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Incidence algebra of a presentation

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Abstract

In this paper we construct an incidence algebra of a poset associated to a presentation by a quiver and relations of a finite dimensional algebra. We provide an exact sequence relying the fundamental groups of the incidence algebra and the presentation. Moreover, we give an algorithm to calculate any fundamental group.

AMS classification : 16E40; 16G20; 06A11; 55Q05

One of the main tools for the study of the category of finite dimensional modules over a basic algebra, over an algebraically closed field k is its presentation by a quiver and relations. This theory is mainly due to P. Gabriel. More precisely, it is well known that for each finite dimensional and basic algebras over an algebraically closed field k , there exists a unique quiver Q and an admissible ideal I of the path quiver algebra kQ , such that A is isomorphic to kQ/I . Such a couple (Q, I) is called a presentation of A by quiver and relations. For each presentation (Q, I) , we can construct an algebraic fundamental group $\Pi_1(Q, I)$.

It has been proved (see [Rey] or [Bus]) that the fundamental group of an incidence algebra has a topological interpretation. Indeed, the algebraic fundamental group is isomorphic to a topological fundamental group of a simplicial complex. In order to give a topological version of all algebraic fundamental groups, we associate in this paper to a presentation (Q, I) , an incidence algebra A which verifies the following :

Theorem. Let (Q, I) be a presentation of an algebra and let A be the associated incidence algebra. There exists a sub-group H of $\Pi_1(Q, I)$ which we describe by generators and relations such that the following sequence is exact :

$$1 \longrightarrow H \longrightarrow \Pi_1(Q, I) \longrightarrow \Pi_1(A) \longrightarrow 1$$

We give a list of sufficient condition to have this subgroup equal to 1 : for example, it happens in the case of Schurian algebras or in the case in which the quiver does not contain sub-quivers in the form of an "eight". In particular, it become easy to provide an other computation of the

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fundamental group of incidence algebra where the quiver is a crown (their fundamental group has been calculated before in [GR])

This paper also describes an algorithm to calculate the fundamental groups of presentations, which allows to present the fundamental groups by generators and relation. In the case Q has more than one vertex, the fundamental group of a presentation (Q, I) is isomorphic to a fundamental group of a couple (Q', I') in which Q' contains a vertex less than Q . Then by applying this process again, we obtain that the fundamental group $\Pi_1(Q, I)$ is isomorphic to a fundamental group of a presentation of an algebra which quiver contains only one vertex; using this we give a presentation by generators and relations. This algorithm is suitable also to calculate Hochschild cohomology group of algebras with presentation verifying $Hom(\Pi_1(Q, I), k^+) \simeq HH^1(kQ/I)$; these algebras are described by I. Assem and M. Saorin in [PS].

This paper gives some result of my Ph.D. and I want to thank all the persons who helped me in its realization : particularly Claude CIBILS, my thesis director and Manolo SAORIN, who is in particular at the origin of the construction of the incidence algebra associated to a presentation described here, and Maria Julia REDONDO.

In this paper, k is an algebraically closed field. Moreover we suppose that the definitions of simplicial complexes, poset, topological fundamental group of a simplicial complex, algebraic fundamental group of a presentation are known. In other case, we find this in [Rey] for example.

1. Algorithm of calculus of $\pi_1(Q, I)$

Before giving the algorithm, we introduce some notations. Let Q be a quiver and x_0 be a vertex of Q . We also consider $(c_i)_{i \in A}$ and $(c'_i)_{i \in A}$ two families of paths of Q indexed by a set A . Then, we denote by $\langle (c_i \sim c'_i)_{i \in A} \rangle$ the smallest equivalence relation on the set $P(Q)$ containing paths of Q and compatible with the concatenation and which verify :

1. if f is an arrow of Q from a to b then $f.f^{-1} \sim b$ and $f^{-1}.f \sim a$.
2. For all i of A , we have : $c_i \sim c'_i$

Moreover, we consider an arrow f_0 of Q from a_0 to b_0 with $a_0 \neq b_0$, and we denote by Q' the quiver obtained from Q by identifying the vertices a_0 and b_0 . More precisely :

- the vertices of Q' are the vertices of Q in which a_0 and b_0 are the same vertex. We denoted c_0 this new vertex.
- the arrows of Q' are the arrows of Q minus f_0 . All the other arrows have the same source and the same terminus than in Q . Let's notice that a parallel arrow of f_0 becomes a loop in Q' .

We also denote by p the "projection" $P(Q)$ on $P(Q')$ defined by :

- $p(s) = s$ for any vertex s not equal to a_0 and b_0 ,
- $p(a_0) = p(b_0) = c_0$,
- $p(f) = f$ for any arrow f not equal to f_0 ,
- $p(f_0) = c_0$.

Finally, let's define R' , the equivalence relation on $kQ' : \langle (p(c_i) \sim p(c'_i))_{i \in A} \rangle$.

Theorem 1.1. In the context defined before, the morphism p induces an isomorphism

$$P_{x_0}(Q)/R \simeq P_{p(x_0)}(Q')/R'$$

in which $P_{x_0}(Q)$ and $P_{p(x_0)}(Q')$ are the sub-sets of walks of respectively $P(Q)$ and $P(Q')$ closed in x_0 and $p(x_0)$.

Proof. First, we construct an application q from $P_{p(x_0)}(Q')$ to $P_{x_0}(Q)$ that verifies $p \circ q = Id$. Let's denote by $\omega' = \alpha_n^{\epsilon_n} \dots \alpha_1^{\epsilon_1}$ a walk of Q' closed in $p(x_0)$. In general, this walk is not a walk of Q because we have not necessary $t(\alpha_i^{\epsilon_i}) = s(\alpha_{i+1}^{\epsilon_{i+1}})$ for all i of $\{1, \dots, n-1\}$. The application q inserts f_0 and f_0^{-1} between some arrows of ω' in order to make ω' a path of Q .

Let's define q more precisely. Let s_i and t_i be respectively the beginning and the end of $\alpha_i^{\epsilon_i}$ considered as a path of Q , and s_{n+1}, t_0 be x_0 . Then we consider $\omega = e_n \cdot \alpha_n^{\epsilon_n} \cdot e_{n-1} \dots e_1 \cdot \alpha_1^{\epsilon_1} \cdot e_0$ a path of Q , in which the e_i are defined as follows :

$$e_i = \begin{cases} t_i & \text{if } t_i = s_{i+1} \\ f_0 & \text{if } t_i = a_0 \text{ and } s_{i+1} = b_0 \\ f_0^{-1} & \text{if } t_i = b_0 \text{ and } s_{i+1} = a_0 \end{cases}$$

Then we have $p(\omega) = \omega'$. We define $q(\omega')$ by $q(\omega') = \omega$. So we have $p \circ q = Id$.

Now we are going to prove that the image of two R' -equivalent walks by q are two R -equivalent walks.

