Computation of 2-groups of positive classes of exceptional number fields

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Abstract. We present an algorithm for computing the 2-group \( C_2^{\text{pos}} F \) of the positive divisor classes in case the number field \( F \) has exceptional dyadic places. As an application, we compute the 2-rank of the wild kernel \( \text{WK}_2(F) \) in \( K_2(F) \).

Résumé. Nous développons un algorithme pour déterminer le 2-groupe \( C_2^{\text{pos}} F \) des classes positives dans le cas où le corps de nombres considéré \( F \) possède des places paires exceptionnelles. Cela donne en particulier le 2-rang du noyau sauvage \( \text{WK}_2(F) \).

1 Introduction

The logarithmic \( \ell \)-class group \( \tilde{C}_\ell F \) was introduced in [10] by J.-F. Jaulent who used it to study the \( \ell \)-part \( \text{WK}_2(F) \) of the wild kernel in number fields: if \( F \) contains a primitive \( 2\ell t \)-th root of unity \((t > 0)\), there is a natural isomorphism

\[
\mu_\ell \otimes \mathbb{Z} \tilde{C}_\ell F \cong \text{WK}_2(F) / \text{WK}_2(F)_{\ell t},
\]

so the \( \ell \)-rank of \( \text{WK}_2(F) \) coincides with the \( \ell \)-rank of the logarithmic group \( \tilde{C}_\ell F \). An algorithm for computing \( \tilde{C}_\ell F \) for Galois extensions \( F \) was developed in [4] and later generalized and improved for arbitrary number fields in [3].

In case the prime \( \ell \) is odd, the assumption \( \mu_\ell \subset F \) may be easily passed if one considers the cyclotomic extension \( F(\mu_\ell) \) and gets back to \( F \) via the so-called transfer (see [12], [15] and [17]). However for \( \ell = 2 \) the connection between symbols and logarithmic classes is more intricate: in the non-exceptional situation (i.e. when the cyclotomic \( \mathbb{Z}_2 \)-extension \( F^c \) contains the fourth root of unity \( i \)) the 2-rank of \( \text{WK}_2(F) \) still coincides with the 2-rank of \( \tilde{C}_2 F \). Even more if the number field \( F \) has no exceptional dyadic place (i.e. if one has \( i \in F_q^c \) for any \( q \mid 2 \)), the same result holds if one replace the ordinary logarithmic class group \( \tilde{C}_2 F \) by a narrow version \( \tilde{C}_2^{\text{pos}} F \). The algorithmic aspect of this is treated in [11].

Last in [13] the authors pass the difficulty in the remaining case by introducing a new 2-class groups \( \tilde{C}_2^{\text{pos}} F \), the \( 2 \)-group of positive divisor classes, which satisfies the rank identity: \( \text{rk}_2 \tilde{C}_2^{\text{pos}} F = \text{rk}_2 \text{WK}_2(F) \).

In this paper we develop an algorithm for computing both \( \tilde{C}_2^{\text{pos}} F \) and \( \tilde{C}_2^{\text{pos}} F \) in case the number field \( F \) does contain exceptional dyadic places.

We conclude with several examples. Combining our algorithm with the work of Belabas and Gangl [1] on the computation of the tame kernel of \( K_2 \) we obtain the complete structure of the wild kernel in some cases.

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2 Positive divisor classes of degree zero

2.1 The group of logarithmic divisor classes of degree zero

Throughout this paper the prime number \( \ell \) equals 2 and we let \( i \) be a primitive fourth root of unity. Let \( F \) be a number field of degree \( n = r + 2c \). According to [9], for every place \( p \) of \( F \) there exists a 2-adic valuation \( \tilde{v}_p \) which is related to the wild 2-symbol in case the cyclotomic \( \mathbb{Z}_2 \)-extension of \( F_p \) contains \( i \). The degree \( \deg(p) \) is a 2-adic integer such that the image of the map \( \text{Log} | \mathcal{P}_F^0 \) is the \( \mathbb{Z}_2 \)-module \( \text{deg}(p) \mathbb{Z}_2 \) (see [10]). (By \( \text{Log} \) we mean the usual 2-adic logarithm.)

The construction of the 2-adic logarithmic valuations \( \tilde{v}_p \) yields

\[
\forall \alpha \in \mathcal{R}_F := \mathbb{Z}_2 \otimes_{\mathbb{Z}} F^{	imes} : \sum_{p \in \mathcal{P}_F^0} \tilde{v}_p(\alpha) \deg(p) = 0, \tag{1}
\]

where \( \mathcal{P}_F^0 \) denotes the set of finite places of the number field \( F \). Setting

\[
\text{div}(\alpha) := \sum_{p \in \mathcal{P}_F^0} \tilde{v}_p(\alpha)p
\]

we obtain by \( \mathbb{Z}_2 \)-linearity:

\[
\text{deg(\text{div}(\alpha)))} = 0. \tag{2}
\]

We define the 2-group of logarithmic divisors of degree 0 as the kernel of the degree map \( \text{deg} \) in the direct sum \( \mathcal{D}_F = \sum_{p \in \mathcal{P}_F^0} \mathbb{Z}_2 p \):

\[
\mathcal{D}_F := \left\{ \sum_{p \in \mathcal{P}_F^0} a_p p \in \mathcal{D}_F \mid \sum_{p \in \mathcal{P}_F^0} a_p \deg(p) = 0 \right\};
\]

and the subgroup of principal logarithmic divisors as the image of the logarithmic map \( \text{div} \):

\[
\mathcal{P}_F := \{ \text{div}(\alpha) \mid \alpha \in \mathcal{R}_F \}.
\]

Because of (2) \( \mathcal{P}_F \) is clearly a subgroup of \( \mathcal{D}_F \). More ever by the so-called extended Gross conjecture, the factorgroup

\[
\mathcal{C}_F := \mathcal{D}_F / \mathcal{P}_F
\]

is a finite 2-group, the 2-group of logarithmic divisor classes. So, under this conjecture, \( \mathcal{C}_F \) is just the torsion subgroup of the group

\[
\mathcal{C}_F := \mathcal{D}_F / \mathcal{P}_F
\]

of logarithmic classes (without any assumption of degree).

