Rationally connected 3-folds and symplectic geometry

Claire Voisin

Rationally connected 3-folds and symplectic geometry

Claire Voisin
CNRS and IHÉS

Pour Jean-Pierre Bourguignon, à l'occasion de ses 60 ans

0 Introduction

Let X be a compact Kähler manifold. Denoting by J the operator of complex structure acting on T_X, Kähler forms on X are symplectic forms which satisfy the compatibility conditions

$$\omega(Ju, Jv) = \omega(u, v),\ u,\ v \in T_{X,x},\ \omega(u, Ju) > 0,\ 0 \neq u \in T_{X,x}.$$

The first condition tells that ω is of type $(1,1)$. The last condition is called the taming condition. The set of Kähler forms is a convex cone, in particular connected, and thus determines a deformation class of symplectic forms on X.

Let X and Y be two complex projective or compact Kähler manifolds. We will say that X and Y are symplectically equivalent if for some symplectic forms α on X, resp. β on Y, which are in the deformation class of a Kähler form on X, resp. Y, there is a diffeomorphism

$$\psi : X \cong Y,$$

such that $\psi^*\beta = \alpha$. Notice that ψ^* induces a bijection between the sets of symplectic forms which are in the deformation class of a symplectic form on Y and X, and thus we may assume that α is a taming form, or even a Kähler form on X.

In the sequel, the compact Kähler manifolds X we will consider are *uniruled* manifolds, which means the following (cf [9]):

Definition 0.1 *A projective complex manifold (or compact Kähler) is uniruled if there exist compact complex manifolds Z and B, and dominating morphisms*

$$f : Z \to X,\ g : Z \to B,$$

where f is non constant on the fibers of g and the generic fiber of g is isomorphic to $\mathbb{P}^1$.

In other words, there is a (maybe singular) rational curve in X passing through any point of X, where a (singular) rational curve is defined as a connected curve whose normalization has only rational components.

The starting point of this work is the following result, due independently to Kollár [8] and Ruan [19] (we refer to [6], [13], [14] for purely symplectic characterizations and studies of uniruledness) :

Theorem 0.2 *Let X and Y be two symplectically equivalent compact Kähler manifolds. Then if X is uniruled, Y is also uniruled.*

We sketch later on the proof of this result, in order to point out why the proof does not extend to cover the rational connectedness property, which we will consider in this paper. Let us recall the definition (cf [2], [10], [9]).

Definition 0.3 *A compact Kähler manifold X is rationally connected if for any two points x, $y \in X$, there exists a (maybe singular) rational curve $C \subset X$ with the property that $x \in C$, $y \in C$.*

Examples of rationally connected varieties are given by smooth Fano varieties, i.e. smooth projective varieties X satisfying the condition that $-K_X$ is ample. (This is the main result of [2], and [10].)

The following conjecture appears in [8]. It was asked to me by Pandharipande and Starr :

Conjecture 0.4 *(Kollár) Assume X is rationally connected. Let Y be a compact Kähler manifold symplectically equivalent to X. Then Y is also rationally connected.*

Remark 0.5 A compact Kähler manifold X whis is rationally connected satisfies $H^2(X, \mathcal{O}_X) = 0$, hence is projective. Thus, under the assumption above, X is projective, and if the answer to conjecture 0.4 is positive, Y is also projective.

This conjecture has an easy positive answer in the case of surfaces, as an immediate consequence of theorem 0.2. Indeed, let X be rationally connected of dimension 2, and let Y be symplectically equivalent to X. Then Y is uniruled, as X is. On the other hand $b_1(Y) = 0$, because $b_1(X) = 0$ and Y is diffeomorphic to X. Thus Y is a rational surface, hence rationally connected.

In this note, we prove the following partial results concerning conjecture 0.4 in dimension 3. I should mention here that in these form the results are partly due to Jason Starr. Indeed, in the original version of this paper, I had worked with a more restricted notion of symplectic equivalence between compact Kähler manifolds, where I considered only symplectic diffeomeorphisms $(X, \alpha) \cong (Y, \beta)$ where α *and* β were taming for the complex structure. Jason Starr showed me how to make the proof of proposition 0.6 work as well when only α is taming, and β is any symplectic form which is a deformation (through a family of symplectic forms) of a Kähler form on Y.

Proposition 0.6 *Let X be rationally connected of dimension 3, and let Y be compact Kähler symplectically equivalent to X. If Y is not rationally connected, X and Y admit almost holomorphic rational maps*

$$\phi : X \dashrightarrow \Sigma,\ \phi' : Y \dashrightarrow \Sigma'$$

to a surface, with rational fibers C, resp. D, of the same class (where we use the symplectomorphism $\psi : X \cong Y$ giving symplectic equivalence to identify $H_2(X, \mathbb{Z})$ and $H_2(Y, \mathbb{Z})$).

Here *almost holomorphic* means that the map is well-defined near a generic fiber. We then consider the case where the above map ϕ is well-defined.

Proposition 0.7 *Under the same assumptions as in proposition 0.6, assume that the rational map ϕ above is well-defined and that either Σ is smooth, or ϕ does not contract a divisor to a point. Then Y is also rationally connected.*

We will use this result together with some birational geometry arguments to prove the following:

Theorem 0.8 *Let X, Y be compact Kähler 3-folds. Assume that X and Y are symplectically equivalent and that one of the two following assumptions hold:*

1. *X is Fano.*
2. *X is rationally connected, and $b_2(X) \leq 2$.*

Then Y is rationally connected.

This anwers conjecture 0.4 when X is a Fano threefold or satisfies $b_2 \leq 2$. The two considered cases have a small overlap. In the class where $b_2(X) \leq 2$, one has all the blow-ups of Fano manifolds with $b_2 = 1$ along a connected submanifold. Thus this is not a bounded family. It is known on the contrary that Fano manifolds form a bounded family (see [2], [10], or [17] for the 3-dimensional case). However the bound for b_2 of a Fano threefold is 10 (cf [17]), showing that the Fano case is far from being included in the second case.

Remark 0.9 Note that for varieties with $b_2 = 1$, conjecture 0.4 obviously has an affirmative answer. Indeed a uniruled projective manifold with $b_2 = 1$ is necessarily Fano. Hence if X is rationally connected with $b_2 = 1$, by theorem 0.2 any projective manifold which is symplectomorphic to it is also uniruled with $b_2 = 1$, hence Fano, hence rationally connected.

To conclude this introduction, let us sketch the proof of theorem 0.2, and explain on an example the difficulty one meets to extend it to the rational connectedness question.

