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Thermal Networks from Phonon Boltzmann's Transport Equation: Part II – Compact Modeling

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Abstract— A multi-point moment matching method is proposed for generating compact models of thermal networks from the gray phonon Boltzmann's transport equation. The resulting algorithm is applied to a reference thermal network for which an highly accurate and compact model is obtained.

I. INTRODUCTION

Various techniques are reported in literature for deriving compact models of dynamic thermal networks [1]. In almost all these works the considered thermal networks are from the heat diffusion equation. However, as proved in [2], at the scales of current semiconductor technology, the heat diffusion equation is not accurate. The Boltzmann's transport equation has then to be used. In [1] thermal networks from the gray Boltzmann's transport equation have been introduced. The problem of generating compact models of the thermal networks from the transport equation is thus open.

In this paper we present a method for deriving compact models of the thermal networks from the transport equation. Firstly we propose a general method for deriving compact models by Galerkin's projection. It is shown that such technique preserves the thermodynamic properties of the thermal networks shown in [2].

Secondly we show how this technique can be applied in order to get compact models which are multi-point moment matching approximations of the thermal networks from the transport equation. In particular a large number of moments in the transfer matrix of the thermal network are matched by the compact model.

Lastly an Arnoldi-like algorithm is derived which avoids the ill-conditioning problems observed when moment matching is obtained by direct computation of the moments [3].

The resulting algorithm does not require to compute solutions to the transport equation in the time domain but just solutions to the transport equation in the complex frequency domain, at the chosen matching points.

The proposed technique is applied to the reference thermal network from the transport equation introduced in [2]. The compact model obtained for this application exhibits levels of accuracy and compactness very similar to those observed for the compact models of the thermal networks from the heat diffusion equation. Thus the compact models of the thermal networks from the transport equation can be derived in an easier way with respect to the compact models of the thermal networks from the heat wave equation [4].

II. COMPACT MODELS BY GALERKIN'S PROJECTION

We refer to the transport equation introduced by the authors in [2]. Let us introduce expansions for $e^+(\mathbf{r}, \boldsymbol{\Omega}, t)$ and $e^-(\mathbf{r}, \boldsymbol{\Omega}, t)$ of the form

$$\begin{aligned} e^+(\mathbf{r}, \boldsymbol{\Omega}, t) &= \mathbf{u}^+(\mathbf{r}, \boldsymbol{\Omega}) \hat{\mathbf{e}}^+(t) \\ e^-(\mathbf{r}, \boldsymbol{\Omega}, t) &= \mathbf{u}^-(\mathbf{r}, \boldsymbol{\Omega}) \hat{\mathbf{e}}^-(t) \end{aligned}$$

in which

$$\begin{aligned} \mathbf{u}^+(\mathbf{r}, \boldsymbol{\Omega}) &= [u_1^+(\mathbf{r}, \boldsymbol{\Omega}), \dots, u_{m^+}^+(\mathbf{r}, \boldsymbol{\Omega})], \\ \mathbf{u}^-(\mathbf{r}, \boldsymbol{\Omega}) &= [u_1^-(\mathbf{r}, \boldsymbol{\Omega}), \dots, u_{m^-}^-(\mathbf{r}, \boldsymbol{\Omega})] \end{aligned}$$

are vectors of *basis functions* and

$$\begin{aligned} \hat{\mathbf{e}}^+(t) &= [\hat{e}_1^+(t), \dots, \hat{e}_{m^+}^+(t)]^T, \\ \hat{\mathbf{e}}^-(t) &= [\hat{e}_1^-(t), \dots, \hat{e}_{m^-}^-(t)]^T \end{aligned}$$

are vectors of freedom degrees. The basis functions in $\mathbf{u}^+(\mathbf{r}, \boldsymbol{\Omega})$ and $\mathbf{u}^-(\mathbf{r}, \boldsymbol{\Omega})$ are assumed to satisfy the homogeneous conditions set for $e^+(\mathbf{r}, \boldsymbol{\Omega}, t)$ and $e^-(\mathbf{r}, \boldsymbol{\Omega}, t)$ both at the boundary of the Σ region and at the interfaces between adjacent homogeneous sub-regions Σ_i .

A compact model can then be obtained, by multiplying the transport equation, by $\mathbf{u}^+(\mathbf{r}, \boldsymbol{\Omega})/c(\mathbf{r})$ and by $\mathbf{u}^-(\mathbf{r}, \boldsymbol{\Omega})/c(\mathbf{r})$, by taking the mean with respect to $\boldsymbol{\Omega}$, by integrating in each homogeneous sub-region Σ_i and by summing all these integrals.

It results in

$$\hat{\mathbf{M}} \frac{d}{dt} \hat{\mathbf{e}}(t) + \hat{\mathbf{K}} \hat{\mathbf{e}}(t) = \hat{\mathbf{g}}(t) \quad (1)$$

in which

$$\hat{\mathbf{e}}(t) = \begin{bmatrix} \hat{\mathbf{e}}^+(t) \\ \hat{\mathbf{e}}^-(t) \end{bmatrix},$$

$$\hat{\mathbf{g}}(t) = \begin{bmatrix} \hat{\mathbf{g}}^+(t) \\ \hat{\mathbf{g}}^-(t) \end{bmatrix},$$

and

$$\begin{aligned} \hat{\mathbf{M}} &= \begin{bmatrix} \hat{\mathbf{M}}^+ & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{M}}^- \end{bmatrix} \\ \hat{\mathbf{K}} &= \begin{bmatrix} \hat{\mathbf{K}}^+ & \hat{\mathbf{N}} \\ -\hat{\mathbf{N}}^T & \hat{\mathbf{K}}^- \end{bmatrix}. \end{aligned}$$

Besides

$$\begin{aligned}
\hat{\mathbf{M}}^+ &= \int_{\Sigma} \frac{\langle \mathbf{u}^+(\mathbf{r}, \boldsymbol{\Omega})^T \mathbf{u}^+(\mathbf{r}, \boldsymbol{\Omega}) \rangle}{c(\mathbf{r})} d\mathbf{r} \\
\hat{\mathbf{M}}^- &= \int_{\Sigma} \frac{\langle \mathbf{u}^-(\mathbf{r}, \boldsymbol{\Omega})^T \mathbf{u}^-(\mathbf{r}, \boldsymbol{\Omega}) \rangle}{c(\mathbf{r})} d\mathbf{r} \\
\hat{\mathbf{K}}^+ &= \int_{\Sigma} \frac{1}{c(\mathbf{r})\tau(\mathbf{r})} \langle (\mathbf{u}^+(\mathbf{r}, \boldsymbol{\Omega}) - \langle \mathbf{u}^+(\mathbf{r}, \boldsymbol{\Omega}) \rangle)^T \cdot (\mathbf{u}^+(\mathbf{r}, \boldsymbol{\Omega}) - \langle \mathbf{u}^+(\mathbf{r}, \boldsymbol{\Omega}) \rangle) \rangle d\mathbf{r} \\
\hat{\mathbf{K}}^- &= \int_{\Sigma} \frac{\langle \mathbf{u}^-(\mathbf{r}, \boldsymbol{\Omega})^T \mathbf{u}^-(\mathbf{r}, \boldsymbol{\Omega}) \rangle}{c(\mathbf{r})\tau(\mathbf{r})} d\mathbf{r} \\
\hat{\mathbf{N}} &= \int_{\Sigma} \frac{v(\mathbf{r}) \langle \boldsymbol{\Omega} \cdot \mathbf{u}^+(\mathbf{r}, \boldsymbol{\Omega})^T \nabla \mathbf{u}^-(\mathbf{r}, \boldsymbol{\Omega}) \rangle}{c(\mathbf{r})} d\mathbf{r} = \\
&= - \int_{\Sigma} \frac{v(\mathbf{r}) \langle \boldsymbol{\Omega} \nabla \mathbf{u}^+(\mathbf{r}, \boldsymbol{\Omega})^T \cdot \mathbf{u}^-(\mathbf{r}, \boldsymbol{\Omega}) \rangle}{c(\mathbf{r})} d\mathbf{r}
\end{aligned} \tag{2}$$

