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| Eric Reynaud. The  $n$ th algebraic fundamental group. 2003. hal-00202442

**HAL Id: hal-00202442**

**<https://hal.science/hal-00202442v1>**

Preprint submitted on 7 Jan 2008

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# The $n^{\text{th}}$ algebraic fundamental group

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One of the main tools for the study of the category of finite dimensional modules over a basic algebra, over an algebraically closed field  $k$  is its presentation by a quiver and relations. This theory is mainly due to P. Gabriel. More precisely, it is well known that for each finite dimensional and basic algebras over an algebraically closed field  $k$ , there exists an unique quiver  $Q$  and an admissible ideal  $I$  of the path quiver algebra  $kQ$ , such that  $A$  is isomorphic to  $kQ/I$ . Such a couple  $(Q, I)$  is called a presentation of  $A$  by quiver and relations or a bound quiver. For each bound quiver  $(Q, I)$ , we can construct an algebraic fundamental group  $\Pi_1(Q, I)$ .

But, in general, different presentations of an algebra can have different fundamental groups. Then an algebra which gives trivial fundamental groups for all its presentations is called simply connected. The importance of these algebras is that, with use of covers, we can often restrict the study of indecomposable modules of an algebra to the indecomposable modules of a simply connected algebra. (see for example [BM]). However, this is not the only way the fundamental group is important. For instance, there are links with simplicial homology and Hochschild cohomology. We can notice for example the theorem of Assem-De la Peña ([AP], p. 200).

It has been proved (see [Rey] or [Bus]) that the fundamental group of an incidence algebra has a topological interpretation. Indeed, the algebraic fundamental group is isomorphic to a topological fundamental group of a simplicial complex. To generalize this result, J.C. Bustamante has constructed a CW-complex for each bound quiver of a triangular algebra which, has the same fundamental group (see [Bus2]).

In this paper, we give a definition of the  $n^{\text{th}}$  fundamental group of a bound quiver and we will prove that there is an isomorphism between this fundamental group and the  $n^{\text{th}}$  homotopy group of the CW-complex. We will first recall the definition of the fundamental group of a bound quiver and the construction of a CW-complex associated. Then we will define the  $n^{\text{th}}$  fundamental group of an incidence algebra and will prove that this group is isomorphic to the  $n^{\text{th}}$  homotopy groups of the CW-complex. Finally, we will give a definition of the  $n^{\text{th}}$  algebraic fundamental group for all bound quivers of a triangular algebra.

## 1. CW-complex associated to a bound quiver.

**1.1 The fundamental group of a bound quiver.** The general definition of the fundamental group depends on a couple  $(Q, I)$  in which  $I$  is an admissible ideal of  $kQ$ , it can be found in [MP] and we recall it below. We notice that for an algebra the presentation as a quiver with relations is not unique in general, that is to say that different ideals  $I$  may exist such that  $A \cong kQ/I$ . So, we do not have in general an unique fundamental group associated to an algebra. Nevertheless, in the case of an incidence algebra it has been proved that the fundamental group does not depend on the presentation of the algebra (see [BG] for example). The class of algebras having this property is larger. For more detail, we can see [BM] on restricted algebras.

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A relation  $\sum_{i=1}^n \lambda_i \omega_i$  is *minimal* if the sum is in  $I$  and if for all non empty proper subset  $J$  of  $\{1, \dots, n\}$  the sum  $\sum_{i \in J} \lambda_i \omega_i$  is not in  $I$ . We note that if the relation is minimal then  $\{\omega_1, \dots, \omega_n\}$  have the same source and the same terminus.

If  $\alpha$  is an arrow from  $x$  to  $y$ , let's denote by  $\alpha^{-1}$  its formal inverse which goes from  $y$  to  $x$ . A walk from  $x$  to  $y$  is a formal product  $\alpha_1^{\pm 1} \alpha_2^{\pm 1} \dots \alpha_n^{\pm 1}$  which begins in  $x$  and ends in  $y$ . The trivial walk  $x$ , which begins in  $x$  and no longer moves is denoted by  $e_x$ . A *closed walk* (or a *loop*) is a walk having the same extremities. A walk is, in fact, a path in the non oriented graph associated to  $Q$ ; in other words, a walk can follow the arrows in any direction.

We consider the smallest equivalence relation  $\sim$  on the walks of  $Q$  containing the following items :

1. if  $\alpha$  is an arrow from  $x$  to  $y$  then  $\alpha \alpha^{-1} \sim e_y$  and  $\alpha^{-1} \alpha \sim e_x$ ,
2. if  $\sum_{i=1}^n \lambda_i \omega_i$  is a minimal relation then  $\omega_1 \sim \dots \sim \omega_n$ ,
3. if  $\alpha \sim \beta$  then, for all  $(\omega, \omega')$ , we have  $\omega \alpha \omega' \sim \omega \beta \omega'$

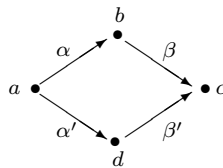
Let  $x_0$  be a vertex of  $Q$ . The set of equivalence classes of loops starting at  $x_0$  does not depend on  $x_0$  since  $Q$  is connected. We denote this set by  $\Pi_1(Q, I)$ . If the quiver  $Q$  comes from a poset  $P$ , the fundamental group associated to  $(Q, I_Q)$  is denoted by  $\Pi_1(P)$ .