For that, it is sufficient to prove this property on the generators of the equivalence relation. First, We consider an arrow f of Q' and the walks $\omega = p_1 \cdot f^{-1} \cdot f \cdot p_2$ and $\omega' = p_1 \cdot p_2$ of Q' closed in $p(x_0)$. Let's prove that $q(\omega)$ and $q(\omega')$ are equivalent. For that, we denote by p'_1 and p'_2 the walks of Q constructed from p_1 and p_2 by inserting f_0 and f_0^{-1} , as explained during the construction of q , so we have :

$$q(\omega) = p'_1 \cdot e_3 \cdot f^{-1} \cdot e_1 \cdot f \cdot e_4 \cdot p'_2 \quad \text{and} \quad q(\omega') = p'_1 \cdot e_2 \cdot p'_2$$

Since $t(f) = s(f^{-1})$ we have $e_1 = t(f)$ and

$$q(\omega) = p'_1 \cdot e_3 \cdot f^{-1} \cdot e_1 \cdot f \cdot e_4 \cdot p'_2 = p'_1 \cdot e_3 \cdot f^{-1} \cdot f \cdot e_4 \cdot p'_2 \sim p'_1 \cdot e_3 \cdot e_4 \cdot p'_2$$

The walks e_2, e_3 and e_4 being composed only with the arrow f_0 and its inverse, we obtain $q(\omega) \sim q(\omega')$. The proof is similar for the insertion of $f \cdot f^{-1}$.

Now we consider two equivalent generators c_1 and c_2 of the equivalence relation R' , and the walks $\omega = p_1 \cdot c_1^{-1} \cdot c_2 \cdot p_2$ and $\omega' = p_1 \cdot p_2$ of Q' closed in $p(x_0)$.

By definition of R' , there exists paths c'_1 and c'_2 of Q equivalent by the relation R such that $p(c'_1) = c_1$ and $p(c'_2) = c_2$. Let's prove that the walks $q(\omega)$ and $q(\omega')$ are equivalent. Let's denote again by p'_1 and p'_2 the walks of Q obtained from p_1 and p_2 by inserting f_0 and f_0^{-1} . Then :

$$q(\omega) = p'_1 \cdot e_3 \cdot c_1'^{-1} \cdot e_1 \cdot c_2' \cdot e_4 \cdot p'_2 \quad \text{and} \quad q(\omega') = p'_1 \cdot e_2 \cdot p'_2$$

The walks c'_1 and c'_2 being equivalent, we have $t(c'_2) = s(c_1'^{-1})$ and $e_1 = t(f)$, and then :

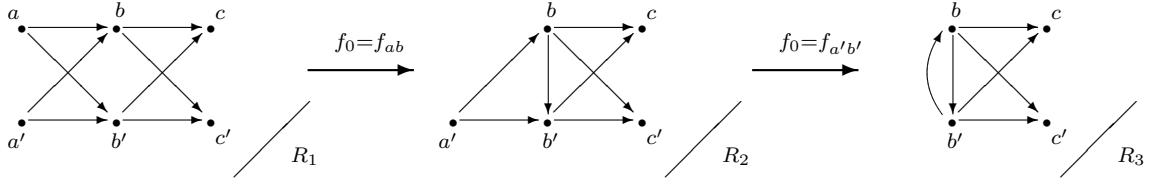
$$q(\omega) = p'_1 \cdot e_3 \cdot c_1'^{-1} \cdot e_1 \cdot c_2' \cdot e_4 \cdot p'_2 = p'_1 \cdot e_3 \cdot c_1'^{-1} \cdot c_2' \cdot e_4 \cdot p'_2 \sim p'_1 \cdot e_3 \cdot e_4 \cdot p'_2$$

Since the walks e_2 , e_3 and e_4 are composed only with the arrow f_0 and its inverse, we obtain $q(\omega) \sim q(\omega')$ again.

By definition of the relation R' , the projection p is also constant on the equivalent classes. Then p and q induce p_* and q_* between the sets $P_{x_0}(Q)/R$ and $P_{p(x_0)}(Q')/R'$. It is sufficient to prove that $q_* \circ p_* = Id$; this being true since the image a walk by the application $q \circ p$ delete the expressions $f_0 \cdot f_0^{-1}$ and $f_0^{-1} \cdot f_0$, but their equivalent classes are the same. \square

The algorithm. Let Q be a quiver and I an ideal of kQ . First, we notice that the equivalence relation defined for the construction of the algebraic fundamental group $\Pi_1(Q, I)$ is indeed the equivalence relation $\langle (\omega_i \sim \omega'_i)_{i \in A} \rangle$ in which, for all i of A , the paths ω_i and ω'_i are paths in the support of a minimal relation. By applying the theorem a number of times equal to the number of vertices of Q minus one, and after the simplification of equivalent loops, we obtain that the algebraic fundamental group $\Pi_1(Q, I)$ is isomorphic to the algebraic fundamental group of a quiver containing only one vertex. This gives a presentation of the group $\Pi_1(Q, I)$ by generators and relations.

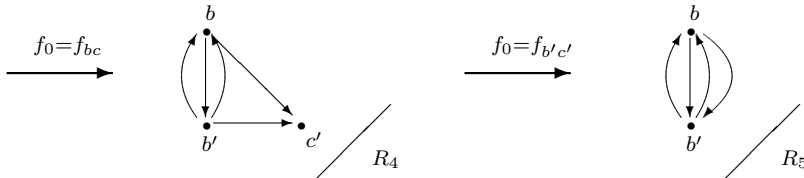
Example 1.2. Let's consider the incidence algebra of the following quiver. We will denote by f_{xy} the arrow from x to y .



With

$$\begin{cases} R_1 = \langle f_{bc}f_{ab} \sim f_{b'c}f_{ab'}, & f_{bc'}f_{ab} \sim f_{b'c'}f_{ab'}, & f_{bc}f_{a'b} \sim f_{b'c}f_{a'b'}, & f_{bc'}f_{a'b} \sim f_{b'c'}f_{a'b'} \rangle \\ R_2 = \langle f_{bc} \sim f_{b'c}f_{bb'}, & f_{bc'} \sim f_{b'c'}f_{bb'}, & f_{bc}f_{a'b} \sim f_{b'c}f_{a'b'}, & f_{bc'}f_{a'b} \sim f_{b'c'}f_{a'b'}, \rangle \\ R_3 = \langle f_{bc} \sim f_{b'c}f_{bb'}, & f_{bc'} \sim f_{b'c'}f_{bb'}, & f_{bc}f_{b'b} \sim f_{b'c}, & f_{bc'}f_{b'b} \sim f_{b'c'}, \rangle \end{cases}$$

We apply the same process on the arrows f_{bc} and $f_{b'c'}$; the new arrows between b and b' will be denoted by $f'_{bb'}$. We obtain :



With

$$\begin{cases} R_4 = \langle b \sim f'_{b'b}f_{bb'}, & f_{bc'} \sim f_{b'c'}f_{bb'}, & f_{b'b} \sim f'_{b'b}, & f_{bc'}f_{b'b} \sim f_{b'c'}, \rangle \\ R_5 = \langle b \sim f'_{b'b}f_{bb'}, & f'_{bb'} \sim f_{bb'}, & f_{b'b} \sim f'_{b'b}, & f'_{bb'}f_{b'b} \sim b', \rangle \end{cases}$$

Finally, we apply the operation one more time with $f_0 = f_{bb'}$ and we denote by $\alpha = p(f_{b'b})$, $\beta = p(f'_{bb'})$ and $\gamma = p(f'_{b'b})$. Then we obtain a quiver containing only one vertex b and three arrows α , β and γ with the relations :

$$R = \langle b \sim \gamma, \beta \sim b, \alpha \sim \gamma, \beta\alpha \sim b \rangle = \langle \alpha \sim \beta \sim \gamma \sim b \rangle$$

The algebraic fundamental group is then trivial which is natural because the corresponding topological fundamental group of this incidence algebra is the one of an octahedra.

