2 = \ell+1$, this gives the desired property. \square

Since, by hypothesis, $a \in \bigcap_{k=0}^m [T_0^{-1}(T_0T_1)^{-k}(I) \cap (T_1T_0)^{-k}(I)]$, the property (H_m) shows that the conclusion of Proposition A is valid with some neighbourhood \mathcal{O} of $t=0$ and some set \mathcal{D}_κ with suitable $\kappa \in (0, 1)$. More exactly, the previous induction proves that \mathcal{O} and κ may be defined as stated before Proposition A. \square

Appendix B. Proof of Propositions 11.4-8.

B.0. Notations. For convenience we present the proofs of Propositions 11.4-8 in the case $d = 1$. The extension to $d \geq 2$ is straightforward for Proposition 11.4,5,6,8 (just replace the inequality $|t\xi(x)| \leq |t| |\xi(x)|$ with the Schwarz inequality $|\langle t, \xi(x) \rangle| \leq \|t\| \|\xi(x)\|$). It is easy for Proposition 11.7 by considering partial derivatives.

We set $\Theta x = F(x, \theta_1)$. So Θ is a random Lipschitz transformation on E , and the transition probability Q can be expressed as : $Qf(x) = \mathbb{E}[f(\Theta x)]$.

For any $\lambda \in (0, 1]$, we set $p_\lambda(x) = 1 + \lambda d(x, x_0)$. Obviously, the space $\mathcal{B}_{\alpha, \gamma}$ is unchanged when $m_{\alpha, \gamma}(f)$ is replaced with

$$m_{\alpha, \gamma}^{(\lambda)}(f) = \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)^\alpha p_\lambda(x)^{\alpha\gamma} p_\lambda(y)^{\alpha\gamma}}, x, y \in E, x \neq y \right\},$$

and, setting

$$|f|_{\alpha, \gamma}^{(\lambda)} = \sup_{x \in E} \frac{|f(x)|}{p_\lambda(x)^{\alpha(\gamma+1)}},$$

the resulting new norm $\|f\|_{\alpha, \gamma}^{(\lambda)} = m_{\alpha, \gamma}^{(\lambda)}(f) + |f|_{\alpha, \gamma}^{(\lambda)}$ is equivalent to $\|\cdot\|_{\alpha, \gamma}$. Consequently, for (α, γ) fixed, Propositions 11.4-8 can be established with $\|f\|_{\alpha, \gamma}^{(\lambda)}$ for any value $\lambda \in (0, 1]$. In most of the next estimates, we shall assume $\lambda = 1$, the possibility of choosing suitable small λ will occur in the proof of (K3) (Prop. 11.4) and $\overline{K3}$ (Prop. 11.8). Anyway, this already appears in the proof of Proposition 11.2, see [?].

Let $\mathcal{C}_\lambda = \max\{\mathcal{C}, 1\} + \lambda d(\Theta x_0, x_0)$. In the sequel, we shall use repeatedly the fact that $p_\lambda(\cdot)$ and $p(\cdot)$ are equivalent functions, and that (see [?] p. 1945)

$$\sup_{x \in E} \frac{p_\lambda(\Theta x)}{p_\lambda(x)} \leq \mathcal{C}_\lambda \leq \mathcal{M}.$$

B.1. A preliminary lemma. The proofs of Propositions 11.4-8 will be based on the following lemma.

Lemma B.1. *Let $q : E \rightarrow \mathbb{C}$ measurable, $Kf(x) = \mathbb{E}[q(\Theta x) f(\Theta x)]$, and let $\lambda \in (0, 1]$. Suppose that there exist constants a, A, b, B such that we have for all $x, y \in E$ satisfying $d(y, x_0) \leq d(x, x_0)$*

$$(i) \quad |q(x)| \leq A p_\lambda(x)^a \quad ; \quad (ii) \quad |q(x) - q(y)| \leq B d(x, y)^\alpha p_\lambda(x)^b.$$

Then we have for $f \in \mathcal{B}_{\alpha, \gamma}$ and x, y as above stated

$$\begin{aligned} |Kf(x)| &\leq A |f|_{\alpha, \gamma}^{(\lambda)} p_\lambda(x)^{a+\alpha(\gamma+1)} \mathbb{E}[\mathcal{M}^{a+\alpha(\gamma+1)}] \\ |Kf(x) - Kf(y)| &\leq A m_{\alpha, \gamma}^{(\lambda)}(f) d(x, y)^\alpha p_\lambda(x)^{a+\alpha\gamma} p_\lambda(y)^{\alpha\gamma} \mathbb{E}[\mathcal{C}^\alpha \mathcal{C}_\lambda^{a+2\alpha\gamma}] \\ &\quad + B |f|_{\alpha, \gamma}^{(\lambda)} d(x, y)^\alpha p_\lambda(x)^b p_\lambda(y)^{\alpha(\gamma+1)} \mathbb{E}[\mathcal{C}^\alpha \mathcal{M}^{b+\alpha(\gamma+1)}]. \end{aligned}$$