Proof of theorem 0.2. Let α be a taming symplectic form on X (one can take here a Kähler form). We will denote in the sequel the degree of curves C in X with respect to α (that is the integrals $\int_C \alpha$) by $deg_\alpha(C)$. Let $\mu_\alpha(X)$ be the minimum of the following set:

$$S_X := \{deg_\alpha(C),\, C \text{ moving rational curve in } X\}.$$

Here by "moving", we mean that the deformations of C sweep-out X. Note that the minimum of the set S_X is well defined, because there are finitely many families of curves of bounded degree in X and the $(1,1)$-part $\alpha^{1,1}$ of α is $> \epsilon\omega$ where ω is any Kähler form on X. Let now C be a moving rational curve on X, which satisfies $deg_\alpha(C) = \mu_\alpha(X)$ and let $[C] \in H_2(X, \mathbb{Z})$ be its homology class. We claim that for $x \in X$, and for adequate cohomology classes $A_1, \ldots, A_r \in H^4(X, \mathbb{Z})$, the Gromov-Witten invariant $GW_{0,[C]}([x], A_1, \ldots, A_r)$ counting genus 0 curves passing through x and meeting representatives B_i of the homology classes Poincaré dual to A_i, is non zero. To see this, we observe that by minimality of $deg_\alpha(C)$, any genus 0 curve of degree $< deg_\alpha(C)$ is not moving, that is, its deformations do not sweep-out X. It follows that for a general point $x \in X$, any genus 0 curve of class $[C]$ and passing through x is irreducible, with normal bundle generated by sections. This implies that the set $Z_{x,[C]}$ of rational curves of classes $[C]$ passing through x has the expected dimension and it is nonempty by assumption. Let r be its dimension, and choose for A_i, $1 \leq i \leq r$ a class h^2, where h is ample line bundle on X. It is then clear that $GW^X_{0,[C]}([x], A_1, \ldots, A_r) \neq 0$, as this number is the degree of a big and nef line bundle on $Z_{x,[C]}$.

As Y is symplectically isomorphic to X, (for some symplectic structures on X, resp. Y, in the deformation class determined by Kähler forms,) we conclude that $GW^Y_{0,\psi_*[C]}([y], A'_1, \ldots, A'_r) \neq 0$, where $A'_i = \psi_* A_i \in H^4(Y, \mathbb{Z})$. But in turn, because Gromov-Witten invariants can be computed using rational curves on Y by excess formulas (see [12], [1], [20]), this implies that there is through any point $y \in Y$ a rational curve of class $\psi_*[C]$. Thus Y is uniruled. ∎

Remark 0.10 The proof above shows in fact a strongest statement, namely the fact that a uniruled compact Kähler manifold X admits non-zero Gromov-Witten invariants in genus 0 passing through one point:

$$GW^X_{0,[C]}([x], A_1, \ldots, A_r) \neq 0.$$

From this point of view, the proof of Theorem 0.8 is somewhat different. Indeed we do not prove that a projective rationally connected 3-fold X admits non-zero Gromov-Witten in genus 0 passing through two points: $GW^X_{0,[C]}([x], [x], A_1, \ldots, A_r) \neq 0$, which would be the natural symplectic analogue of rational connectedness.

Our argument uses Gromov-Witten invariants *in higher genus*, which of course works in the symplectic setting as well. What we show essentially is that there is a covering family of rational curves of class $[C]$ with a non zero 1 point Gromov-Witten invariant: $GW^X_{0,[C]}([x], A_1, \ldots, A_r) \neq 0$, and that there is a non zero Gromov-Witten invariant of the following shape

$$GW^X_{g,[C']}(\underbrace{[C], \ldots, [C]}_{r}, A_1, \ldots, A_N) \neq 0,$$

for some $r > g$ and curve class $[C']$ not proportional to C. We have the same non vanishings for Y.

The second ingredient is the notion of maximal rationally connected fibration due to Kollár-Miyaoka-Mori and Campana in the Kähler context. This last notion does not seem to extend well to the symplectic geometry context. The argument consists roughly in proving that the basis of the maximal rationally connected fibration of Y cannot be a 3-fold by the first non vanishing, and cannot be a surface, which would be uniruled by the second non-vanishing. Finally it cannot be a curve by elementary topological considerations.

Remark 0.11 We used in this sketch of proof the terminology "rational curve in X" to mean "stable n-pointed genus 0 maps $f : C \to X$", which are the correct objects to count in order to compute the Gromov-Witten invariants (cf [4]). However, note that if f is as above, $f(C)$ is a rational curve in the previous sense.

If we want to apply the reasoning to study rational connectedness, we are faced to the following problem: we could as before introduce the minimal degree for which there are rational curves in X passing through any two points of X. On the other hand, it might be that curves of this degree are all reducible, with one component which is highly obstructed, so that one cannot conclude that the corresponding

Gromov-Witten invariant is non zero. In fact, consider the case of a Hirzebruch surface $p : F \to \mathbb{P}^1$ which is a deformation (hence symplectically equivalent to) of a quadric $\mathbb{P}^1 \times \mathbb{P}^1$: Let C_0 be a rational curve which is a section of p with sufficiently negative self-intersection : $C_0^2 < -4$. Then one has in F rational curves consisting of the union of two fibers with the section C_0. Such curves C can be chosen so as to pass through any two points of F, and we may assume they are, among the rational curves satisfying this property, of minimal degree with respect to an adequate polarization. On the other hand, we have $C^2 < 0$ and it is clear that these curves disappear under a deformation from F to $\mathbb{P}^1 \times \mathbb{P}^1$. The corresponding 2-points Gromov-Witten invariant is 0 in this case.

The paper is organized as follows. In section 1, we prove proposition 0.6. In section 2, we study the remaining case, where X is an almost conic bundle (we mean by this that X admits a rational map f to a projective surface Σ, with generic fiber isomorphic to $\mathbb{P}^1$, and that the rational map f is well-defined along the generic fiber). We show that ϕ is actually a morphism (for an adequate choice of birational model of Σ) when $b_2(X) \leq 2$ or X is Fano, unless there are some non trivial genus 0 Gromov-Witten invariants of the form $GW^X_{0,[C']}([C], A_1, \ldots, A_r)$, with $[C']$ not proportional to $[C]$. These Gromov-Witten invariants will be used in the last section to conclude that in this last case, Y is also rationally connected. We also show that when ϕ is well-defined, there are many non zero Gromov-Witten invariants on X, maybe not in genus 0 however.

The proof of theorem 0.8 uses in turn these non zero Gromov-Witten invariants on Y. It is completed in section 3.

Thanks. It is a pleasure to acknowledge discussions with Jason Starr and Rahul Pandharipande, which started me thinking to this question. I thank Dusa McDuff, Yongbin Ruan and Johan de Jong for comments on various versions of the paper. I am mostly indebted to Jason Starr for showing me how to modify my original work to get the present version of the result.

1 Study of the rationally connected fibration of Y

This section has been much simplified and improved thanks to the help of Jason Starr. In the proof of proposition 1.1 below, he showed me how to work with general symplectic equivalence, instead of restricted symplectic equivalence as I did originally.

We will assume that X is a projective rationally connected complex manifold, that Y is compact Kähler and that X and Y are symplectomorphic with respect to some symplectic forms α, β on X, Y respectively, with α a taming form for the complex structure on X and β in the deformation class (as a symplectic form) of a Kähler form on Y. We will denote as before $\psi : X \cong Y$, $\psi^*\beta = \alpha$ such a symplectomorphism. The theory of Gromov-Witten invariants shows that the map ψ identifies the Gromov-Witten invariants of X and Y, computed using holomorphic curves on X and Y.

We start now as in the proof of Theorem 0.2. Introducing as before moving rational curves (or rather genus 0 stable maps) C on X, of minimal degree with respect to α, we concluded that there is a covering family of rational curves (genus 0 stable maps) in Y in the class $\psi_*([C])$.

Our goal in this section is to show the following, (which implies proposition 0.6):

Proposition 1.1 *If Y is not rationally connected, then the covering family of curves C in X is given by an almost holomorphic rational map*

$$\phi : X \dashrightarrow \Sigma$$

to a surface, with rational fibers of class $[C]$. Furthermore, Y also admits an almost holomorphic rational map

$$\phi' : Y \dashrightarrow \Sigma'$$

with rational fiber of class $[D] = \psi_*[C]$.