2. Incidence algebra associated to a presentation

In this paragraph, to each presentation by a quiver and relations (Q, I) of a k -algebra A we are going to construct an incidence algebra A' . The goal of the next paragraphs will be to compare the algebraic fundamental groups of (Q, I) and of A' . We will show that there exists a subgroup H of $\Pi_1(Q, I)$ and an exact sequence $0 \longrightarrow H \longrightarrow \Pi_1(Q, I) \longrightarrow \Pi_1(A') \longrightarrow 0$.

The algebra A' is interesting because it provides when H is trivial, a topological version of the algebraic fundamental group. Indeed, it has been proved (see [Rey] or [Bus]) that the fundamental groups $\Pi_1(A')$ and $\Pi_1(Sim(\Sigma))$ are isomorphic, in which $Sim(\Sigma)$ is the simplicial complex associated to the poset Σ . So if the subgroup H is trivial, the algebraic fundamental group of (Q, I) is isomorphic to the topological fundamental group of a simplicial complex. Finally, let's notice that if the presentation (Q, I) is already a presentation of an incidence algebra A , the simplicial complex associated to the poset of A' is the barycentric decomposition of the simplicial complex associated to the poset of A and then the fundamental groups of A and A' are isomorphic.

We define a poset Σ . In general it will not be the poset of the incidence algebra A' that we will define later.

2.1. The vertices of Σ . We first consider the set $C(Q)/\sim$ where $C(Q)$ is the set of paths of Q and \sim is the equivalence relation used to construct the fundamental group restricted to $C(Q)$. Then, we denote by Σ the set $C(Q)/\sim$ minus the classes containing a path in I . Then, by construction, the paths in a class of Σ are never equivalent to any other path in I .

Since the beginning and the end of equivalent paths are the same, the notions of origin, end, parallelism can be extended to the set Σ . Moreover, since the equivalence relation \sim is compatible with the concatenation, the set Σ inherits a law of composition partially defined. We can notice that this operation is totally defined on $\Sigma \cup \{0\}$ which give to it a structure of semi-group.

2.2. Order relation on Σ . Let (Q, I) be a presentation and Σ the set associated to (Q, I) constructed as bellow.

Proposition - Définition 2.2.1. Let \bar{a} and \bar{b} be the elements of Σ the four assertions are equivalents :

1. $\exists \bar{w}, \bar{w}' \in \Sigma / \bar{b} = \bar{w}.\bar{a}.\bar{w}'$

2. $\forall a \in \bar{a}, \exists b \in \bar{b}, \exists w, w' \in C(Q) \ / \ b = w.a.w'$
3. $\exists a \in \bar{a}, \exists b \in \bar{b}, \exists w, w' \in C(Q) \ / \ b = w.a.w'$
4. $\forall a \in \bar{a}, \forall b \in \bar{b}, \exists w, w' \in C(Q) \ / \ b \sim w.a.w'$

We say in that case that \bar{a} divide \bar{b} and we note denote it by \bar{a}/\bar{b} . This relation is then the restriction to Σ of the relation divide of the semi-group $\Sigma \cup \{0\}$

Proof.

(1 \implies 2) : Let's suppose that there exists \bar{w} and \bar{w}' in Σ that verify $\bar{b} = \bar{w}.\bar{a}.\bar{w}'$. Let's take a, ω and ω' respectively in $\bar{a}, \bar{\omega}$ and $\bar{\omega}'$. It is sufficient to define b by $b = w.a.w'$.

(2 \implies 3) : evident.

(3 \implies 4) : all the paths of \bar{a} (resp. \bar{b}) are equivalent. Moreover the relation \sim is compatible with the multiplication.

(4 \implies 1) : Let's take $b \sim w.a.w'$. Since b is not in I , The construction of Σ tell us that it is the same for $w.a.w'$. Moreover, \bar{w} (resp. \bar{w}') is in Σ . In deed, in the opposite case w will be equivalent to a path in I and the $w.a.w'$ will be equivalent to a path in I which is not permitted. Then $\bar{b} = \bar{w}.\bar{a}.\bar{w}' = \bar{w}.\bar{a}.\bar{w}' \square$

Proposition 2.2.2. The relation divide is an order relation on Σ .

Proof. The reflexivity and the transitivity being obvious, we will prove only the antisym etry. Since \bar{a}/\bar{b} and \bar{b}/\bar{a} , there exists $\alpha_1, \alpha'_1, \alpha_2$ and α'_2 such that $a \sim \alpha_1.b.\alpha'_1$ and $b \sim \alpha_2.a.\alpha'_2$ in which a and b are elements of \bar{a} and \bar{b} . Let's take $w = \alpha_1.\alpha_2$ and $w' = \alpha'_2.\alpha'_1$, we have $a \sim w.a.w'$.

Then, if we use n times the last equation, we have that $a \sim w^n.a.w'^n$ for all n of \mathbb{N} . Since I is admissible, there exists n_0 in \mathbb{N} such that $F^{n_0} \subset I$ in which F^{n_0} is the ideal containing the paths of length n_0 or more. Two cases can happen :

1. The classes \bar{w} and \bar{w}' contain one path whose length is strictly positive. Let's suppose that it is \bar{w} and denote by ω_0 this path. Then, This path a is equivalent to $\omega_0^{n_0}.a.\omega_0'^{n_0}$. But $\omega_0^{n_0}$ being in I , the element a is equivalent to an element in I which is impossible by the construction of the set Σ . Finally, this case can not happen.
2. The classes \bar{w} and \bar{w}' only contain vertices. Since I is admissible, the vertices can not be equivalent to any other path and then \bar{w} and \bar{w}' contain respectively the end t and the beginning s of \bar{a} ; So $w' = s$ and $w = t$. Moreover, for a question of length, the equality $\alpha_1.\alpha_2 = t$ induce $\alpha_1 = \alpha_2 = t$. With a similar proof on w' , we have $a = b$.

\square

Proposition 2.2.3. Let (Q, I) be a presentation of an algebra and Σ the poset a associated to this presentation, then :

1. $\dim(kQ/I) \geq \text{card}(\Sigma)$. Particularly, if kQ/I is finite dimensional then Σ is finite and the incidence algebra associated is also finite dimensional.
2. The quiver Q is connected if and only if all quiver of incidence algebra associated are connected.

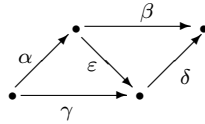
Proof.

(1) : Let $(w_i)_{i \in J}$ be a family of path chosen in each classes of element of Σ . We will prove that it is a free family in kQ/I . Let's suppose that $\sum_{i \in J} \lambda_i w_i \in I$, then there exists $J' \subset J$ non empty such that $\sum_{i \in J'} \lambda_i w_i$ is a minimal relation : so the path w_i , with $i \in J'$ are equivalent. Moreover, since the w_i are chosen in different equivalent classes, the cardinal of J can be only one and the minimal relation can be written $\lambda_k \cdot w_k = 0$ for one k in J . Since w_k is not in I by construction of Σ , the element λ_k is zero. we repeat the operation until all λ_i are zero for $i \in J$. Then, the family $(w_i)_{i \in J}$ is free.