Proof. We have

$$\begin{aligned} |Kf(x)| \leq \mathbb{E}[|q(\Theta x) f(\Theta x)|] &\leq A |f|_{\alpha, \gamma}^{(\lambda)} \mathbb{E}[p_\lambda(\Theta x)^a p_\lambda(\Theta x)^{\alpha(\gamma+1)}] \\ &\leq A |f|_{\alpha, \gamma}^{(\lambda)} p_\lambda(x)^{a+\alpha(\gamma+1)} \mathbb{E}[\mathcal{M}^{a+\alpha(\gamma+1)}]. \end{aligned}$$

Moreover, for $x, y \in E$ satisfying $d(y, x_0) \leq d(x, x_0)$ (thus $p_\lambda(y) \leq p_\lambda(x)$), we have

$$\begin{aligned}
|Kf(x) - Kf(y)| &\leq \mathbb{E} \left[q(\Theta x) |f(\Theta x) - f(\Theta y)| \right] + \mathbb{E} \left[|f(\Theta y)| |q(\Theta x) - q(\Theta y)| \right] \\
&\leq A m_{\alpha, \gamma}^{(\lambda)}(f) \mathbb{E} \left[p_\lambda(\Theta x)^a d(\Theta x, \Theta y)^\alpha p_\lambda(\Theta x)^{\alpha \gamma} p_\lambda(\Theta y)^{\alpha \gamma} \right] \\
&\quad + |f|_{\alpha, \gamma}^{(\lambda)} B \mathbb{E} \left[p(\Theta y)^{\alpha(\gamma+1)} d(\Theta x, \Theta y)^\alpha p_\lambda(\Theta x)^b \right] \\
&\leq A m_{\alpha, \gamma}^{(\lambda)}(f) d(x, y)^\alpha p_\lambda(x)^{a+\alpha \gamma} p_\lambda(y)^{\alpha \gamma} \mathbb{E} \left[\mathcal{C}^\alpha \mathcal{C}_\lambda^{a+2\alpha \gamma} \right] \\
&\quad + B |f|_{\alpha, \gamma}^{(\lambda)} d(x, y)^\alpha p_\lambda(x)^b p_\lambda(y)^{\alpha(\gamma+1)} \mathbb{E} \left[\mathcal{C}^\alpha \mathcal{M}^{b+\alpha(\gamma+1)} \right]. \quad \square
\end{aligned}$$

For the use of Lemma B.1, it is worth noticing that the supremum defining the Hölder constants $m_{\alpha, \gamma}(f)$ or $m_{\alpha, \gamma}^{(\lambda)}(f)$ can be obviously computed over the elements $x, y \in E$ such that $d(y, x_0) \leq d(x, x_0)$. Lemma B.1 will be applied below with $q(\cdot)$ depending on the function ξ . Remember that ξ verifies the following hypothesis :

$$(\mathbf{L})_s \quad \forall (x, y) \in E \times E, \quad |\xi(x) - \xi(y)| \leq S d(x, y) [1 + d(x, x_0) + d(y, x_0)]^s.$$

From $(\mathbf{L})_s$, it follows that there exists $C > 0$ such that we have for $x \in E$

$$|\xi(x)| \leq C p(x)^{s+1},$$

and for $x, y \in E$ satisfying $d(y, x_0) \leq d(x, x_0)$:

$$|\xi(x) - \xi(y)| \leq C d(x, y) p(x)^s \quad \text{and} \quad |\xi(x) - \xi(y)| \leq C d(x, y)^\alpha p(x)^{s+1-\alpha}.$$

B.2. Proof of Proposition 11.4. *This proposition states that (K) of section 4.1 holds w.r.t. $\mathcal{B}_{\alpha, \gamma}$ if $\gamma \geq 1 + s$ and $I = \mathbb{E}[\mathcal{M}^{\alpha(\gamma+1)} + \mathcal{C}^\alpha \mathcal{M}^{2\alpha \gamma}] < +\infty$, $\mathbb{E}[\mathcal{C}^\alpha \max\{\mathcal{C}, 1\}^{2\alpha \gamma}] < 1$.*