**Remark 1.** Let \( F^+ \) be the set of all totally positive elements of \( F^\times \) (i.e. the subgroup \( F^+ := \{ x \in F^\times \mid x_p > 0 \text{ for all real } p \} \)). For

\[
\mathcal{P}_F^+ := \{ \text{div}(\alpha) \mid \alpha \in \mathcal{R}_F^+ := \mathbb{Z}_2 \otimes_{\mathbb{Z}} F^+ \}
\]

the factor group

\[
\mathcal{C}_F^+ := \mathcal{D}_F / \mathcal{P}_F^+
\]

(resp. \( \mathcal{C}_F^{res} := \mathcal{D}_F / \mathcal{P}_F^{res} \))

is the 2-group of narrow logarithmic divisor classes of the number field \( F \) (resp. the 2-group of narrow logarithmic divisor classes of degree 0) introduced in [13] and computed in [14].
2.2 Signs and places

For a field $F$ we denote by $F^c$ (respectively $F^c[i]$) the cyclotomic $\mathbb{Z}_2$-extension (resp. the maximal cyclotomic pro-2-extension) of $F$.

We adopt the notations and definitions in this section from [13].

**Definition 1 (signed places).** Let $F$ be a number field. We say that a non-complex place $p$ of $F$ is signed if and only if $F_p$ does not contain the fourth root $i$. These are the places which do not decompose in the extension $F[i]/F$.

We say that $p$ is logarithmically signed if and only if the cyclotomic $\mathbb{Z}_2$-extension $F_p^c$ does not contain $i$. These are the places which do not decompose in $F^c[i]/F^c$.

**Definition 2 (sets of signed places).** By $PS$, respectively $PLS$, we denote the sets of signed, respectively logarithmically signed, places:

$$PS := \{ p \mid i \not\in F_p \},$$
$$PLS := \{ p \mid i \not\in F_p^c \}.$$ 

A finite place $p \in PLS$ is called exceptional. The set of exceptional places is denoted by $PE$. Exceptional places are even (i.e. finite places dividing 2).

These sets satisfy the following inclusions:

$$P \subset PLS = PE \cup PR \subset Pl(2) \cup Pl(\infty)$$

where $Pl(2)$, $Pl(\infty)$, $PR$ denote the sets of even, infinite and real places of $F$, respectively. From this the finiteness of $PLS$ is obvious.

We recall the canonical decomposition $\mathbb{Q}_2^\times = 2^\mathbb{Z} \times (1 + 4\mathbb{Z}_2) \times \langle -1 \rangle$ and we denote by $\epsilon$ the projection from $\mathbb{Q}_2^\times$ onto $\langle -1 \rangle$.

**Definition 3 (sign function).** For all places $p$ we define a sign function via

$$sg_p : F_p^\times \to \langle -1 \rangle : x \mapsto \begin{cases}
1 & \text{for } p \text{ complex} \\
\text{sign}(x) & \text{for } p \text{ real} \\
\epsilon(Np^{\nu_p(x)}) & \text{for } p \nmid 2 \infty \\
\epsilon(NK_p/Q_2(x)Np^{\nu_p(x)}) & \text{for } p \mid 2
\end{cases}.$$ 

These sign functions satisfy the product formula:

$$\forall x \in F^\times \prod_{p \in Pl_f} sg(x) = 1.$$ 

In addition we have:

**Proposition 1.** The places $p$ of $F$ satisfy the following properties:

(i) if $p \in PLS$ then $(sg_p, \tilde{v}_p)$ is surjective;

(ii) if $p \in PS \setminus PLS$ then $sg_p(\ ) = (-1)^{\tilde{v}_p(\ )}$ and $\tilde{v}_p$ is surjective;

(iii) if $p \not\in PS$ then $sg_p(F_p^\times) = 1$ and $\tilde{v}_p$ is surjective.

**Remark 2.** The logarithmic valuation $\tilde{v}_p$ is surjective in all three cases. Part 2 of the preceding result is often used for testing $p \in PLS$. 

3
2.3 The group of positive divisor classes

For the introduction of that group we modify several notations from \cite{13} in order to make them suitable for actual computations.

Since \( PLS \) is finite we can fix the order of the logarithmically signed places, say \( PLS = \{ p_1, \ldots, p_m \} \), with \( PE = \{ p_1, \ldots, p_e \} \) and \( PR = \{ p_{e+1}, \ldots, p_m \} \). Accordingly we define vectors \( e = (e_1, \ldots, e_m) \in \{ \pm 1 \}^m \).

Remark 3. Since \( \nu \) is obviously a submodule of \( \{ \pm 1 \}^m \).