Here almost holomorphic means that the rational map ϕ is well-defined along the generic fiber of ϕ. Equivalently, choosing a desingularization

$$\tilde{\phi} : \widetilde{X} \to \Sigma, \; \tau : \widetilde{X} \to X$$

of ϕ, where τ is a composition of blow-ups along smooth centers, this means that the exceptional divisors of τ do not dominate Σ. As the fibers of this fibration are rational curves, but X is not necessarily ruled (as it may not exist a line bundle with intersection -1 with fibers), we will say that X is an *almost conic bundle.*

The proof of the proposition is based on the following lemma (here we do not distinguish the image curve and the map, as we know that the map is generically the normalization map):

Lemma 1.2 *Y is rationally connected, unless possibly if the curve D above satisfies $c_1(K_Y) \cdot [D] = -2$ and $GW_{0,[D]}([y]) = 1$.*

Proof. We study the maximal rationally connected fibration of Y, which exists even if Y is only Kähler by [2], and is an almost holomorphic rational map

$$Y \dashrightarrow B.$$

Notice that $dim\, B \leq 2$ because Y is covered by rational curves D of class $[D] = \psi_*[C]$. We use now the following elementary lemma.

Lemma 1.3 *Let X, Y be compact Kähler manifolds which are symplectically equivalent. Assume X is rationally connected. If the basis B of the rationally connected fibration of Y has dimension ≤ 1, Y is rationally connected.*

Proof. We know that $H^1(X, \mathbb{C}) = 0$ because X is rationally connected and this obviously implies $H^0(X, \Omega_X) = 0$, hence $H^1(X, \mathbb{C}) = 0$ by Hodge theory. As Y is diffeomorphic to X, $H^1(Y, \mathbb{C}) = 0$ as well. It follows that if the basis B of the rationally connected fibration of Y has dimension 1, it is isomorphic to $\mathbb{P}^1$. This contradicts [5], which implies that the basis of the rationally connected fibration is not uniruled. ∎

Thus we conclude that if Y is not rationally connected, the basis B of the maximal rationally connected fibration of Y is a surface Σ'. Furthermore the map $\phi' : Y \dashrightarrow \Sigma'$ is almost holomorphic. The surface Σ' is not uniruled by [5], and thus any (connected) rational curve (or rather genus 0 map) $f : \Gamma \to Y$ passing through a general point y of Y (where we may assume, because ϕ' is almost holomorphic that ϕ' is well-defined everywhere along the smooth connected curve $D' := {\phi'}^{-1}(\phi'(y))$) must have image supported on D'. It follows that $[f_*\Gamma] = m[D']$, for some $m \geq 1$.

We apply this to our covering family of rational curves D (genus 0 stable maps) in Y in the class $[D] = \psi_*([C])$ and we conclude that $\psi_*[C] = m[D']$. Next we observe that $GW^Y_{0,[D']}([y]) = 1$, because the only rational curve of class $[D']$ passing through y is D, which is smooth with trivial normal bundle, so that there is fact exactly one genus 0 map f of class $[D']$ passing through y, and as $H^1(N_f(-y)) = 0$, this stable map is computed with multiplicity 1 in $GW^Y_{0,[D']}([y])$.

This implies that $m = 1$, because we find that

$$GW^X_{0,\frac{1}{m}[C]}([x]) \neq 0,$$

so that $m > 1$ would contradict the minimality of $deg_\alpha(C)$. Hence we proved that

$$[D] = [D'],\ GW^Y_{0,[D']}([y]) = 1.$$

Finally, as ϕ' is well-defined along the generic fiber D', we conclude that $N_{D'/Y}$ is trivial, which implies by adjunction that $K_Y \cdot D' = K_Y \cdot [D] = -2$. Thus lemma 1.2 is proved.

■

Proof of proposition 1.1. Notice that, as ψ is a symplectomorphism with respect to symplectic forms α, β of X, resp. Y, which are respective deformations of Kähler forms on X resp. Y, $\psi^* c_1(K_Y) = c_1(K_X)$. This is indeed a standard fact of symplectic geometry: the canonical class of a symplectic manifold X is an invariant of the deformation class of the symplectic form ω on X. Indeed it can be computed using any almost complex structure on X which is tamed by ω or a deformation of ω, the set of such almost complex structures being connected. This almost complex structure makes the tangent bundle into a complex vector bundle and the canonical class is minus the first Chern class of this complex vector bundle.

Furthermore, we have by assumption $[D] = \psi_*([C])$. Thus we have

$$c_1(K_X) \cdot [C] = c_1(K_Y) \cdot [D] = -2,$$

$$GW^X_{0,[C]}([x]) = GW^Y_{0,[D]}([y]) = 1.$$

The first equality together with the fact that the general curve passing through the point X is irreducible, and thus has globally generated normal bundle, implies that for general $x \in X$, the normal bundle of a curve C of class $[C]$ passing through x is trivial, which shows that there are finitely many such curves through x, and that the set of such curves has the expected dimension 0. Thus the number of these curves is equal to $GW^X_{0,[C]}([x])$ and this is equal to 1 by the second equality above. In conclusion we proved that if Σ_0 is the set parameterizing rational curves in X of class $[C]$ and Σ is the union of components of Σ_0 parameterizing moving curves, then the universal curve

$$q' : \mathcal{C} \to \Sigma,\ \Phi' : \mathcal{C} \to X,$$

has the property that Φ' has degree 1. Thus Φ' is birational, and

$$\phi := q' \circ \Phi'^{-1} : X \dashrightarrow \Sigma$$

gives the desired fibration into rational curves.

In order to conclude the proof, it just remains to prove that the rational map $\phi : X \dashrightarrow \Sigma$ is almost holomorphic. Assume this is not the case: let $\tau : X' \to X$ be a composition of blow-ups along smooth centers, such that $\tilde{\phi} := \phi \circ \tau$ is well-defined. Assume there is an exceptional divisor $E \subset X'$ which dominates Σ and is contracted to a curve Z (or a point) in X. Then if C' is the general fiber of $\tilde{\phi}$, C' meets E. On the other hand, $K_{X'} = \tau^* K_X + F$ where F is an effective divisor supported on the exceptional locus, and the multiplicity of E in F is > 0. Thus we find that

$$c_1(K_{X'}) \cdot [C'] = -2 = (\tau^* c_1(K_X) + F) \cdot [C']$$
$$> \tau^* c_1(K_X) \cdot [C'] = c_1(K_X) \cdot [C] = -2,$$

which is a contradiction.

■

2 The case where X is an almost conic bundle

We now study almost conic bundles $\phi : X \dashrightarrow \Sigma$ with generic fiber C. When X is rationally connected, Σ is a rational surface, and thus we may assume to begin that $\Sigma = \mathbb{P}^2$. (Indeed, the fact that ϕ is almost a morphism does not depend on the birational model of the target.) Notice that, because ϕ is almost holomorphic, we have $H \cdot C = 0$, where the line bundle H on X is defined by

$$H := \phi^* \mathcal{O}_{\mathbb{P}^2}(1).$$

The first result is the following:

Proposition 2.1 *Assume that either X is Fano, or $b_2(X) = 2$. Then H is numerically effective, unless we are not in the Fano case, and there exists a curve class $[C']$ not proportional to $[C]$ such that for some cohomology classes $A_1, \ldots, A_r \in H^*(X)$,*

$$GW^X_{0,[C']}([C], A_1, \ldots, A_r) \neq 0.$$

Proof. Suppose first that X is Fano. Then any irreducible curve $Z \subset X$ satisfies $K_X \cdot Z < 0$, hence the Chow variety of its cycle is at least one dimensional because $dim\, X = 3$ (cf [9], theorem 1.15). (This can also be formulated by evaluating the dimension of the space of deformations of the composed map $\widetilde{Z} \to Z \to X$, where $\widetilde{Z} \to Z$ is the normalization.) Thus, Z being irreducible, its cycle can be moved so as to be not contained in the indeterminacy locus of ϕ. Thus $\phi^* H \cdot Z \geq 0$.