The strong ergodicity condition (K1) of section 2 holds by Proposition 11.2. Besides we have for $f \in \mathcal{B}_{\alpha, \gamma}$

$$\pi(|e^{i\langle t, \xi \rangle} - 1| |f|) \leq |f|_{\alpha, \gamma} \pi(|e^{i\langle t, \xi \rangle} - 1| p^{\alpha(\gamma+1)}).$$

Since $\pi(p^{\alpha(\gamma+1)}) < +\infty$ (Prop. 11.1), The continuity condition (K2') of Section 4.1 follows from Lebesgue Theorem. To study the Doeblin-Fortet inequalities (K3), notice that $Q(t) = K$ where K is associated to $q(x) = e^{it\xi(x)}$ with the notations of Lemma B.1. By using $(\mathbf{L})_s$ and the inequality $|e^{ia} - 1| \leq 2|a|^\alpha$, one easily gets (i)-(ii) in Lemma B.1 with $A = 1$, $a = 0$ and $B = D_\lambda |t|^\alpha$, $b = \alpha s$, where D_λ is a positive constant resulting from $(\mathbf{L})_s$ and the equivalence between $p_\lambda(\cdot)$ and $p(\cdot)$. Then we have for $f \in \mathcal{B}_{\alpha, \gamma}$ and $x, y \in E$ as stated in Lemma B.1,

$$|Q(t)f|_{\alpha, \gamma}^{(\lambda)} \leq I |f|_{\alpha, \gamma}^{(\lambda)}$$

$$\frac{|Q(t)f(x) - Q(t)f(y)|}{d(x, y)^\alpha p_\lambda(x)^{\alpha \gamma} p_\lambda(y)^{\alpha \gamma}} \leq \mathbb{E}[\mathcal{C}^\alpha \mathcal{C}_\lambda^{2\alpha \gamma}] m_{\alpha, \gamma}^{(\lambda)}(f) + I D_\lambda |t|^\alpha |f|_{\alpha, \gamma}^{(\lambda)} p_\lambda(x)^{\alpha s - \alpha \gamma} p_\lambda(y)^\alpha.$$

We have $p_\lambda(x)^{\alpha s - \alpha \gamma} p_\lambda(y)^\alpha \leq p_\lambda(x)^{\alpha(s+1-\gamma)} \leq 1$ because $p_\lambda(y) \leq p_\lambda(x)$, and $s+1 \leq \gamma$. Set $E_\lambda = I D_\lambda$. Then the previous estimates prove that $Q(t)$ continuously acts on $\mathcal{B}_{\alpha, \gamma}$, and that

$$m_{\alpha, \gamma}^{(\lambda)}(Q(t)f) \leq \mathbb{E}[\mathcal{C}^\alpha \mathcal{C}_\lambda^{2\alpha \gamma}] m_{\alpha, \gamma}^{(\lambda)}(f) + E_\lambda |t|^\alpha |f|_{\alpha, \gamma}^{(\lambda)}.$$

Now, using the fact that the norms $\|f\|_{\alpha,\gamma}^{(\lambda)}$ and $\|f\| = m_{\alpha,\gamma}^{(\lambda)}(f) + \pi(|f|)$ are equivalent [?] (Prop. 5.2), one obtains with some new constant E'_λ :

$$\begin{aligned} m_{\alpha,\gamma}^{(\lambda)}(Q(t)f) &\leq \mathbb{E}[\mathcal{C}^\alpha \mathcal{C}_\lambda^{2\alpha\gamma}] m_{\alpha,\gamma}^{(\lambda)}(f) + E'_\lambda |t|^\alpha \left(m_{\alpha,\gamma}^{(\lambda)}(f) + \pi(|f|) \right) \\ &\leq \left(\mathbb{E}[\mathcal{C}^\alpha \mathcal{C}_\lambda^{2\alpha\gamma}] + E'_\lambda |t|^\alpha \right) m_{\alpha,\gamma}^{(\lambda)}(f) + E'_\lambda |t|^\alpha \pi(|f|). \end{aligned}$$

