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Electrostatics in a wormhole geometry

Bernard Linet *

Laboratoire de Mathématiques et Physique Théorique

CNRS/UMR 6083, Fédération Denis Poisson

Université François Rabelais, 37200 TOURS, France

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Abstract

We determine in closed form the electrostatic potential generated by a point charge at rest in a simple model of static spherically symmetric wormhole. From this, we deduce the electrostatic self-energy of this point charge.

The physical effects in wormholes has been a subject of considerable investigation. The present work is devoted to solving the electrostatic equation in closed form in the background metric of Morris and Thorne [1] which describes a simple wormhole. We take up again the question of the electrostatic self-energy analysed by Khusnutdinov and Bakhmatov [2] in the framework of the multipole formalism. We emphasize that we consider electrostatics in a static spherically symmetric wormhole connecting two different asymptotically flat spacetimes. Of course, electromagnetism in a static spherically symmetric wormhole connecting two distant regions in the same spacetime is radically different [3].

In the paper of Morris and Thorne [1], a simple geometry of wormhole is described by the metric

$$ds^2 = -dt^2 + dl^2 + \left(l^2 + w^2\right)\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{1}$$

in the coordinate system (t, l, θ, φ) with $-\infty < l < \infty$. The throat is located at $l = 0$. The two different asymptotically flat spacetimes are defined for $l \to \pm\infty$.

To solve in the next the electrostatic equation, it is convenient to write metric (1) in isotropic coordinates (t, r, θ, φ). The radial cordinate r is related to the proper radial coordinate l by

$$r = \frac{1}{2}\left(l + \sqrt{l^2 + w^2}\right) \tag{2}$$

with the range $0 < r < \infty$. Now the throat is located at $r = w/2$. We note that

$$\frac{dr}{dl} = \frac{l + \sqrt{l^2 + w^2}}{2\sqrt{l^2 + w^2}} \quad \text{satisfying} \quad \frac{dr}{dl} > 0$$

*E-mail: linet@lmpt.univ-tours.fr

and thus coordinate transformation (2) is valid eveywhere. The inverse transformation is given by

$$l = r - \frac{w^2}{4r}. \tag{3}$$

Taking into account (2), metric (1) is written as

$$ds^2 = -dt^2 + \left(1 + \frac{w^2}{4r^2}\right)^2 \left(dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right) \tag{4}$$

in the coordinate system (t, r, θ, φ) with $r > 0$.

We set the Cartesian coordinates (x^i), $i = 1, 2, 3$, associated to the spherical coordinates (r, θ, φ). Metric (4) takes then the form

$$ds^2 = -dt^2 + \left(1 + \frac{w^2}{4r^2}\right)^2 \left((dx^1)^2 + (dx^2)^2 + (dx^3)^2\right) \tag{5}$$

with $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$.

In the background metric (5), we consider a test electromagnetic field generated by a point charge e at rest. The charge density ρ of this point charge located at x_0^i has the expression

$$\rho(x^i) = e\frac{\delta^{(3)}\left(x^i - x_0^i\right)}{\sqrt{-g}} \tag{6}$$

where g is the determinant of the metric (5),

$$\sqrt{-g} = \left(1 + \frac{w^2}{4r^2}\right)^3.$$

We note $(r_0, \theta_0, \varphi_0)$ the position of the point charge with $r_0 > 0$.

The Maxwell equations in metric (5) for the electric component A of the electromagnetic potential reduce with source (6) to the electrostatic equation

$$\triangle A + h(r)\frac{x^i}{r}\partial_i A = -\frac{4\pi e}{1 + w^2/4r^2}\,\delta^{(3)}(x^i - x_0^i) \tag{7}$$

where $\triangle$ is the Laplacian operator and the function h has the expression

$$h(r) = \frac{1}{1 + \dfrac{w^2}{4r^2}}\frac{d}{dr}\left(1 + \frac{w^2}{4r^2}\right) = -\frac{w^2}{2r\left(r^2 + w^2/4\right)}. \tag{8}$$

The aim of this work is to determine in closed form the solution to equation (7) which is smooth everywhere except of course at the point x_0^i.

In a recent paper [4], we have shown that if h obeys certain differential relations then there exists a solution to equation (7) which can be written in the form

$$A(x^i) = eNg(r)g(r_0)F(s(x^i, x_0^i)) \tag{9}$$

with

$$s(x^i, x_0^i) = \frac{\Gamma(x^i, x_0^i)}{k(r)k(r_0)} \tag{10}$$

where $\Gamma(x^i, x_0^i) = (x^1 - x_0^1)^2 + (x^2 - x_0^i)^2 + (x^3 - x_0^3)^2$. The functions g, k and F are to be determined and N is a numerical factor.

