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Algebraic fundamental group and simplicial complexes

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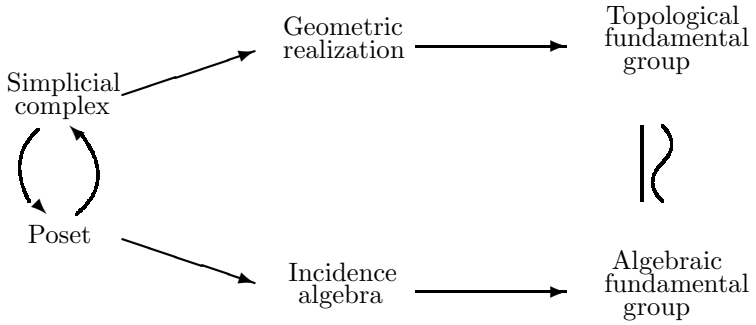
Abstract

In this paper we prove that the fundamental group of a simplicial complex is isomorphic to the algebraic fundamental group of its incidence algebra, and we derive some applications.

AMS classification : 16E40 ; 16G20 ; 06A11 ; 55Q05

Let k be a field and A be a basic and split finite dimensional k-algebra, which means that $A/r = k \times k \times \ldots \times k$ where r is the radical of A. There exists a unique quiver Q and usually several admissible ideals I of the algebra kQ such that $A = kQ/I$ (see [6]). In the 1980s, an algebraic fundamental group has been defined which depends on the presentation of A, that is to say on the choice of the ideal I (see [13]). For incidence algebras, that is algebras obtained from a simplicial complex, it has been proved that the presentation does not influence the fundamental group ([15]). Then it is a natural question to compare it with the fundamental group of the geometric realisation. Note also that in [4,8] the analogous question concerning homology is solved.

Actually, we prove that the fundamental groups considered for a finite and connected simplicial complex are isomorphic. The following diagram summarizes the situation :



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The isomorphism above enables us to adapt results of algebraic topology to the purely algebraic setting : for instance the isomorphism recently proved between $Hom(\Pi_1(Q, I_Q), k^+)$ and $HH^1(kQ/I_Q)$ where Q is an ordered quiver, I_Q the associated parallel path and k^+ the additive group of a field ([8], see also [7]) is a consequence of the classic result in algebraic topology which states that the abelianisation of the Π_1 and the first homology group of a simplicial complex are isomorphic. In another direction, we also derive an algebraic Van Kampen theorem.

For our purpose we consider in the first part of this paper simplicial complexes and usual fundamental groups ; as well as posets, incidence algebras and algebraic fundamental groups. We also explain the relations between posets and simplicial complexes. In the last section, we provide the isomorphism and give applications.

This work is part of my thesis in Montpellier and I thank C. Cibils for his help, his apposite advice and his patience. When finishing this paper, I learned that J.C. Bustamente had independently considered a similar context.

All simplicial complexes will be finite, connected and not empty.

1. Fundamental group and incidence algebras

We consider the classical definition of a simplicial complex (see for instance [11]). A simplicial complex is the union of elements $\{a_i\}_{i\in I}$ called the vertices, and finite sets of vertices called the simplexes, such that if S is a simplex all subsets of S are also simplexes. Each set containing only one element is a simplex. A simplicial complex is said to be finite if I is finite, and is connected if for each couple of vertices (s, t), there exist vertices $s_0, \dots, s_n$ such that $s_0 = s$, $s_n = t$ and $\{s_{i-1}, s_i\}$ is a simplex for each i in $\{1, \dots, n\}$.

The geometric realization of a simplicial complex C with vertices $\{a_i\}_{i\in I}$ is as follows. Let $\{A_i\}_{i\in I}$ be points of $\mathbb{R}^n$, such that if $\{a_{\alpha_1}, \dots, a_{\alpha_p}\}$ is a simplex of C, then the points $A_{\alpha_1}, \dots, A_{\alpha_p}$ are linearly independent. The set of points whose barycentric coordinates are strictly positive is called a face. Note that we prefer to define a face as the points with strictly positive barycentric coordinates instead of positive barycentric coordinates : this way, the geometric realization becomes the disjoint union of its faces and there exists only one face containing it. Moreover, if S and S' are two simplexes with empty intersection, the intersection of the corresponding faces is empty. A geometric realization of C denoted by $|C|$, is the union of the faces associated to simplexes of a simplicial complex. A closed face is the closure of a face.