and

$$\begin{aligned}
\hat{\mathbf{g}}^+(t) &= \int_{\Sigma} \frac{\langle \mathbf{u}^+(\mathbf{r}, \boldsymbol{\Omega})^T g^+(\mathbf{r}, \boldsymbol{\Omega}, t) \rangle}{c(\mathbf{r})} d\mathbf{r}, \\
\hat{\mathbf{g}}^-(t) &= \int_{\Sigma} \frac{\langle \mathbf{u}^-(\mathbf{r}, \boldsymbol{\Omega})^T g^-(\mathbf{r}, \boldsymbol{\Omega}, t) \rangle}{c(\mathbf{r})} d\mathbf{r}.
\end{aligned}$$

We notice that in the definition of $\hat{\mathbf{N}}$ all the terms on the boundary $\partial\Sigma$ and on the interfaces between the homogeneous sub-regions Σ_i cancel because of the assumed boundary conditions for the basis functions in $\mathbf{u}^+(\mathbf{r}, \boldsymbol{\Omega})$ and $\mathbf{u}^-(\mathbf{r}, \boldsymbol{\Omega})$.

From the definition of the powers and temperature rises at the ports of the thermal networks it follows

$$\hat{\mathbf{g}}(t) = \hat{\mathbf{F}}\mathbf{u}(t) \tag{4}$$

in which

$$\mathbf{u}(t) = \begin{bmatrix} \mathbf{P}^+(t) \\ \mathbf{T}^-(t) \end{bmatrix}$$

and

$$\hat{\mathbf{F}} = \begin{bmatrix} \hat{\mathbf{F}}^+ & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{F}}^- \end{bmatrix}$$

with

$$\begin{aligned}
\hat{\mathbf{F}}^+ &= \int_{\Sigma} \frac{\langle \mathbf{u}^+(\mathbf{r}, \boldsymbol{\Omega})^T \mathbf{f}^+(\mathbf{r}, \boldsymbol{\Omega}, t) \rangle}{c(\mathbf{r})} d\mathbf{r}, \\
\hat{\mathbf{F}}^- &= \int_{\Sigma} \frac{\langle \mathbf{u}^-(\mathbf{r}, \boldsymbol{\Omega})^T \mathbf{f}^-(\mathbf{r}, \boldsymbol{\Omega}, t) \rangle}{c(\mathbf{r})} d\mathbf{r}.
\end{aligned}$$

Besides

$$\mathbf{y}(t) = \hat{\mathbf{F}}^T \hat{\mathbf{e}}(t) \tag{5}$$

in which

$$\mathbf{y}(t) = \begin{bmatrix} \mathbf{T}^+(t) \\ \mathbf{P}^-(t) \end{bmatrix}.$$

III. THERMODYNAMIC PROPERTIES

The introduced compact model preserves the thermodynamic properties of the thermal network. In fact the following properties hold:

Property 1 (Passivity): For each time interval $t_1 \leq t \leq t_2$ it results in

$$\hat{W}(t_2) \leq \hat{W}(t_1) + \int_{t_1}^{t_2} \mathbf{T}(t)^T \mathbf{P}(t) dt \tag{6}$$

$\hat{W}(t)$ being the nonnegative quantity

$$\hat{W}(t) = \frac{1}{2} \hat{\mathbf{e}}(t)^T \hat{\mathbf{M}} \hat{\mathbf{e}}(t).$$

Proof: Multiplying Eq. (1) by $\hat{\mathbf{e}}(t)^T$ it results in

$$\frac{d}{dt} \frac{1}{2} \hat{\mathbf{e}}(t)^T \hat{\mathbf{M}} \hat{\mathbf{e}}(t) + \hat{\mathbf{e}}(t)^T \hat{\mathbf{K}} \hat{\mathbf{e}}(t) = \hat{\mathbf{e}}(t)^T \hat{\mathbf{g}}(t). \tag{7}$$

In results in

$$\hat{\mathbf{e}}(t)^T \hat{\mathbf{M}} \hat{\mathbf{e}}(t) = \hat{\mathbf{e}}^+(t)^T \hat{\mathbf{M}}^+ \hat{\mathbf{e}}^+(t) + \hat{\mathbf{e}}^-(t)^T \hat{\mathbf{M}}^- \hat{\mathbf{e}}^-(t) \geq 0$$

since $\hat{\mathbf{M}}^+$ and $\hat{\mathbf{M}}^-$ are symmetric, positive definite matrices. Similarly it results in

$$\hat{\mathbf{e}}(t)^T \hat{\mathbf{K}} \hat{\mathbf{e}}(t) = \hat{\mathbf{e}}^+(t)^T \hat{\mathbf{K}}^+ \hat{\mathbf{e}}^+(t) + \hat{\mathbf{e}}^-(t)^T \hat{\mathbf{K}}^- \hat{\mathbf{e}}^-(t) \geq 0$$

since $\hat{\mathbf{K}}^+$ and $\hat{\mathbf{K}}^-$ are symmetric, positive definite matrices. Besides from Eqs. (4), (5) it results in

$$\begin{aligned}
\hat{\mathbf{e}}(t)^T \hat{\mathbf{g}}(t) &= \hat{\mathbf{e}}^+(t)^T \hat{\mathbf{F}}^+ \mathbf{P}^+(t) + \hat{\mathbf{e}}^-(t)^T \hat{\mathbf{F}}^- \mathbf{T}^-(t) \\
&= \mathbf{T}^+(t)^T \mathbf{P}^+(t) + \mathbf{P}^-(t)^T \mathbf{T}^-(t) \\
&= \mathbf{T}(t)^T \mathbf{P}(t).
\end{aligned}$$

Thus, integrating Eq. (6) with respect to t for $t_1 \leq t \leq t_2$ the thesis follows. \blacksquare

