Algebraic fundamental group and simplicial complexes

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Abstract

In this paper we prove that the fundamental group of a simplicial complex is isomorphic to the algebraic fundamental group of its incidence algebra, and we derive some applications.

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Let $k$ be a field and $A$ be a basic and split finite dimensional $k$-algebra, which means that $A/r = k \times k \times \ldots \times k$ where $r$ is the radical of $A$. There exists a unique quiver $Q$ and usually several admissible ideals $I$ of the algebra $kQ$ such that $A = kQ/I$ (see [6]). In the 1980s, an algebraic fundamental group has been defined which depends on the presentation of $A$, that is to say on the choice of the ideal $I$ (see [13]). For incidence algebras, that is algebras obtained from a simplicial complex, it has been proved that the presentation does not influence the fundamental group ([15]). Then it is a natural question to compare it with the fundamental group of the geometric realisation. Note also that in [4,8] the analogous question concerning homology is solved.

Actually, we prove that the fundamental groups considered for a finite and connected simplicial complex are isomorphic. The following diagram summarizes the situation :

\[
\begin{array}{ccc}
\text{Simplicial complex} & \xrightarrow{\text{Geometric realization}} & \text{Topological fundamental group} \\
\text{( )} & \text{Poset} & \xrightarrow{\text{Incidence algebra}} \text{Algebraic fundamental group} \\
\end{array}
\]

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The isomorphism above enables us to adapt results of algebraic topology to the purely algebraic setting: for instance the isomorphism recently proved between $\text{Hom}(\Pi_1(Q, I_Q), k^+)$ and $\text{HH}^1(kQ/I_Q)$ where $Q$ is an ordered quiver, $I_Q$ the associated parallel path and $k^+$ the additive group of a field ([8], see also [7]) is a consequence of the classic result in algebraic topology which states that the abelianisation of the $\Pi_1$ and the first homology group of a simplicial complex are isomorphic. In another direction, we also derive an algebraic Van Kampen theorem.

For our purpose we consider in the first part of this paper simplicial complexes and usual fundamental groups; as well as posets, incidence algebras and algebraic fundamental groups. We also explain the relations between posets and simplicial complexes. In the last section, we provide the isomorphism and give applications.

This work is part of my thesis in Montpellier and I thank C. Cibils for his help, his apposite advice and his patience. When finishing this paper, I learned that J.C. Bustamente had independently considered a similar context.

All simplicial complexes will be finite, connected and not empty.

1. Fundamental group and incidence algebras

We consider the classical definition of a simplicial complex (see for instance [11]). A simplicial complex is the union of elements $\{a_i\}_{i \in I}$ called the vertices, and finite sets of vertices called the simplexes, such that if $S$ is a simplex all subsets of $S$ are also simplexes. Each set containing only one element is a simplex. A simplicial complex is said to be finite if $I$ is finite, and is connected if for each couple of vertices $(s, t)$, there exist vertices $s_0, \ldots, s_n$ such that $s_0 = s$, $s_n = t$ and $\{s_{i-1}, s_i\}$ is a simplex for each $i$ in $\{1, \ldots, n\}$.

The geometric realization of a simplicial complex $C$ with vertices $\{a_i\}_{i \in I}$ is as follows. Let $\{A_i\}_{i \in I}$ be points of $\mathbb{R}^n$, such that if $\{a_{i_1}, \ldots, a_{i_p}\}$ is a simplex of $C$, then the points $A_{i_1}, \ldots, A_{i_p}$ are linearly independent. The set of points whose barycentric coordinates are strictly positive is called a face. Note that we prefer to define a face as the points with strictly positive barycentric coordinates instead of positive barycentric coordinates: this way, the geometric realization becomes the disjoint union of its faces and there exists only one face containing it. Moreover, if $S$ and $S'$ are two simplexes with empty intersection, the intersection of the corresponding faces is empty. A geometric realization of $C$ denoted by $|C|$, is the union of the faces associated to simplexes of a simplicial complex. A closed face is the closure of a face.