The first step is to determine the elementary function in the Hadamard sense A_e to equation (7). This solution is defined by requiring in the vicinity of the point x_0^i the following development:

$$A_e(x^i) = \frac{1}{\sqrt{\Gamma(x^i, x_0^i)}} \left(U_0(r, r_0) + U_1(r, r_0)\Gamma(x^i, x_0^i) + \cdots \right). \tag{11}$$

In the case of metric (5), we have from (7) the first coefficient

$$U_0(r, r_0) = \frac{e}{\sqrt{1 + w^2/4r^2}\sqrt{1 + w^2/4r_0^2}}.$$

By inserting expression (8) into the differential relation of our previous paper [4], we obtain

$$h' + \frac{1}{2}h^2 + \frac{2}{r}h = \frac{w^2}{2(r^2 + w^2/4)^2}.$$

In consequence, we know to be able to find a solution in form (9) by considering the case II of our paper. By comparing, we put $a = w/2$, $b = w/2$ and $A = -4$. The functions g and k are then given by

$$k(r) = \frac{r^2 + w^2/4}{w/2}, \quad g(r) = \frac{r}{r^2 + w^2/4}, \tag{12}$$

and the differential equation for F is

$$s(1 - s)F'' + (-3s + 3/2)F' - F = 0. \tag{13}$$

According to (10) with (12), we have $0 \leq s \leq 1$.

We are now in a position to determine in closed form the elementary solution in the Hadamard sense A_e to equation (13) by taking as solution F_e to equation (13)

$$F_e(s) = \frac{1}{\sqrt{s(1 - s)}} \tag{14}$$

since

$$F_e(s) \sim \frac{1}{\sqrt{s}} + \frac{1}{2}\sqrt{s} \quad \text{as} \quad s \to 0. \tag{15}$$

Using (9), we get finally

$$A_e(x^i) = \frac{1}{2}ewg(r)g(r_0)F_e\left(\frac{\Gamma(x^i, x_0^i)}{k(r)k(r_0)}\right) \tag{16}$$

with $N = w/2$ to satisfy (11). We can directly verify that A_e given by expression (16) obeys the partial differential equation (7).

However, we are in a case where solution (16) presents another singularity. Indeed, it defines a new point charge e diametrically opposed to x_0^i on the sphere $r = r_1$ with $r_1 = w^2/4r_0$. This corresponds to $s = 1$ in expression (14) since

$$F_e(s) \sim \frac{1}{\sqrt{1-s}} \quad \text{as} \quad s \to 1.$$

We must look for another solution F_v to equation (13) which leads only a point charge located at x_0^i. We take the solution

$$F_v(s) = \frac{1}{2\sqrt{s(1-s)}} + \frac{\arcsin(1-2s)}{\pi\sqrt{s(1-s)}}. \tag{17}$$

The regularity at $s = 1$ of expression (17) results from the fact that $\arcsin(-1) = -\pi/2$. We note the expansion of function (17)

$$F_v(s) \sim \frac{1}{\sqrt{s}} - \frac{2}{\pi} + \frac{1}{2}\sqrt{s} - \frac{4}{3\pi}s \quad \text{as} \quad s \to 0. \tag{18}$$

This choice of function F_v yields the electrostatic potential V generated by a point charge e located at x_0^i

$$V(x^i) = \frac{1}{2}ewg(r)g(r_0)F_v\left(\frac{\Gamma(x^i, x_0^i)}{k(r)k(r_0)}\right). \tag{19}$$

We point out that the electric flux through the sphere at the infinity $r \to \infty$ of the electrostatic potential (19) is

$$4\pi e\left[\frac{1}{2} + \frac{1}{\pi}\arcsin\left(\frac{r_0^2 - w^2/4}{r_0^2 + w^2/4}\right)\right]. \tag{20}$$

We have the opportunity to add to potential (19) the electrostatic monopole solution to equation (7) which is smooth everywhere but we have no prescription to fix this homogeneous solution.

We turn now to calculate the electrostatic self-energy. We develop expression (19) in the vicinity of the point x_0^i by using expansion (18). We get

$$V(x^i) \sim \frac{U_0(r, r_0)}{\sqrt{\Gamma(x^i, x_0^i)}} - \frac{ew}{\pi}g(r)g(r_0) \quad \text{as} \quad x^i \to x_0^i. \tag{21}$$

Thus, the electrostatic self-energy W of the point charge e at rest in the wormhole geometry (5) can be immediately deduces from development (21)

$$W(r_0) = -\frac{1}{2\pi}e^2wg^2(r_0) = -\frac{e^2wr_0^2}{2\pi\left(r_0^2 + w^2/4\right)^2}. \tag{22}$$

The electrostatic self-energy (22) can be expressed with the proper radial coordinate l in the form

$$W(l_0) = -\frac{e^2 w}{2\pi\left(l_0^2 + w^2\right)} \tag{23}$$

Result (23) coincides with the one of Khusnutdinov and Bakhmatov [2].

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