Property 2 (Reciprocity): In the Laplace transform domain s , for any couple of situations $\mathbf{P}_1(s)$, $\mathbf{T}_1(s)$, and $\mathbf{P}_2(s)$, $\mathbf{T}_2(s)$, it results in

$$\mathbf{T}_1(s)^T \mathbf{P}_2(s) = \mathbf{T}_2(s)^T \mathbf{P}_1(s)$$

Proof: Let it be

$$\hat{\mathbf{J}} = \begin{bmatrix} \mathbf{I}_{m^+} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_{m^-} \end{bmatrix}. \tag{8}$$

Multiplying by $\hat{\mathbf{J}} \hat{\mathbf{e}}_2(s)^T$ the Laplace transform of Eq. (1) it follows

$$s \hat{\mathbf{e}}_2(s)^T \hat{\mathbf{J}} \hat{\mathbf{M}} \hat{\mathbf{e}}_1(s) + \hat{\mathbf{e}}_2(s)^T \hat{\mathbf{J}} \hat{\mathbf{K}} \hat{\mathbf{e}}_1(s) = \hat{\mathbf{e}}_2(s)^T \hat{\mathbf{J}} \hat{\mathbf{F}} \mathbf{u}_1(s). \tag{9}$$

Similarly, by exchanging indexes 1 and 2 it follows

$$s \hat{\mathbf{e}}_1(s)^T \hat{\mathbf{J}} \hat{\mathbf{M}} \hat{\mathbf{e}}_2(s) + \hat{\mathbf{e}}_1(s)^T \hat{\mathbf{J}} \hat{\mathbf{K}} \hat{\mathbf{e}}_2(s) = \hat{\mathbf{e}}_1(s)^T \hat{\mathbf{J}} \hat{\mathbf{F}} \mathbf{u}_2(s). \tag{10}$$

Since the following matrices are symmetric

$$\hat{\mathbf{J}} \hat{\mathbf{M}} = \begin{bmatrix} \hat{\mathbf{M}}^+ & \mathbf{0} \\ \mathbf{0} & -\hat{\mathbf{M}}^- \end{bmatrix} = \hat{\mathbf{M}}^T \hat{\mathbf{J}} \tag{11}$$

$$\hat{\mathbf{J}} \hat{\mathbf{K}} = \begin{bmatrix} \hat{\mathbf{K}}^+ & \hat{\mathbf{N}} \\ \hat{\mathbf{N}}^T & -\hat{\mathbf{K}}^- \end{bmatrix} = \hat{\mathbf{K}}^T \hat{\mathbf{J}} \tag{12}$$

it results in

$$\begin{aligned}
\hat{\mathbf{e}}_1(s)^T \hat{\mathbf{J}} \hat{\mathbf{M}} \hat{\mathbf{e}}_2(s) &= \hat{\mathbf{e}}_2(s)^T \hat{\mathbf{J}} \hat{\mathbf{M}} \hat{\mathbf{e}}_1(s) \\
\hat{\mathbf{e}}_1(s)^T \hat{\mathbf{J}} \hat{\mathbf{K}} \hat{\mathbf{e}}_2(s) &= \hat{\mathbf{e}}_2(s)^T \hat{\mathbf{J}} \hat{\mathbf{K}} \hat{\mathbf{e}}_1(s).
\end{aligned}$$

Moreover since

$$\hat{\mathbf{J}}\hat{\mathbf{F}} = \begin{bmatrix} \hat{\mathbf{F}}^+ & \mathbf{0} \\ \mathbf{0} & -\hat{\mathbf{F}}^- \end{bmatrix} = \hat{\mathbf{F}}\mathbf{J} \quad (13)$$

in which

$$\mathbf{J} = \begin{bmatrix} \mathbf{I}_{n^+} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_{n^-} \end{bmatrix}, \quad (14)$$

from Eq. (5) it results in

$$\hat{\mathbf{e}}_2(s)^T \hat{\mathbf{J}}\hat{\mathbf{F}}\mathbf{u}_1(s) = \hat{\mathbf{e}}_2(s)^T \hat{\mathbf{F}}\hat{\mathbf{J}}\mathbf{u}_1(s) = \mathbf{y}_2(s)^T \mathbf{J}\mathbf{u}_1(s)$$

Similarly

$$\hat{\mathbf{e}}_1(s)^T \hat{\mathbf{J}}\hat{\mathbf{F}}\mathbf{u}_2(s) = \mathbf{u}_1(s)^T \mathbf{J}\mathbf{u}_2(s).$$

Thus, by subtracting Eq. (10) from Eq. (9) it follows

$$\mathbf{y}_2(s)^T \mathbf{J}\mathbf{u}_1(s) = \mathbf{y}_2(s)^T \mathbf{J}\mathbf{u}_2(s)$$

or equivalently

$$\begin{aligned} \mathbf{P}_1^+(s)^T \mathbf{T}_2^+(s) - \mathbf{T}_1^-(s)^T \mathbf{P}_2^-(s) &= \\ &= \mathbf{P}_2^+(s)^T \mathbf{T}_1^+(s) - \mathbf{T}_2^-(s)^T \mathbf{P}_1^-(s) \end{aligned}$$

Thus

$$\begin{aligned} \mathbf{P}_1^+(s)^T \mathbf{T}_2^+(s) + \mathbf{P}_1^-(s)^T \mathbf{T}_2^-(s) &= \\ &= \mathbf{P}_2^+(s)^T \mathbf{T}_1^+(s) + \mathbf{P}_2^-(s)^T \mathbf{T}_1^-(s) \end{aligned}$$

and the thesis follows. \blacksquare

IV. MOMENT MATCHING

Let $\mathbf{w}(\mathbf{r}, \boldsymbol{\Omega}, t)$ be the row vector of the energy density rises corresponding to the source densities in the row vector $\mathbf{f}(\mathbf{r}, \boldsymbol{\Omega}) \delta(t)$, in which

$$\mathbf{f}(\mathbf{r}, \boldsymbol{\Omega}) = [\mathbf{f}^+(\mathbf{r}, \boldsymbol{\Omega}) \quad \mathbf{f}^-(\mathbf{r}, \boldsymbol{\Omega})].$$

Let $\mathbf{w}(\mathbf{r}, \boldsymbol{\Omega}, s)$ be the Laplace transform of $\mathbf{w}(\mathbf{r}, \boldsymbol{\Omega}, t)$. In each homogeneous sub-region Σ_i it is

$$\begin{aligned} s\mathbf{w}(\mathbf{r}, \boldsymbol{\Omega}, s) + v(\mathbf{r})\boldsymbol{\Omega} \cdot \nabla \mathbf{w}(\mathbf{r}, \boldsymbol{\Omega}, s) + \\ + \frac{\mathbf{w}(\mathbf{r}, \boldsymbol{\Omega}, s) - \langle \mathbf{w}(\mathbf{r}, \boldsymbol{\Omega}, s) \rangle}{\tau(\mathbf{r})} = \mathbf{f}(\mathbf{r}, \boldsymbol{\Omega}) \end{aligned} \quad (15)$$

At each interface between homogeneous sub-regions Σ_i it is

$$\begin{aligned} [v(\mathbf{r})\mathbf{w}^n(\mathbf{r}, \boldsymbol{\Omega}, s)] &= \mathbf{0}, \\ \left[\frac{\mathbf{w}_t(\mathbf{r}, \boldsymbol{\Omega}, s)}{c(\mathbf{r})} \right] &= \mathbf{0}. \end{aligned}$$

