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Cosmological constant and lensing

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Abstract

The effect of the cosmological constant on the curvature of light due to an isolated spherical mass is recalculated without using the lens equation and compared to a lensing cluster.

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Key-Words: cosmological parameters – lensing

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1 Introduction

Until september 2007, the official opinion had it that the deflection of light by an isolated spherical mass is independent of the cosmological constant. This was corrected in the beautiful work by Rindler & Ishak [1]. One month later, Ishak, Rindler et al. [2] showed that this dependence is not academic by exhibiting several lensing clusters, that do constrain the cosmological constant within today's bounds from the standard model of cosmology.

Two consecutive papers by Lake [3] and Sereno [4] confirm the dependence. Sereno however finds a different expression for the deflection. Moreover he computes how the time delay is affected by the cosmological constant, which is particularly interesting in the light of the impressive progress [5] in the observation of such a time delay.

The aim of the present paper is to to give a detailed, self-contained account of how to compute the deflection in presence of a cosmological constant and only in terms of quantities observed in lensing clusters. In particular we do not use the lens equation. Our computations are to first order in Newton's constant and in the deflection angle. We do not neglect higher terms in the cosmological constant as they do contribute significantly in certain lensing clusters.

2 The set up

Consider a universe, which is empty except for one static, spherical, non-rotating mass M, the lens L. A source S, at rest with respect to the lens, emits photons, which are observed at nostra terra T also assumed at rest. We neglect the masses of the source and of our local group. We use polar coordinates (r, θ, φ) centered at the lens. Because of spherical symmetry the photons' trajectory is in a plane that we take to be $\theta = \pi/2$. The angle φ is measured with respect to the axis defined by the source, $\varphi_S = 0$. With a cosmological constant, the gravitational field outside the mass M is given by the Kottler metric,

$$\mathrm{d}\tau^2 = B\,\mathrm{d}t^2 - B^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\varphi^2, \quad \theta = \pi/2, \quad B = 1 - \frac{2GM}{r} - \tfrac{1}{3}\Lambda r^2. \tag{1}$$

Our initial conditions for the photon are position $(r_S, \varphi_S = 0)$ and velocity characterised by the coordinate angle $\epsilon_S := r_S\,|\mathrm{d}\varphi/\mathrm{d}r\,(r_S)|$. Let r_0 be the peri-lens. We will suppose $2GM/r_0 \ll 1$ and $\epsilon_S \ll 1$ and keep only terms linear in these two quantities. We will also suppose that $\Lambda r^2/3 < 9/10$ to avoid the coordinate singularity at the equator of the de Sitter sphere. The final conditions are here on earth (r_T, φ_T) where the photons are detected together with their (coordinate) angle $\epsilon_T := r_T\,|\mathrm{d}\varphi/\mathrm{d}r\,(r_T)|$.

We are interested in situations where more than one images of the source are seen. Because of the spherical symmetry there can be only one or two images depending on the distances of earth and source from the lens. There is however one exception, when earth, lens and source are aligned. Then $r_T = r_S$, $\varphi_T = \pi$ and we get an infinite number of images forming Einstein's ring.

We will concentrate on the non-aligned case and denote by $\epsilon'_S,\ r'_0,\ \epsilon'_T$ the parameters of the lower trajectory, figure 1.

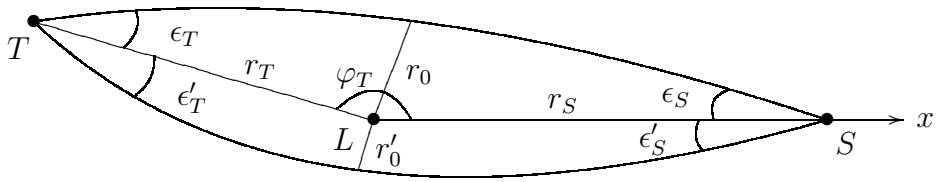

Figure 1: A double image

Our tasks are:

- Integrate the photons' geodesics and compute φ_T as a function of $r_S,\ \epsilon_S,\ r_T$ and φ'_T as a function of $r_S,\ \epsilon'_S,\ r_T$. The surprising, but well known, fact is that in this calculation the cosmological constant drops out, which is not the case for massive geodesics.
- Compute the coordinate distance r_T as a function of $r_S,\ \epsilon_T,\ \epsilon'_T$, such that the two trajectories meet, $\varphi'_T = \varphi_T$.
- Compute the physically measured angles α_T and α'_T from the coordinate angles ϵ_T and ϵ'_T. Here the cosmological constant re-enters the scene [1].
- Compute the area distance d_L of the lens as seen from earth from the coordinate distance r_T.
- Compute the area distance d_S of the source as seen from earth from the coordinate distances r_T, r_S and the angles α_T and α'_T. Here the cosmological constant will also play a role.