Note that since we assume that the simplicial complexes are finite, their geometric realization exist. Indeed let C be a complex with set of vertices $a_1, \dots, a_n$. The geometric realization $|C|$ of a simplicial complex C is a subset of $\mathbb{R}^n$ and inherits the topology of $\mathbb{R}^n$. Behind the usual definition of the fundamental group $\Pi_1(|C|)$ obtained through homotopy classes of closed paths, we recall the construction of the edge-paths group of $|C|$. This provides another description of the fundamental group $\Pi_1(|C|)$ which is useful for our purpose (see [11] for example).

An *edge-path* of C is a finite sequence of vertices $a_{i_r} \dots a_{i_1}$ such that for all j in $\{1, \dots, r-1\}$, the set $\{a_{i_j}, a_{i_{j+1}}\}$ is a simplex. If $w = a_{i_r} \dots a_{i_1}$ is an edge-path, let w^{-1} denote the edge-path $a_{i_1} \dots a_{i_r}$. An edge-path is said to be *closed* (or *an edge-loop*) if the first and the last

vertices are the same. Let $w = a_{i_r} \dots a_{i_1}$ and $w' = a'_{i'_{r'}} \dots a'_{i'_1}$ be two edge-paths ; if $a_{i'_{r'}} = a_{i_1}$, the product $w.w'$ is defined and is equal to $a_{i_r} \dots a_{i_1} a'_{i'_{r'}} \dots a'_{i'_1}$

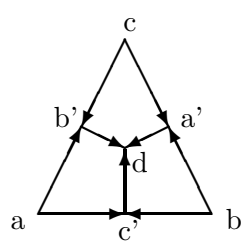
This is an *allowable operation* on edge-paths : if three consecutive vertices of the edge-path are in the same simplex, the middle vertex can be removed. Conversely, we can add a vertex between two others, if these three vertices are in a same simplex of C. Moreover, it is possible to change $a_{i_0} a_{i_0}$ by a_{i_0} and conversely. This generates an equivalence relation on the set of edge-paths. Let $\overline{w}$ denote the equivalence class of the edge-path w. As two equivalent edge-paths have the same extremities, the product defined before, when it exists, is defined also on the equivalence classes, as well as on the set of edge-loops starting at a fixed point.

Proposition 1.1. ([11] 6.3.1 and 6.3.2.) Let C be a simplicial complex and x_0 a vertex of C. The set of equivalence classes of edge-loops starting at one point x_0 is a group for the product defined before. Since C is connected, this group does not depend on x_0 and is denoted by $\Pi_1(C)$. The fundamental groups $\Pi_1(C)$ and $\Pi_1(|C|)$ are isomorphic.

Hereafter, $\Pi_1(C)$ will be either the approximation of the fundamental group $\Pi_1(C)$ or the fundamental group $\Pi_1(|C|)$ itself.

Given a *poset* (i.e. a partially ordered set), there is an associated *ordered quiver*, that is to say a finite oriented graph without loops and such that if there exists an arrow from a to b, there does not exist another path from a to b. To each element of the poset corresponds a vertex of the graph. Moreover, let S_1 and S_2 be vertices in the graph ; there exists an arrow from S_1 to S_2 if and only if the element associated to S_1 in the poset is smaller than the element associated to S_2 and if there does not exist an element of the poset strictly between these two elements. The graph obtained is an ordered quiver.

For example, the graph that corresponds to the poset a, b, c, a', b', c', d with $a < b' < d$, $c < b' < d$, $a < c' < d$, $b < c' < d$, $b < a' < d$ and $c < a' < d$ is :



Conversely, by this operation all ordered graphs arise from a poset. There is a bijection between the set of ordered graphs and the set of posets. Moreover, a poset is said to be connected if its ordered quiver is connected. All the posets considered will be connected, finite and non empty.

Let now Q be a quiver, and k be a field. We denote kQ the k-vector space with basis the paths of Q (the paths of length 0 being the vertices), with the multiplication given by the composition of two paths if possible and 0 otherwise. Two paths of Q are *parallel* if they have the same beginning and the same end. The k-space generated by the set of differences of two parallel paths is a two-sided ideal of kQ, denoted I_Q and called *parallel ideal*. The quotient algebra kQ/I_Q is the *incidence algebra* of Q.

The general definition of the fundamental group depends on a couple (Q, I) where I is an admissible ideal of kQ, it can be found in [8, 15, 7] and we recall it below. We notice that for

an algebra the presentation as a quiver with relations is not unique in general, that is to say that different ideals I may exist such that $A \cong kQ/I$. We do not have in general a unique fundamental group associated to an algebra. Nevertheless, in the case of an incidence algebra it has been proved that the fundamental group does not depend on the presentation of the algebra (see [15] for example).