Definition 4 (positive divisor classes). With the notations above:

\[
\mathcal{D}_F := \{ (a, e) \in \mathcal{D}_{F}^{PE} \times \{ \pm 1 \}^m \mid \text{sg}(a, e) = 1 \} \tag{4}
\]

For \( \alpha \in \mathcal{R}_F := \mathcal{Z}_2 \otimes \mathcal{Z}_2 \mathcal{F}^\times \), let \( \text{div}'(\alpha) \) denotes the image of \( \text{div}(\alpha) \) in \( \mathcal{D}_{F}^{PE} \) and \( \text{sg}(\alpha) \) the vector of signs \( (\text{sg}_{p_1}(\alpha), \ldots, \text{sg}_{p_m}(\alpha)) \) in \( \{ \pm 1 \}^m \). Then

\[
\mathcal{P}_{F}^{pos} := \{ (\text{div}'(\alpha), \text{sg}(\alpha)) \in \mathcal{D}_{F}^{PE} \times \{ \pm 1 \}^m \mid \alpha \in \mathcal{R}_F \} \tag{5}
\]

is obviously a submodule of \( \mathcal{D}_{F}^{pos} \), which is called the principal submodule.

Definition 4 (positive divisor classes). With the notations above:

(i) The group of positive logarithmic divisor classes is the factor group

\[
\mathcal{C}_{F}^{pos} = \mathcal{D}_{F}^{pos} / \mathcal{P}_{F}^{pos} .
\]

(ii) The subgroup of positive logarithmic divisor classes of degree zero is the kernel \( \mathcal{C}_{F}^{pos} \) of the degree map \( \deg \) in \( \mathcal{D}_{F}^{pos} \):

\[
\mathcal{C}_{F}^{pos} := \{ (a, e) + \mathcal{P}_{F}^{pos} \mid \deg(a) \in \deg(\mathcal{D}_{F}(PE)) \}.
\]

Remark 3. The group \( \mathcal{C}_{F}^{pos} \) is infinite whenever the number field \( F \) has no exceptional places, since in this case \( \deg(\mathcal{C}_{F}^{pos}) \) is isomorphic to \( \mathcal{Z}_2 \). The finiteness of \( \mathcal{C}_{F}^{pos} \) in case \( PE \neq \emptyset \) follows from the so-called generalized Gross conjecture.

For the computation of \( \mathcal{C}_{F}^{pos} \) we need to introduce primitive divisors.

Definition 5. A divisor \( b \) of \( F \) is called a primitive divisor if \( \deg(b) \) generates the \( \mathcal{Z}_2 \)-module \( \deg(\mathcal{D}_{F}) = 4[F \cap \mathcal{Q}_2 : \mathcal{Q}]_2 \).

We close this section by presenting a method for exhibiting such a divisor:

Let \( q_1, \ldots, q_s \) be all dyadic primes; and \( p_1, \ldots, p_s \) be a finite set of non-dyadic primes which generates the 2-group of 2-ideal-classes \( \mathcal{C}_{F}^{2} \) (i.e. the quotient of the usual 2-class group by the subgroup generated by ideals above 2).

Then every \( p \in \{ q_1, \ldots, q_s, p_1, \ldots, p_t \} \) with minimal 2-valuation \( \nu_2(\deg(p)) \) is primitive.
2.4 Galois interpretations and applications to $K$-theory

Let $F^{lc}$ be the locally cyclotomic 2-extension of $F$ (i.e. the maximal abelian pro-2-extension of $F$ which is completely split at every place over the cyclotomic $\mathbb{Z}_2$-extension $F^c$). Then by $\ell$-adic class field theory (cf. [3]), one has the following interpretations of the logarithmic class groups:

$$\text{Gal}(F^{lc}/F) \simeq \mathcal{C}_F$$ and $$\text{Gal}(F^{lc}/F^c) \simeq \tilde{\mathcal{C}}_F.$$  

Remark 4. Let us assume $i \notin F^c$. Thus we may list the following special cases:

(i) In case $PLS = \emptyset$, the group $\mathcal{C}_F^{pos} \simeq \mathbb{Z}_2 \oplus \tilde{\mathcal{C}}_F^{pos}$ of positive divisor classes has index 2 in the group $\mathcal{C}_F \simeq \mathbb{Z}_2 \oplus \tilde{\mathcal{C}}_F$ of logarithmic classes of arbitrary degree; as a consequence its torsion subgroup $\tilde{\mathcal{C}}_F^{pos}$ has index 2 in the finite group $\tilde{\mathcal{C}}_F$ of logarithmic classes of degree 0 yet computed in [3].

(ii) In case $PE = \emptyset$, the group $\mathcal{C}_F^{pos} \simeq \mathbb{Z}_2 \oplus \tilde{\mathcal{C}}_F^{pos}$ has index 2 in the group $\mathcal{C}_F^{res} \simeq \mathbb{Z}_2 \oplus \tilde{\mathcal{C}}_F^{res}$ of narrow logarithmic classes of arbitrary degree; and its torsion subgroup $\tilde{\mathcal{C}}_F^{pos}$ has index 2 in the finite group $\tilde{\mathcal{C}}_F^{res}$ of narrow logarithmic classes of degree 0 introduced in [13] and computed in [11].

Definition 6. We adopt the following conventions from [3, 7, 13, 14]:

(i) $F$ is exceptional whenever one has $i \notin F^c$ (i.e. $[F^c[i]:F] = 2$);

(ii) $F$ is logarithmically signed whenever one has $i \notin F^{lc}$ (i.e. $PLS \neq \emptyset$);

(iii) $F$ is primitive whenever one at least between the exceptional places does not split in (the first step of the cyclotomic $\mathbb{Z}_2$-extension) $F^c/F$.