At $\partial\Sigma$ one of the following boundary conditions hold

$$\begin{aligned} \mathbf{w}^n(\mathbf{r}, \boldsymbol{\Omega}, s) &= \mathbf{0}, \\ \mathbf{w}^t(\mathbf{r}, \boldsymbol{\Omega}, s) &= \mathbf{0}. \end{aligned}$$

Let us introduce matching points α_r with $r = 1 \dots l$. Expanding $\mathbf{w}(\mathbf{r}, \boldsymbol{\Omega}, s)$ in a Taylor series around α_p it results in

$$\mathbf{w}(\mathbf{r}, \boldsymbol{\Omega}, s) = \sum_0^{+\infty} \mathbf{w}_p(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r)(s - \alpha_r)^p.$$

By substituting this expansion into Eq. (15) it straightforwardly follows that in each homogeneous sub-region Σ_i it is

$$\begin{aligned} \alpha_r \mathbf{w}_0(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) + v(\mathbf{r})\boldsymbol{\Omega} \cdot \nabla \mathbf{w}_0(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) + \\ + \frac{\mathbf{w}_0(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) - \langle \mathbf{w}_0(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) \rangle}{\tau(\mathbf{r})} = \mathbf{f}(\mathbf{r}, \boldsymbol{\Omega}) \end{aligned} \quad (16)$$

$$\begin{aligned} \alpha_r \mathbf{w}_p(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) + v(\mathbf{r})\boldsymbol{\Omega} \cdot \nabla \mathbf{w}_p(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) + \\ + \frac{\mathbf{w}_p(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) - \langle \mathbf{w}_p(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) \rangle}{\tau(\mathbf{r})} = -\mathbf{w}_{p-1}(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) \end{aligned} \quad (17)$$

At each interface between homogeneous sub-regions Σ_i it is

$$[v(\mathbf{r})\mathbf{w}_p^n(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r)] = \mathbf{0}, \quad (18)$$

$$\left[\frac{\mathbf{w}_p^t(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r)}{c(\mathbf{r})} \right] = \mathbf{0}. \quad (19)$$

At each point of $\partial\Sigma$ one of the following boundary conditions hold

$$\mathbf{w}_p^n(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) = \mathbf{0}, \quad (20)$$

$$\mathbf{w}_p^t(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) = \mathbf{0}. \quad (21)$$

Let us assume that the basis functions in the vectors $\mathbf{u}^+(\mathbf{r}, \boldsymbol{\Omega})$ and $\mathbf{u}^-(\mathbf{r}, \boldsymbol{\Omega})$ spam the space spanned by the elements of $\mathbf{w}_p^+(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r)$ and $\mathbf{w}_p^-(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r)$ respectively, for $p = 0 \dots k_r$ and $r = 1 \dots l$. Precisely let us assume that

$$\mathbf{w}_p(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) = \mathbf{u}(\mathbf{r}, \boldsymbol{\Omega}) \hat{\mathbf{W}}_p(\alpha_r) \quad (22)$$

in which

$$\begin{aligned} \mathbf{u}(\mathbf{r}, \boldsymbol{\Omega}) &= \begin{bmatrix} \mathbf{u}^+(\mathbf{r}, \boldsymbol{\Omega}) & \mathbf{0} \\ \mathbf{0} & \mathbf{u}^-(\mathbf{r}, \boldsymbol{\Omega}) \end{bmatrix} \\ \hat{\mathbf{W}}_p(\alpha_r) &= \begin{bmatrix} \hat{\mathbf{W}}_p^+(\alpha_r) \\ \hat{\mathbf{W}}_p^-(\alpha_r) \end{bmatrix}, \end{aligned}$$

for $p = 0 \dots k_r$ and $r = 1 \dots l$. In this way the compact model of the thermal network turns out to be a Padè type approximant of the thermal network. This is proved by the following results.

Lemma 1: It is

$$\hat{\mathbf{W}}_p(\alpha_r) = (-\hat{\mathbf{K}}(\alpha_r)^{-1} \hat{\mathbf{M}})^p \hat{\mathbf{K}}(\alpha_r)^{-1} \hat{\mathbf{F}}$$

in which

$$\hat{\mathbf{K}}(\alpha_r) = \alpha_r \hat{\mathbf{M}} + \hat{\mathbf{K}}.$$

for each $p = 0 \dots k_r$ and $r = 1 \dots l$.

Proof: Multiplying Eq. (16) by $\mathbf{u}(\mathbf{r}, \boldsymbol{\Omega})^T / c(\mathbf{r})$, integrating with respect to $\boldsymbol{\Omega}$ and with respect to \mathbf{r} in each homogeneous sub-region Σ_i it is

$$\begin{aligned} \alpha_r \int_{\Sigma_i} \frac{4\pi \langle \mathbf{u}(\mathbf{r}, \boldsymbol{\Omega})^T \mathbf{w}_0(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) \rangle}{c(\mathbf{r})} d\mathbf{r} + \\ + \int_{\Sigma_i} \frac{4\pi v(\mathbf{r}) \langle \boldsymbol{\Omega} \cdot \mathbf{u}(\mathbf{r}, \boldsymbol{\Omega})^T \nabla \mathbf{w}_0(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) \rangle}{c(\mathbf{r})} d\mathbf{r} + \\ + \int_{\Sigma_i} \frac{4\pi \langle \mathbf{u}(\mathbf{r}, \boldsymbol{\Omega})^T (\mathbf{w}_0(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) - \langle \mathbf{w}_0(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) \rangle) \rangle}{c(\mathbf{r})\tau(\mathbf{r})} d\mathbf{r} = \\ = \int_{\Sigma_i} \frac{4\pi \langle \mathbf{u}(\mathbf{r}, \boldsymbol{\Omega})^T \mathbf{f}(\mathbf{r}, \boldsymbol{\Omega}) \rangle}{c(\mathbf{r})} d\mathbf{r}. \end{aligned}$$

Summing the equations for all Σ_i , using boundary conditions (18)-(21) and substituting Eq. (22) it follows

$$\hat{\mathbf{K}}(\alpha_r)\hat{\mathbf{W}}_0(\alpha_r) = \hat{\mathbf{F}}. \quad (23)$$

Similarly, multiplying Eq. (17) by $\mathbf{u}(\mathbf{r}, \boldsymbol{\Omega})^T/c(\mathbf{r})$, integrating with respect to $\boldsymbol{\Omega}$ and with respect to \mathbf{r} in each homogeneous sub-region Σ_i it is

$$\begin{aligned} & \alpha_r \int_{\Sigma_i} \frac{4\pi \langle \mathbf{u}(\mathbf{r}, \boldsymbol{\Omega})^T \mathbf{w}_p(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) \rangle}{c(\mathbf{r})} d\mathbf{r} + \\ & + \int_{\Sigma_i} \frac{4\pi v(\mathbf{r}) \langle \boldsymbol{\Omega} \cdot \mathbf{u}(\mathbf{r}, \boldsymbol{\Omega})^T \nabla \mathbf{w}_p(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) \rangle}{c(\mathbf{r})} d\mathbf{r} + \\ & + \int_{\Sigma_i} \frac{4\pi \langle \mathbf{u}(\mathbf{r}, \boldsymbol{\Omega})^T (\mathbf{w}_0(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) - \langle \mathbf{w}_p(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) \rangle) \rangle}{c(\mathbf{r})\tau(\mathbf{r})} d\mathbf{r} = \\ & = - \int_{\Sigma_i} \frac{4\pi \langle \mathbf{u}(\mathbf{r}, \boldsymbol{\Omega})^T \mathbf{w}_{p-1}(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) \rangle}{c(\mathbf{r})} d\mathbf{r}. \end{aligned}$$