(2) : This proof will be easier after the demonstration the surjection between the two fundamental group. \square

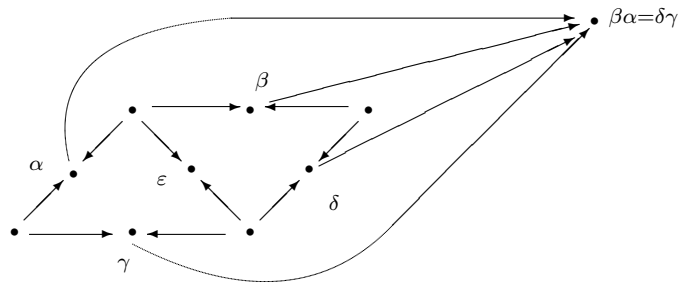
2.3. Incidence algebra associated to a presentation. We will say that a presentation (Q, I) verifies the property P if one element of a class of $C(Q)/\simeq$ is in I induces that all elements in that class are in I . Then the incidence algebra A' associated to a presentation (Q, I) is the incidence algebra of Σ if the presentation satisfies the property P . If the presentation does not satisfies the property P , we design by A' the incidence algebra associated to the presentation (Q, \tilde{I}) in which \tilde{I} is the ideal generated by the minimal relation of length at least one ; this being possible since (Q, \tilde{I}) verifies P .

2.4. Exemple Let Q be the following quiver and I the ideal of kQ :



$$I = \langle \epsilon\alpha, \delta\epsilon, \beta\alpha - \delta\gamma \rangle$$

Except the arrows and the vertices, there exists only one path in Q which is $\beta\alpha = \delta\gamma$. The incidence quiver is :



3. Relation between the two fundamental groups.

Let (Q, I) be a presentation of an algebra by quiver and relations. In the previous paragraph, we have construct an incidence algebra A' with the presentation (Q, I) . These two objects, the presentation and the incidence algebra, have their own algebraic fundamental groups, and our aim is to compare them. We will first prove that there exists always a surjection from $\Pi_1(Q, I)$ to $\Pi_1(A')$. Then to study the default of injectivity, we will introduce a subgroup H of $\Pi_1(Q, I)$ such that $\Pi_1(Q, I)/H$ is isomorphic to $\Pi_1(A')$. In other words, we give a system of generators of the kernel of the surjection.

3.1. Surjection from $\Pi_1(Q, I)$ to $\Pi_1(A')$

Theorem 3.1.1. Let (Q, I) be presentation of an algebra and A' the incidence algebra associated, then there exists a surjection :

$$\phi_* : \Pi_1(Q, I) \twoheadrightarrow \Pi_1(\Sigma).$$

Proof. First, let's suppose that the presentation (Q, I) verifies the property P . Let's denote by Q_Σ the ordered quiver associated to Σ . To simplify the notations of the proof, a vertex of Q_Σ can be also a class of paths of $C(Q)/\sim$ or sometimes a path of Q if this path is alone in its class. It happens for example for the classes of vertices or arrows. Let's consider now the application ϕ defined from $P(Q)$, the set of walks of Q , to $P(Q_\Sigma)$, the set of walks of Q_Σ , characterized by :

- If a is a vertex of Q (considered as a constant walk) then $\phi(a) = a$. We notice that since the ideal is admissible, a is not in I ; the class of a is then in Σ .
- Let α be an arrow of Q from a to b , then a and b are divisors of α . Since the ideal I is admissible, the arrows of Q are not in I ; then α is in Σ . Since it does not exist, for the order relation 'divide', element strictly between a and α (resp. between b and α), there exists in Q_Σ two arrows α_a and α_b from respectively the vertices a and b to the arrows α . We can then define $\phi(\alpha)$ by $\phi(\alpha) = \alpha_b^{-1}\alpha_a$.

$$\text{In } Q : \quad a \xrightarrow{f} b \quad \text{In } Q_\Sigma : \quad a \xrightarrow{\alpha_a} \alpha_b \xleftarrow{\alpha_b} b$$

- The application ϕ is compatible with the composition of paths, then if $\alpha_n^{\varepsilon_n} \dots \alpha_1^{\varepsilon_1}$ is a walk of $P(Q)$ then

$$\phi(\alpha_n^{\varepsilon_n} \dots \alpha_1^{\varepsilon_1}) = \phi(\alpha_n)^{\varepsilon_n} \dots \phi(\alpha_1)^{\varepsilon_1}$$

Let X_0 be the vertex of Q chosen to calculate $\Pi_1(Q, I)$, the vertex $\phi(X_0) = X_0$ of Q_Σ will be taken to calculate $\Pi_1(Q_\Sigma, I_{Q_\Sigma})$. Moreover, we denote by ϕ again the morphism induced by ϕ from the group of paths of Q closed in X_0 to $\Pi_1(Q_\Sigma, I_{Q_\Sigma})$.

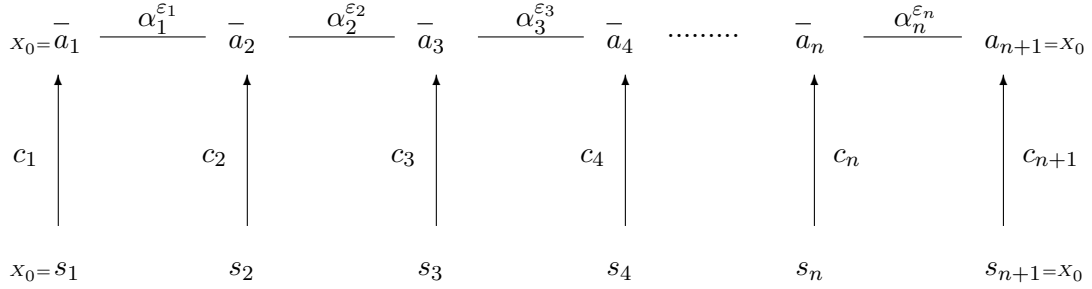
We will first prove that the application ϕ is surjective. Let $p = \alpha_n^{\varepsilon_n} \dots \alpha_1^{\varepsilon_1}$ be a walk of Q_Σ crossing successively the vertices $\overline{a_1}, \dots, \overline{a_{n+1}}$ with $\overline{a_1} = \overline{a_{n+1}} = X_0$; in other words $\overline{a_i}$ is the source of $\omega_i^{\varepsilon_i}$. Let's prove that p is equivalent to a path p' crossing the sources $s_1 \dots s_{n+1}$ of classes of paths $\overline{a_1}, \dots, \overline{a_{n+1}}$.

For all i in $\{1, \dots, n\}$, the vertex $\overline{s_i}$ divides $\overline{a_i}$, so there exists a path c_i of Q_Σ linking $\overline{s_i}$ to $\overline{a_i}$. Moreover, since X_0 is also a vertex of Q , the path c_1 and c_{n+1} are the trivial path X_0 .