The following theorem is a consequence of the results in [3, 7, 9, 10, 13, 14]:

Theorem 1. Let $WK_2(F)$ (resp. $K_2^c(F) := \cap_{n \geq 2} K_2^c(F)$) be be the 2-part of the wild kernel (resp. the 2-subgroup of infinite height elements) in $K_2(F)$.

(i) In case $i \in F^{lc}$ (i.e. in case $PLS = \emptyset$), we have both:

$$\text{rk}_2 WK_2(F) = \text{rk}_2 \mathcal{C}_F = \text{rk}_2 \tilde{\mathcal{C}}_F^{res}.$$  

(ii) In case $i \notin F^{lc}$ but $F$ has no exceptional places (i.e. $PE = \emptyset$), we have:

$$\text{rk}_2 WK_2(F) = \text{rk}_2 \tilde{\mathcal{C}}_F^{res}.$$  

(iii) In case $PE \neq \emptyset$, then we have:

$$\text{rk}_2 WK_2(F) = \text{rk}_2 \mathcal{C}_F^{res}.$$  

And in this last situation there are two subcases:

(a) If $F$ is primitive, i.e. if the set $PE$ of exceptional dyadic places contains a primitive place, we have:

$$K_2^c(F) = WK_2(F).$$  

(b) If $F$ is imprimitive and $K_2^c(F) = \oplus_{n=1}^{n} \mathbb{Z}/2^n \mathbb{Z}$, we get:

i. $WK_2(F) = \mathbb{Z}/2^{n+1} \mathbb{Z} \oplus (\oplus_{i=1}^{n} \mathbb{Z}/2^n \mathbb{Z})$ if $\text{rk}_2(\mathcal{C}_F^{pos}) = \text{rk}_2(\tilde{\mathcal{C}}_F^{pos})$;

ii. $WK_2(F) = \mathbb{Z}/2^n \mathbb{Z} \oplus (\oplus_{i=1}^{n} \mathbb{Z}/2^n \mathbb{Z})$ if $\text{rk}_2(\mathcal{C}_F^{pos}) < \text{rk}_2(\tilde{\mathcal{C}}_F^{pos})$.  

5
3 Computation of positive divisor classes

We assume in the following that the set $PE$ of exceptional places is not empty.

3.1 Computation of exceptional units

Classically the group of logarithmic units is the kernel in $R_F$ of the logarithmic valuations (see [9]):

$$\tilde{E}_F = \{ x \in R_F \mid \forall p : \tilde{v}_p(x) = 0 \}$$

In order to compute positive divisor classes in case $PE$ is not empty, we ought to introduce a new group of units:

**Definition 7.** We define the group of logarithmic exceptional units as the kernel of the non-exceptional logarithmic valuations:

$$\tilde{E}_{exc}^e = \{ x \in R_F \mid \forall p \notin PE : \tilde{v}_p(x) = 0 \}$$ (6)

We just know that is a subgroup of the 2-group of 2-units $E_F' = \mathbb{Z}_2 \otimes E_F'$. If we assume that there are exactly $s$ places in $F$ containing 2 we have, say:

$$E_F' = \mu_F \times \langle \varepsilon_1, \cdots, \varepsilon_{r+c-1+s} \rangle$$

For the calculation of $\tilde{E}_{exc}^e$ we use the same precision $\eta$ as for our 2-adic approximations used in the course of the calculation of $\tilde{C}_F$. Then we obtain a system of generators of $\tilde{E}_{exc}^e$ by computing the nullspace of the matrix

$$B = \begin{pmatrix} \tilde{v}_p(\varepsilon_j) & 2^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 2^n \end{pmatrix}$$

with $r+c-1+s+e$ columns and $e$ rows, where $e$ is the cardinality of $PE$ and the precision $\eta$ is determined as explained in [3].

We assume that the nullspace is generated by the columns of the matrix

$$B' = \begin{pmatrix} C \\ - \\ - \\ D \end{pmatrix}$$

where $C$ has $r+c-1+s$ and $D$ exactly $e$ rows. It suffices to consider $C$. Each column $(n_1, \cdots, n_{r+c-1+s})^t$ of $C$ corresponds to a unit

$$\prod_{i=1}^{r+c-1+s} \varepsilon_i^{n_i} \in \tilde{E}_{exc}^e \cap R_F^{2^n}$$
so that we can choose

\[ \tilde{\varepsilon} := \prod_{i=1}^{r+c-1+s} \varepsilon_i^{n_i} \]

as an approximation for an exceptional unit. This procedure yields \( k \geq r + c + e \) exceptional units, say: \( \tilde{\varepsilon}_1, \ldots, \tilde{\varepsilon}_k \). By the so-called generalized conjecture of Gross we would have exactly \( r + c + e \) such units. So we assume in the following that the procedure does give \( k = r + c + e \) (otherwise we would refute the conjecture). Hence, from now on we may assume that we have determined exactly \( r + c + e \) generators \( \tilde{\varepsilon}_1, \ldots, \tilde{\varepsilon}_{r+c+1} \) of \( \tilde{E}^{\text{exc}}_F \), and we write:

\[ \tilde{E}^{\text{exc}}_F = \langle -1 \rangle \times \langle \tilde{\varepsilon}_1, \ldots, \tilde{\varepsilon}_{r+c+1} \rangle \]

**Definition 8.** The kernel of the canonical map \( R_F \to D^{\ell}_F^{\text{pos}} \) is the subgroup of *positive logarithmic units*:

\[ \tilde{E}^{\text{pos}}_F = \{ \tilde{\varepsilon} \in \tilde{E}^{\text{exc}}_F \mid \forall p \in \text{PLS} \quad \text{sg}_p(\tilde{\varepsilon}) = +1 \} \]

The subgroup \( \tilde{E}^{\text{pos}}_F \) has finite index in the group \( \tilde{E}^{\text{exc}}_F \) of exceptional units.