Note that since we assume that the simplicial complexes are finite, their geometric realization exist. Indeed let $C$ be a complex with set of vertices $a_1, \ldots, a_n$. The geometric realization $|C|$ of a simplicial complex $C$ is a subset of $\mathbb{R}^n$ and inherits the topology of $\mathbb{R}^n$. Behind the usual definition of the fundamental group $\Pi_1(|C|)$ obtained through homotopy classes of closed paths, we recall the construction of the edge-paths group of $|C|$. This provides another description of the fundamental group $\Pi_1(|C|)$ which is useful for our purpose (see [11] for example).

An edge-path of $C$ is a finite sequence of vertices $a_{i_r} \ldots a_{i_1}$ such that for all $j$ in $\{1, \ldots, r-1\}$, the set $\{a_{i_j}, a_{i_{j+1}}\}$ is a simplex. If $w = a_{i_r} \ldots a_{i_1}$ is an edge-path, let $w^{-1}$ denote the edge-path $a_{i_1} \ldots a_{i_r}$. An edge-path is said to be closed (or an edge-loop) if the first and the last
vertices are the same. Let \( w = a_i \ldots a_{i_1} \) and \( w' = a'_{i'} \ldots a'_{i_{1'}} \) be two edge-paths; if \( a_{i_{1'}} = a_{i_1} \), the product \( w \cdot w' \) is defined and is equal to \( a_i \ldots a_{i_1} a'_{i'} \ldots a'_{i_{1'}} \).

This is an allowable operation on edge-paths: if three consecutive vertices of the edge-path are in the same simplex, the middle vertex can be removed. Conversely, we can add a vertex between two others, if these three vertices are in the same simplex of \( C \). Moreover, it is possible to change \( a_{i_0} a_{i_0} \) by \( a_{i_0} \) and conversely. This generates an equivalence relation on the set of edge-paths. Let \( w \) denote the equivalence class of the edge-path \( w \).

**Proposition 1.1.** ([11] 6.3.1 and 6.3.2.) Let \( C \) be a simplicial complex and \( x_0 \) a vertex of \( C \). The set of equivalence classes of edge-loops starting at one point \( x_0 \) is a group for the product defined before. Since \( C \) is connected, this group does not depend on \( x_0 \) and is denoted by \( \Pi_1(C) \). The fundamental groups \( \Pi_1(C) \) and \( \Pi_1(|C|) \) are isomorphic.

Hereafter, \( \Pi_1(C) \) will be either the approximation of the fundamental group \( \Pi_1(C) \) or the fundamental group \( \Pi_1(|C|) \) itself.

Given a poset (i.e. a partially ordered set), there is an associated ordered quiver, that is to say a finite oriented graph without loops and such that if there exists an arrow from \( a \) to \( b \), there does not exist another path from \( a \) to \( b \). To each element of the poset corresponds a vertex of the graph. Moreover, let \( S_1 \) and \( S_2 \) be vertices in the graph; there exists an arrow from \( S_1 \) to \( S_2 \) if and only if the element associated to \( S_1 \) in the poset is smaller than the element associated to \( S_2 \) and if there does not exist an element of the poset strictly between these two elements. The graph obtained is an ordered quiver.

For example, the graph that corresponds to the poset \( a, b, c, a', b', c', d \) with \( a < b' < d \), \( c < b' < d \), \( a < c' < d \), \( b < c' < d \), \( b < a' < d \) and \( c < a' < d \) is:

```
    c
   /|\
  b' / | \ a'
   /  |  \
  a    d
   /  |  \
  b   c
```

Conversely, by this operation all ordered graphs arise from a poset. There is a bijection between the set of ordered graphs and the set of posets. Moreover, a poset is said to be connected if its ordered quiver is connected. All the posets considered will be connected, finite and non empty.

Let now \( Q \) be a quiver, and \( k \) be a field. We denote \( kQ \) the \( k \)-vector space with basis the paths of \( Q \) (the paths of length 0 being the vertices), with the multiplication given by the composition of two paths if possible and 0 otherwise. Two paths of \( Q \) are parallel if they have the same beginning and the same end. The \( k \)-space generated by the set of differences of two parallel paths is a two-sided ideal of \( kQ \), denoted \( I_Q \) and called parallel ideal. The quotient algebra \( kQ/I_Q \) is the incidence algebra of \( Q \).