Summing the equations for all Σ_i , using boundary conditions (18)-(21) and substituting Eq. (22) it follows

$$\hat{\mathbf{K}}(\alpha_r)\hat{\mathbf{W}}_p(\alpha_r) = -\hat{\mathbf{M}}\hat{\mathbf{W}}_{p-1}(\alpha_r). \quad (24)$$

From Eqs. (23), (24) the thesis follows. ■

Lemma 2: It is

$$\begin{aligned} & \int_{\Sigma} \frac{4\pi \langle \mathbf{f}(\mathbf{r}, -\boldsymbol{\Omega})^T \mathbf{w}_p(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) \rangle}{c(\mathbf{r})} d\mathbf{r} = \\ & = - \int_{\Sigma} \frac{4\pi \langle \mathbf{w}_0(\mathbf{r}, -\boldsymbol{\Omega}, \alpha_r)^T \mathbf{w}_{p-1}(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) \rangle}{c(\mathbf{r})} d\mathbf{r} \quad (25) \end{aligned}$$

$$\begin{aligned} & \int_{\Sigma} \frac{4\pi \langle \mathbf{w}_{q-1}(\mathbf{r}, -\boldsymbol{\Omega}, \alpha_r)^T \mathbf{w}_p(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) \rangle}{c(\mathbf{r})} d\mathbf{r} = \\ & = \int_{\Sigma} \frac{4\pi \langle \mathbf{w}_q(\mathbf{r}, -\boldsymbol{\Omega}, \alpha_r)^T \mathbf{w}_{p-1}(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) \rangle}{c(\mathbf{r})} d\mathbf{r} \quad (26) \end{aligned}$$

Proof: Multiplying Eq. (16) by $\mathbf{w}_p(\mathbf{r}, -\boldsymbol{\Omega}, \alpha_r)^T/c(\mathbf{r})$, integrating with respect to $\boldsymbol{\Omega}$ and with respect to \mathbf{r} in each homogeneous sub-region Σ_i it is

$$\begin{aligned} & \alpha_r \int_{\Sigma_i} \frac{4\pi \langle \mathbf{w}_p(\mathbf{r}, -\boldsymbol{\Omega}, \alpha_r)^T \mathbf{w}_0(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) \rangle}{c(\mathbf{r})} d\mathbf{r} + \\ & + \int_{\Sigma_i} \frac{4\pi v(\mathbf{r}) \langle \boldsymbol{\Omega} \cdot \mathbf{w}_p(\mathbf{r}, -\boldsymbol{\Omega}, \alpha_r)^T \nabla \mathbf{w}_0(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) \rangle}{c(\mathbf{r})} d\mathbf{r} + \\ & + \int_{\Sigma_i} \frac{4\pi \langle \mathbf{w}_p(\mathbf{r}, -\boldsymbol{\Omega}, \alpha_r)^T (\mathbf{w}_0(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) - \langle \mathbf{w}_0(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) \rangle) \rangle}{c(\mathbf{r})\tau(\mathbf{r})} d\mathbf{r} = \\ & = \int_{\Sigma_i} \frac{4\pi \langle \mathbf{w}_p(\mathbf{r}, -\boldsymbol{\Omega}, \alpha_r)^T \mathbf{f}(\mathbf{r}, \boldsymbol{\Omega}) \rangle}{c(\mathbf{r})} d\mathbf{r}. \end{aligned}$$

Similarly multiplying Eq. (17) by $\mathbf{w}_0(\mathbf{r}, -\boldsymbol{\Omega}, \alpha_r)^T/c(\mathbf{r})$, integrating with respect to $\boldsymbol{\Omega}$ and with respect to \mathbf{r} in each homogeneous sub-region Σ_i it is

$$\begin{aligned} & \alpha_r \int_{\Sigma_i} \frac{4\pi \langle \mathbf{w}_0(\mathbf{r}, -\boldsymbol{\Omega}, \alpha_r)^T \mathbf{w}_p(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) \rangle}{c(\mathbf{r})} d\mathbf{r} + \\ & + \int_{\Sigma_i} \frac{4\pi v(\mathbf{r}) \langle \boldsymbol{\Omega} \cdot \mathbf{w}_0(\mathbf{r}, -\boldsymbol{\Omega}, \alpha_r)^T \nabla \mathbf{w}_p(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) \rangle}{c(\mathbf{r})} d\mathbf{r} + \\ & + \int_{\Sigma_i} \frac{4\pi \langle \mathbf{w}_0(\mathbf{r}, -\boldsymbol{\Omega}, \alpha_r)^T (\mathbf{w}_0(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) - \langle \mathbf{w}_p(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) \rangle) \rangle}{c(\mathbf{r})\tau(\mathbf{r})} d\mathbf{r} = \\ & = - \int_{\Sigma_i} \frac{4\pi \langle \mathbf{w}_0(\mathbf{r}, -\boldsymbol{\Omega}, \alpha_r)^T \mathbf{w}_{p-1}(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) \rangle}{c(\mathbf{r})} d\mathbf{r}. \end{aligned}$$

By comparing these equations, and by summing over all sub-regions Σ_i , Eq. (25) descends.

Multiplying Eq. (17) by $\mathbf{w}_q(\mathbf{r}, -\boldsymbol{\Omega}, \alpha_r)^T/c(\mathbf{r})$, integrating with respect to $\boldsymbol{\Omega}$ and with respect to \mathbf{r} in each homogeneous sub-region Σ_i it is

$$\begin{aligned} & \alpha_r \int_{\Sigma_i} \frac{4\pi \langle \mathbf{w}_q(\mathbf{r}, -\boldsymbol{\Omega}, \alpha_r)^T \mathbf{w}_p(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) \rangle}{c(\mathbf{r})} d\mathbf{r} + \\ & + \int_{\Sigma_i} \frac{4\pi v(\mathbf{r}) \langle \boldsymbol{\Omega} \cdot \mathbf{w}_q(\mathbf{r}, -\boldsymbol{\Omega}, \alpha_r)^T \nabla \mathbf{w}_p(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) \rangle}{c(\mathbf{r})} d\mathbf{r} + \\ & + \int_{\Sigma_i} \frac{4\pi \langle \mathbf{w}_q(\mathbf{r}, -\boldsymbol{\Omega}, \alpha_r)^T (\mathbf{w}_0(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) - \langle \mathbf{w}_p(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) \rangle) \rangle}{c(\mathbf{r})\tau(\mathbf{r})} d\mathbf{r} = \\ & = - \int_{\Sigma_i} \frac{4\pi \langle \mathbf{w}_q(\mathbf{r}, -\boldsymbol{\Omega}, \alpha_r)^T \mathbf{w}_{p-1}(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) \rangle}{c(\mathbf{r})} d\mathbf{r}. \end{aligned}$$