A relation $\sum_{i=1}^{n} \lambda_i \omega_i$ is *minimal* if the sum is in I and if for all non empty proper subset J of $\{1, \ldots, n\}$ the sum $\sum_{i \in J} \lambda_i \omega_i$ is not in I. We note that if the relation is minimal then $\{\omega_1, \ldots, \omega_n\}$ have the same source and the same terminus.

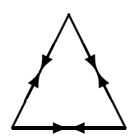
If α is an arrow from x to y, let α^{-1} denote its formal inverse which goes from y to x. A walk from x to y is a formal product $\alpha_1^{\pm 1} \alpha_2^{\pm 1} \ldots \alpha_n^{\pm 1}$ which begins in x and ends in y. The trivial walk x, which begins in x and no longer moves is denoted e_x. A *closed walk* (or a *loop*) is a walk having the same extremities. A walk is, in fact, a path in the non oriented graph associated to Q ; in other words, a walk can follow the arrows in any direction.

We consider the smaller equivalence relation $\sim$ on the walks of Q containing the following items :

1. if α is an arrow from x to y then $\alpha\alpha^{-1} \sim e_y$ and $\alpha^{-1}\alpha \sim e_x$,
2. if $\sum_{i=1}^{n} \lambda_i \omega_i$ is a minimal relation then $\omega_1 \sim \ldots \sim \omega_n$,
3. if $\alpha \sim \beta$ then, for all (ω, ω'), we have $\omega\, \alpha\, \omega' \sim \omega\, \beta\, \omega'$

Let x_0 be a vertex of Q. The set of equivalence classes of loops starting at x_0 does not depend on x_0 since Q is connected. We denote this set by $\Pi_1(Q, I)$. If the quiver Q comes from a poset P, the fundamental group associated to (Q, I_Q) is denoted $\Pi_1(P)$.

We remark that if I is the parallel ideal, the second item means parallel paths are equivalent. For example if Q is defined by the following quiver, the fundamental group $\Pi_1(Q, I_Q)$ is isomorphic to $\mathbb{Z}$.



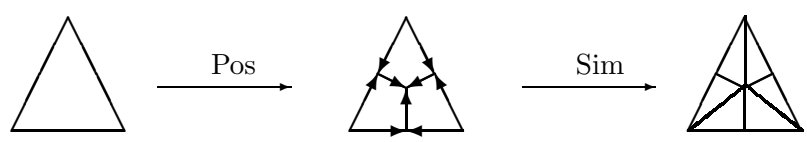
We note that the hypothesis I admissible is not fully used neither to define the fundamental group nor to prove that it is a group. So, in this paper, we will consider the fundamental group of a couple (Q, I) where I satisfies $F^n \subset I \subset F$ for an integer n. This will be used to adapt Van Kampen's theorem to purely algebraic fundamental groups.

Let C be a simplicial complex. The set of non empty simplexes of C ordered by inclusion is a poset which we will denote $Pos(C)$. This Pos defines an application from simplicial complexes to posets which is injective, but not surjective, since there is no simplicial complex which gives the poset $a < b$. Due to the construction of the quiver from a simplicial complex, an arrows can only go from a vertex corresponding to a p-face to a vertex corresponding to a q-face with $p > q$, then the path algebra of this quiver is then of finite dimension.

We provide now the construction of a simplicial complex from a poset. These procedures are of course not inverse one of each other, their composition is the barycentric decomposition.

To each poset P we associate a simplicial complex $Sim(P)$, where a n-simplex is a subset of P containing $n+1$ elements and totally ordered. The application Sim is surjective but not injective, for instance the simplicial complexes which are associated to $a < b < c$, $a < b < d$ and to $a < c < b$, $a < d < b$ are the same.

Let C be a simplicial complex, $|Sim(Pos(C))|$ is the geometric realization of the barycentric decomposition of C. Then for example let C be the set of the non empty parts of $\{a, b, c\}$; its geometric realization being the triangle drawn on the next figure. Then, $Pos(C)$ contains all the elements of C that is to say $T = \{a, b, c\}$ $A_1 = \{b, c\}$, $A_2 = \{a, c\}$, $A_3 = \{a, b\}$, $S_1 = \{a\}$, $S_2 = \{b\}$ and $S_3 = \{c\}$ and its order is defined by $S_i \leq A_j \leq T$ for all $i, j \in \{1, 2, 3\}$ and $i \neq j$. The associated quiver is drawn on the next figure. Then $Sim(Pos(C))$ is the complex containing the total ordered subsets of P that is to say : for all $i, j \in \{1, 2, 3\}$, the 0-simplexes are $\{S_i\}$, $\{A_i\}$, $\{T\}$, the 1-simplexes are $\{S_i, A_j\}$ $i \neq j$, $\{S_i, T\}$, $\{A_i, T\}$, the 2-simplexes are $\{S_i, A_j, T\}$ $i \neq j$. By identifying poset and associated ordered quiver, simplicial complexes and their geometric realization, the situation can be summarized by the following diagram :



2. Equivalence between algebraic and topological approaches.

The aim of this section is to prove that the fundamental group $\Pi_1(|C|)$ defined on the geometric realization of a finite simplicial complex C is isomorphic to the fundamental group $\Pi_1(Pos(C))$ of the incidence algebra of the poset deduced from the complex.