Since $\mathcal{C}_\lambda \leq \mathcal{M}$ and $\mathcal{C}_\lambda \rightarrow \max\{\mathcal{C}, 1\}$ when $\lambda \rightarrow 0$, it follows from Lebesgue theorem that one can choose λ such that $\mathbb{E}[\mathcal{C}^\alpha \mathcal{C}_\lambda^{2\alpha\gamma}] < 1$. Now let $\tau > 0$ such that $\kappa := \mathbb{E}[\mathcal{C}^\alpha \mathcal{C}_\lambda^{2\alpha\gamma}] + E'_\lambda \tau^\alpha < 1$. Then, if $|t| \leq \tau$, we have

$$m_{\alpha,\gamma}^{(\lambda)}(Q(t)f) \leq \kappa m_{\alpha,\gamma}^{(\lambda)}(f) + E'_\lambda \tau^\alpha \pi(|f|).$$

Since $\pi(|Q(t)f|) \leq \pi(|Qf|) = \pi(|f|)$, this gives $\|Q(t)f\| \leq \kappa \|f\| + (1 + E'_\lambda \tau^\alpha) \pi(|f|)$, and this easily leads to (K3), with $\mathcal{O} = (-\tau, \tau)$.

Finally, since the canonical embedding from $(\mathcal{B}_{\alpha,\gamma}, \|\cdot\|_{\alpha,\gamma})$ into $(\mathcal{B}_{\alpha,\gamma}, \pi(\cdot))$ is compact (this easily follows from Ascoli and Lebesgue theorem, see [?] Lemma 5.4), Property (K4) is then a consequence of (K3) and [?] [?]. \square

In the next proofs, except for Proposition 11.8, the technical parameter λ used above will be neglected, namely we shall assume $\lambda = 1$, and the effective computation of the constants occuring in the proofs will be of no relevance. So, to simplify the next estimates, we shall still denote by C the constant in the above inequalities resulting from $(\mathbf{L})_s$, even if it is slightly altered through the computations (the effective constants will actually depend on parameters as $\alpha, t_0 \in \mathbb{R}$ fixed, $k \in \mathbb{N}$ fixed, $s, S \dots$).

Proposition 11.5 will follow from Lemma B.4 with $k = 0$.

B.3. Proof of Proposition 11.6. *This proposition states that $\|Q(t) - Q\|_{\mathcal{B}_{\alpha,\gamma}, \mathcal{B}_{\alpha,\gamma'}} = O(|t|)$ if $\gamma > 0$, $\gamma' \geq \gamma + \frac{s+1}{\alpha}$, and $I = \mathbb{E}[\mathcal{M}^{\alpha(\gamma'+1)} + \mathcal{C}^\alpha \mathcal{M}^{\alpha(\gamma'+\gamma)} + \mathcal{C}^\alpha \mathcal{M}^{\alpha\gamma'}] < +\infty$.*

Let $K = Q(t) - Q(0)$. Then K is associated to $q(x) = e^{it\xi(x)} - 1$. Using $(\mathbf{L})_s$ and the inequality $|e^{ia} - 1| \leq |a|$, one easily gets (i)-(ii) in Lemma B.1 with $A = C|t|$, $a = s+1$, and $B = C|t|$ and $b = s+1 - \alpha$. So

$$|Kf(x)| \leq IC |t| |f|_{\alpha,\gamma} p(x)^{s+1+\alpha(\gamma+1)},$$

and, by using the fact that $p(y) \leq p(x)$,

$$\begin{aligned} |Kf(x) - Kf(y)| &\leq IC |t| m_{\alpha,\gamma}(f) d(x,y)^\alpha p(x)^{s+1+\alpha\gamma} p(y)^{\alpha\gamma} \\ &\quad + IC |t| |f|_{\alpha,\gamma} d(x,y)^\alpha p(x)^{s+1-\alpha} p(y)^{\alpha\gamma} p(x)^\alpha \\ &\leq IC |t| \|f\|_{\alpha,\gamma} d(x,y)^\alpha p(x)^{\alpha\gamma'} p(y)^{\alpha\gamma'}. \end{aligned} \quad \square$$

B.4. Proof of Proposition 11.7. *This proposition states that $\mathcal{C}(m)$ ($m \in \mathbb{N}^*$) holds with $\mathcal{B} = \mathcal{B}_{\alpha,\eta}$ and $\tilde{\mathcal{B}} = \mathcal{B}_{\alpha,\eta'}$ if $\eta \geq 1 + s$, $\eta' > \eta + \frac{m(s+1)}{\alpha}$, $\mathbb{E}[\mathcal{M}^{\alpha(\eta'+1)} + \mathcal{C}^\alpha \mathcal{M}^{2\alpha\eta'}] < +\infty$ and $\mathbb{E}[\mathcal{C}^\alpha \max\{\mathcal{C}, 1\}^{2\alpha\eta'}] < 1$.*

Let $k \in \mathbb{N}$. Let us recall that we set $Q_k(t)(x, dy) = i^k \xi(y)^k e^{it\xi(y)} Q(x, dy)$ ($x \in E, t \in \mathbb{R}$). For $u \in \mathbb{R}$, we set $e^{iu\xi(\cdot)} = e_u(\cdot)$. Let $t, t_0 \in \mathbb{R}$, $h = t - t_0$. We suppose that $|h| \leq 1$.