By comparing this equations with the the transpose of the equation obtained by exchanging indices p and q , and by summing over all sub-regions Σ_i , Eq. (26) descends. ■

Let $\mathbf{H}(s)$ and $\hat{\mathbf{H}}(s)$ be the transfer matrices of the thermal network and of its compact model respectively, transforming $\mathbf{u}(s)$ into $\mathbf{y}(s)$. Let us consider their moments $\mathbf{H}_p(\alpha_r)$ and $\hat{\mathbf{H}}_p(\alpha_r)$ in the Taylor series expansions

$$\begin{aligned} \mathbf{H}(s) &= \sum_p^{\infty} \mathbf{H}_p(\alpha_r)(s - \alpha_r)^p, \\ \hat{\mathbf{H}}(s) &= \sum_p^{\infty} \hat{\mathbf{H}}_p(\alpha_r)(s - \alpha_r)^p. \end{aligned}$$

The following multi-point moment matching properties hold.

Theorem 1: It is

$$\mathbf{H}_p(\alpha_r) = \hat{\mathbf{H}}_p(\alpha_r)$$

for $p = 0 \dots k_r$ and $r = 1 \dots l$.

Proof: It is

$$\mathbf{H}(s) = \int_{\Sigma} \frac{4\pi \langle \mathbf{f}(\mathbf{r}, \boldsymbol{\Omega})^T \mathbf{w}(\mathbf{r}, \boldsymbol{\Omega}, s) \rangle}{c(\mathbf{r})} d\mathbf{r}$$

so that, by using Eq. (22),

$$\begin{aligned} \mathbf{H}_p(\alpha_r) &= \\ &= \int_{\Sigma} \frac{4\pi \langle \mathbf{f}(\mathbf{r}, \boldsymbol{\Omega})^T \mathbf{w}_p(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) \rangle}{c(\mathbf{r})} d\mathbf{r} = \\ &= \int_{\Sigma} \frac{4\pi \langle \mathbf{f}(\mathbf{r}, \boldsymbol{\Omega})^T \mathbf{u}(\mathbf{r}, \boldsymbol{\Omega}) \rangle}{c(\mathbf{r})} d\mathbf{r} \hat{\mathbf{W}}_p(\alpha_r) = \\ &= \hat{\mathbf{F}}^T \hat{\mathbf{W}}_p(\alpha_r) = \\ &= \hat{\mathbf{H}}_p(\alpha_r) \end{aligned}$$

for $p = 0 \dots k_r$ and $r = 1 \dots l$. ■

Theorem 2: It is

$$\mathbf{H}_p(\alpha_r) = \hat{\mathbf{H}}_p(\alpha_r)$$

for $p = k_r + 1 \dots 2k_r + 1$ and $r = 1 \dots l$.

Proof: Let it be

$$\begin{aligned} p &= p_1 + p_2, \\ 1 &\leq p_1 \leq k_r + 1, \\ 0 &\leq p_2 \leq k_r. \end{aligned}$$

From Lemma 1, recalling Eqs. (8)-(13), it is

$$\begin{aligned}
\mathbf{J}\hat{\mathbf{H}}_p(\alpha_r) &= \\
&= \mathbf{J}\hat{\mathbf{F}}^T(-\hat{\mathbf{K}}(\alpha_r)^{-1}\hat{\mathbf{M}})^p \hat{\mathbf{K}}(\alpha_r)^{-1}\hat{\mathbf{F}} = \\
&= \hat{\mathbf{F}}^T(-\hat{\mathbf{J}}\hat{\mathbf{K}}(\alpha_r)^{-1}\hat{\mathbf{M}})(-\hat{\mathbf{K}}(\alpha_r)^{-1}\hat{\mathbf{M}})^{p-1} \hat{\mathbf{K}}(\alpha_r)^{-1}\hat{\mathbf{F}} = \\
&= \hat{\mathbf{F}}^T(-\hat{\mathbf{K}}(\alpha_r)^{-T}\hat{\mathbf{J}}\hat{\mathbf{M}})(-\hat{\mathbf{K}}(\alpha_r)^{-1}\hat{\mathbf{M}})^{p-1} \hat{\mathbf{K}}(\alpha_r)^{-1}\hat{\mathbf{F}} = \\
&= \hat{\mathbf{F}}^T(-\hat{\mathbf{K}}(\alpha_r)^{-T}\hat{\mathbf{M}})\hat{\mathbf{J}}(-\hat{\mathbf{K}}(\alpha_r)^{-1}\hat{\mathbf{M}})^{p-1} \hat{\mathbf{K}}(\alpha_r)^{-1}\hat{\mathbf{F}} = \\
&= \dots \\
&= \hat{\mathbf{F}}^T(-\hat{\mathbf{K}}(\alpha_r)^{-T}\hat{\mathbf{M}})^{p_1-1}\hat{\mathbf{J}}(-\hat{\mathbf{K}}(\alpha_r)^{-1}\hat{\mathbf{M}})^{p_2+1} \hat{\mathbf{K}}(\alpha_r)^{-1}\hat{\mathbf{F}} = \\
&= -\hat{\mathbf{F}}^T(-\hat{\mathbf{K}}(\alpha_r)^{-T}\hat{\mathbf{M}})^{p_1-1} \hat{\mathbf{K}}(\alpha_r)^{-T}(\hat{\mathbf{J}}\hat{\mathbf{M}}) \cdot \\
&\quad \cdot (-\hat{\mathbf{K}}(\alpha_r)^{-1}\hat{\mathbf{M}})^{p_2} \hat{\mathbf{K}}(\alpha_r)^{-1}\hat{\mathbf{F}} = \\
&= -((-\hat{\mathbf{K}}(\alpha_r)^{-1}\hat{\mathbf{M}})^{p_1-1} \hat{\mathbf{K}}(\alpha_r)^{-1}\hat{\mathbf{F}})^T(\hat{\mathbf{J}}\hat{\mathbf{M}}) \cdot \\
&\quad \cdot (-\hat{\mathbf{K}}(\alpha_r)^{-1}\hat{\mathbf{M}})^{p_2} \hat{\mathbf{K}}(\alpha_r)^{-1}\hat{\mathbf{F}} = \\
&= -\hat{\mathbf{W}}_{p_1-1}(\alpha_r)^T \hat{\mathbf{J}}\hat{\mathbf{M}}\hat{\mathbf{W}}_{p_2}(\alpha_r)
\end{aligned}$$