**1.2 CW-complex associated to a bound quiver.** Let's recall first how to construct a simplicial complex  $C$  from a bound quiver  $(Q, I)$  where  $kQ/I$  is an incidence algebra. The  $n$ -simplexes of  $C$  are the sets of  $n$  vertices of  $Q$  which are in a same path. The construction that follows proposed by J.C. Bustamante (see [Bus2]) is a generalization of this. From a bound quiver  $(Q, I)$ , he constructs a CW-complex  $\mathcal{B}(Q, I)$  :

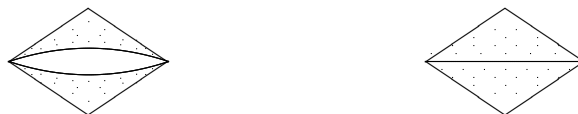
Let's consider the previous equivalent relation  $\sim$  defined to construct the fundamental group and let's restrict it to  $C(Q)$ , the set of paths on  $Q$ . Then two paths are equivalent if they are equivalent walks :

1. The 0-cells of  $\mathcal{B}(Q, I)$  are the vertices of  $Q$ .
2. The 1-cells are the classes  $\tilde{\omega}$  of paths of  $Q$  where  $\tilde{\omega}$  contains at least a path which is not in  $I$ . We attach a 1-cell  $\tilde{\omega}$  on the 0-cells which correspond to the source and the tail of all paths in  $\tilde{\omega}$ .
3. The  $n$ -cells, for  $n \geq 2$ , are exactly  $(\tilde{\omega}_n, \dots, \tilde{\omega}_1)$  such that it exists for all  $i$  in  $\{1, \dots, n\}$  a path  $\omega_i$  in  $\tilde{\omega}_i$  which verifies  $\omega_n \dots \omega_2 \omega_1 \neq 0$  in  $kQ/I$ . Then we attach this cell on the  $n + 1$  cells of dimension  $n - 1$  :
  - $(\tilde{\omega}_{n-1}, \dots, \tilde{\omega}_1)$
  - $(\tilde{\omega}_n, \dots, \omega_{i+1} \omega_i \dots \tilde{\omega}_1)$  for all  $i$  in  $\{1, \dots, n - 1\}$
  - $(\tilde{\omega}_n, \dots, \tilde{\omega}_2)$

**Example.** Let's consider the quiver  $Q$  :



If  $I = \{0\}$  then the 0-cells are  $a, b, c, d$  the 1-cells are  $\alpha, \alpha', \beta, \beta'$  and the 2-cells are  $\beta\alpha, \beta'\alpha'$ . If  $I$  is generated by  $\beta\alpha - \beta'\alpha'$  then the 0-cells are  $a, b, c, d$  the 1-cells are  $\alpha, \alpha', \beta, \beta'$  and there only one 2-cells  $\beta\alpha = \beta'\alpha'$ . Then CW-complexes are respectively :



For more details and other examples, see [Bus2].

**1.3 Comparison of the fundamental groups.** J.C. Bustamante has proved that there exists an isomorphism between the fundamental group of a bound quiver and the fundamental group of the CW-complex associated, in the case where the bound quiver is a bound quiver of a triangular algebra, in other words such that the quiver of the presentation does not contain oriented cycles.

**1.4 Incidence algebra of an ordered quiver, of a poset.**

Given a *poset* (i.e. a partially ordered set), there is an associated *ordered quiver*, that is to say a finite oriented graph without loops and such that if there exists an arrow from  $a$  to  $b$ , there does not exist another path from  $a$  to  $b$ . To each element of the poset corresponds a vertex of the graph. Moreover, let  $S_1$  and  $S_2$  be vertices in the graph; there exists an arrow from  $S_1$  to  $S_2$  if and only if the element associated to  $S_1$  in the poset is smaller than the element associated to  $S_2$  and if there does not exist an element of the poset strictly between these two elements. The graph obtained is an ordered quiver.

Let  $Q$  be a quiver, and  $k$  be a field. We denote  $kQ$  the  $k$ -vector space with basis the paths of  $Q$  (the paths of length 0 being the vertices), with the multiplication given by the composition of two paths if possible and 0 otherwise. Two paths of  $Q$  are *parallel* if they have the same beginning and the same end. The  $k$ -space generated by the set of differences of two parallel paths is a two-sided ideal of  $kQ$ , denoted  $I_Q$  and called *parallel ideal*. The quotient algebra  $kQ/I_Q$  is the *incidence algebra* of  $Q$ .

**1.5 Incidence algebra associated to a bound quiver.** Let's consider the poset  $P_{(Q,I)}$  containing the cells of  $\mathcal{B}(Q,I)$  with the order relation :  $c_1 \leq c_2$  if and only if  $c_1 \subset \overline{c_2}$ . In other words, if  $c_2$  is attached on the cell  $c_1$  then  $c_1 \leq c_2$ . In that case we have :

**Theorem.** Let  $(Q,I)$  be a bound quiver such that  $kQ/I$  is a triangular algebra and  $A_{(Q,I)}$  be the incidence algebra associated to  $P_{(Q,I)}$ . The fundamental groups, of the bound quiver  $(Q,I)$  on one hand, and the incidence algebra  $A_{(Q,I)}$  on other hand, are isomorphic.

Let's remark that if  $(Q,I)$  is a bound quiver coming from a simplicial complex, this theorem already exists (see [Rey]). The proof of this theorem is a consequence of the next paragraph and of the theorem of J.C. Bustamante that prove the isomorphism between the fundamental group of a bound quiver and the fundamental group of the CW-Complex associated.

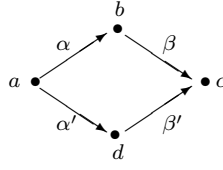
**2. The  $n^{\text{th}}$ -group of homotopy of an incidence algebra.**

**2.1 The  $N$ -walk-matrices.** Let's define first the notion of walk-matrix which will allow us to construct the  $n^{\text{th}}$  fundamental group. Let  $(Q,I)$  be a bound quiver and let  $N, n_1, n_2, \dots, n_N$  be integers. Let's denote by  $Q_0$  the vertices of  $Q$ . A  $N$ -matrix of  $(Q,I)$  of length  $(n_1, \dots, n_N)$  is a matrix  $n_1 \times \dots \times n_N$  containing vertices of  $Q$  that is a map  $f$  :

$$f : \begin{matrix} [[1, n_1]] \times [[1, n_2]] \times \dots \times [[1, n_N]] & \rightarrow & Q_0 \\ (i_1, i_2, \dots, i_N) & \mapsto & f(i_1, i_2, \dots, i_N) \end{matrix}$$

where  $[[1, n]]$  is the set  $\{1, 2, \dots, n\}$ . By convention, the 0-matrices are the matrices which contains only one element. A  $N$ -matrix is a  $N$ -walk-matrix if for all  $(i_1, \dots, i_N)$ , all vertices  $f(j_1, \dots, j_N)$  with  $i_k \leq j_k \leq i_k + 1$  are contained in a same path. In other words, the sub-matrix of  $f$  of length  $(2, 2, \dots, 2)$  contain only vertices included in a same path. Let's denote by  $\mathcal{M}_N(Q)$  the set of  $N$ -walk-matrices.