Lemma B.4. For $k \in \mathbb{N}$, we have $Q_k \in \mathcal{C}^0(\mathbb{R}, \mathcal{B}_{\alpha, \gamma}, \mathcal{B}_{\alpha, \gamma'})$ if $\gamma \geq 1 + s$, $\gamma' > \gamma + \frac{(s+1)k}{\alpha}$, and $I = \mathbb{E}[\mathcal{M}^{\alpha(\gamma'+1)} + \mathcal{C}^\alpha \mathcal{M}^{\alpha(\gamma'+\gamma)}] < +\infty$.

Proof. Let $K = Q_k(t) - Q_k(t_0)$. Then K is associated to $q(x) = (i\xi(x))^k (e_t(x) - e_{t_0}(x))$. Let $0 < \varepsilon < \alpha$. Using the inequality $|e^{ia} - 1| \leq 2|a|^\varepsilon$, one gets (i) in Lemma B.1 with $A = C|h|^\varepsilon$ and $a = (s+1)(k+\varepsilon)$. Using also $|e^{ia} - 1| \leq 2|a|^\alpha$, we have for $k \geq 1$ and for $x, y \in E$ such that $d(y, x_0) \leq d(x, x_0)$ (thus $p(y) \leq p(x)$):

$$\begin{aligned} |q(x) - q(y)| &\leq |\xi(x)^k - \xi(y)^k| |e_t(x) - e_{t_0}(x)| + |\xi(y)^k| |(e_t(x) - e_{t_0}(x)) - (e_t(y) - e_{t_0}(y))| \\ &\leq C |\xi(x) - \xi(y)| p(x)^{(s+1)(k-1)} |h|^\varepsilon p(x)^{(s+1)\varepsilon} \\ &\quad + C p(x)^{(s+1)k} \left(|e_h(x) - e_h(y)| + |e_h(y) - 1| |e_{t_0}(x) - e_{t_0}(y)| \right) \\ &\leq C |h|^\varepsilon d(x, y)^\alpha p(x)^{s+1-\alpha} p(x)^{(s+1)(k-1+\varepsilon)} \\ &\quad + C p(x)^{(s+1)k} \left(|h|^\alpha d(x, y)^\alpha p(x)^{\alpha s} + |h|^\varepsilon p(x)^{(s+1)\varepsilon} |t_0|^\alpha d(x, y)^\alpha p(x)^{\alpha s} \right) \\ &\leq C |h|^\varepsilon d(x, y)^\alpha p(x)^{(s+1)(k+\varepsilon)-\alpha} + C |h|^\varepsilon d(x, y)^\alpha p(x)^{(s+1)(k+\varepsilon)+\alpha s}. \end{aligned}$$

Hence (ii) in Lemma B.1 holds with $B = C|h|^\varepsilon$ and $b = (s+1)(k+\varepsilon) + \alpha s$. If $k = 0$, the previous computation, which starts from $|q(x) - q(y)| \leq |(e_t(x) - e_{t_0}(x)) - (e_t(y) - e_{t_0}(y))|$, yields the same conclusion. By hypothesis, one can choose ε such that $\gamma' \geq \gamma + \frac{(s+1)(k+\varepsilon)}{\alpha}$, and Lemma B.1 yields for $f \in \mathcal{B}_{\alpha, \gamma}$

$$|Kf(x)| \leq IC |h|^\varepsilon \|f\|_{\alpha, \gamma} p(x)^{(s+1)(k+\varepsilon)+\alpha(\gamma+1)} = IC |h|^\varepsilon \|f\|_{\alpha, \gamma} p(x)^{\alpha(\gamma'+1)},$$

and, by using the fact that $s \leq \gamma - 1$ and $p(y) \leq p(x)$,

$$\begin{aligned} |Kf(x) - Kf(y)| &\leq IC |h|^\varepsilon m_{\alpha, \gamma}(f) d(x, y)^\alpha p(x)^{(s+1)(k+\varepsilon)+\alpha\gamma} p(y)^{\alpha\gamma} \\ &\quad + IC |h|^\varepsilon \|f\|_{\alpha, \gamma} d(x, y)^\alpha p(x)^{(s+1)(k+\varepsilon)+\alpha(\gamma-1)} p(y)^{\alpha\gamma} p(x)^\alpha \\ &\leq IC |h|^\varepsilon \|f\|_{\alpha, \gamma} d(x, y)^\alpha p(x)^{\alpha\gamma'} p(y)^{\alpha\gamma'} \\ &\quad + IC |h|^\varepsilon \|f\|_{\alpha, \gamma} d(x, y)^\alpha p(x)^{\alpha\gamma'} p(y)^{\alpha\gamma'}. \quad \square \end{aligned}$$