We prove first that for any poset P we have $\Pi_1(P) \simeq \Pi_1(Sim(P))$. We will use the approximation of the topological fundamental group considered before in order to provide an isomorphism between $\Pi_1(Sim(P))$ and $\Pi_1(P)$.

Theorem 2.1. Let P be a poset, the groups $\Pi_1(P)$ and $\Pi_1(Sim(P))$ are isomorphic.

Proof. Due to proposition 1, it is sufficient to prove that $\Pi_1(P)$ is isomorphic to the edge-paths group of $|Sim(C)|$

For each vertex s of the poset P, the set $\{s\}$ is totally ordered and it corresponds to a 0-simplex of the simplicial complex $Sim(P)$. We denote by s' this 0-simplex of $Sim(P)$.

Let ϕ be the map from the set of the associated quiver walks to the set of edge-paths of $Sim(P)$ be defined by

$$\phi(\alpha_n^{\epsilon_n} \dots \alpha_1^{\epsilon_1}) = s'_{n+1} \dots s'_1$$

where s_i and s_{i+1} are the poset elements which are the origin and the end of the walk $\alpha_i^{\epsilon_i}$.

The map ϕ is well defined. Indeed, $s'_{n+1} \dots s'_1$ is an edge-path because for all i in $\{1, \dots, n\}$, the set $\{s_i, s_{i+1}\}$ is totally ordered since α_i is an arrow of the extremities s_i, s_{i+1} and therefore $\{s'_i, s'_{i+1}\}$ is a simplex of $Sim(P)$.

We assert now that the images by ϕ of equivalent walks are equivalent. We have to prove this fact on the generators of the equivalence relation. Let f be an arrow from s_1 to s_2, then $\phi(f.f^{-1}) = s'_1.s'_2.s'_1$ which is equivalent to $s'_1 = \phi(e_{v_1})$. Let c_1 and c_2 be two parallel paths crossing respectively the vertex $s_1, t_1, t_2, \dots, t_n, s_2$ and $s_1, u_1, u_2, \dots, u_n, s_2$. Then the

sets $\{s_1, t_1, t_2, \ldots, t_n, s_2\}$ and $\{s_1, u_1, u_2, \ldots, u_n, s_2\}$ are totally ordered and the edge paths $\phi(c_1) = s'_1.t'_1.t'_2.\ldots.t'_n, s'_2$ and $\phi(c_2) = s'_1.t'_1.t'_2.\ldots.t'_n, s'_2$ are both equivalent to $s'_1.s'_2$.

The third relation is immediate because the equivalence relation on the edge paths set is compatible with the product of the group.

Let $\phi_* : \Pi_1(P) \to \Pi_1(Sim(P))$ denote the application induced by ϕ. Note that the image of a loop is also a loop.

First of all ϕ_* is a morphism. Indeed :

If $\begin{cases} \phi(p) = x_0 v_n \ldots v_1 x_0 \\ \phi(q) = x_0 w_n \ldots w_1 x_0 \end{cases}$ then $\begin{cases} \phi(p).\phi(q) = x_0 v_n \ldots v_1 x_0 x_0 w_n \ldots w_1 x_0 \\ \phi(p.q) = x_0 v_n \ldots v_1 x_0 w_n \ldots w_1 x_0 \end{cases}$

These two edge-paths are equivalent so $\phi_*(p)\phi_*(q) = \phi_*(p.q)$.

To prove that ϕ_* is bijective, we are going to construct its inverse ψ_*.

Let $s'_{n+1} \ldots s'_1$ be an edge-path. We fix i in $\{1, \ldots, n\}$. The set $\{s'_i, s'_{i+1}\}$ being a simplex, the set $\{s_i, s_{i+1}\}$ is totally ordered. Then, there exists a maximal (for the inclusion) totally ordered set containing it and having s_i and s_{i+1} as extremities. The choice of this is not important because all paths corresponding to these sets have the same origin and the same end and therefore are parallel. This maximal set corresponds to a path or to an inverse path $w_i^{\epsilon_i}$ of the associated quiver with origin s_i and end s_{i+1}. So we can define a morphism ψ from edge paths group to $\Pi_1(C)$ by $\psi(s'_{n+1} \ldots s'_1) = w_n^{\epsilon_n} \ldots w_1^{\epsilon_1}$ if $n \geq 1$ and $\psi(s'_1) = s_1$.