### 3.2 The algorithm for computing \( \tilde{C}^{\ell}_F^{\text{pos}} \)

We assume \( PE \neq \emptyset \) and that the logarithmic 2-class group \( \tilde{C}^{\ell}_F \) is isomorphic to the direct sum

\[ \tilde{C}^{\ell}_F \cong \bigoplus_{i=1}^{\nu} \mathbb{Z}/2^{n_i} \mathbb{Z} \]

subject to \( 1 \leq n_1 \leq \cdots \leq n_{\nu} \). Let \( a_i \) (\( 1 \leq i \leq \nu \)) be fixed representatives of the \( \nu \) generating divisor classes. Then any divisor \( a \) of \( D_F^{\ell} \) can be written as

\[ a = \sum_{i=1}^{\nu} a_i a_i + \lambda b + \text{div}(\alpha) \]

with suitable integers \( a_i \in \mathbb{Z}_2 \), a primitive divisor \( b \), \( \lambda = \text{deg}(\alpha) \) and an appropriate element \( \alpha \) of \( R_F \). With each divisor \( a_i \) we associate a vector

\[ e_i := (\text{sg}(a_i, 1), 1, \cdots, 1) \in \{ \pm 1 \}^m , \]

where \( m \) again denotes the number of divisors in \( \text{PLS} \). Clearly, that representation then satisfies \( \text{sg}(a_i, e_i) = 1 \), hence the element \( (a_i, e_i) \) belongs to \( D^{\ell}_{F}^{\text{pos}} \). Setting \( e_b = (\text{sg}(b, 1), 1, \cdots, 1) \) as above and writing

\[ e' := \text{sg}(\alpha) \times \prod_{i=1}^{\nu} e^{a_i} \times e \times e^\lambda_b \]

for abbreviation any element \( (a, e) \) of \( D^{\ell}_{F}^{\text{pos}} \) can then be written in the form

\[ (a, e) = \left( \sum_{i=1}^{\nu} a_i a_i + \lambda b + \text{div}(\alpha), e' \times \prod_{i=1}^{\nu} e^{a_i} \times \text{sg}(\alpha) \times e^\lambda_b \right) \]

\[ = \sum_{i=1}^{\nu} a_i (a_i, e_i) + \lambda (b, e_b) + (0, e') + (\text{div}(\alpha), \text{sg}(\alpha)) . \]
The multiplications are carried out coordinatewise. The vector \( \mathbf{e}' \) is therefore contained in the \( \mathbb{Z}_2 \)-module generated by \( \mathbf{g}_i \in \mathbb{Z}^m \) \((1 \leq i \leq m)\) with \( \mathbf{g}_1 = (1, \cdots, 1) \), whereas \( \mathbf{g}_i \) has first and \( i \)-th coordinate -1, all other coordinates 1 for \( i > 1 \).

As a consequence, the set
\[
\{(a_j, e_j) \mid 1 \leq j \leq \nu\} \cup \{(0, g_i) \mid 2 \leq i \leq m\} \cup \{(b, e)\}
\]
contains a system of generators of \( C^{\ell}_{pos} \ell \) (note that \( (0, g_1) \) is trivial in \( C^{\ell}_{pos} \ell \)).

We still need to expose the relations among those. But the latter are easy to characterize. We must have
\[
\begin{align*}
\sum_{j=1}^{\nu} a_j (a_j, e_j) + \sum_{i=2}^{m} b_i (0, g_i) + \lambda (b, e_b) & \equiv 0 \mod \tilde{P}^{\ell}_{pos} \\
\sum_{j=1}^{\nu} a_j (a_j, e_j) + \sum_{i=2}^{m} b_i (0, g_i) + \lambda (b, e_b) & = (\text{div}(\alpha), \text{sg}(\alpha)) + \sum_{p \in PE} (d_p p, 1)
\end{align*}
\]
with indeterminates \( a_j, b_i, d_p \) from \( \mathbb{Z}_2 \). Considering the two components separately, we obtain the conditions
\[
\sum_{j=1}^{\nu} a_j a_j + \lambda b \equiv \sum_{p \in PE} d_p p \mod \tilde{P}^{\ell}_{F} \quad (7)
\]
and
\[
\prod_{j=1}^{\nu} e_j^{a_j} \times \prod_{i=2}^{m} g_i^{b_i} \times e_b^{\lambda} = \text{sg}(\alpha) \quad . \quad (8)
\]

Let us recall that we have already ordered \( PLS \) so that exactly the first \( e \) elements \( p_1, \cdots, p_e \) belong to \( PE \). Then the first one of the conditions above is tantamount to
\[
\sum_{j=1}^{\nu} a_j a_j \equiv \sum_{i=1}^{e} d_{p_i} \left( p_i - \frac{\deg p_i}{\deg b} b \right) \mod \tilde{P}^{\ell}_{F} \quad .
\]

The divisors
\[
p_i - \frac{\deg p_i}{\deg b} b
\]
on the right-hand side can again be expressed by the \( a_j \). For \( 1 \leq i \leq e \) we let
\[
\text{div}(\alpha_i) + p_i - \frac{\deg p_i}{\deg b} b = \sum_{j=1}^{\nu} c_{ij} a_j \quad .
\]

The calculation of the \( \alpha_i, c_{ij} \) is described in [15].