Lemma B.4'. For $k \in \mathbb{N}$, we have $Q_k \in \mathcal{C}^1(\mathbb{R}, \mathcal{B}_{\alpha, \gamma}, \mathcal{B}_{\alpha, \gamma'})$ with $Q'_k = Q_{k+1}$ if $\gamma \geq 1 + s$, $\gamma' > \gamma + \frac{(s+1)(k+1)}{\alpha}$, and $I = \mathbb{E}[\mathcal{M}^{\alpha(\gamma'+1)} + \mathcal{C}^\alpha \mathcal{M}^{\alpha(\gamma'+\gamma)}] < +\infty$.

Proof. Let $K = Q_k(t) - Q_k(t_0) - h Q_{k+1}(t_0)$, and $q(x) = (i\xi(x))^k (e_t(x) - e_{t_0}(x) - ih\xi(x)e_{t_0}(x))$. For $u \in \mathbb{R}$, we set $\phi(u) = e^{iu} - 1 - iu$. Let $0 < \varepsilon < \alpha$. We shall use the following usual inequalities

$$|\phi(u)| \leq 2|u|^{1+\varepsilon}, \quad |\phi(u) - \phi(v)| \leq 2|u - v| (|u|^\varepsilon + |v|^\varepsilon).$$

Writting $q(x) = (i\xi(x))^k e_{t_0}(x) \phi(h\xi(x))$, one easily gets (i) in Lemma B.1 with $A = C|h|^{1+\varepsilon}$ and $a = (s+1)(k+1+\varepsilon)$. Proceeding as in the previous proof, one obtains for $x, y \in E$ such that $d(y, x_0) \leq d(x, x_0)$

$$\begin{aligned} |q(x) - q(y)| &\leq |\xi(x)^k - \xi(y)^k| |\phi(h\xi(x))| + |\xi(y)^k| |e_{t_0}(x)\phi(h\xi(x)) - e_{t_0}(y)\phi(h\xi(y))| \\ &\leq C d(x, y)^\alpha p(x)^{s+1-\alpha} p(x)^{(s+1)(k-1)} |h|^{1+\varepsilon} p(x)^{(s+1)(1+\varepsilon)} \end{aligned}$$

$$\begin{aligned}
& + C p(x)^{(s+1)k} \left(|\phi(h\xi(x)) - \phi(h\xi(y))| + |\phi(h\xi(y))| |e_{t_0}(x) - e_{t_0}(y)| \right) \\
\leq & C |h|^{1+\varepsilon} d(x, y)^\alpha p(x)^{s+1-\alpha+(s+1)(k+\varepsilon)} + C |h|^{1+\varepsilon} p(x)^{(s+1)k} \times \\
& \left(|\xi(x) - \xi(y)| p(x)^{(s+1)\varepsilon} + p(x)^{(s+1)(1+\varepsilon)} |t_0|^\alpha d(x, y)^\alpha p(x)^{\alpha s} \right) \\
\leq & C |h|^{1+\varepsilon} d(x, y)^\alpha p(x)^{s+1-\alpha+(s+1)(k+\varepsilon)} \\
& + C |h|^{1+\varepsilon} d(x, y)^\alpha p(x)^{(s+1)k} \left(p(x)^{s+1-\alpha+(s+1)\varepsilon} + p(x)^{(s+1)(1+\varepsilon)+\alpha s} \right).
\end{aligned}$$

We have $s + 1 - \alpha + (s + 1)(k + \varepsilon) = (s + 1)(k + 1 + \varepsilon) - \alpha \leq (s + 1)(k + 1 + \varepsilon) + \alpha s$, and finally one gets (ii) in Lemma B.1 with $B = C |h|^{1+\varepsilon}$ and $b = (s + 1)(k + 1 + \varepsilon) + \alpha s$. To prove $Q_k \in \mathcal{C}^1(\mathbb{R}, \mathcal{B}_{\alpha, \gamma}, \mathcal{B}_{\alpha, \gamma'})$, one can then apply Lemma B.1 by proceeding exactly as in the previous proof (replace $|h|^\varepsilon$ with $|h|^{1+\varepsilon}$, and k with $k + 1$, with ε such that $\alpha \gamma' \geq \alpha \gamma + (s + 1)(k + 1 + \varepsilon)$). \square