Finally we will compare the theory to the observation of the lensing cluster SDSS J1004 +4112 with a quasar as source. The angles α_T and α'_T are measured as well as the mass of the lens and the redshifts z_L of the lens and z_S of the source [6, 7].

- We will compute the area distances d_L and d_S from their redshifts z_L and z_S using the Hubble diagram for a flat 3-space with cosmological constant and 27 % of matter.

3 Integrating the geodesics

We start with the list of the non-vanishing Christoffel symbols for the Kottler metric with $\theta = \pi/2$ and denote $' := \mathrm{d}/\mathrm{d}r$,

$$\Gamma^t{}_{tr} = B'/(2B), \quad \Gamma^r{}_{tt} = BB'/2, \quad \Gamma^r{}_{rr} = -B'/(2B), \tag{2}$$

$$\Gamma^r{}_{\varphi\varphi} = -rB, \quad \Gamma^\varphi{}_{r\varphi} = 1/r. \tag{3}$$

The geodesic equations read:

$$\ddot{t} + B'/B\,\dot{t}\dot{r} = 0, \tag{4}$$
$$\ddot{r} + \tfrac{1}{2}BB'\dot{t}^2 - \tfrac{1}{2}BB'\dot{r}^2 - rB\dot{\varphi}^2 = 0, \tag{5}$$
$$\ddot{\varphi} + 2r^{-1}\dot{r}\dot{\varphi} = 0, \tag{6}$$

where we denote the affine parameter by p and $\dot{} := \mathrm{d}/\mathrm{d}p$. We immediately get three first integrals:

$$\dot{t} = 1/B, \quad r^2\dot{\varphi} = J, \quad \dot{r}^2/B + J^2/r^2 - 1/B = -E. \tag{7}$$

The last two come from invariance of the metric under rotations and time translations and the integration constants J and E have the meaning of angular momentum and energy per unit of mass. For the photon, $E = 0$. Eliminating affine parameter and (coordinate) time we get:

$$\frac{\mathrm{d}r}{\mathrm{d}\varphi} = \pm r\sqrt{r^2/J^2 - B}. \tag{8}$$

At the peri-lens, $\mathrm{d}r/\mathrm{d}\varphi(r_0) = 0$ and therefore $J = r_0 B(r_0)^{-1/2}$. Substituting J into equation (8), the cosmological constant drops out and we have:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}r} = \pm\frac{1}{r\sqrt{r^2/r_0^2 - 1}}\left[1 - \frac{2GM}{r} - \frac{2GM}{r_0}\frac{r}{r + r_0}\right]^{-1/2}. \tag{9}$$

From now on we will omit terms of order $(GM/r_0)^2$, ϵ_S^2 and $(GM/r_0)\epsilon_S$ and write equalities up to this order with a $\sim$ sign. For example:

$$\epsilon_S = r_S|\mathrm{d}\varphi/\mathrm{d}r(r_S)| \sim (r_S^2/r_0^2 - 1)^{-1/2} \sim r_0/r_S. \tag{10}$$

Note that for the upper trajectory, $\mathrm{d}\varphi/\mathrm{d}r$ is negative for r between r_S and r_0, positive between r_0 and r_T. Therefore

$$\varphi_T = \int_{r_0}^{r_S}\left|\frac{\mathrm{d}\varphi}{\mathrm{d}r}\right|\mathrm{d}r + \int_{r_0}^{r_T}\left|\frac{\mathrm{d}\varphi}{\mathrm{d}r}\right|\mathrm{d}r. \tag{11}$$

Using $\int x^{-1}(x^2-1)^{-1/2}\,\mathrm{d}x = -\arcsin 1/x$, $\int x^{-2}(x^2-1)^{-1/2}\,\mathrm{d}x = (x^2-1)^{1/2}/x$, $\int(x+1)^{-1}(x^2-1)^{-1/2}\,\mathrm{d}x = [(x-1)/(x+1)]^{1/2}$, we get to linear order:

$$\varphi_T \sim \pi - \epsilon_S\left(1 + \frac{r_S}{r_T}\right) + \frac{4GM}{\epsilon_S r_S}. \tag{12}$$

For the lower trajectory the signs of $\mathrm{d}\varphi/\mathrm{d}r$ are opposite and we have

$$\varphi_T' \sim \pi + \epsilon_S'\left(1 + \frac{r_S}{r_T}\right) - \frac{4GM}{\epsilon_S' r_S}. \tag{13}$$

We can trade ϵ_S for ϵ_T using $r_0 \sim \epsilon_S r_S \sim \epsilon_T r_T$ and likewise for the primed quantities.