We will prove now that this application is constant on the equivalence class. In deed $\psi(s's')$ is a path from s to s, so $\psi(s's') = s = \psi(s')$. Moreover, let take s', t', u' such that $\{s', t', u'\}$ is a simplex, so $\psi(s't'u') = \psi(s't').\psi(t'u')$ and $\psi(s'u')$ are paths from u to s. Therefore they are parallel.

Finally, we will verify that $\phi_* \, o \, \psi_* = \psi_* \, o \, \phi_* = Id$. Let f be an arrow of the ordered quiver associated to P from s_1 to s_2 then $\phi_*(f) = s'_1 s'_2$ and $\psi_* \, o \, \phi_*(f)$ is a path from s_1 to s_2. Since the quiver is ordered, it does not exist any other path than f, so $\psi_* \, o \, \phi_*(f) = f$ and $\psi_* \, o \, \phi_* = Id$. For the other equality, let's consider an edge path $s'_1 s'_2$. Then $\psi_*(s'_1 s'_2)$ is a path from s_1 to s_2 and then $\phi_* \, o \, \psi_*(s'_1 s'_2)$ is an edge path beginning with s'_1 and ending with s'_2 such that all vertices of this edge-path are in a same simplex. So it is equivalent to $s'_1 s'_2$.

Theorem 2.2. Let C be a simplicial complex. The fundamental groups $\Pi_1(|C|)$ and $\Pi_1(Pos(C))$ are isomorphic.

Proof. The fundamental groups associated to the simplicial complex and to its barycentric decomposition are isomorphic. Then,

$$\Pi_1(|C|) \simeq \Pi_1(Sim(Pos(C)).$$

Moreover, the previous theorem shows that the group $\Pi_1(Pos(C))$ and $\Pi_1(Sim(Pos(C)))$ are isomorphic.

3. Applications

We first show that in a particular case of incidence algebra the result obtained by I. Assem and J.A. De La Peña ([1], p.200) is a consequence of a classic fact in algebraic topology ; so this proof has the advantage to link this result to algebraic topology.

Theorem 3.1. Let P be a poset and $A = kQ/I$ its incidence algebra over a field k. Then

$$Hom(\Pi_1(P), k^+) \simeq HH^1(A)$$

where k^+ is the additive group of the field k and $HH^1(A)$ is the first Hochschild cohomology group of A.

Proof. Let C be a simplicial complex and $C_\bullet(\Lambda)$ be the chain complex over a ring Λ associated to the simplicial homology of C ; the latter homology will be denoted by $H_\bullet(C, \Lambda)$. Moreover, the cohomology of the complex $Hom_\Lambda(C_\bullet(\Lambda), A)$, where A is a Λ-module, is denoted by $H^\bullet(C, \Lambda, A)$.

For any simplicial complex, the abelianisation Π_1^{ab} of Π_1 is isomorphic to the first homology group (see for example [11], 6.4.7). If we denote by C the simplicial complex associated to Q, we have

$$(\Pi_1(C))^{ab} \simeq H_1(C, \mathbb{Z})$$

and

$$Hom((\Pi_1(C))^{ab}, k^+) \simeq Hom(H_1(C, \mathbb{Z}), k^+).$$

Note that since k^+ is abelian we have :

$$Hom(\Pi_1(C), k^+) \simeq Hom(H_1(C, \mathbb{Z}), k^+).$$

Making use of theorem we have :

$$Hom(\Pi_1(Q, I_Q), k^+) \simeq Hom(H_1(C, \mathbb{Z}), k^+) \qquad (*)$$

Adjunction, for two rings R and S and bimodules A_R, ${}_RB_S$, C_S, provides an isomorphism, (see [17], p.37, for example) :

$$Hom_S(A \underset{R}{\otimes} B, C) \simeq Hom_R(A, Hom_S(B, C))$$

In our case, we consider $B = C = S = k$, $R = \mathbb{Z}$ and $A = H_1(C, \mathbb{Z})$. Identifying commutative groups and $\mathbb{Z}$-modules, this isomorphism becomes

$$Hom_k(H_1(C, \mathbb{Z}) \underset{\mathbb{Z}}{\otimes} k, k) \simeq Hom(H_1(C, \mathbb{Z}), k^+).$$