The general definition of the fundamental group depends on a couple \( (Q, I) \) where \( I \) is an admissible ideal of \( kQ \), it can be found in [8, 15, 7] and we recall it below. We notice that for
an algebra the presentation as a quiver with relations is not unique in general, that is to say that different ideals $I$ may exist such that $A \cong kQ/I$. We do not have in general a unique fundamental group associated to an algebra. Nevertheless, in the case of an incidence algebra it has been proved that the fundamental group does not depend on the presentation of the algebra (see [15] for example).

A relation $\sum_{i=1}^{n} \lambda_i \omega_i$ is minimal if the sum is in $I$ and if for all non empty proper subset $J$ of $\{1, \ldots, n\}$ the sum $\sum_{i \in J} \lambda_i \omega_i$ is not in $I$. We note that if the relation is minimal then $\{\omega_1, \ldots, \omega_n\}$ have the same source and the same terminus.

If $\alpha$ is an arrow from $x$ to $y$, let $\alpha^{-1}$ denote its formal inverse which goes from $y$ to $x$. A walk from $x$ to $y$ is a formal product $\alpha_1^{\pm 1} \alpha_2^{\pm 1} \ldots \alpha_n^{\pm 1}$ which begins in $x$ and ends in $y$. The trivial walk $x$, which begins in $x$ and no longer moves is denoted $e_x$. A closed walk (or a loop) is a walk having the same extremities. A walk is, in fact, a path in the non oriented graph associated to $Q$; in other words, a walk can follow the arrows in any direction.

We consider the smaller equivalence relation $\sim$ on the walks of $Q$ containing the following items :

1. if $\alpha$ is an arrow from $x$ to $y$ then $\alpha \alpha^{-1} \sim e_y$ and $\alpha^{-1} \alpha \sim e_x$,
2. if $\sum_{i=1}^{n} \lambda_i \omega_i$ is a minimal relation then $\omega_1 \sim \ldots \sim \omega_n$,
3. if $\alpha \sim \beta$ then, for all $(\omega, \omega')$, we have $\omega \alpha \omega' \sim \omega \beta \omega'$

Let $x_0$ be a vertex of $Q$. The set of equivalence classes of loops starting at $x_0$ does not depend on $x_0$ since $Q$ is connected. We denote this set by $\Pi_1(Q, I)$. If the quiver $Q$ comes from a poset $P$, the fundamental group associated to $(Q, I_Q)$ is denoted $\Pi_1(P)$.

We remark that if $I$ is the parallel ideal, the second item means parallel paths are equivalent. For example if $Q$ is defined by the following quiver, the fundamental group $\Pi_1(Q, I_Q)$ is isomorphic to $\mathbb{Z}$.

\begin{center}
\begin{tikzpicture}
\node (A) at (0,0) {$x$};
\node (B) at (1,1) {$y$};
\node (C) at (2,0) {$z$};
\node (D) at (2,-1) {$w$};
\draw[->] (A) -- (B);
\draw[->] (B) -- (C);
\draw[->] (C) -- (D);
\end{tikzpicture}
\end{center}

We note that the hypothesis $I$ admissible is not fully used neither to define the fundamental group nor to prove that it is a group. So, in this paper, we will consider the fundamental group of a couple $(Q, I)$ where $I$ satisfies $F^n \subset I \subset F$ for an integer $n$. This will be used to adapt Van Kampen’s theorem to purely algebraic fundamental groups.

Let $C$ be a simplicial complex. The set of non empty simplexes of $C$ ordered by inclusion is a poset which we will denote $\text{Pos}(C)$. This $\text{Pos}$ defines an application from simplicial complexes to posets which is injective, but not surjective, since there is no simplicial complex which gives the poset $a < b$. Due to the construction of the quiver from a simplicial complex, an arrows can only go from a vertex corresponding to a $p$-face to a vertex corresponding to a $q$-face with $p > q$, then the path algebra of this quiver is then of finite dimension.

We provide now the construction of a simplicial complex from a poset. These procedures are of course not inverse one of each other, their composition is the barycentric decomposition.