The walks p is then equivalent to :

$$p = (c_{n+1}\alpha_n^{\varepsilon_n} c_n) \dots (c_3^{-1}\alpha_2^{\varepsilon_2} c_2)(c_2^{-1}\alpha_1^{\varepsilon_1} c_1)$$

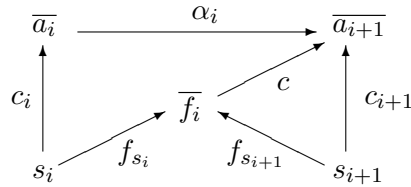
The situation is summarized in the following schema :



Let i be fixed in $\{1, \dots, n\}$. We suppose moreover that a_i/a_{i+1} , that is to say that $\varepsilon_i = 1$; the other case is solved similarly. Since $\overline{a_i}$ is linked to $\overline{a_{i+1}}$ in Q_Σ by α_i , it does not exist element in Σ strictly between $\overline{a_i}$ and $\overline{a_{i+1}}$. Then it exist an arrows f_i from Q such that $\overline{a_{i+1}} = \overline{a_i} \cdot f_i$ or $\overline{a_{i+1}} = \overline{f_i} \cdot \overline{a_i}$.

In the second case $\overline{a_{i+1}} = \overline{f_i} \cdot \overline{a_i}$, the paths c_i and c_{i+1} have the same origin s_i and then the paths $\alpha_i c_i$ and c_{i+1} are parallel and then $c_{i+1}^{-1} \alpha_i c_i$ is equivalent to s_i .

We consider now the first case : $\overline{a_{i+1}} = \overline{a_i} \cdot f_i$. The arrow f_i of Q has s_{i+1} as source and s_i as terminus, then there exists arrows f_{s_i} and $f_{s_{i+1}}$ of Q_Σ linking respectively s_i and s_{i+1} to the vertex f_i ; these arrows verify $\phi(f_i) = f_{s_i}^{-1} f_{s_{i+1}}$. Moreover $\overline{f_i}$ divide $\overline{a_{i+1}}$, then there exists a path c of Q_Σ from f_i to a_{i+1} .



The paths c_{i+1} and $c f_{s_{i+1}}$ in one hand and $\alpha_i c_i$ and $c f_{s_i}$ in other hand are parallel. The walks $c_{i+1}^{-1} \alpha_i c_i$ is then equivalent to $f_{s_{i+1}}^{-1} c^{-1} c f_{s_i}$ and then to $f_{s_{i+1}}^{-1} f_{s_i}$ which is exactly the image by ϕ of f_i^{-1} . We notice that if we have considered $\overline{a_{i+1}}/\overline{a_i}$, we would conclude that the path is equivalent to the image of the arrow f_i .

We also notice for later that by using similar process, that the walk of Q_Σ from a to b :

$$a \rightarrow f_1 \rightarrow \overline{f_2 f_1} \rightarrow \overline{f_3 f_2 f_1} \rightarrow \dots \rightarrow \overline{f_n f_{n-1} \dots f_1} \rightarrow \overline{f_n \dots f_2} \rightarrow \overline{f_n \dots f_3} \rightarrow \dots \rightarrow b$$

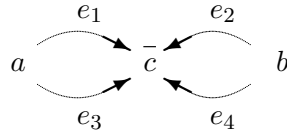
is equivalent to $\phi(c)$ in which $c = f_n f_{n-1} \dots f_1$.

We will prove now that ϕ is constant on the \sim -equivalence classes. Then it induces the morphism ϕ_* . We are going to verify first that for all arrows f from a to b , we have $\phi(f f^{-1}) \sim \phi(b)$ and $\phi(f^{-1} f) \sim \phi(a)$. Indeed $\phi(f^{-1} f) = f_a^{-1} f_b f_b^{-1} f_a \sim a = \phi(a)$; The proof is the same for the other equality.

Then, it is sufficient to verify that the images of two paths c_1 and c_2 of Q contained in a same minimal relation are equivalent. Let's notice first that since (Q, I) verifies the relation P , the paths c_1 and c_2 are represented in Σ . Let's denote $c_1 = \alpha_n \dots \alpha_1$ and $c_2 = \beta_m \dots \beta_1$, and let a and b be respectively the beginning and the end of the two paths c_1 and c_2 . Let's define now the following paths :

- e_1 the path of Q_Σ crossing the vertices $a, \alpha_1, \alpha_2\alpha_1, \dots, c_1$
- e_2 the path of Q_Σ crossing the vertices $b, \alpha_n, \alpha_n\alpha_{n-1}, \dots, c_1$
- e_3 the path of Q_Σ crossing the vertices $a, \beta_1, \beta_2\beta_1, \dots, c_2$
- e_4 the path of Q_Σ crossing the vertices $b, \beta_m, \beta_m\beta_{m-1}, \dots, c_2$

The paths c_1 and c_2 being equivalent, they are represented in Σ by the same vertex \bar{c} . The situation is resumed in the following scheme, the arrows being paths of Q_Σ :



If we use the remark of the end the surjectivity of ϕ , we have $e_2^{-1}e_1 \sim \phi(c_1)$ and $e_4^{-1}e_3 \sim \phi(c_2)$, Therefore $\phi(c_1)$ and $\phi(c_2)$ are equivalent. We obtain a surjective application ϕ_* from $\Pi_1(Q, I)$ to $\Pi_1(Q_\Sigma)$.

Let's suppose now that the presentation (Q, I) does not verify the property P . We can deduce from above that there exists a surjection from $\Pi_1(Q, \tilde{T})$ to A' , and by construction of \tilde{T} the fundamental groups $\Pi_1(Q, \tilde{T})$ and $\Pi_1(Q, I)$ are isomorphic. The theorem is then proved. \square

Proof of proposition III.2.3. We are going to prove now the property which says that a quiver Q is connected if and only if the poset Σ is connected.

Let's suppose that Q is connected and consider \bar{a} and \bar{b} two vertices of Σ . Moreover, we denote by \bar{s}_a and \bar{s}_b the sources of \bar{a} and \bar{b} . Since \bar{s}_a/\bar{a} and \bar{s}_b/\bar{b} , there exists paths of Σ denoted by c_1 and c_2 linking respectively \bar{s}_a to \bar{a} and \bar{s}_b to \bar{b} . Moreover, the vertices $s_a = \bar{s}_a$ and $s_b = \bar{s}_b$ of Σ are vertices of Q . Therefore, since Q is connected, there exists a walk ω of Q linking s_a and s_b . Finally $c_2\phi(\omega)c_1^{-1}$ is a walk linking \bar{a} and \bar{b} . The poset Σ is then connected.

Conversely, let's suppose that Σ is connected and let's denote by x and y two vertices of Q , they are also in Σ . Since Σ is connected, there exists a walk ω' from x to y in Q . The application ϕ being surjective, there exists a walk of Q from x to y and such that $\phi(\omega) \sim \omega'$. Then, Q is connected. \square

3.2. Generators of the kernel.

Let (Q, I) be a presentation by quiver and relations, and \sim be the equivalence relation defined in the construction of the algebraic fundamental group $\Pi(Q, I)$. We also consider the equivalence relation R in $P(Q)/\sim$, the set of isoclasses by the equivalence relations \sim of walks of Q , the smallest equivalence relation compatible with the multiplication and verifying the following property : if two parallel paths c_1 and c_2 of Q divide two \sim -equivalent path not in I , then the classes \bar{c}_1 and \bar{c}_2 are R -equivalent.