Consequently, the coefficient vectors \((a_1, \cdots, a_\nu, \lambda)\) can be chosen as \( \mathbb{Z}_2 \)-linear combinations of the rows of the following matrix \( A \in \mathbb{Z}_2^{(\nu+e) \times (\nu+1)} \):
Obviously, the family \((g_i)_{2 \leq i \leq m}\) is free over \(\mathbb{F}_2\) implying that the exponents \(b_i\) are uniquely defined. Consequently, if the \(k\)-th coordinate of the product \(\text{sg}(\alpha) \times \prod_{j=1}^{\nu} e_j^{a_j} \times e_b^\lambda\) is \(-1\) we must have \(b_k = 1\), otherwise \(b_k = 0\) for \(2 \leq k \leq m\). (We note that the product over all coordinates is always \(1\).) Therefore, we denote by \(b_{2j}, \cdots, b_{m,j}\) the exponents of the relation belonging to the \(j\)-th column of \(A\) for \(j = 1, \cdots, \nu\).

Unfortunately, the elements \(\alpha\) are only given up to exceptional units. Hence, we must additionally consider the signs of the exceptional units of \(F\). For

\[
\tilde{E}_F^e = \langle -1 \rangle \times \langle \tilde{e}_1, \cdots, \tilde{e}_{r+e-1+e} \rangle
\]

we put:

\[
\text{sg}(\tilde{e}_j) = \prod_{i=1}^{m} g_i^{b_{i,j+e+e}}.
\]  

Using the notations of (11) and (12) the rows of the following matrix \(A' \in \mathbb{Z}_{2}^{(r+e+r+c) \times (\nu + m)}\) generate all relations for the \((a_j, e_j), (b, e_b), (0, g_i)\).

\[
A' = \begin{pmatrix}
\begin{array}{cccc|c}
A & b_{2,1} & \cdots & b_{m,1} \\
& \ddots & \ddots & \ddots \\
& \ddots & \ddots & \ddots \\
& & b_{2,\nu+e} & \cdots & b_{m,\nu+e} \\
& & - & \cdots & b_{2,\nu+e+1} \\
& & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots \\
b_{2,\nu+e+r+c} & \cdots & b_{m,\nu+e+r+c} \\
\end{array}
\end{pmatrix}.
\]
3.3 The algorithm for computing $\tilde{C}^F_{\text{pos}}$

We assume that $PE = \{p_1, \ldots, p_n\} \neq \emptyset$ is ordered by increasing 2-valuations $v_2(\deg p_i)$; that the group $C^F_{\text{pos}}$ of positive divisor classes is isomorphic to the direct sum

$$C^F_{\text{pos}} \cong \oplus_{i=1}^w \mathbb{Z}/2^{m_i}\mathbb{Z};$$

and that we know a full set of representatives $(b_i, f_i)$ $(1 \leq i \leq w)$ for all classes.

Then each $(b, f) \in \tilde{D}^F_{\text{pos}}$ satisfies $\deg(b) \in \deg(D(F(PE)))$ and

$$b \equiv \sum_{i=1}^w b_i b_i \mod (D(F(PE)) + \tilde{P}_F).$$

Obviously, we obtain

$$0 \equiv \deg(b) \equiv \sum_{i=1}^w b_i \deg(b_i) \mod \deg(D(F(PE))).$$

We reorder the $b_i$ if necessary so that

$$v_2(\deg(b_1)) \leq v_2(\deg(b_i)) \ (2 \leq i \leq w)$$

is fulfilled. We put

$$t := \max(\min(\{v_2(\deg(p)) \mid p \in D(F(PE))\}) - v_2(\deg(b_1)), 0) = \max(v_2(\deg(p_1)) - v_2(\deg(b_1)), 0)$$

and

$$\delta := b_1 + \sum_{i=2}^w \frac{\deg(b_i)}{\deg(b_1)} b_i.$$

Then we get:

$$b \equiv \sum_{i=2}^w b_i \left( b_i - \frac{\deg(b_1)}{\deg(b_1)} b_1 \right) + \delta b_1 \mod (D(F(PE)) + \tilde{P}_F)$$

and so

$$\deg b \equiv 0 \equiv \sum b_i \times 0 + \delta \deg b_1 \mod \deg D(F(PE)).$$

From this it is immediate that a full set of representatives of the elements of $\tilde{C}^F_{\text{pos}}$ is given by

$$\left( b_i - \frac{\deg(b_1)}{\deg(b_1)} b_1, f_i \times f_1^{-\deg(b_1)/\deg(b_1)} \right) \ \text{for} \ 2 \leq i \leq w$$

and

$$\left( b'_1 := 2^t b_1 - 2^t \frac{\deg(b_1)}{\deg(p_1)} b_1, f'_1 \right).$$

Let us denote the class of $(c, f)$ in $\tilde{C}^F_{\text{pos}}$ by $[c, f]$. 