Thus recalling Eqs. (2), (3) and (22) it results in

$$\begin{aligned}
\mathbf{J}\hat{\mathbf{H}}_p(\alpha_r) &= \\
&= -\hat{\mathbf{W}}_{p_1-1}(\alpha_r)^T \hat{\mathbf{J}}\hat{\mathbf{M}}\hat{\mathbf{W}}_{p_2}(\alpha_r) = \\
&= -\hat{\mathbf{W}}_{p_1-1}(\alpha_r)^T \int_{\Sigma} \frac{4\pi \langle \mathbf{u}(\mathbf{r}, -\boldsymbol{\Omega})^T \mathbf{u}(\mathbf{r}, \boldsymbol{\Omega}) \rangle}{c(\mathbf{r})} d\mathbf{r} \hat{\mathbf{W}}_{p_2}(\alpha_r) = \\
&= - \int_{\Sigma} \frac{4\pi \langle \mathbf{w}_{p_1-1}(\mathbf{r}, -\boldsymbol{\Omega}, \alpha_r)^T \mathbf{w}_{p_2}(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) \rangle}{c(\mathbf{r})} d\mathbf{r}
\end{aligned}$$

Then from Lemma 2

$$\begin{aligned}
\mathbf{J}\hat{\mathbf{H}}_p(\alpha_r) &= \\
&= - \int_{\Sigma} \frac{4\pi \langle \mathbf{w}_{p_1-1}(\mathbf{r}, -\boldsymbol{\Omega}, \alpha_r)^T \mathbf{w}_{p_2}(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) \rangle}{c(\mathbf{r})} d\mathbf{r} = \\
&= - \int_{\Sigma} \frac{4\pi \langle \mathbf{w}_{p_1-2}(\mathbf{r}, -\boldsymbol{\Omega}, \alpha_r)^T \mathbf{w}_{p_2+1}(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) \rangle}{c(\mathbf{r})} d\mathbf{r} = \\
&= \dots \\
&= - \int_{\Sigma} \frac{4\pi \langle \mathbf{w}_0(\mathbf{r}, -\boldsymbol{\Omega}, \alpha_r)^T \mathbf{w}_{p-1}(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) \rangle}{c(\mathbf{r})} d\mathbf{r} = \\
&= \int_{\Sigma} \frac{4\pi \langle \mathbf{f}(\mathbf{r}, -\boldsymbol{\Omega})^T \mathbf{w}_p(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r) \rangle}{c(\mathbf{r})} d\mathbf{r} = \\
&= \mathbf{J}\mathbf{H}_p(\alpha_r).
\end{aligned}$$

from which the thesis follows. \blacksquare

V. ALGORITHM

By choosing the functions in $\mathbf{u}(\mathbf{r}, \boldsymbol{\Omega})$ in such a way that Eq. (22) hold for a choice of the matching points α_r and matching orders k_r , with $r = 1 \dots l$, a compact model is obtained which approximates the response and preserves the thermodynamic properties of the thermal network from the transport equation. This is obtained without computing transient solutions of the transport equation, but just solving the transport equation in the complex frequency domain at the chosen matching points.

In order to avoid ill-conditioning problems [3] the functions in $\mathbf{u}(\mathbf{r}, \boldsymbol{\Omega})$ are not just taken equal to the chosen moments $\mathbf{w}_p(\mathbf{r}, \boldsymbol{\Omega}, \alpha_r)$ with $p = 0 \dots k_r$ and $r = 1 \dots l$. Instead the following Arnoldi-like algorithm is repeated for each of the $n^+ + n^-$ elements $\mathbf{f}(\mathbf{r}, \boldsymbol{\Omega})$ and for each expansion point α_r with $r = 1 \dots l$.

$$r(\mathbf{r}, \boldsymbol{\Omega}) = f_i(\mathbf{r}, \boldsymbol{\Omega})$$

For $p = 0 \dots k_r$

Solve Eqs. (27) - (29) for $v(\mathbf{r}, \boldsymbol{\Omega})$

For $q = 1 \dots p - 1$

$$v(\mathbf{r}, \boldsymbol{\Omega}) = v(\mathbf{r}, \boldsymbol{\Omega}) - u_q(\mathbf{r}, \boldsymbol{\Omega}) \int_{\Sigma} \frac{4\pi \langle v(\mathbf{r}, \boldsymbol{\Omega}) u_q(\mathbf{r}, \boldsymbol{\Omega}) \rangle}{c(\mathbf{r})} d\mathbf{r}$$

End

$$u_p(\mathbf{r}, \boldsymbol{\Omega}) = \frac{v(\mathbf{r}, \boldsymbol{\Omega})}{\int_{\Sigma} \frac{4\pi \langle v^2(\mathbf{r}, \boldsymbol{\Omega}) \rangle}{c(\mathbf{r})} d\mathbf{r}}$$

$$r(\mathbf{r}, \boldsymbol{\Omega}) = u_p(\mathbf{r}, \boldsymbol{\Omega})$$

End

Append $[u_0(\mathbf{r}, \boldsymbol{\Omega}) \dots u_{k_r}(\mathbf{r}, \boldsymbol{\Omega})]$ to $\mathbf{u}(\mathbf{r}, \boldsymbol{\Omega})$ \blacksquare

In each homogeneous sub-region Σ_i it is

$$\begin{aligned}
\alpha_r v(\mathbf{r}, \boldsymbol{\Omega}) + v(\mathbf{r}) \boldsymbol{\Omega} \cdot \nabla v(\mathbf{r}, \boldsymbol{\Omega}) + \\
+ \frac{v(\mathbf{r}, \boldsymbol{\Omega}) - \langle v(\mathbf{r}, \boldsymbol{\Omega}) \rangle}{\tau(\mathbf{r})} = r(\mathbf{r}, \boldsymbol{\Omega}) \quad (27)
\end{aligned}$$

At each interface between homogeneous sub-regions Σ_i it is

$$[v(\mathbf{r})v^n(\mathbf{r}, \boldsymbol{\Omega})] = 0, \quad [v^t(\mathbf{r}, \boldsymbol{\Omega})/c(\mathbf{r})] = 0. \quad (28)$$

At $\partial\Sigma$ one of the following boundary conditions hold

$$v^n(\mathbf{r}, \boldsymbol{\Omega}) = 0, \quad v^t(\mathbf{r}, \boldsymbol{\Omega}) = 0. \quad (29)$$

VI. APPLICATION

Let us consider again the application example investigated in [2]. It results in

$$\begin{aligned}
w(\mathbf{r}, \boldsymbol{\Omega}, s) = \frac{c}{4\pi} R \left(\frac{K(\rho, \mu, p|l_1)}{l_1^3} + \right. \\
\left. + \int_0^{+\infty} \frac{dK}{d\xi}(\rho, \mu, p|\xi) F(\xi, p|l_1) d\xi \right) (1+p)
\end{aligned}$$

in which

$$\mu = \frac{\mathbf{r}}{|\mathbf{r}|} \cdot \boldsymbol{\Omega}, \quad p = s\tau,$$

and $K(\rho, \mu, p|\xi)$ is the energy density distribution of the transport equation without regeneration given by

$$K(\rho, \mu, p|\xi) = \begin{cases} \frac{1 - e^{-(1+p)(\rho\mu + \sqrt{\xi^2 - \rho^2(1-\mu^2)})}}{1+p} & \left(\frac{\rho}{\xi}, \mu\right) \in \mathcal{C}_1 \\ \frac{e^{-(1+p)(\rho\mu - \sqrt{\xi^2 - \rho^2(1-\mu^2)})}}{1+p} - \frac{e^{-(1+p)(\rho\mu + \sqrt{\xi^2 - \rho^2(1-\mu^2)})}}{1+p} & \left(\frac{\rho}{\xi}, \mu\right) \in \mathcal{C}_2 \\ 0 & \left(\frac{\rho}{\xi}, \mu\right) \in \mathcal{C}_3 \end{cases}$$

Using this expression we can compute $\mathbf{u}(\mathbf{r}, \boldsymbol{\Omega})$ and thus we can evaluate the integrals defining the compact models.