Now one can prove Proposition 11.7. Let $\varepsilon > 0$ be such that $\eta + \frac{m(s+1)}{\alpha} + (2m+1)\varepsilon \leq \eta'$. Let $I = [\eta, \eta']$, $\mathcal{B}_\theta = \mathcal{B}_{\alpha, \theta}$, $T_0(\theta) = \theta + \varepsilon$ and $T_1(\theta) = \theta + \frac{s+1}{\alpha} + \varepsilon$. With these choices, proposition 11.7 is a direct consequence of lemmas B.4-4'. \square

B.5. Proof of Proposition 11.8. *This proposition states that, if we have, for some $\gamma' > \gamma > 1 + s : \mathbb{E}[\mathcal{M}^{\alpha(\gamma'+1)} + C^\alpha \mathcal{M}^{\alpha(\gamma'+\gamma)}] < +\infty$ and $\mathbb{E}[\mathcal{C}^\alpha \max\{\mathcal{C}, 1\}^{2\alpha\gamma}] < 1$, then condition (S) holds on $\mathcal{B}_{\alpha, \gamma}$ if and only if ξ is non-arithmetic w.r.t. $\mathcal{B}_{\alpha, \gamma}$. If moreover (CLT) holds, then the two previous equivalent conditions hold if ξ is nonlattice.*

This is a direct consequence of Propositions 5.3-4 and the following lemma. Notice that one may suppose that γ' is such that $1 + s + (\gamma' - \gamma) \leq \gamma$.

Lemma B.5. *Under the above hypotheses, Condition (\widehat{K}) of section 5.2 is fulfilled on $\mathcal{B}_{\alpha, \gamma}$ w.r.t. the auxiliary semi-norm $q(\cdot) = |\cdot|_{\alpha, \gamma'}$. Moreover $\mathcal{B}_{\alpha, \gamma}$ satisfies Condition (b2), and we have $c^{-1} \pi(|\cdot|) \leq q(\cdot) \leq c \|\cdot\|_{\alpha, \gamma}$ for some $c > 0$.*

Proof. Condition (K1) holds by Proposition 11.2. Since $|\cdot|_{\alpha, \gamma'} \leq \|\cdot\|_{\alpha, \gamma'}$, we have $(\widehat{K2})$ by Lemma B.4 (case $k = 0$). To prove $(\widehat{K3})$ and $(\widehat{K4})$, we observe that the norm $\|\cdot\|_{\alpha, \gamma}$ and the auxiliary semi-norm $|\cdot|_{\alpha, \gamma'}$ may be replaced with any equivalent norm and semi-norm on $\mathcal{B}_{\alpha, \gamma}$; of course $(\widehat{K2})$ then remains valid. Given a real parameter $\lambda \in (0, 1)$ on which conditions will be imposed later, let us consider on $\mathcal{B}_{\alpha, \gamma}$ the norm

$$\|f\|_{\alpha, \gamma, \gamma'}^{(\lambda)} = m_{\alpha, \gamma}^{(\lambda)}(f) + |f|_{\alpha, \gamma'}^{(\lambda)}$$

with $m_{\alpha, \gamma}^{(\lambda)}(f)$ and $|f|_{\alpha, \gamma'}^{(\lambda)}$ defined in B.0. It can be easily shown that $\|\cdot\|_{\alpha, \gamma}$ and $\|\cdot\|_{\alpha, \gamma, \gamma'}^{(\lambda)}$ are equivalent ([?] Prop. 5.2). Besides, instead of $|\cdot|_{\alpha, \gamma'}$, we shall consider the equivalent auxiliary semi-norm $|\cdot|_{\alpha, \gamma'}^{(\lambda)}$. Finally we have to establish that, for any compact set K_0 in \mathbb{R} :

- $\exists \kappa < 1, \exists C > 0, \forall n \geq 1, \forall f \in \mathcal{B}_{\alpha, \gamma}, \forall t \in K_0, \|Q(t)^n f\|_{\alpha, \gamma, \gamma'}^{(\lambda)} \leq C \kappa^n \|f\|_{\alpha, \gamma, \gamma'}^{(\lambda)} + C |f|_{\alpha, \gamma'}^{(\lambda)}$
- $\forall t \in K_0, r_{ess}(Q(t)) \leq \kappa$.