Then $(*)$ becomes

$$Hom(\Pi_1(Q, I_Q), k^+) \simeq Hom_k(H_1(C, \mathbb{Z}) \underset{\mathbb{Z}}{\otimes} k, k) \qquad (*)$$

Moreover, the universal coefficients theorem in homology and in cohomology (see for example [10], p.176-179) are as follows. Let Λ be a principal ring and C be a flat chain complex over Λ, and let A be a Λ-module, then

$$\begin{cases} 0 \to H_n(C) \otimes_\Lambda A \to H_n(C \otimes_\Lambda A) \to Tor_1^\Lambda(H_{n-1}(C), A) \to 0 \\ 0 \to Ext_\Lambda^1(H_{n-1}(C), A) \to H^n(Hom_\Lambda(C, A)) \to Hom_\Lambda(H_n(C), A) \to 0 \end{cases}$$

are exact.

As $C_\bullet(\mathbb{Z})$ is free and therefore flat, the hypotheses are verified with $C = C_\bullet(\mathbb{Z})$, $\Lambda = \mathbb{Z}$, $A = k$ and $n = 1$ in the first theorem and $C = C_\bullet(k)$, $\Lambda = k$, $A = k$ et $n = 1$ in the second.

Moreover, $Ext^1_k(H_{n-1}(C), A) = 0$ because k is a field, and $Tor^\Lambda_1(H_0(C), k) = 0$ because $H_0(\mathbb{Z})$ is free (see for example [10], p.63, cor.2.4.7).

Then we have

$$\begin{cases} H_1(C, \mathbb{Z}) \otimes_{\mathbb{Z}} k \simeq H_1(C \otimes_{\mathbb{Z}} k, \mathbb{Z}) \\ H^1(C, k, k)) \simeq Hom_k(H_1(C, k), k) \end{cases}$$

Since the complexes $C_\bullet(\mathbb{Z}) \otimes_{\mathbb{Z}} k$ and $C_\bullet(k)$ are isomorphic, the equation $(*)$ becomes

$$Hom(\Pi_1(Q, I), k^+) \simeq Hom_k(H_1(C \underset{\mathbb{Z}}{\otimes} k, \mathbb{Z}), k) \simeq Hom_k(H_1(C, k), k) \simeq H^1(C, k, k)$$

We conclude using the Gerstenhaber-Schack theorem ([4] 1.4, or [8]) which shows that the cohomologies $H^i(C, k, k)$ and $HH^i(kQ/I)$ are isomorphic.

We introduce now the notion of completed quiver which is interesting in view of the following results, and also in order to adapt Van Kampen's theorem to an algebraic setting. A quiver is said to be a completed quiver if

1. it does not contain cycles.
2. each path of length at least two has a parallel arrow,
3. there are no couples of parallel arrows.

Theorem 3.2. Let Q be a quiver without cycles and without parallel arrows.

1. There exists an ordered quiver Q^o obtained by considering the set of paths of length at least 2 and deleting all arrows parallel to these paths.
2. There exists an completed quiver Q^c obtained from Q by considering the set of paths of length at least 2 and adding a parallel arrow for each of these paths, unless there already is one, either added or in Q.
3. The three fundamental groups of the quivers Q, Q^o and Q^c with their own parallel ideals which are not necessary admissible, are isomorphic :

$$\Pi_1(Q, I_Q) \simeq \Pi_1(Q^o, I_{Q^o}) \simeq \Pi_1(Q^c, I_{Q^c})$$

Proof.

First, we give more details on the construction of Q^c and Q^o. The vertices of Q^c and Q^o are the same as those of Q. Moreover, there exists an arrow from a to b, $a \neq b$, in Q^c if and only if there exists a path (possibly an arrow) from a to b in Q, and there exists an arrow in Q^o if and only if there exists an arrow in Q from a to b which is not parallel to another path in Q.

We prove that for any path in Q^c (resp. in Q) there exists a parallel path constructed with arrows that come from Q (resp. Q^o). Let $q = \alpha_1 \dots \alpha_n$ be a path of Q^c. For any α_i that does not come from Q, there exists by construction a path ω_i in Q^c, already present in Q, parallel to α_i. If α_i is an arrow which comes from Q, let us set $\omega_i = \alpha_i$. Then the path q is parallel to the path $q' = \omega_1 \dots \omega_n$ which is issued from Q.

The same process is applicable to the quivers Q and Q^o.