**Example.** let  $Q$  be the quiver :



and  $I$  be the ideal generated by  $\alpha\beta - \alpha'\beta'$ . Then, for instance :

$$[a] \quad \begin{bmatrix} b \\ d \end{bmatrix} \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \begin{bmatrix} a & b & c & d & a \\ a & b & a & d & a \\ a & a & a & a & a \end{bmatrix}$$

are respectively 0-matrix, 1-matrix, 1-matrix and 2-matrix of length  $( ), (2), (3)$  and  $(3,5)$ . The second matrix is not a walk-matrix because  $b$  and  $d$  are not vertices of a same path. The others are walk-matrices. For example, the fourth is a 4-walk-matrix because each set of vertices  $\{a, b, a, b\}$   $\{b, c, b, a\}$   $\{c, d, a, d\}$   $\{d, a, d, a\}$   $\{a, b, a, a\}$   $\{b, a, a, a\}$   $\{a, d, a, a\}$  and  $\{d, a, a, a\}$  are vertices of a path. Lastly, we can remark that a  $N$ -matrix can be considered as a  $N + 1$ - matrix. For example the matrix number 2 can be considered either as a 1 matrix of length  $(2)$  or a 2-matrix of length  $(2,1)$  or a 3-matrix of length  $(2,1,1)$  etc...

Let's generalize now the notions of source and tail. For that, let's denote by  $s_1, \dots, s_N$  the applications sources and by  $t_1, \dots, t_N$  the applications tails defined from  $\mathcal{M}_N(Q)$  to  $\mathcal{M}_{N-1}(Q)$  by :

$$\begin{cases} s_i(M)(x_1, \dots, x_N) = M(x_1, \dots, x_{i-1}, 1, x_i, \dots, x_N) \\ t_i(M)(x_1, \dots, x_N) = M(x_1, \dots, x_{i-1}, n_i, x_i, \dots, x_N) \end{cases}$$

for all  $N$ -matrices  $M$  and for all  $x_1, \dots, x_N$  such that the previous expressions have a sense; the integer  $n_i$  being the dimension of the matrix  $M$  on the  $i^{\text{th}}$  component. By convention  $s = s_N$  and  $t = t_N$  and when we talk about source and tail, it concerns the images by  $s$  and  $t$ . Two  $N$ -matrices  $M$  and  $M'$  are said equal alongside if  $s_i(M) = s_i(M')$  and  $t_i(M) = t_i(M')$  for all  $i$  in  $\{1, \dots, N\}$ .

**Example.** Let's consider the bound quiver defined in the previous example and the 2-walk-matrix :

$$M = \begin{bmatrix} a & b & c & d \\ b & c & a & c \end{bmatrix}$$

Then :

$$s_1(M) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \quad s(M) = s_2(M) = \begin{bmatrix} a \\ b \end{bmatrix} \quad t_1(M) = \begin{bmatrix} b \\ c \\ a \\ c \end{bmatrix} \quad t(M) = t_2(M) = \begin{bmatrix} d \\ c \end{bmatrix}$$

Let's define now an operation on  $N$ -walk-matrices which will be the intern law of composition in the  $n^{\text{th}}$  fundamental group. Let  $M_1$  and  $M_2$  be two  $N$ -matrices of length  $(n_1, \dots, n_N)$  and  $(n'_1, \dots, n'_N)$  such that  $t(M_1) = s(M_2)$ . In particular, we have  $n_k = n'_k$  for  $k$  lower or equal than  $N - 1$ . Then let's denote by  $M_1 \times M_2$  the matrix obtained when we "pile"  $M_1$  on  $M_2$  on the  $N^{\text{th}}$  component and when we delete one of the common  $(N - 1)$ -matrix. In other words, the product  $M_1 \times M_2$  is defined by :

$$M_1 \times M_2(x_1, \dots, x_{N-1}, t) = \begin{cases} M_1(x_1, \dots, x_{N-1}, t) & \text{if } t \in \{1, \dots, n_N\} \\ M_2(x_1, \dots, x_{N-1}, t - n_N) & \text{if } t \in \{n_N + 1, \dots, n_N + n'_N\} \end{cases}$$

for all  $x_1, \dots, x_{N-1}$  such that the previous expressions have a sense.

We define also an other operation on the  $N$ -matrices. Let  $M_1$  and  $M_2$  be  $N$ -walk-matrices of common length  $(n_1, \dots, n_N)$ . Let's denote by  $M = [M_1, M_2]$  the  $(N + 1)$ -matrix defined by  $M(\dots, 1) = M_1(\dots)$  and  $M(\dots, 2) = M_2(\dots)$ . We remark that in general  $[M_1, M_2]$  and  $M_1 \times M_2$  are not walk-matrices.

**Examples.**

$$\begin{aligned} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \times \begin{bmatrix} c \\ d \\ e \end{bmatrix} &= \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} & \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} \times \begin{bmatrix} d & g \\ e & h \\ f & i \end{bmatrix} &= \begin{bmatrix} a & g \\ b & h \\ c & i \end{bmatrix} \\ & & \left[ \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} d \\ e \\ f \end{bmatrix} \right] &= \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} \end{aligned}$$

**2.2 The  $n^{\text{th}}$  fundamental group.** Let's define in this paragraph the  $n^{\text{th}}$  fundamental group. For that we must first construct an equivalence relation  $\sim$  on the  $N$ -walk-matrices. Let's define it inductively on  $N$ . If  $N$  is 0 two walk-matrices are equivalent if they are equal. Let's suppose the construction of  $\sim$  on  $\mathcal{M}_{N-1}(Q)$  done for a fixed non zero integer  $N$ . The equivalent relation  $\sim$  on  $\mathcal{M}_N(Q)$  is then generated by :