Example 3.2.1. Let (Q, I) be the following presentation :

$$Q = \begin{array}{ccc} \bullet & \xrightarrow{\alpha} & \bullet & \xrightarrow{\gamma} & \bullet \\ & \xleftarrow{\beta} & & & \end{array} \quad I = \langle \gamma\alpha - \gamma\beta \rangle$$

The paths α and β are parallel and divide respectively $\gamma\alpha$ and $\gamma\beta$ which are \sim -equivalent. Therefore, the classes of paths $\bar{\alpha}$ and $\bar{\beta}$ are R -equivalent.

Example 3.2.2. We consider now the quiver Q containing only one vertex s and only one arrow f , and the ideal I restricted to $\{0\}$:

$$Q = \begin{array}{ccc} & & \curvearrowright f \\ & & \downarrow \\ \bullet & & \downarrow \\ s & & \downarrow \end{array} \quad I = \{0\}$$

The paths s and f are parallel and divide f which is equivalent to itself, therefore the classes \bar{s} and \bar{f} are R -equivalent.

Let H be the normal sub-group of $\Pi_1(Q, I)$ associated to the relation R , that is to say the smallest normal sub-group H of $\Pi_1(Q, I)$ containing the walks $p^{-1}\bar{c}_1^{-1}\bar{c}_2p$, with p a walk which begins in X_0 , the vertex chosen to calculate $\Pi_1(Q, I)$, and with c_1 and c_2 parallel paths dividing respectively two paths \sim -equivalent non equal to zero in kQ/I . The sub-group H can be written as follows :

$$H = \langle \bar{p}^{-1}\bar{c}_1^{-1}\bar{c}_2\bar{p} \ / \ \bar{p} \text{ a walk which begins in } X_0, \bar{c} \in \Sigma, \text{ and } \bar{c}_1/\bar{c}, \bar{c}_2/\bar{c}, \bar{c}_1/\bar{c}_2 \rangle$$

3.3. The exact sequence.

Theorem 3.3.1. Let (Q, I) be a presentation of an algebra and A' the associated incidence algebra. Then the group $\Pi_1(Q, I)/R$ and $\Pi_1(Q_\Sigma)$ are isomorphic. In other words, the sequence

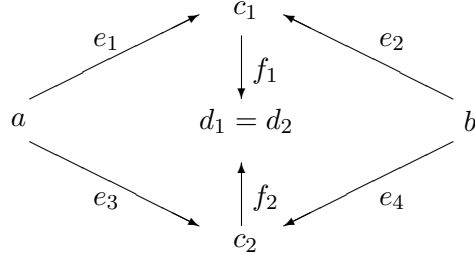
$$0 \longrightarrow H \longrightarrow \Pi_1(Q, I) \longrightarrow \Pi_1(A') \longrightarrow 0$$

is exact.

Proof. We will use the same notation than the ones in the proof of the surjectivity. Moreover, it is sufficient to prove this for a presentation that verifies the property P since $\Pi_1(Q, I)$ and $\Pi_1(Q, \tilde{I})$ are isomorphic.

Let's prove first that ϕ_* is constant on the equivalent classes of the relation R ; therefore, it will induce ϕ_{**} . We are going to prove that the images of two parallel paths c_1 and c_2 in Q dividing respectively the equivalent paths d_1 and d_2 not in I , are equivalent. We notice that c_1 and c_2 are not in I , because in the reverse case d_1 and d_2 would be in I which is not possible. Then the classes \bar{c}_1 and \bar{c}_2 are in Σ .

Since c_1/d_1 and c_2/d_2 , there exists f_1 a path of Q_Σ from \bar{c}_1 to \bar{d}_1 and f_2 a path of Q_Σ from \bar{c}_2 to \bar{d}_2 . Moreover, we use the same definition than in the proof of the surjectivity for the paths e_1, e_2, e_3 and e_4 . We obtain the following situation; the arrows being paths of Q_Σ :



By using the remark at the end of the surjectivity of ϕ , we have :

$$\phi(c_1) \sim e_2^{-1}e_1 \sim e_2^{-1}f_1^{-1}f_1e_1 \sim e_4^{-1}f_2^{-1}f_2e_3 \sim e_4^{-1}e_3 \sim \phi(c_2)$$

The construction of the antecedent used in the proof of the surjectivity defined an application ψ from the set of walks of Q_Σ to quotient set of the walks of Q by the two equivalence relations. We begin to define this application more precisely. Let's consider the application ψ defined from $P(Q_\Sigma)$, the set of paths of Q_Σ , to $(P(Q)/\sim)/R$, and generated by the following relation :

1. $\psi(e) = s(e)$ for all vertices e of Q_Σ .
2. If $\alpha_n^{\varepsilon_n} \dots \alpha_1^{\varepsilon_1}$ is a closed walk in x_0 then

$$\psi(\alpha_n^{\varepsilon_n} \dots \alpha_1^{\varepsilon_1}) = \psi(\alpha_n)^{\varepsilon_n} \dots \psi(\alpha_1)^{\varepsilon_1}.$$

3. If α is an arrow of Q_Σ from \bar{a} to \bar{b} then there exists an arrows f of Q verifying $\bar{b} = \bar{f}.\bar{a}$ or $\bar{b} = \bar{a}.\bar{f}$. If $\bar{b} = \bar{f}.\bar{a}$ then $\psi(\alpha)$ is the origin of \bar{a} else $\psi(\alpha)$ is equal to f^{-1} .

Let x_0 be the vertex chosen to calculate $\Pi_1(Q, I)$, the vertex $x_0 = \phi(x_0)$ will be also chosen to calculate $\Pi_1(Q_\Sigma, I_{Q_\Sigma})$. Moreover, we will denote one more time by ψ , the application ψ restricted to the set of closed walks in x_0 . We notice that the image of ψ is included in $\Pi(Q_\Sigma, I_{Q_\Sigma})$.

Let's prove that ψ is well defined.

1. If there exists f and g verifying $\bar{b} = \bar{a}.\bar{f} = \bar{g}.\bar{a}$, then $s(\bar{a}) = s(\bar{f}) = s(\bar{g})$, $t(\bar{f}) = s(\bar{a})$ and $t(\bar{a}) = s(\bar{g})$; the applications *source* and *terminus* being always used on the classes of walks of Q and not on the vertices of Σ . Then f is a loop of extremity $s(\bar{a})$. The vertex $s(\bar{a})$ of Q and the arrow f are parallel and divide f , not equal to zero since the ideal I is admissible, then they are R -equivalent; in fact I being admissible, arrows and vertices of Q can not be in I . With a similar proof on g we deduce that the two definitions in that case coincide.
2. If there exists f and g verifying $\bar{b} = \bar{f}.\bar{a} = \bar{g}.\bar{a}$ or $\bar{b} = \bar{a}.\bar{f} = \bar{a}.\bar{g}$, then f and g parallel arrows dividing b , not in I because b is in Σ , therefore they are R -equivalent.