The fundamental groups $\Pi_1(Q, I_Q)$, $\Pi_1(Q^o, I_{Q^o})$ and $\Pi_1(Q^c, I_{Q^c})$ are isomorphic. Since any path in Q^c is parallel to a path that comes from Q, every walk of $\Pi_1(Q^c, I_{Q^c})$ is equivalent to a walk that contains only arrows issuing from Q ; then, the equivalent classes will not be changed by adding the parallel arrows of Q^c.

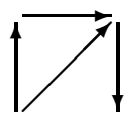
The proof of the isomorphism between $\Pi_1(Q^o, I_{Q^o})$ and $\Pi_1(Q, I_Q)$ is the same.

Q^o is ordered. The quiver Q^o does not contain cycles, and there is no arrow parallel to a path by construction.

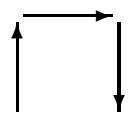
Q^c is completed. The quiver Q^o does not contain cycles either, and if there exists a path p from a to b and no arrow from a to b, then there is a path in Q, parallel to p. This is in contradiction with the construction of Q^o.

Example 3.3.

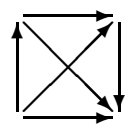
Q =



Q^o =



Q^c =



Theorem 3.4. (Van Kampen) Let Q be a connected ordered quiver and let Q_1, Q_2 be two connected subquivers such that $Q_1^c \cup Q_2^c = Q^c$ and such that Q_0, the ordered quiver associated to the completed quiver $Q_1^c \cap Q_2^c$, is also connected. The inclusions i_1 (resp. i_2) from Q_0^c to Q_1^c (resp. Q_2^c) induce i_{1*} (resp i_{2*}) from $\Pi_1(Q_0, I_{Q_0})$ to $\Pi_1(Q_1, I_{Q_1})$ (resp. $\Pi_1(Q_2, I_{Q_2})$).

Therefore $\Pi_1(Q, I_Q)$ is the free product of $\Pi_1(Q_1, I_{Q_1})$ and $\Pi_1(Q_2, I_{Q_2})$ by adding the relations $i_{1*}(\alpha) = i_{2*}(\alpha)$ for all α in $\Pi(Q_0, I_{Q_0})$

Proof. We use Van Kampen's theorem for a simplicial complex (see [11], p243, for example) :

K is a connected complex and K_0, K_1, K_2 are connected subcomplexes, such that $K_1 \cup K_2 = K$ and $K_1 \cap K_2 = K_0$; a_0 is a vertex of K_0 ; the injection maps $j_r : K_0 \to K_r$, $r = 1, 2$, induce

$$j_{r*} : \Pi(K_0, a_0) \mapsto \Pi(K_r, a_0)$$

Then $\Pi_1(K)$ is the free product of $\Pi_1(K_1)$ and $\Pi_1(K_2)$ by means of the mapping $j_{1*}(\alpha) \mapsto j_{2*}(\alpha)$ for all α in $\Pi(Q_0, I_{Q_0})$.

To Q,Q_0,Q_1,Q_2, there correspond simplicial complexes that we denote K,K_0,K_1,K_2. By hypothesis, these simplicial complexes are connected. Moreover, as there is a bijection between the paths of Q_i^c of length n and the n-dimensional simplexes of K_i, we have $K = K_1 \cup K_2$ and $K_0 = K_1 \cap K_2$.

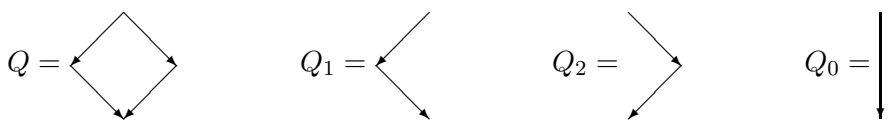
Let us denote by ϕ_* the isomorphism defined in theorem between $\Pi_1(Q, I_Q)$ and $\Pi_1(K)$. Then it is easy to see that the following diagrams are commutative (n=1,2) :

$$\begin{tikzcd}
\Pi_1(Q_0, I_{Q_0}) \arrow[r, "i_{k*}"] \arrow[d, "\phi_*"'] \arrow[dr, phantom, "\circlearrowright"] & \Pi_1(Q_n, I_{Q_n}) \arrow[d, "\phi_*"] \\
\Pi_1(K_0) \arrow[r, "j_{k*}"'] & \Pi_1(K_n)
\end{tikzcd}$$

Therefore $\Pi_1(Q, I_Q)$ is the free product of $\Pi_1(Q_1, I_{Q_1})$ and $\Pi_1(Q_2, I_{Q_2})$ by adding the relations $i_{1*}(\alpha) = i_{2*}(\alpha)$ for all α in $\Pi(Q_0, I_{Q_0})$