For the two trajectories to meet, $\varphi_T = \varphi_T'$, we must have:

$$\frac{r_T}{r_S} \sim \frac{4GM}{\epsilon_T\epsilon_T' r_T} - 1. \tag{14}$$

4 From coordinate angles to physical angles

Let us compute the relation between the coordinate angle ϵ_T and the corresponding physical angle α_T. Officially since october 1983, lengths are measured in terms of proper time of flight of photons, one nano-second $\sim$ 0.3 m $\sim$ 1 foot. The coordinate time $\mathrm{d}t$ it takes the photon to travel from (r_T, φ_T) to $(r_T - \mathrm{d}r, \varphi_T)$ is computed from $0 = B(r_T)\,\mathrm{d}t^2 - B(r_T)^{-1}\,\mathrm{d}r^2$, see figure 2.

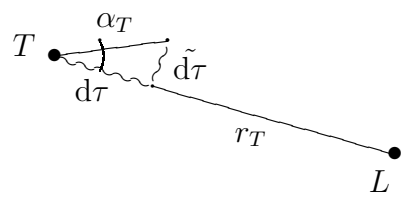

Figure 2: Measuring the angle α_T in ns/ns

Likewise the coordinate time $\mathrm{d}\tilde{t}$ it takes the photon to travel from $(r_T - \mathrm{d}r, \varphi_T)$ to $(r_T - \mathrm{d}r, \varphi_T - \mathrm{d}\varphi)$ is computed from $0 = B(r_T)\,\mathrm{d}\tilde{t}^2 - r_T^2\,\mathrm{d}\varphi^2$. The we have,

$$\tan\alpha_T = \frac{\mathrm{d}\tilde{\tau}}{\mathrm{d}\tau} = \frac{\mathrm{d}\tilde{t}\sqrt{B(r_T)}}{\mathrm{d}t\sqrt{B(r_T)}} = \frac{r_T\,\mathrm{d}\varphi}{\mathrm{d}r/\sqrt{B(r_T)}} = \epsilon_T\sqrt{B(r_T)} \sim \epsilon_T\sqrt{1-\Lambda r_T^2/3} \sim \alpha_T. \quad (15)$$

Similarly we get $\alpha'_T \sim \epsilon'_T\sqrt{1-\Lambda r_T^2/3}$. Rindler & Ishak [1] compute the physical angle by means of the usual formula

$$\cos\alpha_T = \frac{(\mathrm{d}r, 0)\cdot(\mathrm{d}r, \mathrm{d}\varphi)}{[(\mathrm{d}r, 0)\cdot(\mathrm{d}r, 0)]^{1/2}[(\mathrm{d}r, \mathrm{d}\varphi)\cdot(\mathrm{d}r, \mathrm{d}\varphi)]^{1/2}}\,, \quad (16)$$

where the scalar product $\cdot$ comes from the negative of the spatial part of the Kottler metric. The two ways to compute the physical angle agree to all orders.

Finally we note that in Schwarzschild's solution and far out in the asymptotic region, physical and coordinate angles coincide.

5 From coordinate distances to area distances

Imagine a standard candle radiating photons isotropically at the position of the lens. Its area distance d_L as seen from the earth is defined by the relation between the infinitesimal solid angle $\mathrm{d}\Omega$ in which photons are radiated and the infinitesimal area $\mathrm{d}S$ of the light-sensitive plate on which they are collected on earth, $\mathrm{d}S/\mathrm{d}\Omega = 4\pi d^2$. The isotropy of the standard candle is of course with respect to physically measured angles α which in this position do coincide with the polar angles θ and φ, see figure 3.