Suppose now that $b_2(X) = 2$ but X is not Fano. We have to show that either H is numerically effective, or there exists a curve class $[C']$ not proportional to $[C]$ such that for some cohomology classes $A_1, \ldots, A_r \in H^*(X)$,

$$GW^X_{0,[C']}([C], A_1, \ldots, A_r) \neq 0.$$

As K_X is not nef, there exists a Mori contraction $c : X \to X'$, with $(Pic\, X')\otimes\mathbb{Q} = \mathbb{Q}$ and $-K_{X/X'}$ relatively ample. We consider the three possible dimensions of X' (cf [16]).

1) $dim\, X' = 1$, that is $X' = \mathbb{P}^1$. In this case, the contraction is given by a pencil whose fibers are Del Pezzo surfaces. Let $L = c^*\mathcal{O}_{\mathbb{P}^1}(1)$. If $L \cdot C = 0$, then L is proportional to H (because $b_2(X) = 2$), and this contradicts the fact that the Iitaka dimension of H is at least 2. In the other case, we observe that the fibers of c are uniruled. Fix a polarization h on X and introduce the minimal degree with respect to h of rational curves contained in the fibers of c and sweeping-out X. Let $[C']$ be a class curve such that $L.[C'] = 0$ and achieving this minimal degree. All curves of class $[C']$ are supported on fibers of c. Exactly as in the proof of theorem 0.2, one then shows that for a covering family of rational curves C' of this minimal degree, the generic member is irreducible with semipositive normal bundle. Using now the fact that C intersects non trivially the generic fiber of c, one concludes immediately that there is a non zero Gromov-Witten invariant

$$GW^X_{0,[C']}([C], A_1, \ldots, A_r).$$

2) $dim\, X' = 2$. We have $c^* Pic\, X' = \mathbb{Z}L$, where L is ample on X', and if $C \cdot L = 0$ we conclude as before that L is proportional to H. In this case H is numerically effective. In the other case, the map $c : X \to X'$ has for generic fiber a rational curve C' with trivial normal bundle and satisfying $K_X \cdot C' = -2$. Furthermore there are only finitely many 2-dimensional fibers of c. If C is generic, there is thus exactly a 1-dimensional family of fibers C' meeting C, and this is exactly the expected dimension. It thus follows that there is a non trivial Gromov-Witten invariant

$$GW^X_{0,[C']}([C], A_1),$$

where $A_i = h^2 \in H^4(X, \mathbb{Z})$ for some ample class $h \in H^2(X, \mathbb{Z})$.

3) $dim\, X' = 3$. In this case c is a divisorial contraction. Note that C is not proportional to the contracted extremal ray, because C is a moving curve. A look at the list of divisorial contractions (cf [15]) shows the following (see [18]): Let E be the exceptional divisor of the contraction, so that E is either a ruled surface contracted to a smooth curve, or $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ contracted to a point. Let $[C']$ be the class of the fiber of the contracting ruling in the first case, or the class of a line in the second case, or the class of one of the two rulings in the third case. Then for any curve class γ such that $\gamma \cdot E \neq 0$, one has $GW^X_{0,[C]'}(\gamma, A_1, \ldots, A_r) \neq 0$, for an adequate number r, which will be in fact 0 or 1.

On the other hand $E \cdot C = 0$ is impossible, because in this case E and H would be proportional in $Pic\, X$, and E is contractible while the Iitaka dimension of H is at least 2. We deduce from this that one has $GW^X_{0,[C']}([C], A_1, \ldots, A_r) \neq 0$, where $A_i = h^2 \in H^4(X, \mathbb{Z})$ for some ample class $h \in H^2(X, \mathbb{Z})$.

∎

From this, we get the following result:

Corollary 2.2 *Assume that either X is Fano, or $b_2(X) = 2$. Then there exists a well-defined morphism $\phi : X \to \Sigma$ with fiber C, where Σ is a normal surface, unless we are not in the Fano case and there exists a curve class $[C']$ not proportional to $[C]$ such that for some cohomology classes $A_1, \ldots, A_r \in H^*(X)$,*

$$GW^X_{0,[C']}([C], A_1, \ldots, A_r) \neq 0.$$

Proof. We use the contraction theorem (cf. [7], or [15], p 162) which tells that such a morphism exists if and only if H is numerically effective and the curves $Z \subset X$ satisfying $Z.H = 0$ also satisfy $Z \cdot K_X < 0$.

Indeed, by the previous theorem, we know that H is numerically effective, unless there exists a curve class $[C']$ not proportional to $[C]$ such that for some cohomology classes $A_1, \ldots, A_r \in H^*(X)$,

$$GW^X_{0,[C']}([C], A_1, \ldots, A_r) \neq 0.$$

Thus, in order to apply the contraction theorem, we just have to show that for any curve $Z \subset X$ satisfying the condition $Z \cdot H = 0$, one has $K_X \cdot Z < 0$.

In the Fano case, this is obvious. When $b_2(X) = 2$, the orthogonal of H in $H_2(X, \mathbb{Q})$ is generated by the class of C, which satisfies the condition $C \cdot K_X = -2$. ∎

We will use the following observation:

Lemma 2.3 *Assuming ϕ is well defined and either $b_2(X) \leq 2$ or X is Fano, we may furthermore assume (by changing Σ if necessary) that ϕ does not contract a divisor to a point of Σ.*

Proof. First of all, note that if $b_2(X) \leq 2$, ϕ cannot contract a divisor D to a point of Σ. Indeed, such a divisor would satisfy $D \cdot C = 0$, hence would be proportional to H. But the Iitaka dimension of H is 2, while no multiple of D moves, which is a contradiction.

Consider now the Fano case. Let x be a point of Σ, and let E be the pure 2-dimensional part of $\phi^{-1}(x)$, (counted with multiplicities). We claim that $-E$ is numerically effective on the fibers of ϕ and non trivial on $\phi^{-1}(x)$.

Assuming the claim, $H - \epsilon E$ remains numerically effective for a sufficiently small ϵ. On the other hand, curves Z satisfying $Z \cdot (H - \epsilon E) = 0$ satisfy the condition $K_X \cdot Z < 0$ for the same reasons as before, hence we can apply the contraction theorem to $H - \epsilon E$, which does not contract E anymore. This leads eventually to a morphism ϕ' which does not contract any divisor to a point.

To see the claim, we observe that $-E \cdot F = 0$ for any irreducible curve F contained in a fiber of ϕ but not contained in $\phi^{-1}(x)$. Furthermore $-E_{|E}$ is effective and non trivial on each component of E. This implies that $-E \cdot F \geq 0$ for any irreducible curve $F \subset E$ whose deformations cover a 2-dimensional component of $\phi^{-1}(x)$. Consider now any irreducible curve $F \subset X$ contained in $\phi^{-1}(x)$. As X is Fano of dimension 3, the cycle of any such F deforms to cover at least a divisor in X (cf [9], Theorem 1.15). On the other hand, all such deformations remain contained in a fiber of ϕ. It follows that either the cycle of F deforms to cover a 2-dimensional component of $\phi^{-1}(x)$, so that $-E \cdot F \geq 0$ as shown previously, or the cycle of F can be moved to be supported in another fiber, in which case we have $-E \cdot F = 0$.