We have $Q(t) = K$ with $q(x) = e^{it\xi(x)}$ satisfying Conditions (i)-(ii) of Lemma B.1 with $A = 1, a = 0, B = C_\lambda |t|^\alpha$ ($C_\lambda > 0$) and $b = \alpha s$. Let $f \in \mathcal{B}_{\alpha, \gamma}$. Because of the presence of γ' in the above norm, Lemma B.1 cannot be directly applied here. However one can follow the proof

of lemma B.1 and see that

$$|Q(t)f|_{\alpha,\gamma'}^{(\lambda)} \leq \mathbb{E}[\mathcal{M}^{\alpha(\gamma'+1)}] |f|_{\alpha,\gamma'}^{(\lambda)} \leq I |f|_{\alpha,\gamma'}^{(\lambda)},$$

and that for $x, y \in E$ such that $d(y, x_0) \leq d(x, x_0)$, we have

$$\begin{aligned} |Q(t)f(x) - Q(t)f(y)| &\leq m_{\alpha,\gamma'}^{(\lambda)}(f) d(x, y)^\alpha p_\lambda(x)^{\alpha\gamma} p_\lambda(y)^{\alpha\gamma} \mathbb{E}[\mathcal{C}^\alpha \mathcal{C}_\lambda^{2\alpha\gamma}] \\ &\quad + D'_\lambda |t|^\alpha |f|_{\alpha,\gamma'}^{(\lambda)} d(x, y)^\alpha p_\lambda(x)^{\alpha s} p_\lambda(y)^{\alpha(\gamma'+1)} \mathbb{E}[\mathcal{C}^\alpha \mathcal{M}^{\alpha(\gamma'+1)+\alpha s}]. \end{aligned}$$

Thus

$$\frac{|Q(t)f(x) - Q(t)f(y)|}{d(x, y)^\alpha p_\lambda(x)^{\alpha\gamma} p_\lambda(y)^{\alpha\gamma}} \leq \mathbb{E}[\mathcal{C}^\alpha \mathcal{C}_\lambda^{2\alpha\gamma}] m_{\alpha,\gamma'}^{(\lambda)}(f) + I D'_\lambda |t|^\alpha |f|_{\alpha,\gamma'}^{(\lambda)} p_\lambda(x)^{\alpha s - \alpha\gamma} p_\lambda(y)^{\alpha(\gamma' - \gamma + 1)}.$$

Besides we have $p_\lambda(x)^{\alpha s - \alpha\gamma} p_\lambda(y)^{\alpha(\gamma' - \gamma + 1)} \leq p_\lambda(x)^{\alpha(s+1+\gamma' - 2\gamma)} \leq 1$, and, by Lebesgue theorem, we have $\kappa := \mathbb{E}[\mathcal{C}^\alpha \mathcal{C}_\lambda^{2\alpha\gamma}] < 1$ for sufficiently small λ . The previous estimates then gives the desired Doeblin-Fortet inequalities. Finally, since the canonical embedding from $(\mathcal{B}_{\alpha,\gamma}, \|\cdot\|_{\alpha,\gamma})$ into $(\mathcal{B}_{\alpha,\gamma}, |\cdot|_{\alpha,\gamma'})$ is compact (this easily follows from Ascoli theorem, see [?] Lemma 5.4), the property $r_{ess}(Q(t)) \leq \kappa$ is then a consequence of [?] [?]. This proves the first assertion of Lemma B.5.

Condition (b2) is obvious. The fact that $\|\cdot\|_{\alpha,\gamma,\gamma'}^{(\lambda)}$ dominates $|\cdot|_{\alpha,\gamma'}$ and is equivalent to $\|\cdot\|_{\alpha,\gamma}$ easily gives the second inequality stated on $q(\cdot) = |\cdot|_{\alpha,\gamma'}$. To prove the first one, observe that $\gamma' \leq 2\gamma$, therefore $\mathbb{E}[\mathcal{C}^\alpha \max\{\mathcal{C}, 1\}^{\alpha\gamma'}] < 1$, so Proposition 11.1 yields $\pi(d(\cdot, x_0)^{\alpha(\gamma'+1)}) < +\infty$. From the inequality $|f| \leq |f|_{\alpha,\gamma'} p^{\alpha(\gamma'+1)}$ ($f \in \mathcal{B}_{\alpha,\gamma}$), it follows that $\pi(|f|) \leq c |f|_{\alpha,\gamma'}$ with $c = \pi(p^{\alpha(\gamma'+1)})$. \square

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