Let's prove now that the application ψ is constant on the equivalent class and then induces an application ψ_* from $\Pi_1(Q_\Sigma)$ to $\Pi_1(Q, I)/H$

We will verify this on the generators of the equivalence relation. Let α be an arrow of Q_Σ from the vertex \bar{a} to the vertex \bar{b} . Then there exists f verifying $\bar{b} = \bar{f}.\bar{a}$ or $\bar{b} = \bar{a}.\bar{f}$. In the first case, insert $\alpha^{-1}\alpha$ in a walk of Q_Σ do not change its image since $\psi(\alpha)$ is a vertex. In the second case, this is equivalent to insert $f.f^{-1}$ in the walk, which do not change its equivalent class. We do the same thing on $\alpha\alpha^{-1}$.

Let's consider now two parallel paths c and c' of Q_Σ . We will prove that changing $\psi(c)$ by $\psi(c')$ in a walk of Q does not change its equivalent class. Let $\bar{a}_1, \dots, \bar{a}_n$ and $\bar{a}'_1, \dots, \bar{a}'_{n'}$ be respectively the vertices crossed by c and c' and let's denote by $\bar{a} = \bar{a}_1 = \bar{a}'_1$. For all i , the vertex \bar{a}_i is linked to the vertex \bar{a}_{i+1} then there exists an arrow f of Q such that $\bar{a}_{i+1} = \bar{a}_i.\bar{f}$ or $\bar{a}_{i+1} = \bar{f}.\bar{a}_i$. Then

$$\begin{array}{ll} \exists r, s \in \mathbb{N} & \exists f_1 \dots f_{r+s} \quad \text{such that} \quad \bar{a}_n = \bar{f}_{r+1} \dots \bar{f}_{r+s}.\bar{a}.\bar{f}_1 \dots \bar{f}_r \\ \exists r', s' \in \mathbb{N} & \exists f'_1 \dots f'_{r'+s'} \quad \text{such that} \quad \bar{a}_{n'} = f'_{r'+1} \dots f'_{r'+s'}.\bar{a}.\bar{f}'_1 \dots \bar{f}'_{r'} \end{array}$$

We have then $\psi(c)^{-1} = \bar{f}_1.\bar{f}_2 \dots \bar{f}_r$ and $\psi(c')^{-1} = \bar{f}'_1.\bar{f}'_2 \dots \bar{f}'_{r'}$. These paths are parallel and theirs \sim -classes divide $\bar{a}_n = \bar{a}'_{n'}$, therefore they are R -equivalent.

Finally let's prove that the applications ψ_* and ϕ_{**} verify $\psi_* \circ \phi_{**} = Id$. Then, ϕ_{**} will be injective. Since ψ_* and ϕ_{**} are two morphisms, it is sufficient to prove this on the vertices and on the arrows of Q_Σ . If a is a vertex of Q , then $\psi_* \circ \phi_{**}(a) = a$. If f is an arrow of Q from a to b then

$$\psi_* \circ \phi_{**}(f) = \psi_*(f_b^{-1}.f_a) = \psi_*(f_b)^{-1}.\psi_*(f_a) = (f^{-1})^{-1}.a = f$$

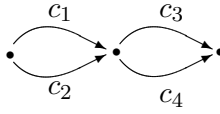
□

3.4. Cases in which the morphism is an isomorphism.

We will give sufficient conditions to have an isomorphism between the two fundamental groups described in the previous paragraphs, that is to say to have a sub-group H restricted to the neutral element.

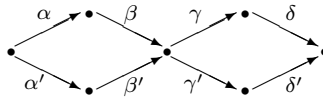
Proposition 3.4.1. Let (Q, I) be a presentation of an algebra and Q_Σ the associated incidence quiver. Then if one of the following conditions is satisfied, the sub-group H is trivial and then $\Pi_1(Q_\Sigma) \simeq \Pi_1(Q, I)$.

1. The ideal I is restricted to 0.
2. If c_1 and c_2 are parallel and not equivalent then the paths $\omega_1.c_1.\omega_2$ and $\omega_3.c_2.\omega_4$ are not equivalent, for all paths $\omega_1, \omega_2, \omega_3$ and ω_4 such that the previous products has sense.
3. If c_1 and c_2 are parallel then the paths $\omega_1.c_1.\omega_2$ and $\omega_3.c_2.\omega_4$ are not parallel, for all paths $\omega_1, \omega_2, \omega_3$ and ω_4 which are not all of them trivial and such that the previous products has sense. This condition is independent to I .
4. The couple (Q, I) is the presentation of an incidence algebra, more generally a presentation of a Schurian algebra.
5. The quiver Q does not contain cycles and sub-quivers in the form of eight, that is to say sub-quivers in the form :



in which arrows represent in fact paths.

Remark. We have just seen that H is trivial in the case of Schurian algebra. Nevertheless, this is not the case for constricted algebra. Indeed, let's consider the following quiver Q :



and the ideal $I = \langle \delta\gamma\beta\alpha - \delta'\gamma'\beta'\alpha' \rangle$. The algebra kQ/I is a restricted algebra non Schurian and the sub-group H is isomorphic to \mathbb{Z} .

Proof de la proposition.

1 : in the case in which $I = 0$, two equivalent \sim -paths are equal. Then if c_1 and c_2 are parallel dividing a same path d , then c_1 and c_2 are equal, and $\overline{c_1} = \overline{c_2}$ bring no generators to H .

2 : this is a translation of the definition.

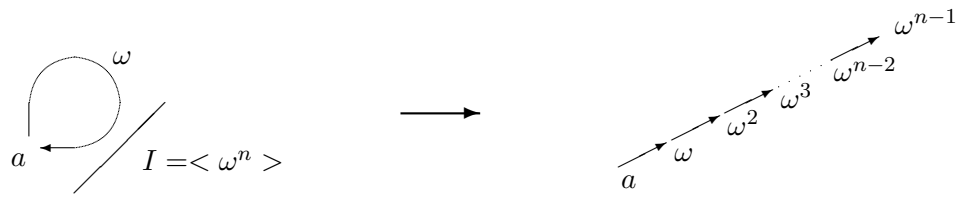
3 : if the multiples of c_1 and c_2 are never parallel, they can not be equivalents.

4 : let c_1 and c_2 be two parallel paths dividing d_1 and d_2 equivalent and not equal to zero. Therefore c_1 and c_2 are parallel and not equal to zero. Since the algebra is a Shurian algebra, this two paths are equivalent.