Hereafter the case $l_1 = 1/2$, $l_2 = 1$, $l_{12} = 2$ is considered. A compact model has been generated choosing as expansion points and orders those used by the authors for the thermal networks from the heat diffusion equation [5]. We have assumed $l = 15$ so that, being $n^+ = 2$ and $n^- = 0$, it results in $m^+ = m^- = 30$. The Nyquist plot of the impedance matrices of the thermal network and of its compact model are compared in Fig. 2. The power impulse thermal response

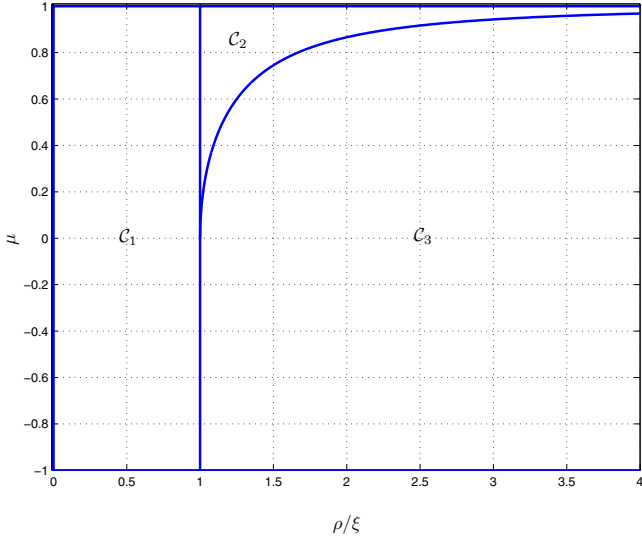


Fig. 1. Regions C_1 , C_2 , C_3 . The curve separating C_2 from C_3 has equation $\rho/\xi = 1/\sqrt{1-\mu^2}$ with $\mu > 0$.

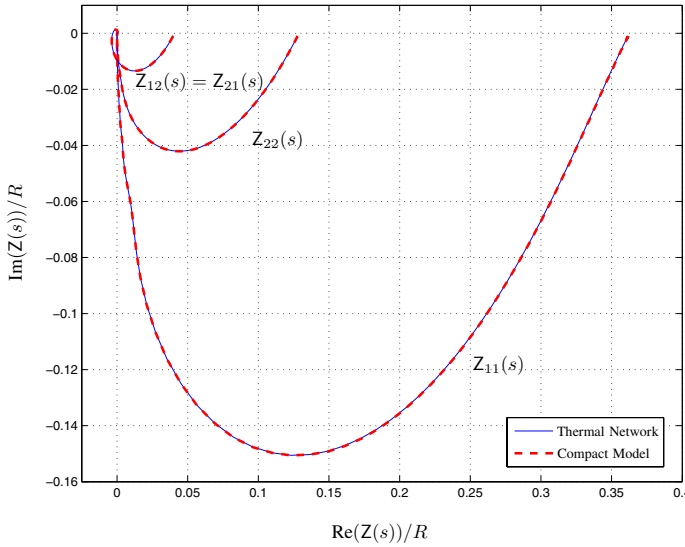


Fig. 2. Nyquist plots of the thermal impedances of the thermal network and its compact model.

matrices and the power step thermal response matrices are compared in Figs. 3, 4. A 0.1% error in the approximation is obtained both in the time domain and in the frequency domain.

VII. CONCLUSIONS

A multi-point moment matching method has been proposed for generating compact models of thermal networks from the gray phonon Boltzmann's transport equation. The resulting algorithm has been applied to a reference thermal network for which an highly accurate and compact model has been obtained.

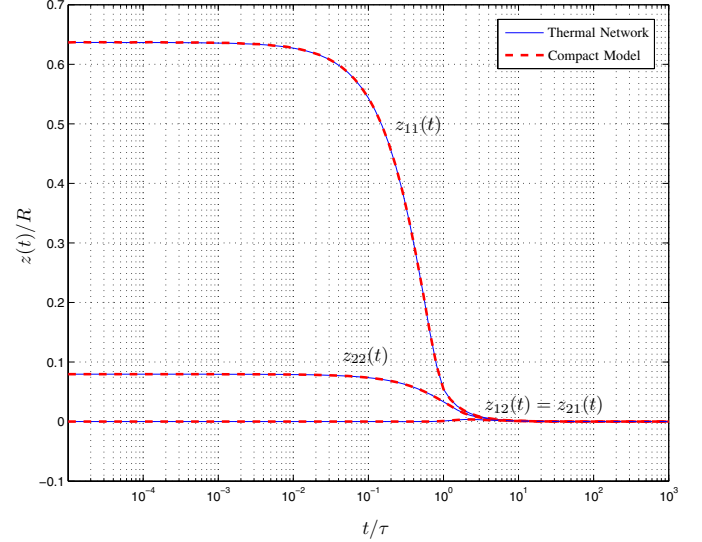


Fig. 3. Plots of the power impulse thermal responses of the thermal network and its compact model.

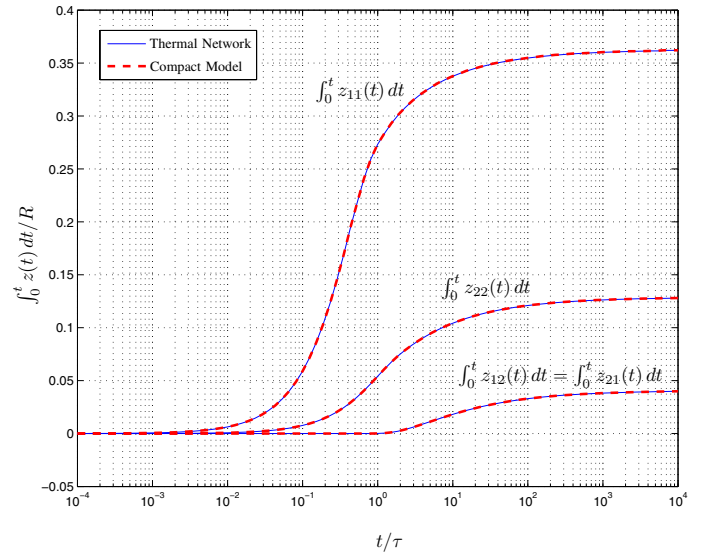


Fig. 4. Plots of the power step thermal responses of the thermal network and its compact model.

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