1. Let  $M_1$  and  $M_2$  two  $N$ -walk-matrices of same length  $(n_1, \dots, n_N)$ . If it exists a  $N + 1$ -walk-matrix  $F$  such that its source is  $M_1$  and its tail is  $M_2$  and such that the  $N$ -matrices  $F(\dots, t)$  for all integer  $t$  are equal alongside then  $M_1$  and  $M_2$  are equivalent.
2. Let  $M$  a  $N$ -walk-matrix of length  $\{n_1, \dots, n_N\}$ . Let's consider  $M'$  the matrix obtained from  $M$  by deleting an hyperplan of vertices. More precisely it exists  $i$  in  $\{1, \dots, N\}$  and  $k$  in  $\{1, \dots, n_i\}$  such that

$$M'(x_1, \dots, x_N) = \begin{cases} M(x_1, \dots, x_{i-1}, & x_i & , x_{i+1}, \dots, x_N) & \text{if } 1 \leq x_i < k \\ M(x_1, \dots, x_{i-1}, & x_i + 1 & , x_{i+1}, \dots, x_N) & \text{if } k \leq x_i \leq n_i - 1 \end{cases}$$

If  $M'$  is also a walk-matrix such that the sources and tails of  $M$  and  $M'$  are equivalent ( $s_1(M_1) \sim s_1(M_2)$ ,  $s_2(M_1) \sim s_2(M_2)$ , ...) then  $M$  and  $M'$  are equivalent.

**Example 1.** Let's consider again the bound quiver used in examples of the previous paragraph. Then the 1-matrices :

$$M_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad M_2 = \begin{bmatrix} a \\ d \\ c \end{bmatrix}$$

are equivalent because then 2-matrix

$$M = \begin{bmatrix} a & a & a \\ b & a & d \\ c & c & c \end{bmatrix}$$

is a walk-matrix such that  $s(M) = M_1$ ,  $t(M) = M_2$  and the columns of  $M$  are equal alongside. We have used the condition 1 in the definition.

**Example 2.** With the same bound quiver, by using the second condition several times :

$$\begin{bmatrix} a \\ b \\ c \\ d \\ a \end{bmatrix} \sim \begin{bmatrix} a \\ c \\ d \\ a \end{bmatrix} \sim \begin{bmatrix} a \\ d \\ a \end{bmatrix} \sim \begin{bmatrix} a \\ a \end{bmatrix} \sim [ a ]$$

**Example 3.** Let's use again the same bound quiver and let's consider these 2-matrices :

$$M_1 = \begin{bmatrix} a & a & a \\ a & b & b \\ a & a & c \end{bmatrix} \quad M_2 = \begin{bmatrix} a & a \\ a & b \\ a & c \end{bmatrix} \quad M_3 = \begin{bmatrix} a & a & a \\ a & a & c \end{bmatrix}$$

Then the matrices  $M_1$ ,  $M_2$  and  $M_3$  are equivalent because  $M_2$  is obtained from  $M_1$  by deleting the second columns and  $M_3$  is obtained from  $M_1$  by deleting the second line; moreover the matrices  $M_2$  and  $M_3$  satisfy the condition 2 of the definition that is  $M_2$  and  $M_3$  are walk-matrices and the sources and tails are equivalent :

$$\begin{bmatrix} a \\ a \\ a \end{bmatrix} \sim \begin{bmatrix} a \\ a \end{bmatrix} \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix} \sim \begin{bmatrix} a \\ c \end{bmatrix} \quad \begin{bmatrix} a \\ a \\ c \end{bmatrix} \sim \begin{bmatrix} a \\ c \end{bmatrix}$$

Let's remark that if the hyperplan is chosen inside the matrix, that is to say if  $2 \leq k \leq n_i - 1$ , and if the matrix  $M'$  is a walk-matrix then  $M$  and  $M'$  are equivalent. In fact, deleting an hyperplan of vertices in  $M$  either delete an hyperplan to the sources or terminus of  $M$  or let them unchanged. Moreover, if  $N = 1$ , deleting an hyperplan of vertices does not change the source and the tail, then inductively we prove that  $M \sim M'$ .

Let's remark also that this equivalence relation is compatible with the operation  $\times$  previously defined. Moreover, it is completely defined on  $\mathcal{M}_N^x(Q)$ , the set of  $N$ -walk-matrices such that the side is formed only with the vertex  $x$  of  $Q$ . Then the  $n^{\text{th}}$  fundamental group of the presentation  $(Q, I)$  of an incidence algebra  $A$  is equal by definition to :

$$\Pi_n(A) = (\mathcal{M}_N^x(Q) / \sim, \times)$$

If  $Q$  is connected, what we suppose from now, the  $n^{\text{th}}$  fundamental group does not depend of the vertex  $x$  chosen.

### 2.3 Simplicial complexes and incidence algebras.

Let's recall first the definition of the  $n^{\text{th}}$  homotopy group of a topological space  $X$ . Let's denote by  $B^n = \{x \in \mathbb{R}^n / \|x\| \leq 1\}$ ,  $S^n = \{x \in \mathbb{R}^n / \|x\| = 1\}$ . Moreover let  $M_x$ ,  $x$  in  $X$ , be the set of maps from  $B^n$  to  $X$  constant to  $x$  on  $S^n$ . Two maps  $f_1$  and  $f_2$  in  $M_x$  are homotopic if it exists a continuous map  $F$  from  $B^n \times [0, 1]$  to  $X$  such that  $F(\cdot, 0) = f_0(\cdot)$  and  $F(\cdot, 1) = f_1(\cdot)$ . The homotopy relation is an equivalent relation and the  $n^{\text{th}}$  homotopy group of  $X$  is defined as the quotient of  $M_x$  by the homotopy relation.