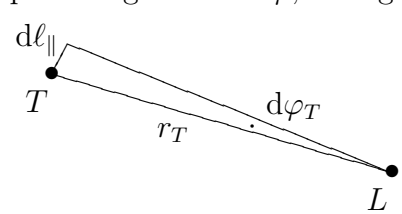

Figure 3: Area distance of the lens as seen from earth

The area is again measured by means of the time of flight of test photons traveling along its edges. For the edge in the plane $\theta = \pi/2$ this coordinate time of flight is given by $0 = B(r_T)\,\mathrm{d}t^2 - r_T^2\,\mathrm{d}\varphi^2$. The proper time is therefore $\mathrm{d}\tau = r_T \mathrm{d}\varphi$. For symmetry reasons we get the same proper time for an edge orthogonal to the plane $\theta = \pi/2$ and consequently

$$d_L = r_T. \tag{17}$$

For the area distance d_S of the source as seen from the earth, the calculation is more involved. The isotropy of the standard candle in the plane $\theta = \pi/2$ is defined with respect to the physical angle α_S whose period is $2\pi\, B(r_S)^{1/2}$. The proper time of flight length of the infinitesimal edge at earth in this plane is $\mathrm{d}\ell_\parallel = r_T|\mathrm{d}\varphi_T|$, see figure 4.

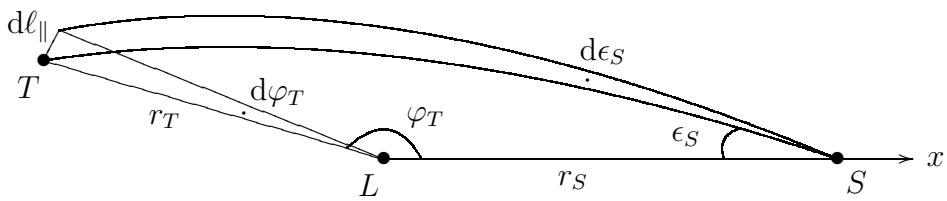

Figure 4: Area distance of the source as seen from earth

Differentiating equation (12) with respect to ϵ_S and using equation (14) we get

$$\begin{aligned} \mathrm{d}\varphi_T &\sim -\left(1+\frac{r_S}{r_T}\right)\left(1+\frac{\epsilon_T'}{\epsilon_T}\right)\mathrm{d}\epsilon_S \sim -\left(1+\frac{r_S}{r_T}\right)\left(1+\frac{\alpha_T'}{\alpha_T}\right)\frac{\mathrm{d}\alpha_S}{\sqrt{B(r_S)}} \\ &\sim -\left(1+\frac{r_S}{r_T}\right)\left(1+\frac{\alpha_T'}{\alpha_T}\right)\frac{\mathrm{d}\alpha_S}{\sqrt{1-\Lambda r_S^2/3}}\,. \end{aligned} \tag{18}$$

For an edge orthogonal to the plane $\theta = \pi/2$, the calculation is easier because of the axial symmetry around the axis lens - source. An infinitesimal rotation by a coordinate angle $\mathrm{d}\eta$ of period 2π results in an infinitesimal length at the earth of

$$\mathrm{d}\ell_\perp \sim r_T \sin(\pi - \varphi_T)\,\mathrm{d}\eta \sim r_T\left(1+\frac{r_T}{r_S}\right)\frac{|\alpha_T - \alpha_T'|}{\sqrt{B(r_T)}}\,\mathrm{d}\eta. \tag{19}$$

Replacing the coordinate angle $\mathrm{d}\eta$ by a physical angle we finally get the area distance from $\mathrm{d}S = \mathrm{d}\ell_\parallel \mathrm{d}\ell_\perp$:

$$d_S \sim \frac{r_T + r_S}{\sqrt{1-\Lambda r_S^2/3}}\sqrt{\left(1+\frac{\alpha_T'}{\alpha_T}\right)\frac{r_T}{r_S}\,\frac{|\alpha_T - \alpha_T'|}{\sqrt{1-\Lambda r_T^2/3}}}\,. \tag{20}$$

Note that a singularity occurs when earth, lens and source are aligned, $\alpha_T = \alpha_T'$. This comes from the focusing of the lens.