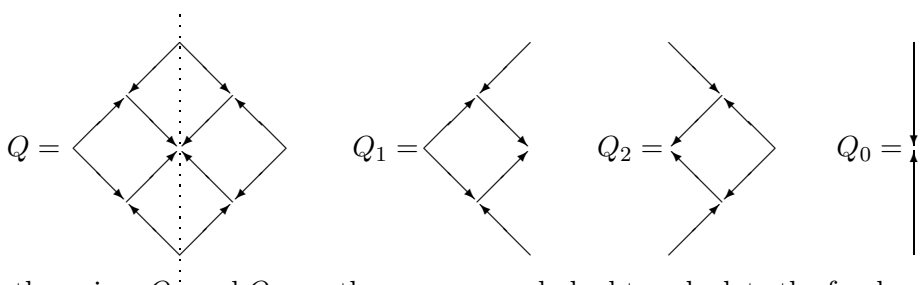
Corollary 3.5. If $\Pi_1(Q_0, I_{Q_0}) = 0$, then $\Pi_1(Q, I_Q)$ is the free product of $\Pi_1(Q_1, I_{Q_1})$ and $\Pi_1(Q_2, I_{Q_2})$.

Example 3.6. We begin with an easy example to show how the theorem works

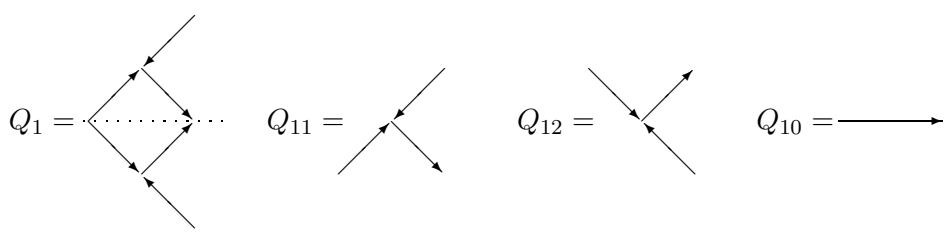


As the fundamental group of a tree is zero, we deduce from the corollary that $\Pi_1(Q, I_Q) = 0$.

Example 3.7. Now, let's consider the quiver Q drawn on the next figure. The dotted line indicates the limits of quivers Q_1 and Q_2. Then the quivers Q, Q_1, Q_2 and Q_0 are :

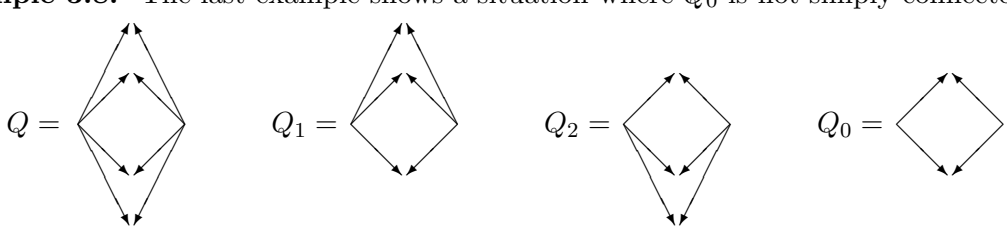


Since the quiver Q_1 and Q_2 are the same, we only had to calculate the fundamental group of Q_1. Let's use again the Van Kampen theorem, and decompose the quiver Q_1 in Q_{11} and Q_{12}. The intersection quiver will be denoted by Q_{10} :

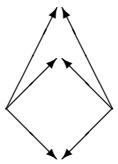
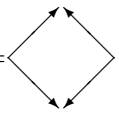
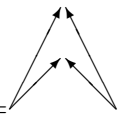


The fundamental groups of quivers Q_{10}, Q_{11} and Q_{12} is 0. Then it is the same for the quiver Q_1 and therefore for Q itself.

Example 3.8. The last example shows a situation where Q_0 is not simply connected :



Once again, the quivers Q_1 and Q_2 are the same and we use again the Van Kampen theorem to calculate it :

$Q_1 =$  $Q_{11} =$  $Q_{12} =$  $Q_{10} =$

Since the fundamental group Q_{10} is 0, Q_1 is the free product of Q_{11} and Q_{12} which are isomorphic to $\mathbb{Z}$. Then, if Q_{11} and Q_{12} are generated by a and b, Q_1 is the group generated by $\{a, b\}$. In the same way, Q_2 is generated by two elements c and d. If we add the relation $i_{1*}(Q_0) - i_{2*}(Q_0) = 0$, we obtain that $d = b$ and therefore the fundamental group of Q is the group generated by $\{a, b, c\}$.

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