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Now we establish a matrix of relations for the generating classes. For this we consider relations:
\[
\sum_{i=2}^{w} a_i \left[ b_i - \frac{\deg(b_i)}{\deg(b_1)} b_1, f_i \times f_1^{-\frac{\deg(b_i)}{\deg(b_1)}} \right] + a_1 \left[ 2^t b'_1, f'_1 \right] = 0 ,
\]
hence
\[
\sum_{i=2}^{w} a_i [b_i, f_i] + \left( 2^t a_1 - \sum_{i=2}^{w} \frac{\deg(b_i)}{\deg(b_1)} a_i \right) [b_1, f_1] = 0 .
\]
A system of generators for all relations can then be computed analogously to the previous section. We calculate a basis of the nullspace of the matrix \( A'' = (a''_{ij}) \in \mathbb{Z}^{w \times 2w} \) with first row
\[
\left( 2^t, \frac{\deg(b_2)}{\deg(b_1)}, \ldots, \frac{\deg(b_w)}{\deg(b_1)}, 2^m_1, 0, \ldots, 0 \right)
\]
and in rows \( i = 2, \ldots, w \) all entries are zero except for \( a''_{ii} = 1 \) and \( a''_{i,w+1} = 2^m_i \).

We note that we are only interested in the first \( w \) coordinates of the obtained vectors of that nullspace.

### 4 Examples

The methods described here are implemented in the computer algebra system Magma [2]. Many of the fields used in the examples were results of queries to the QaoS number field database [5, section 6]. More extensive tables of examples can be found at:

http://www.math.tu-berlin.de/~pauli/K

In the tables abelian groups are given as a list of the orders of their cyclic factors.

- \([::]\) denotes the index \( (K_2(O_F) : WK_2(F)) \) (see [1, equation (6)]);
- \( d_F \) denotes the discriminant for a number field \( F \);
- \( \mathcal{C}t_F \) denotes the class group, \( P \) the set of dyadic places;
- \( \mathcal{C}t_F' \) denotes the 2-part of \( \mathcal{C}t/(P) \);
- \( \widetilde{\mathcal{C}}t_F \) denotes the logarithmic classgroup;
- \( \mathcal{C}t_F^{\text{pos}} \) denotes the group of positive divisor classes;
- \( \widetilde{\mathcal{C}}t_F^{\text{pos}} \) denotes the group of positive divisor classes of degree 0;
- \( rk_2 \) denotes the 2-rank of the wild kernel \( WK_2 \).

K. Belabas and H. Gangl have developed an algorithm for the computation of the tame kernel \( K_2(O_F) \) [1]. The following table contains the structure of \( K_2(O_F) \) as computed by Belabas and Gangl and the 2-rank of the wild kernel \( WK_2 \) calculated with our methods for some imaginary quadratic fields. We also give the structure of the wild kernel if it can be deduced from the structure of \( K_2(O_F) \) and of the rank of the wild kernel computed here or in [13].
### 4.1 Imaginary Quadratic Fields

| $d_F$ | $C\ell_F$ | $K_2O_F$ | $[\cdot]$ | $|P|/|PE|$ | $C\ell'_F$ | $C\ell_F$ | $C\ell_F^{pos}$ | $C\ell'_F^{pos}$ | $rk_2$ | $WK_2$ |
|-------|------------|----------|----------|-------------|------------|-----------|-------------|-------------|--------|--------|
| -184  | 4          | 2        | 1        | 1           | 2          | 1         | 2           | 2           | 1      | 2      |
| -248  | 8          | 2        | 1        | 1           | 4          | 2         | 4           | 2,2         | 1      | 2      |
| -399  | 2,8        | 2,12     | 2        | 2           | 2          | 2         | 2,2         | 1           | 4      |        |
| -632  | 8          | 2        | 1        | 1           | 4          | 2         | 4           | 2,2         | 1      | 2      |
| -759  | 2,12       | 2,18     | 6        | 2           | 2          | 2         | 2           | 2,2         | 1      | 2      |
| -799  | 16         | 2,4      | 2        | 2           | 2          | 2         | 2,4         | 2           | 2      | 2,2    |
| -959  | 36         | 2,4      | 2        | 2           | 2          | 4         | 4,8         | 4           | 1      | 4      |

### 4.2 Real Quadratic Fields

| $d_F$ | $C\ell_F$ | $[\cdot]$ | $|P|/|PE|$ | $C\ell'_F$ | $C\ell_F$ | $C\ell_F^{pos}$ | $C\ell'_F^{pos}$ | $rk_2$ | $WK_2$ |
|-------|------------|----------|----------|------------|-----------|-------------|-------------|--------|--------|
| 776   | 2          | 4        | 1        | 1           | 2          | 2           | 2,2         | 2           | 2      |        |
| 904   | 8          | 4        | 1        | 1           | 4          | 2           | 4           | 2,2         | 1      |        |
| 29665 | 2,16       | 8        | 2        | 2           | 2          | 2           | 2,2         | 2           | 2      |        |
| 34689 | 32         | 8        | 2        | 2           | 2          | 2           | 2,2         | 2           | 2      |        |
| 69064 | 4,8        | 4        | 1        | 1           | 2          | 4,8         | 2,2,4       | 2,2,2       | 4      |        |
| 90321 | 2,2,8      | 24       | 2        | 2           | 2          | 2           | 2,2,2       | 2,2,2       | 2      |        |
| 104584| 4,8        | 4        | 1        | 1           | 2          | 2,8         | 2,4,2,8     | 2,2,4       | 2      |        |
| 245854| 4,8        | 4        | 1        | 1           | 2          | 2,8         | 2,4,2,8     | 2,2,4       | 3      |        |
| 300040| 2,2,8      | 4        | 1        | 1           | 2          | 2,8         | 2           | 2,2,2       | 2      |        |
| 374105| 32         | 8        | 2        | 2           | 2          | 2,2         | 2           | 2,2,2       | 2      |        |
| 171865| 2,32       | 8        | 2        | 2           | 4          | 4           | 2,2,4       | 2,2,4       | 3      |        |
| 285160| 2,32       | 4        | 1        | 1           | 32         | 32          | 32          | 32          | 1      |        |
| 318097| 64         | 8        | 2        | 2           | 2,2,4      | 2,2,4       | 2           | 2,2,4      | 2      |        |
| 409221| 64         | 12       | 1        | 1           | 64         | 64          | 2,2,2,4     | 2,2,4      | 2      |        |
| 651784| 2,32       | 4        | 1        | 1           | 2,16       | 2,8         | 2,2,16      | 2,2,2,8    | 3      |        |
The studied fields are given by a generating polynomial $f$ and have Galois group of their normal closure isomorphic to $C_3$ (cyclic) or $S_3$ (dihedral); $r$ denotes the number of real places.