Let's talk now about the relation between the topological and the algebraic approaches :

**Theorem.** Let  $(Q, I)$  be the presentation of an incidence algebra  $A$ , let's denote by  $C$  the simplicial complex associated, then :

$$\Pi_n(A) \simeq \Pi_n(C)$$

In light of the previous theorem, it will be clear why  $\mathcal{M}_N^x(Q) / \sim$  is called the  $n^{\text{th}}$  homotopy group. We already know that the fundamental group  $\Pi_1(Q, I)$  defined with the walks is isomorphic to the topological fundamental group  $\Pi_1(C)$  (see [Bus] or [Rey]). We can deduce that the new definition of the first fundamental group of an incidence algebra coincides with the standard one. It seems interesting now to give an explicit isomorphism between the two definitions of the fundamental group of an incidence algebra. Let's consider a

path  $\alpha_n \dots \alpha_2 \alpha_1$  of  $Q$ . We denote by

$$\phi(\alpha_n \dots \alpha_2 \alpha_1) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n+1} \end{bmatrix}$$

where  $a_i$  and  $a_{i+1}$  are respectively the source and the tail of  $\alpha_i$ . This map  $\phi$  will induce the isomorphism.

### 2.4 Proof of the theorem 2.3.

The proof of this theorem is inspired by the proof of the approximation theorem of a simplicial complex (see [HW] for example). We will recall some definitions and results before prove this theorem.

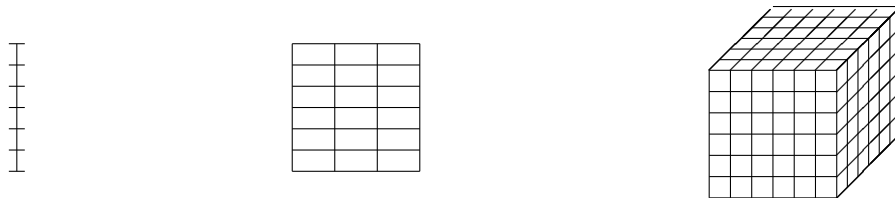
Let's recall first that for all element  $x$  in the geometric realization  $|C|$  of a simplicial complex  $C$ , it exists only one face containing  $x$ . This face is called the carrying face. Moreover two maps  $f$  and  $g$  from  $[0, 1]^n$  to the geometric realization  $|C|$  of a simplicial complex are said  $C$ -approximated if for all  $x$  in  $[0, 1]^n$ , it exists a simplex  $s$  in  $C$  such that  $f(x)$  and  $g(x)$  are contained in the adherence of the realization of  $s$ . We know moreover that two maps  $C$ -approximated are homotopic (see for example [Hil] p102). Let's define also, for all vertices  $a$  of a simplicial complex, the star of  $a$  denoted by  $st(a)$ , being the union of the faces having  $a$  as vertex. When  $Q$  is finite, the realization  $|st(a)|$  of  $st(a)$  is open in  $|C|$  (see [Hil] p 102) and  $\bigcap_{i \in I} |st(a_i)| \neq \emptyset$  if and only if  $(a_i)_{i \in I}$  are on a same simplex (see [Hil] p.92).

Moreover, let's denote by  $C_{(n_1, \dots, n_N)}^N$  the polyhedra obtained from  $[0, 1]^N$  by "cutting" it on each dimension  $i$  in  $n_i$  equal pieces. More precisely :

1. The vertices are the points of coordinates  $\left(\frac{k_1}{n_1}, \frac{k_2}{n_2}, \dots, \frac{k_N}{n_N}\right)$  with  $k_i$  in  $\{0, \dots, n_i\}$  for all  $i$  in  $\{1, \dots, n\}$ ,
2. the 1-faces are the segments linking two vertices which coordinates are all similar except one which differs of  $\frac{1}{n_i}$ ,
3. the 2-faces are delimited by 4 1-faces which form squares,
4. the 3-faces are delimited by 6 2-faces which form cubes, etc...

We will do, like for simplicial complexes, the difference between the theoric polyhedra  $C_x^N$ , with  $x = (n_1, \dots, n_N)$ , that is to say the set containing the vertices, and informations that can determine the faces, and  $|C_x^N|$  the realization of  $C_x^N$ . Let's remark that by construction  $|C_x^N| = [0, 1]^N$ . We denote also  $C_n^N = C_{(n, n, \dots, n)}^N$ . We extend the definition of a star of a vertex and the notion of carrying face to such polyhedra.

**Example.** For example these 3 next polyhedra are respectively  $C_6^1$ ,  $C_{(6,3)}^2$  and  $C_6^3$



Moreover a morphism from  $C_x^N$  in the simplicial complex  $C$ , with  $x = (n_1, \dots, n_N)$  is a map  $u$  which preserves the faces, that is to say if  $a_1, \dots, a_m$  are on a same face of  $C_x^N$  then  $u(a_1), \dots, u(a_m)$  are in a same simplex of  $C$ . Let's denote by  $Hom(C_x^N, C)$  their set.

Let's remark that each morphism from  $C_x^N$  to  $C$  defines an unique  $N$ -walk-matrix and reciprocally : let  $u$  be a morphism from  $C_x^N$  to  $C$  and let's denote  $s(i_1, \dots, i_N)$  the vertex of coordinates  $\left(\frac{i_1}{n_1}, \dots, \frac{i_N}{n_N}\right)$ . Then the  $N$ -walk-matrix  $f$  defined by :

$$f(i_1, \dots, i_n) = u(s(i_1, \dots, i_N))$$



**Example.** Let's consider the morphism  $u$  from  $C_{(2,3)}^2$  to a simplicial complex  $C$  such that  $u(s(i, j)) = a_{ij}$  where  $s(i, j) = s_{i,j}$  is the vertex of  $C_{(2,3)}^2$  at the position  $(i, j)$ , and  $a_{ij}$  is a vertex of  $C$ . Then the associated matrix is :

$$\begin{array}{cccc}
 & s_{1,3} & s_{2,3} & s_{3,3} & s_{4,3} \\
 & \begin{array}{|c|c|c|} \hline & & \\ \hline \end{array} & & & \\
 s_{1,2} & \begin{array}{|c|c|c|} \hline & s_{2,2} & s_{3,2} & \\ \hline \end{array} & & & s_{4,2} \\
 & & & & \\
 & s_{1,1} & s_{2,1} & s_{3,1} & s_{4,1}
 \end{array}
 \longrightarrow
 \begin{bmatrix}
 a_{13} & a_{23} & a_{33} & a_{43} \\
 a_{12} & a_{22} & a_{32} & a_{42} \\
 a_{11} & a_{21} & a_{31} & a_{41}
 \end{bmatrix}$$