6 From redshifts to area distances

Here comes the shaky part of the reasoning. Indeed, as long as we do not have a solution of Einstein's equation interpolating between Kottler's and Friedmann's solutions, we do not know above what length scale the masses of the other galaxies and their expansion must be taken into account. Justified only by this ignorance, we make the crude assumption that the measured redshifts are exclusively due to expansion, although we had put the source and the earth at rest in the Kottler metric. We will use the Hubble diagram from the standard model of cosmology with a flat 3-space, a cosmological constant $\Lambda = 1.5 \cdot 10^{-52}$ m^{-2}, a Hubble parameter $H_0 = 2.5 \cdot 10^{-18}$ s^{-1} and 27 % of matter in order to compute the area distances. A Runge-Kutta integration of Einstein's equation is used to obtain the scale factor $a(t)$. For numerical convenience, we choose the initial condition $a_0 = 1.2 \cdot 10^{26}$ m. Then the area distance d follows from the redshift z:

$$d(z) = \int_0^z \frac{\mathrm{d}a(t(\tilde{z}))/\mathrm{d}t}{a(t(\tilde{z}))}\,\mathrm{d}\tilde{z}\,, \tag{21}$$

where by abuse of notation we write $t(z)$ for the inverse function of $z(t) := 1/a(t) - 1$. A rough fit to the numerical solution is given by [8],

$$a(t) \sim a_0(pH_0t)^{1/p}, \quad p = 0.69\,. \tag{22}$$

It is good to 3 % up to $z = 2$ and yields

$$d(z) \sim \frac{(z+1)^{1-p} - 1}{(1-p)H_0}. \tag{23}$$

Because of its singularity we cannot use the area distance d_S (20) to estimate the *position* of the source. The singularity is interpreted as magnification rather than an actual get-together. But then we need another assumption to estimate the position of the source. The simplest assumption coming to mind is

$$d_S = \frac{r_T + r_S}{\sqrt{1 - \Lambda r_S^2/3}} \tag{24}$$

which would be the correct area distance in absence of the lens. Of course one can argue that when we send the mass of the source to zero, we should send Λ to zero at the same time, $d_S = r_T + r_S$.

7 SDSS J1004+4112

Let us see how our assumptions compare to observation. Consider the lensing cluster of SDSS J1004+4112 and the quasar as source [6, 7]. As we have at least 4 images, the cluster cannot be spherically symmetric. We will again close our eyes and consider only the images C and D with $\alpha_T = 10'' \pm 10$ % and $\alpha'_T = 5'' \pm 10$ %. The mass of the cluster is $M = (1 \pm 0.2) \cdot 10^{44}$ kg. The cluster has a redshift of $z_L = 0.68$ yielding $d_L = r_T = 7.0 \cdot 10^{25}$

m from the numerical integration. For the quasar we have $z_S = 1.734$ and $d_S = 13.7 \cdot 10^{25}$ m, which translates into $r_S = 7.5 \cdot 10^{20}$ m with the singular area distance equation (20). However with this value the condition (14)

$$\frac{r_T}{r_S} \sim \frac{4GM}{\alpha_T \alpha'_T r_T} (1 - \Lambda r_T^2/3) - 1. \tag{25}$$

for the two photon trajectories to meet cannot be satisfied whatever the value of Λ, zero or positive. If we use the non-singular area distance (24) we get $r_S = 5.6 \cdot 10^{25}$ m and the condition for the trajectories to intersect can be met within the experimental error bars for the cluster mass M and the angles α_T and α'_T only if the cosmological constant satisfies

$$\Lambda > 0.81 \cdot 10^{-52} \text{ m}^{-2}. \tag{26}$$

On the other hand, if we assume $d_S = r_T + r_S$ we get $r_S = 6.7 \cdot 10^{25}$ m and the condition for the trajectories to intersect can be met within the error bars only if the cosmological constant satisfies

$$\Lambda = (2.5 \pm 1.5) \cdot 10^{-52} \text{ m}^{-2}. \tag{27}$$

In both cases we have not varied the cosmological constant in the Hubble diagram and it is encouraging that both constraints, (26) and(27) are compatible with the present observational bounds, $\Lambda = (1.5 \pm 0.7) \cdot 10^{-52}$ m^{-2}, from the standard model of cosmology. Note that Ishak, Rindler et al. [2], presumingly assuming $d_S = r_T + r_S$, find an upper bound from two other lensing clusters.

8 Conclusions

We agree with Rindler & Ishak, gravitational lensing depends on the cosmological constant. Our formula (25) for this dependence agrees with theirs [1] in the aligned case. We also agree with them [2], this dependence is not negligible for certain clusters and further confrontation with observational data is necessary.

Of course the main problem remains to find a solution of Einstein's equation that interpolates between a homogeneous family of static, curved Kottler solutions and the expanding, flat Friedmann solution. We hope that Rindler & Ishak's ground breaking work will give new impetus to this fundamental problem.

Many years ago, dark energy was proposed as an alternative to the cosmological constant. And still, nobody has ever told us how dark energy modifies the Schwarzschild solution.

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