∎

We consider now the case where ϕ is well defined (but Σ may be singular). Our main result is the following:

Theorem 2.4 *Let X be a rationally connected 3-fold which admits a morphism $\phi : X \to \Sigma$ to a normal surface Σ, with generic fiber a rational curve C. Assume that either Σ is smooth, or ϕ does not contract a divisor to a point of Σ. Then there* *exist integers g, r with $g < r$, cohomology classes $A_1, \ldots, A_N \in H^4(X, \mathbb{Z})$ and a homology class $[C'] \in H_2(X, \mathbb{Z})$ not proportional to $[C]$ such that*

$$GW^X_{g,[C']}(\underbrace{[C], \ldots, [C]}_{r}, A_1, \ldots, A_N) \neq 0.$$

Before giving the proof, let us establish a few lemmas.

Lemma 2.5 *Σ contains a complete linear system of generically smooth curves Z of genus g, which do not meet generically the singular locus of Σ, and satisfy*

$$r = h^0(\Sigma, \mathcal{O}_\Sigma(Z)) - 1 = h^0(Z, \mathcal{O}_Z(Z)) > g. \tag{2.1}$$

Proof. If Σ is smooth, Σ is rational and the result is obvious (we can even take $g = 0$). In general, we start from a "very moving" generic smooth rational curve $\Gamma_0 \subset X$. Recall that "very moving" means that the normal bundle $N_{\Gamma_0/X}$ is ample. Using the assumption that no divisor is contracted to a point by ϕ or that Σ is smooth, one concludes that for Γ_0 generic, $\phi(\Gamma_0) =: \Gamma'_0$ avoids the singular locus of Σ.

Let $\mathcal{L} := \mathcal{O}_\Sigma(\Gamma'_0)$. Observe that $H^1(\Sigma, \mathcal{O}_\Sigma) = 0$, because Σ admits a desingularization which is rationally connected. It follows that the restriction map:

$$H^0(\Sigma, \mathcal{L}) \to H^0(\Gamma'_0, N_{\Gamma'_0/\Sigma})$$

is surjective. Observe now that because the equisingular deformations of Γ'_0 in Σ (which are singular rational curves) cover Σ, one has $K_\Sigma \cdot \Gamma'_0 < 0$.

In fact we may even assume $K_\Sigma \cdot \Gamma'_0 < -1$, replacing if necessary Γ_0 by a ramified cover of it, which by ampleness of the normal bundle can be deformed to an embedding.

It thus follows that

$$deg\, N_{\Gamma'_0/\Sigma} = deg\, K_{\Gamma'_0} \otimes K_\Sigma^{-1}{}_{|\Gamma'_0} \geq deg\, K_{\Gamma'_0} + 2.$$

This inequality implies that the linear system $H^0(\Gamma'_0, N_{\Gamma'_0/\Sigma})$ has no base-point on Γ'_0 so that a generic deformation Z of Γ'_0 is smooth. Letting g be the arithmetic genus of Γ'_0, that is the genus of a generic deformation Z of Γ'_0 in Σ, we now find that Z satisfies the desired property

$$r = h^0(\Sigma, \mathcal{O}_\Sigma(Z)) - 1 = h^0(Z, \mathcal{O}_Z(Z)) > g = h^0(Z, K_Z),$$

because $deg\, \mathcal{O}_Z(Z) \geq deg\, K_Z + 2$ by adjunction and because $K_\Sigma \cdot Z < -1$. ∎

Remark 2.6 The inequality $deg\, \mathcal{O}_Z(Z) \geq deg\, K_Z + 2$ also implies that $h^1(Z, \mathcal{O}_Z(Z)) = 0$, a fact which will be used later on.

Let $x_1, \ldots, x_r$ be r generic points of Σ. Then there is a unique curve $Z \subset \Sigma$ belonging to the linear system $|\, \mathcal{L}\, |$ and passing through $x_1, \ldots, x_r$. This curve is smooth and by Bertini the surface $X_Z := \phi^{-1}(Z)$ is smooth. Choose now a section $\Gamma \subset X_Z$ of the morphism $\phi_Z := \phi_{|X_Z} : X_Z \to Z$. Let $C_i := \phi^{-1}(x_i)$. Let us prove now the following:

Lemma 2.7 *$\mathcal{L}$, $x_1, \dots, x_r$, Γ being as above, for any $k > 0$, any stable map $f : \Gamma_1 \to X$ of class*

$$[\Gamma] + k[C]$$

meeting the r generic fibers $C_1, \dots C_r$ of ϕ has the property that $\phi \circ f(\Gamma_1) = Z$.

Proof. This is almost obvious. We just have to be a little careful with the singularities of Σ. Let us thus introduce a desingularization $\tau : \Sigma' \to \Sigma$ of Σ. Let $\mathcal{L}' := \tau^*\mathcal{L}$ and $\tilde{x}_1, \dots, \tilde{x}_r$ the points of Σ' over the generic points $x_1, \dots, x_r$ of Σ.

Then if $f : \Gamma_1 \to X$ is a curve as above, denote by $\widetilde{\Gamma}_1' \subset \Sigma'$ the proper transform of $\Gamma_1' := \phi \circ f(\Gamma_1) \subset \Sigma$ (counted with multiplicities) in Σ'. We observe that because the class of $f(\Gamma_1)$ is $[\Gamma] + k[C]$ and $\phi(C)$ is a point, $\widetilde{\Gamma}_1'$ belongs to one of the linear systems

$$\mid \tau^*\mathcal{L} - E \mid$$

on Σ', where E is an effective divisor supported on the exceptional locus of the desingularization. The linear system above has dimension $\leq r$, with equality if and only if E is empty. As $\widetilde{\Gamma}_1'$ passes through r generic points of Σ', it follows that the linear system $\mid \tau^*\mathcal{L} - E \mid$ has dimension r. Thus E is empty, and the curve Γ_1' does not meet the singular locus of Σ. Hence $\Gamma_1' \in \mid \mathcal{L} \mid$, and as it passes through $x_1, \dots, x_r$, it must be equal to Z. ■

Consider the morphism $\phi_Z : X_Z \to Z$. The smooth fiber of ϕ_Z is a $\mathbb{P}^1$, and the singular fibers are chains of $\mathbb{P}^1$'s. Note that by successive contractions of -1-curves not meeting Γ, one can construct from F_Z a geometrically ruled surface X_Z^0. The curve Γ is then the inverse image of a curve (still denoted Γ) in X_Z^0. Γ is a section of the structural morphism $p : X_Z^0 = \mathbb{P}(\mathcal{E}) \to Z$, where $\mathcal{E} := p_*\mathcal{O}_{X_Z^0}(\Gamma)$ is a rank 2 vector bundle on Z. We shall denote by $\sigma : X_Z \to X_Z^0$ such a contraction morphism. It will be convenient to choose the following basis E_i of the lattice

$$H^2(X_Z, \mathbb{Z})/\sigma^* H^2(X_Z^0, \mathbb{Z}) = \sigma^* H^2(X_Z^0, \mathbb{Z})^\perp.$$

We factor $\sigma : X_Z \to X_Z^0$ as a sequence of m blow-ups at one point. Let $\sigma_i : X_Z \to X_Z^i$ be the successive surfaces appearing in this factorization. Then we define for $i \geq 1$, $[E_i] := \sigma_i^*[E]$, where E is the exceptional curve of the blow-up $X_Z^i \to X_Z^{i-1}$. The classes $[E_i]$ are effective, and they satisfy

$$[E_i]^2 = -1, \ [E_i] \cdot K_{X_Z} = -1.$$

Proof of theorem 2.4. We will denote by $j : X_Z \hookrightarrow X$ the inclusion. For a curve Γ contained in X_Z, we will denote by $[\Gamma]_{X_Z} \in H^2(X_Z, \mathbb{Z})$ its cohomology class in X_Z and $[\Gamma] \in H^4(X, \mathbb{Z})$ its cohomology class in X. Hence $[\Gamma] = j_*[\Gamma]_{X_Z}$.