**4.3 Examples of Degree 3**
| \( f \) | \( d_F \) | \( r \) | \( \text{Gal} \) | \( \mathcal{C} \ell_F \) | \( |P| \) | \( |PE| \) | \( \mathcal{C} \ell' \) | \( \mathcal{C} \ell_F \) | \( \mathcal{C} \ell_{F}^{\text{pos}} \) | \( \mathcal{C} \ell_F^{\text{pos}} \) | \( r_{k_2} \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( x^4 - 59x^2 - 120x - 416 \) | 860400 | 2 | \( D_4 \) | \[16\] | 8 | 2 | 2 | [] | [] | [2] | [2] | 1 |
| \( x^4 - x^3 - 2x^2 + 5x + 1 \) | 3967 | 2 | \( S_4 \) | [] | 8 | 2 | 2 | [] | [] | [2] | [2] | 1 |
| \( x^4 - x^3 + 86x^2 - 66x + 1791 \) | 701125 | 0 | \( D_4 \) | \[2,8\] | 1 | 1 | 1 | [2] | [2] | [2] | [2] | 2 |
| \( x^4 + 14 \) | 702464 | 0 | \( D_4 \) | \[4,4\] | 1 | 1 | 1 | [4] | [2] | [2] | [2,2] | 1 |
| \( x^4 + 58x^2 + 1 \) | 705600 | 0 | \( E_4 \) | \[4,8\] | 2 | 2 | 2 | [4] | [2] | [4] | [2,2] | 1 |
| \( x^4 - 2x^3 + 59x^2 - 24x + 738 \) | 728128 | 0 | \( D_4 \) | \[32\] | 2 | 2 | 2 | [2] | [4] | [2] | [2] | 1 |
| \( x^4 + 21x^2 + 120 \) | 730080 | 0 | \( D_4 \) | \[4,8\] | 6 | 2 | 2 | [2] | [4] | [2,2] | [2,2] | 2 |
| \( x^4 - 5x + 30 \) | 766125 | 0 | \( S_4 \) | \[2,16\] | 20 | 1 | 1 | [4] | [2,2] | [2,2] | [2,2] | 0 |
| \( x^4 + 58x^2 + 1 \) | 705600 | 0 | \( E_4 \) | \[4,8\] | 2 | 2 | 2 | [4] | [2] | [4] | [2,2] | 1 |
| \( x^5 + x^4 + x^3 - 8x^2 - 12x + 16 \) | -4424116 | 3 | \( S_5 \) | \[4\] | 64 | 3 | 2 | [] | [] | [2] | [2] | 1 |
| \( x^5 + x^4 - 13x^3 - 26x^2 - 8x - 1 \) | -3504168 | 3 | \( S_5 \) | \[4\] | 8 | 2 | 2 | [] | [] | [2,2] | [2,2] | 2 |
| \( x^5 - 10x^3 + 9x^2 + 7x - 1 \) | -3477048 | 3 | \( S_5 \) | \[4\] | 16 | 2 | 2 | [] | [] | [2] | [2] | 0 |
| \( x^5 + 2x^4 + 6x^3 + 11x^2 - 2x - 9 \) | -3420711 | 3 | \( S_5 \) | \[4\] | 16 | 2 | 2 | [] | [] | [2] | [2] | 1 |
| \( x^5 - 14x^3 + 26x^2 - 11x - 1 \) | -3356683 | 3 | \( S_5 \) | \[4\] | 8 | 1 | 1 | [4] | [4] | [4] | [4] | 1 |
| \( x^5 + 2x^4 + 9x^3 + 3x^2 + 10x - 24 \) | 2761273 | 1 | \( S_5 \) | \[10\] | 8 | 3 | 3 | [] | [] | [2] | [2] | 0 |
| \( x^5 + x^4 - 3x^3 + 15x^2 + 36x - 18 \) | 3825936 | 1 | \( D_5 \) | \[11\] | 288 | 3 | 1 | [] | [] | [2] | [2] | 0 |
| \( x^5 + 2x^4 + 12x^3 + 14x^2 - 12x - 16 \) | 4892116 | 1 | \( S_5 \) | \[12\] | 8 | 3 | 3 | [] | [] | [2] | [2] | 1 |
| \( x^5 + 2x^4 - 8x^3 - 4x^2 + 7x + 1 \) | 13664837 | 5 | \( S_5 \) | \[4\] | 64 | 2 | 2 | [] | [] | [2] | [2] | 1 |
| \( x^5 + 2x^4 - 11x^3 - 27x^2 - 10x + 1 \) | 17371748 | 5 | \( S_5 \) | \[2\] | 64 | 2 | 2 | [] | [] | [2] | [2] | 1 |
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