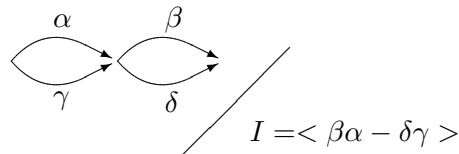
5 : let c_1 and c_2 be two parallel path dividing two paths d_1 and d_2 equivalent. Since it does not exist quivers in the form of eight and loops then there exists ω and ω' such that $d_1 = \omega.c_1.\omega'$ and $d_2 = \omega.c_2.\omega'$. Then since d_1 and d_2 are equivalent then $\omega^{-1}.d_1.\omega'^{-1}$ and $\omega'^{-1}.d_2.\omega^{-1}$ are equivalent and then c_1 and c_2 too. \square

4. Examples and influence of the presentation.

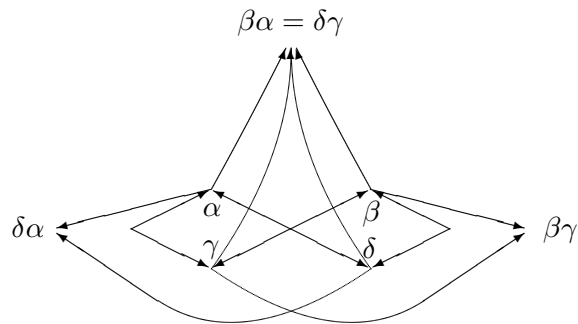
Example 4.1. The aims of the these two examples is to show simple cases in which the subgroup H is not trivial.



In this example, the vertex a and the arrows ω divide ω . Then H is equal to the fundamental group of (Q, I) . On the right, we find the associated incidence quiver which the fundamental group is trivial. Let's consider now the following quiver :

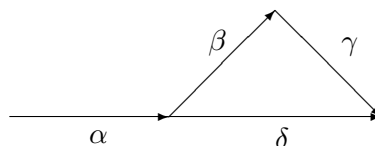


The quiver of the associated incidence algebra is :

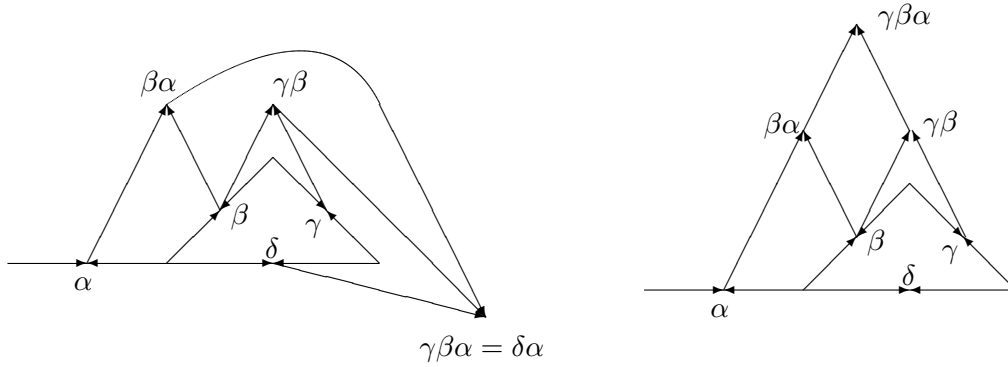


In that case, the arrows α , γ and β , δ divide $\beta\alpha = \delta\gamma$. The relation R is generated by $\alpha R \gamma$ and $\beta R \delta$.

Example 4.2. This example shows that the incidence algebra associated to a presentation (Q, I) is not an invariant of the algebra. Indeed, let's consider the quiver Q :



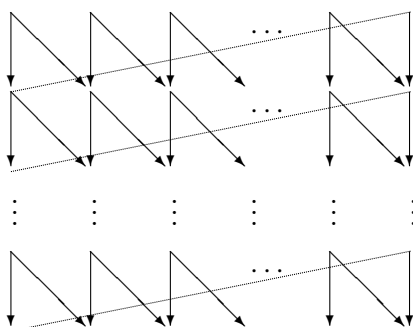
and the two ideals $I_1 = \langle \gamma\beta\alpha - \delta\alpha \rangle$ and $I_2 = \langle \delta\alpha \rangle$ of kQ . The two algebra kQ/I_1 and kQ/I_2 are isomorphic. Let's construct now the incidence quivers $\Sigma_{(Q,I_1)}$ and $\Sigma_{(Q,I_2)}$ which follows :



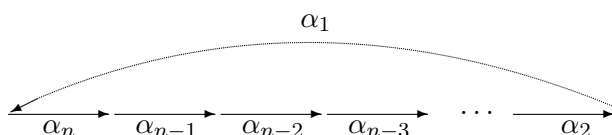
For the presentation (Q, I_1) , the fundamental group is trivial, then the fundamental group of the associated incidence quiver is restricted to 1 since the theorem of the exact sequence. For the second presentation (Q, I_2) , the fundamental group of the associated incidence quiver is isomorphic to \mathbb{Z} . To see that, we are going to calculate the sub-group H of $\Pi_1(Q, I_2)$. Only the paths δ and $\gamma\beta$ are parallel and not equivalent. The subgroup H is then generated by the relation $\beta\alpha - \delta\gamma\alpha$. We choose the vertex α as base point to calculate the fundamental group. But this relation is trivial in $\Pi_1(Q, I_2)$ and then H is restricted to 1. The fundamental group of $\Sigma_{(Q,I_2)}$ is then isomorphic to $\Pi_1(Q, I_2)$ since the theorem of the exact sequence which is isomorphic to \mathbb{Z} .

Then, there exists two presentations of an algebra whose incidence quivers give different fundamental groups ; The fundamental groups of the associated incidence algebras also depend on the presentation of the algebra.

Example 4.3. In this example, we recalculate in a simple way the fundamental group of a type of a particular incidence algebra and then its first group of Hochschild cohomology with the theorem of Assem and De La Peña. This incidence algebras are such that their quiver is a crown, that is to say a quiver as follows :



It has been considered for example in [GS] and their fundamental group has been calculated in [GR]. For that, let consider the following cyclic quiver Q_n :



Let p and q be natural integer such that $q < n$. We also consider the ideal I_{pq} of kQ_n generated by the paths of Q_n of length $p \times n + q$. Then the incidence quiver associated to (Q_n, I_{pq}) is the crown previously described with n vertices on the horizontal and $n \times p + q$ vertices on the vertical. The vertices of the first line represent the vertices of Q_n , the vertices of the second line are the arrows of Q_n , the vertices of the third line are the path of length 2 of Q_n , etc.

Then, it is sufficient to calculate the fundamental group of Q_n and the group H . The fundamental group does not change with I_{pq} since I_{pq} contains only minimal relation with one term. It is then isomorphic to \mathbb{Z} . For the subgroup H , two cases can happen. If $p = 0$, it does not exist not equal to zero and parallel paths then H is 1 and $\Pi_1(Q_c, I_{pq})$ is isomorphic to \mathbb{Z} . If $p \neq 0$, the path $\alpha_1 \dots \alpha_n$ and the origin of α_1 divide $\alpha_1 \dots \alpha_n$, the subgroup H contains a generator of the cyclic group $\Pi_1(Q_c, I_{pq})$, then $\Pi_1(Q_c, I_{pq})/H$ is trivial.

Then, the fundamental of a crown $C_{n,p,q}$ with n columns and with $pn + q$ lines is :

$$\Pi_1(C_{n,p,q}, I_{C_{n,p,q}}) = \begin{cases} \mathbb{Z} & \text{if } p = 0 \\ 0 & \text{if } p > 0 \end{cases}$$

For an other demonstration, see [GR].

Moreover since $Hom(\Pi_1(Q, I), k^+) \simeq HH^1(kQ/I)$ in the case of incidence algebra, we can deduce that the first Hochschild cohomology group of the incidence algebra of the crown is k if $p = 0$ and 0 otherwise.

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