For each morphism  $u$  from  $C_x^N$  to  $C$ , we will construct a continuous map  $|u|$  from  $|C_x^N| = [0, 1]^N$  to  $|C|$  such that if  $a_1, \dots, a_p$  are vertices of the carrying face of  $x$  in  $C_x^N$  then  $|u|(x)$  is inside the geometric simplex  $\{u(a_1), \dots, u(a_p)\}$ . So if  $x$  is in the face  $s$  of  $|C_x^N|$  then  $|u|(x)$  is in the image by  $u$  of  $s$ . Let's construct  $|u|$  by recurrence. If  $N$  is 0, the polyhedra  $C_x^N$  contains only one vertex  $a$  and the morphisme  $|u|$  is defined by  $|u|(a) = u(a)$ . Let's suppose that the construction is effective for  $N$  and let's prove it for  $N + 1$ . Let  $u$  be a morphism from  $C_{(n_1, \dots, n_{N+1})}^{N+1}$  to  $C$ . Let's denote for all  $k$  in  $\{0, \dots, n_{N+1}\}$  :

$$u_k(x_1, \dots, x_n) = u\left(x_1, \dots, x_n, \frac{k}{n_{N+1}}\right)$$

The maps  $u_k$  are morphisms from  $C_{(n_1, \dots, n_N)}^N$  to  $C$ , then it exists, by hypothesis, a map  $|u_k|$ . Let's denote for all  $t$  in  $[0, 1]$  :

$$|u|\left(x_1, \dots, x_n, \frac{k+t}{n}\right) = |u_k|(x_1, \dots, x_n) \times (1-t) + |u_{k+1}|(x_1, \dots, x_n) \times t$$

where  $k$  is an integer in  $\{1, \dots, n_{N+1}\}$ , and  $x_1, \dots, x_n$  are reals in  $[0, 1]$ . We will prove that the map  $u \mapsto |u|$  induces an isomorphism between the fundamental groups. For that, let's recall some lemmas.

**Lemma 1 - Lebesgue's number -.** Let  $X$  be a metric compact and  $(U_i)_{i \in I}$  be a cover of open sets of  $X$ , then there exists  $\delta$  (Lebesgue's number) such that all subsets contained in  $X$  which have a diameter lower than  $\delta$  are included in one of the  $U_i$  (see for example [Mau], p.20)

**Lemma 2.** Let  $f$  be a continuous map from  $[0, 1]^N$  to  $|C|$  where  $C$  is a simplicial complex, then it exists  $n$  in  $\mathbb{N}$  such that for all  $n'$  greater than  $n$ , it exists a morphism  $u$  from  $C_{n'}^N$  to  $C$  such that  $|u|$  and  $f$  are homotopic. (see for example [HW], easy adaptation of simplicial complexes)

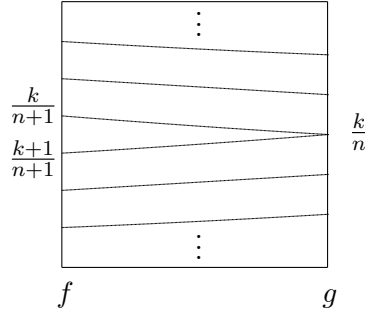
**Proof of the theorem.** Let's denote by  $n_1, \dots, n_N$  arbitrary integers and let's recall that to each  $N$ -walk-matrix  $M$  of length  $(n_1 + 1, \dots, n_N + 1)$ , we associate a morphism from  $C_{(n_1, \dots, n_N)}^N$  to  $C$  and reciprocally.

Let's denote by  $\phi$  the map from  $\mathcal{M}_N(Q)$  to  $\mathcal{C}^0([0, 1]^N, |C|)$ , the set of continuous map from  $[0, 1]^N$  to  $|C|$ , defined by  $\phi(M) = |u|$ , where  $u$  is the morphism from  $C_{(n_1, \dots, n_N)}^N$  to  $C$  associated to  $M$ .

✂ Let's prove first that the images by  $\phi$  of two equivalent  $N$ -matrices are homotopic. It is sufficient to verify that, on the generators of the equivalence relation. For that, let's consider  $M$  and  $M'$  two  $N$ -walk-matrices equal alongside and such that  $[M, M']$  is a  $(N + 1)$ -matrix. Let's denote also by  $u$  and  $u'$  the morphisms from  $C_{(n_1, \dots, n_N)}^N$  to  $C$  corresponding respectively to the matrices  $M$  and  $M'$  and let's prove that  $|u|$  and  $|u'|$  are homotopic. This is due to the fact that  $[|u, u'|]$  defines an homotopic map which transforms  $|u|$  into  $|u'|$ ; then  $|u|$  and  $|u'|$  are  $C$ -homotopic. For the second relation which generates the equivalent relation it is sufficient to prove, because of the symmetry between the variables, that :

$$|[M_1, \dots, M_k, M, M_{k+1}, \dots, M_n]| \sim |[M_1, \dots, M_k, M_{k+1}, \dots, M_n]|$$

Let's remark first that :  $|[M_1, \dots, M_k, M, M_{k+1}, \dots, M_n]| \sim |[M_1, \dots, M_k, M_k, M_{k+1}, \dots, M_n]|$  because these two matrices form a  $(N + 1)$ -walk-matrix and their images are homotopic by using the previous point. Let's denote  $f = |[M_1, \dots, M_k, M_k, M_{k+1}, \dots, M_n]|$  and  $g = |[M_1, \dots, M_k, M_{k+1}, \dots, M_n]|$ , and let's prove that  $f$  and  $g$  are homotopic. For that it is enough to continuously bring the two  $M_k$  in one. If we "look" only on the last dimension, we can summarize the situation by the followed scheme :