Let g, r, $x_1, \dots, x_r$, Z, $\Gamma \subset X_Z$ be as in lemmas 2.5, 2.7. Let $C_1, \dots, C_r$ be the generic fibers $\phi^{-1}(x_i)$ of ϕ. We now consider curves (stable maps) of genus g and class $[\Gamma] + k[C]$ in X, where k will be chosen sufficiently large.

The expected dimension of the family of such curves is equal to

$$-K_X \cdot ([\Gamma] + k[C]) = 2k - K_X \cdot [\Gamma] = 2k + \chi(\Gamma, N_{\Gamma/X})$$

$$= 2k + \chi(\Gamma, N_{\Gamma/X_Z}) + \chi(Z, N_{Z/\Sigma}) = 2k + \chi(\Gamma, N_{\Gamma/X_Z^0}) + r$$

$$= 2k + r + \chi(Z, \mathcal{E}) + g - 1 = 2k + r + deg\,\mathcal{E} + 1 - g.$$

If we consider the family of such curves meeting $C_1, \ldots, C_r$, its expected dimension is $N := 2k + deg\,\mathcal{E} + 1 - g$, and by lemma 2.7, we know that these curves are all contained in a given surface X_Z, where Z is a generic member of the linear system $\mid \mathcal{L} \mid$ on Σ. Note that N is the expected dimension of the space of deformations of a smooth curve of class $[\Gamma] + k[C]$ in X_Z. If k satisfies the condition $\Gamma^2 + 2k > 2g$, choose a section Γ_k of $X_Z \to Z$ of class $[\Gamma] + k[C]$ in X_Z.

Then as N_{Γ_k/X_Z} has degree $> 2g - 2$, it satisfies

$$H^1(\Gamma_k, N_{\Gamma_k/X_Z}) = 0.$$

As furthermore $H^1(\Gamma_k, (N_{X_Z/X})_{|\Gamma_k}) = H^1(N_{Z/\Sigma}) = 0$ by remark 2.6, one concludes that $H^1(\Gamma_k, N_{\Gamma_k/X}) = 0$, so that the deformation space of Γ_k in X is locally smooth of the right dimension $N + r$. Furthermore, if $y_1, \ldots, y_N \in \Gamma_k$ are generic, and $D := \{y_1, \ldots, y_N\}$, the restriction map:

$$H^0(\Gamma_k, N_{\Gamma_k/X_Z}) \to H^0(D, (N_{\Gamma_k/X_Z})_{|D})$$

is an isomorphism. Choosing N curves $B_1, \ldots, B_N \subset X$ meeting X_Z in $y_1, \ldots, y_N$ respectively, we find that Γ_k is an isolated point in the family of curves of genus g meeting $C_1, \ldots, C_r$ and $B_1, \ldots, B_N$. This gives at least one positive contribution to $GW^X_{g,[\Gamma_k]}(\underbrace{[C], \ldots, [C]}_{r}, [B_1], \ldots, [B_N])$.

However, in order to compute the Gromov-Witten invariant above, we need to control all curves in X_Z whose class in X is equal to $[\Gamma_k] = j_*[\Gamma_k]_{X_Z}$.

From lemma 2.7, we know that any curve in X of class $[\Gamma_k]$ which meets $C_1, \ldots, C_r$ is contained in X_Z. In order to conclude the proof, we have to compute the contribution to $GW^X_{g,[\Gamma_k]}(\underbrace{[C], \ldots, [C]}_{r}, [B_1], \ldots, [B_N])$ of all the families of curves $f : \Gamma_1 \to X_Z$, where Γ_1 is (maybe nodal) of arithmetic genus g, such that the class in X of $f(\Gamma_1)$ (counted with multiplicities) is equal to $[\Gamma_k]$, with k large.

For this, we need the following lemma

Lemma 2.8 *Classes in the kernel of $j_* : H_2(X_Z, \mathbb{Z}) \to H_2(X, \mathbb{Z})$ are integral combinations of the classes $\frac{1}{2}[C] - [E_i]$.*

Proof. $H_2(X_Z, \mathbb{Z})$ is generated over $\mathbb{Z}$ by the classes $[C]$ of the fiber of ϕ_Z, the class $[\Gamma]$ of a section of ϕ_Z and the classes $[E_i]$.

If $\alpha \in Ker\, j_*$, write

$$\alpha = n[C] + m[\Gamma] + \sum_i n_i(\frac{1}{2}[C] - [E_i]),\ n,\ m,\ n_i \in \mathbb{Z}.$$

Then we must have $m = 0$ because $\phi_*(j_*\alpha) = 0 = m[Z]$. Next we have $K_X \cdot [E_i] = -1$, because $K_{X_Z} \cdot [E_i] = -1$ and K_X has the same restriction as K_{X_Z} on the fibres of ϕ_Z. Furthermore $K_X \cdot [C] = -2 \neq 0$, and $K_X \cdot (\frac{1}{2}[C] - [E_i]) = 0$. Thus

$$j_*\alpha = 0 \Rightarrow K_X \cdot \alpha = 0 \Rightarrow n = 0.$$

Hence we proved that α is a combination of the $\frac{1}{2}[C] - [E_i]$ with integral coefficients n_i. Note that if such a combination belongs to $H_2(X_Z, \mathbb{Z})$, the $n_i \in \mathbb{Z}$ satisfy the condition that 2 divides $\sum_i n_i$.

■

We need thus to study maps $f : \Gamma_1 \to X_Z$ where Γ_1 is a nodal curve of genus g, $f_*[\Gamma_1]_{fund} = \gamma := [\Gamma_k] + \sum_i n_i(\frac{1}{2}[C] - [E_i])$. Note that for each such map, $\phi_Z \circ f : \Gamma_1 \to Z$ is an isomorphism on the (unique) genus g component of Γ_1 and contracts all the other components of Γ_1, which must be rational. As $deg\, N_{Z/\Sigma} > 2g-2$, it follows that $H^1(\Gamma_1, f^*N_{Z/\Sigma}) = 0$, and as an easy consequence, for fixed γ, the contribution of this family to $GW^X_{g,[\Gamma_k]}(\underbrace{[C],\ldots,[C]}_{r},[B_1],\ldots,[B_N])$ is equal to

$$GW^{X_Z}_{g,\gamma}([B_1]_{|X_Z},\ldots,[B_N]_{|X_Z}).$$

Of course $[B_i]_{|X_Z}$ is a multiple of the class of a point of X_Z. It thus remains to prove that for k large enough and any $\gamma = [\Gamma_k] + \sum_i n_i(\frac{1}{2}[C] - [E_i])$,

$$GW^{X_Z}_{g,\gamma}(\underbrace{[pt],\ldots,[pt]}_{N}) \geq 0.$$

Note that by deforming X_Z, we may assume the successive blow-ups starting from X^0_Z are at m distinct points $z_1,\ldots,z_m \in X^0_Z = \mathbb{P}(\mathcal{E})$.