More precisely, consider the map  $\psi$  defined as :

$$\psi(x_1, \dots, x_n, t) = \begin{cases} f(x_1, \dots, x_{n-1}, & x_n \cdot (1-t) + x_n \cdot t \cdot \frac{n+1}{n} & ) & \text{if } x \leq \frac{k}{n+1} \\ f(x_1, \dots, x_{n-1}, & x_n \cdot (1-t) + t \cdot (1 - (1-x_n) \cdot \frac{n+1}{n}) & ) & \text{if } x \geq \frac{k+1}{n+1} \\ f(x_1, \dots, x_{n-1}, & x_n \cdot (1-t) + t \cdot \frac{k}{n} & ) & \text{in other case} \end{cases}$$

✂ Then  $\phi$  induces a map  $\phi^*$  from  $\mathcal{M}_N(Q)/\sim$  to  $C^0([0, 1]^N, |C|)/\sim$ . Let's prove that  $\phi^*$  is injective. For that let's consider two  $N$ -matrices  $M_1, M_2$  and let's denote by  $u_1, u_2$  the morphisms from  $C_{(n_1, \dots, n_N)}^N$  to  $C$  associated to  $M_1$  and  $M_2$ . Moreover, let's suppose that  $|u_1|$  and  $|u_2|$  are homotopic. We will show that  $M_1$  and  $M_2$  are equivalent. First, let's notice that, even if we must add hyperplans to the matrices  $M_1$  and  $M_2$ , we can suppose that their lengths are the same and that they are square matrices, that is to say that their dimension are in the form of  $(n, \dots, n)$  where  $n$  is an integer.

Since  $|u_1|$  and  $|u_2|$  are homotopic, it exists a continuous map  $f$  from  $[0, 1]^{N+1}$  to  $|C|$  verifying for all  $x_1, \dots, x_N$  in  $[0, 1]$  :

$$\begin{cases} f(x_1, \dots, x_N, 0) = |u_1|(x_1, \dots, x_N) \\ f(x_1, \dots, x_N, 1) = |u_2|(x_1, \dots, x_N) \\ f(x_1, \dots, x_N, t) = |u_1|(x_1, \dots, x_N) = |u_2|(x_1, \dots, x_N) & \text{if one of the } x_i \text{ is 0 or 1.} \end{cases}$$

In consequence of the lemme 2, its exists an integer  $m$  and a morphism  $u$  from  $C_m^{N+1}$  to  $C$  such that  $|u|$  and  $f$  are homotopic. In consequence of the lemme 2, we must also chose  $m$  how we want if it is greater than a certain value. So we can suppose that  $m$  is a multiple of  $n$ . Let's denote  $u'_1(x_1, \dots, x_N) = u(x_1, \dots, x_N, 0)$  and  $u'_2(x_1, \dots, x_N) = u(x_1, \dots, x_N, 1)$ . Since  $u'_1 \sim u'_2$ , it is sufficient to prove that  $u_1 \sim u'_1$  and  $u_2 \sim u'_2$ . Since the proof of these two assertions are identical, we will prove only the first one.

Since  $|u_1| = f(\dots, 0) = |u'_1|$ , the morphisms  $u_1$  and  $u_2$  have the same image on the vertices common between  $C_n^N$  and  $C_m^N$  so on  $C_n^N$  since  $m$  is a multiple of  $n$ . In fact, by construction of  $|u_1|$  (resp.  $|u'_1|$ ), for all vertex  $a$  of the polyhedra,  $u_1(a)$  (resp.  $u'_1(a)$ ) and  $|u_1|(a)$  (resp.  $|u'_1|(a)$ ) are equal. So  $u'_1$  is obtained from  $u_1$  by inserting a fix number (equal to  $m/n-1$ ) of hyperplans of vertices between each existing hyperplan. For example if  $N = 2$  and  $m = 3n$ , the scheme is the next one, the vertices encircled coming from  $u_1$ , the other coming from  $u'_1$ .

$$\left( \begin{array}{cccccccc} \textcircled{a_{11}} & a_{12} & a_{13} & \textcircled{a_{14}} & a_{15} & a_{16} & \textcircled{a_{17}} & \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & \cdots \\ \textcircled{a_{41}} & a_{42} & a_{43} & \textcircled{a_{44}} & a_{45} & a_{46} & \textcircled{a_{47}} & \\ & & & \vdots & & & & \end{array} \right)$$

To show that  $u_1$  and  $u'_1$  are equivalent, it is enough to show that the vertices of  $C_m^N$  are contained in a  $N$ -face of  $C_n^N$  are in a same simplex. Let  $b$  be a vertex of  $C_m^N$  contained in a face  $C_n^N$  whose vertices are  $a_1, \dots, a_p$ . By construction of  $|u_1|$ , we have that  $|u_1|(b)$  is contained in the simplex  $s$  whose vertices are  $u_1(a_1), \dots, u_1(a_p)$ . Then

$$u'_1(b) = |u'_1(b)| \in s$$

So  $\phi^*$  is injective.

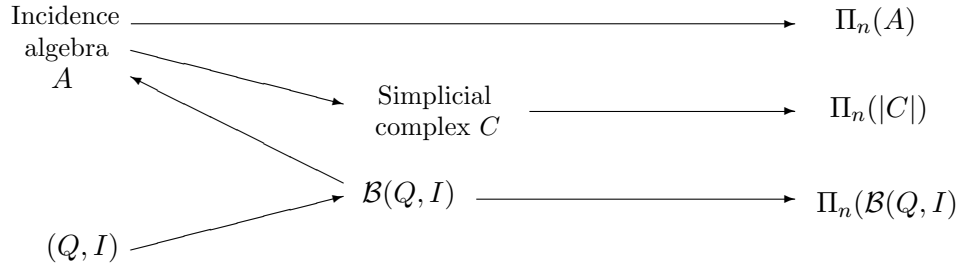
✂ Moreover the lemme 2 prove that  $\phi^*$  is surjective. Finally, let's restrict the source set to the set of  $N$ -walk-matrices equal alongside to the vertex  $x$ . The product  $\times$  partially defined on  $\mathcal{M}_N(Q)$  become an intern law of composition and the map  $\phi^*$  becomes a group morphism and then an isomorphism.

**2.5 Case where the poset is infinite.** Can we extend the previous theorem in the case of infinite poset? In some cases, yes. More precisely, if all totally ordered subset of the poset are finite, then the fundamental groups, algebraic and topologic, are isomorphic.

Let's recall first the construction of the geometric realization  $|C|$  of an infinite simplicial complex. Let's denote by  $C_0$  the set of vertices of  $C$ . Note that the simplexes of the simplicial complexes  $C$  will contain only a finite number of vertices. First, let's describe the points of  $|C|$ , we will give the topology later. The points of  $|C|$  are the family  $(\lambda_i)_{i \in C_0}$  of  $[0, 1]$ , where the indices of non zero  $\lambda_i$  must be included in a same simplex and verify  $\sum_{i \in C_0} \lambda_i = 1$ . Let's remark that this equality has a sense since the set of non zero  $\lambda_i$  is finite. To each simplex  $s = (a_1, \dots, a_n)$  correspond a subset  $\bar{s}$  of  $|C|$  formed by the point  $(\lambda_i)_{i \in C_0}$  such that  $\lambda_i = 0$  if  $i$  is not in  $\{a_1, \dots, a_n\}$ . The injection  $f$  defined from  $\bar{s}$  to  $\mathbb{R}^n$  by  $f((\lambda_i)_{i \in C_0}) = (\lambda_{a_1}, \dots, \lambda_{a_n})$  allows to bring the topology of  $\mathbb{R}^n$  on  $\bar{s}$ . Finally, let's defined the topology of  $|C|$  by describing its closed subset : a subset  $A$  of  $|C|$  is closed if and only if  $A \cap \bar{s}$  is closed in  $\bar{s}$  for all simplex  $s$  of  $C$ . This construction of the geometric realization coincides, in the case where the simplicial complex is finite, to the existing definition (See [HW], p 46). In the case of infinite simplicial complexes, the notion of  $st(a)$  for a vertex  $a$  can be extend and is again an open subset of  $|C|$  ([HW],p 47). Then the previous proof work even if the simplicial complex is infinite.

### 3. The $n$ -group of homotopy.

**3.1 Motivation and definition.** We have seen that we can construct from a presentation  $(Q, I)$  of a triangular algebra, a CW-complex  $\mathcal{B}(Q, I)$ . Moreover, it exists an incidence algebra  $A$  associated to  $(Q, I)$  having the same fundamental group. To this algebra  $A$ , we can associated a  $n$ -group of algebraic homotopy isomorphic to the  $n$ -group of homotopy of the simplicial complex associated to  $A$ . The following diagram illustrates this situation :



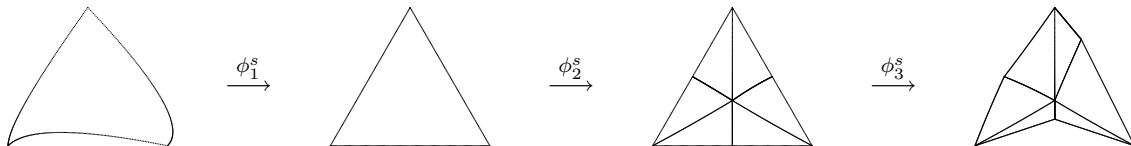
We have proved in the previous paragraph that  $\Pi_n(A)$  and  $\Pi_n(|C|)$  are isomorphic. But the situation is better than that because the 3 fundamental groups are isomorphic :

**Theorem.** Let  $kQ/I$  be a triangular algebra,  $\mathcal{B}(Q, I)$  be the CW-complex associated,  $A$  be its incidence algebra and  $C$  be the simplicial complex associated to  $A$ , then :

$$\pi_n(A) \simeq \pi_n(|C|) \simeq \pi_n(\mathcal{B}(Q, I))$$

So, it seems natural to define the  $n^{\text{th}}$  algebraic fundamental group of a presentation  $(Q, I)$  of a triangular algebra, as the  $n^{\text{th}}$  fundamental group of the incidence algebra associated. In particular this theorem prove again the theorem of the paragraph I.4. that is to say the fundamental groups  $\Pi_1(Q, I)$  and  $\Pi_1(A)$  are isomorphic.

**3.2 Proof of the theorem 3.1.** Let's denote by  $\Delta_n$  the geometric  $n$ -simplex of  $\mathbb{R}^n$  defined as the lower convex containing the points  $(1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, \dots, 0, 1)$ ; the set  $\Delta_n$  can be considered as a CW-complex. Let  $s$  be a  $n$ -cell of  $\mathcal{B}(Q, I)$ . The CW-complex  $\mathcal{B}(Q, I)$  being regular, the set  $\bar{s}$  is homeomorphic to the closed unit ball of dimension  $n$  and then is homeomorphic to  $\Delta^n$ . Moreover, we can chose an homeomorphism which conserves the subcells, that is to say the image of a  $k$ -cell of  $\Delta_n$  is a cell of dimension  $k$  of  $\mathcal{B}(Q, I)$ . Let's denote by  $\phi_1^s$  this homeomorphism and let's consider  $\phi_2^s$  the homeomorphism induced by the identity between  $\Delta_n$  and its barycentric subdivision  $\Delta'_n$ . Finally, let's denote by  $s'$  the subcomplex of  $|C|$  formed by the vertices associated to sub-cells of  $s$ . Since  $s'$  and  $\Delta_n$  are two geometric realizations of a same CW-complex, they are isomorphic; let's denote by  $\phi_3^s$  this homeomorphism. Let's show the action of  $\phi_1^s, \phi_2^s$  and  $\phi_3^s$  on an example :



Finally, let's denote by  $\phi$  the homeomorphism from  $\mathcal{B}(Q, I)$  to  $|C|$  defined by  $\phi(x) = \phi_s(x)$  for all non degenerated cells  $s$  and all points  $x$  in  $\bar{s}$ . This continuous map is in fact an homeomorphism, the homotopy group associated to  $\mathcal{B}(Q, I)$  on one hand and  $|C|$  on other hand are isomorphic.

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