We have the following:

Lemma 2.9 *m being fixed, for k sufficiently large, for a fixed choice of distinct points $z_1,\ldots,z_m \in \mathbb{P}(\mathcal{E})$, for any choice of integers $n_1,\ldots,n_m \in \mathbb{Z}$, any linear system L on the surface X'_Z which is $\mathbb{P}(\mathcal{E})$ blown-up at $z_1,\ldots,z_m$, of class*

$$c_1(L) = \gamma = [\Gamma_k] + l[C] - \sum_i n_i[E_i],\ l = \frac{1}{2}\sum_i n_i$$

satisfies $h^0(X'_Z, L) \leq N+1-g$, and when equality holds, the generic member of this linear system is smooth.

Assuming this lemma, it follows that for each γ as above, the dimension of the space of divisors in X'_Z of class γ has dimension $\leq N$. Thus the dimension of the space of divisors of class γ passing through N generically chosen points is 0. Furthermore, when equality holds, the finitely many divisors of class γ passing through N generically chosen points are smooth. It follows that the stable maps $f : \Gamma_1 \to X'_Z$ of class $f_*[\Gamma_1]_{fund} = \gamma$ passing through N generically chosen points have finitely many possible images which are smooth curves of genus g. Thus each of these f's must be an isomorphism, and there are also finitely many such stable maps f. It follows that $GW^{X'_Z}_{g,\gamma}(\underbrace{[pt],\ldots,[pt]}_{N}) \geq 0$. The proof of Theorem 2.4 is thus finished, modulo the proof of lemma 2.9. ■

Proof of lemma 2.9. Note that if $n_i \leq 0$, n_iE_i is contained in the fixed part of $\mid L \mid$. Thus it suffices to prove the result assuming $n_i \geq 0$, and $l \leq \frac{1}{2}\sum_i n_i$. Next, note that because $\gamma \cdot [C] = 1$, any section of L vanishing to order n_i at z_i vanishes to order $n_i - 1$ along the fiber C_{z_i} passing through z_i. This way, we are now reduced to the case where $n_i = 0$ or $n_i = 1$, and $l \leq \frac{1}{2}\sum_i n_i$. Notice that, in both reduction steps, if either one of the $n_i < 0$ or $n_i \geq 2$, the inequality becomes a strict inequality.

We have thus to show that for k large enough, for any choice of s points $z_{i_1},\ldots,z_{i_s}$ among $z_1,\ldots,z_m$, for $L \in Pic\, X'_Z$, with

$$c_1(L) = [\Gamma] + k[C] + l[C] - \sum_{j\leq s}[E_{i_j}],\ l \leq \frac{s}{2},$$

we have $h^0(X'_Z, L) \leq N + 1 - g$, while for $l < \frac{s}{2}$, we have $h^0(X'_Z, L) < N + 1 - g$. Note that for $l = 0$, $s = 0$, we can take for L the line bundle $\mathcal{O}_{X_Z}(\Gamma_k)$ which has $N + 1 - g$ sections.

The points $z_{i_j} \in \mathbb{P}(\mathcal{E})$ determine a vector bundle $\mathcal{E}'$ on Z, defined as the kernel of the evaluation map $p_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) = \mathcal{E} \to \oplus \mathcal{O}(1)_{|z_{i_j}}$. Then sections of L on X'_Z identify via p_* to sections of $\mathcal{E}'(D)$ on Z, for some $D \in Pic^{k+l}(Z)$. There are finitely many bundles $\mathcal{E}'$, and thus for k large enough, and any $l \geq 0$, $deg\, D = k + l$, we have $H^1(Z, \mathcal{E}'(D)) = 0$. As $deg\, \mathcal{E}' = deg\, \mathcal{E} - s$, it follows that

$$h^0(Z, \mathcal{E}'(D)) = \chi(Z, \mathcal{E}'(D)) = deg\, \mathcal{E}'(D) + 2 - 2g$$

$$= deg\, \mathcal{E} - s + 2k + 2l + 2 - 2g \leq deg\, \mathcal{E} + 2 - 2g + 2k = h^0(X_Z, \Gamma_k) = N + 1 - g,$$

with equality only when $2l = s$.

When equality holds, we have seen that all the n_i must be equal to 0 or 1, and the fact that the generic curve of class γ is smooth is deduced from the fact that with the notation above, the bundle $\mathcal{E}'(D)$ is generated by sections, for $D \in Pic^{k+l}(Z)$. ∎

3 Proofs of the main results

Proof of Proposition 0.7. Here $\psi : X \cong Y$ is a symplectomorphism with respect to some symplectic forms α, β on X, resp. Y, where α tames the complex structure on X and β is a deformation (as a symplectic form) of a Kähler form on Y. We assume that the conclusion of proposition 0.6 holds, but furthermore the rational map $\phi : X \dashrightarrow \Sigma$ is well-defined, and that either ϕ does not contract a divisor, or Σ is smooth. We can thus apply the conclusion of Theorem 2.4. This tells us that there exist integers $g < r$, cohomology classes $A_1, \ldots, A_N \in H^4(X, \mathbb{Z})$ and a homology class $[C'] \in H_2(X, \mathbb{Z})$ not proportional to $[C]$ such that

$$GW^X_{g,[C']}(\underbrace{[C], \ldots, [C]}_{r}, A_1, \ldots, A_N) \neq 0.$$

It follows that the curve class $[D'] = \psi_*[C']$ and the cohomology classes $A_i := \psi_* A_i \in H^*(Y)$ satisfy:

$$GW^Y_{g,[D']}(\underbrace{[D], \ldots, [D]}_{r}, A'_1, \ldots, A'_N) \neq 0.$$

But then this means that there exist a curve D' of genus g in Y, of class not proportional to $[D]$, meeting r generic fibers $D_1, \ldots, D_r$ of ϕ'. This implies that the surface Σ' contains genus g curves $D'' := \phi'(D')$ passing through r generic points, with $r > g$. In fact we will rather consider in the following lemma these curves as stable maps from a nodal curve to Σ. The normal bundle should be thought as $N_{\phi'}$.

Lemma 3.1 *If Σ' satisfies this property, the Kodaira dimension of Σ' is $-\infty$.*

Proof. Indeed, the generic curve D'' above has genus g and satisfies

$$h^0(N_{D''/\Sigma'}/Tors) \geq r > g,$$

where $Tors$ is the torsion of $N_{D''/\Sigma'}$. It follows that D'' contains at least one moving irreducible component D''_0 which has genus g_0, and satisfies

$$h^0(D''_0, N_{D''_0/\Sigma'}/Tors) > g_0.$$

We claim that this implies $deg\,(N_{D''_0/\Sigma'}/Tors) > 2g_0 - 2$. Assuming the claim, it follows that $deg\,(N_{D''_0/\Sigma'}) > 2g_0 - 2$, hence by adjunction that $K_{\Sigma'} \cdot D''_0 < 0$. This implies that $h^0(\Sigma', K^{\otimes l}_{\Sigma' |D''_0}) = 0, \forall l > 0$, and as D''_0 is moving, this implies that $h^0(\Sigma', K^{\otimes l}_{\Sigma'}) = 0, \forall l > 0$.

To see the claim, observe that Riemann-Roch gives

$$\chi(D''_0, N_{D''_0/\Sigma'}/Tors) = deg\,(N_{D''_0/\Sigma'}/Tors) + 1 - g_0.$$

Thus, if $deg\,(N_{D''_0/\Sigma'}/Tors) \leq 2g_0 - 2$ and $h^0(D''_0, N_{D''_0/\Sigma'}/Tors) > g_0$, we find that $h^1(D''_0, N_{D''_0/\Sigma'}/Tors) \neq 0$. Thus by Serre duality,

$$h^0(D''_0, (N_{D''_0/\Sigma'}/Tors)^* \otimes K_{D''_0}) \neq 0.$$

But then this implies, because D''_0 is irreducible, that

$$h^0(D''_0, N_{D''_0/\Sigma'}/Tors) \leq h^0(D''_0, K_{D''_0}) = g_0,$$

which is a contradiction. ■

Thus we conclude in this case that Σ' is (birationally) a ruled surface, and it follows that the basis of the rationally connected fibration of Y has dimension ≤ 1. By lemma 1.3, Y is rationally connected. ■

Proof of theorem 0.8. We assume that X and Y are symplectically equivalent and that, either X is Fano, or X is rationally connected with $b_2(X) \leq 2$. Thus there is a symplectomorphism $\psi : X \cong Y$ between X endowed with a Kähler form α and Y endowed with a symplectic form β which is a deformation of Kähler form.

We want to show that Y is rationally connected. We argue by contradiction, and assume that Y is not rationally connected. Applying lemma 1.2, we find that there are curve classes $[C)$, $[D]$ on X resp. Y, satisfying the following properties:

1. $c_1(K_Y) \cdot [D] = -2 = c_1(K_X) \cdot [C]$.
2. $GW^Y_{0,[D]}([y]) = 1 = GW^X_{0,[C]}([x])$.
3. The class $[C]$ is of minimal degree with respect to α, among those class curves satisfying the property $GW^X_{0,[C]}([x]) \neq 0$.

Furthermore, as proved in proposition 1.1, the manifolds X and Y are in this case almost conic bundles with fiber D, resp. C of class $[D]$, resp. $[C]$ where $[D] = \psi_*[C]$. Let us denote by $\phi : X \dashrightarrow \Sigma$, and $\phi' : Y \dashrightarrow \Sigma'$ the almost conic bundle structures on X and Y respectively.

Our assumption is that $b_2(X) \leq 2$ or X is Fano. Hence we can apply to X the corollary 2.2, because X is an almost conic bundle with fiber C. Thus we conclude, with the notations of this section, that the morphism $\phi : X \dashrightarrow \Sigma$ with fiber C is well-defined, unless there exists a curve class $[C']$ not proportional to $[C]$ such that for some cohomology classes $A_1, \ldots, A_r \in H^*(X'_{n-1})$,

$$GW^X_{0,[C']}([C], A_1, \ldots, A_r) \neq 0.$$

However, in the later case, we conclude, by denoting $[D'] = \psi_*[C']$, $A'_i = \psi_* A_i$, that

$$GW^Y_{0,[D']}([D], A'_1, \ldots, A'_r) \neq 0.$$

It follows that there exists a rational curve of class $[D']$ which meets a generic curve $D \subset Y$ and as $[D']$ is not proportional to D, we conclude that $\phi'(D')$ is not a point. Hence it follows that the surface Σ' is swept-out by rational curves and the basis of the rationally connected fibration of Y has dimension ≤ 1, which implies by lemma 1.3 that Y is rationally connected, a contradiction.

Thus the morphism $\phi : X \to \Sigma$ with fiber C is well-defined. Furthermore, by lemma 2.3, we may assume that ϕ does not contract a divisor to a point. By proposition 0.7, Y is then rationally connected, which is a contradiction.

References

[1] Kai Behrend and Barbara Fantechi. The intrinsic normal cone. *Invent. Math.*, 128(1):45–88, 1997.

[2] Frédéric Campana. Connexité rationnelle des variétés de Fano. *Ann. Sci. École Norm. Sup. (4)*, 25(5):539–545, 1992.

[3] Frédéric Campana, Thomas Peternell. Rigidity theorems for primitive Fano threefolds. *Comm. Analysis Geom.* 2 (1994), 173-201.

[4] William Fulton and Rahul Pandharipande. Notes on stable maps and quantum cohomology. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 45–96. Amer. Math. Soc., Providence, RI, 1997.

[5] Tom Graber, Joe Harris, and Jason Starr. Families of rationally connected varieties. *J. Amer. Math. Soc.*, 16(1):57–67 (electronic), 2003.

[6] Jianxun Hu, Tian-Jun Li, and Yongbin Ruan. Birational cobordism invariance of uniruled symplectic manifolds. *Invent. Math.*, 2008.

[7] Yujiro Kawamata, Katsumi Matsuda, Kenji Matsuki. Introduction to the minimal model problem. *Algebraic geometry, Sendai 1985*, 283-360, Adv. Stud. Pure Math. 10, North-Holland, Amsterdam (1987).

[8] János Kollár. Low degree polynomial equations: arithmetic, geometry and topology. *European Congress of Mathematics, Vol. I*, (Budapest, 1996), 255–288, Progr. Math., 168, Birkhäuser, Basel, 1998.

[9] János Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics].* Springer-Verlag, Berlin, 1996.

[10] János Kollár, Yoichi Miyaoka, and Shigefumi Mori. Rational connectedness and boundedness of Fano manifolds. *J. Differential Geom.*, 36(3):765–779, 1992.

[11] János Kollár, Yoichi Miyaoka, and Shigefumi Mori. Rationally connected varieties. *J. Algebraic Geom.*, 1(3):429–448, 1992.

[12] Jun Li and Gang Tian. Comparison of algebraic and symplectic Gromov-Witten invariants. *Asian J. Math.*, 3(3):689–728, 1999.

[13] Dusa McDuff. The structure of rational and ruled symplectic 4-manifolds. *J. Amer. Math. Soc.*, 3(3):679–712, 1990.

[14] Dusa McDuff. Hamiltonian S^1 manifolds are uniruled, preprint. 2007.

[15] Yoichi Miyaoka and Thomas Peternell. *Geometry of higher-dimensional algebraic varieties*, volume 26 of *DMV Seminar*. Birkhäuser Verlag, Basel, 1997.

[16] Shigefumi Mori. Threefolds whose canonical bundles are not numerically effective. *Ann. of Math. (2)*, 116(1):133–176, 1982.

[17] Shigefumi Mori and Shigeru Mukai. Classification of Fano 3-folds with $B_2 \geq 2$. *Manuscripta Math.*, 36(2):147–162, 1981/82.

[18] Yongbin Ruan. Symplectic topology and extremal rays. *Geom. Funct. Anal.*, 3(4):395–430, 1993.

[19] Yongbin Ruan. Virtual neighborhoods and pseudo-holomorphic curves. In *Proceedings of 6th Gökova Geometry-Topology Conference*, volume 23, pages 161–231, 1999.

[20] Bernd Siebert. Algebraic and symplectic Gromov-Witten invariants coincide, *Ann. Inst. Fourier* 49 (1999) 1743–1795.