To each poset $P$ we associate a simplicial complex $\text{Sim}(P)$, where a n-simplex is a subset of $P$ containing $n+1$ elements and totally ordered. The application $\text{Sim}$ is surjective but not injective, for instance the simplicial complexes which are associated to $a < b < c$, $a < b < d$ and to $a < c < b$, $a < d < b$ are the same.
Let $C$ be a simplicial complex, $|\text{Sim}(\text{Pos}(C))|$ is the geometric realization of the barycentric decomposition of $C$. Then for example let $C$ be the set of the non empty parts of $\{a, b, c\}$; its geometric realization being the triangle drawn on the next figure. Then, $\text{Pos}(C)$ contains all the elements of $C$ that is to say $T = \{a, b, c\}$ $A_1 = \{b, c\}$, $A_2 = \{a, c\}$, $A_3 = \{a, b\}$, $S_1 = \{a\}$, $S_2 = \{b\}$ and $S_3 = \{c\}$ and its order is defined by $S_i \leq A_j \leq T$ for all $i, j \in \{1, 2, 3\}$ and $i \neq j$. The associated quiver is drawn on the next figure. Then $\text{Sim}(\text{Pos}(C))$ is the complex containing the total ordered subsets of $P$ that is to say : for all $i, j \in \{1, 2, 3\}$, the 0-simplexes are $\{S_i\}$, $\{A_i\}$, $\{T\}$, the 1-simplexes are $\{S_i, A_j\}$ $i \neq j$, $\{S_i, T\}$, $\{A_i, T\}$, the 2-simplexes are $\{S_i, A_j, T\}$ $i \neq j$. By identifying poset and associated ordered quiver, simplicial complexes and their geometric realization, the situation can be summarized by the following diagram:

![Diagram](image_url)

2. Equivalence between algebraic and topological approaches.

The aim of this section is to prove that the fundamental group $\Pi_1(|C|)$ defined on the geometric realization of a finite simplicial complex $C$ is isomorphic to the fundamental group $\Pi_1(\text{Pos}(C))$ of the incidence algebra of the poset deduced from the complex.

We prove first that for any poset $P$ we have $\Pi_1(P) \simeq \Pi_1(\text{Sim}(P))$. We will use the approximation of the topological fundamental group considered before in order to provide an isomorphism between $\Pi_1(\text{Sim}(P))$ and $\Pi_1(P)$.

**Theorem 2.1.** Let $P$ be a poset, the groups $\Pi_1(P)$ and $\Pi_1(\text{Sim}(P))$ are isomorphic.

**Proof.** Due to proposition 1, it is sufficient to prove that $\Pi_1(P)$ is isomorphic to the edge-paths group of $|\text{Sim}(C)|$

For each vertex $s$ of the poset $P$, the set $\{s\}$ is totally ordered and it corresponds to a 0-simplex of the simplicial complex $\text{Sim}(P)$. We denote by $s'$ this 0-simplex of $\text{Sim}(P)$.

Let $\phi$ be the map from the set of the associated quiver walks to the set of edge-paths of $\text{Sim}(P)$ be defined by

$$\phi(\alpha_n^{\varepsilon_n} \ldots \alpha_1^{\varepsilon_1}) = s'_{n+1} \ldots s'_1$$

where $s_i$ and $s_{i+1}$ are the poset elements which are the origin and the end of the walk $\alpha_i^{\varepsilon_i}$.

The map $\phi$ is well defined. Indeed, $s'_{n+1} \ldots s'_1$ is an edge-path because for all $i$ in $\{1, \ldots, n\}$, the set $\{s_i, s_{i+1}\}$ is totally ordered since $\alpha_i$ is an arrow of the extremities $s_i, s_{i+1}$ and therefore $\{s'_i, s'_{i+1}\}$ is a simplex of $\text{Sim}(P)$.

We assert now that the images by $\phi$ of equivalent walks are equivalent. We have to prove this fact on the generators of the equivalence relation. Let $f$ be an arrow from $s_1$ to $s_2$, then $\phi(f, f^{-1}) = s'_1, s'_2, s'_1$ which is equivalent to $s'_1 = \phi(e_v)$. Let $c_1$ and $c_2$ be two parallel paths crossing respectively the vertex $s_1, t_1, t_2, \ldots, t_n, s_2$ and $s_1, u_1, u_2, \ldots, u_n, s_2$. Then the
sets \( \{s_1, t_1, t_2, \ldots, t_n, s_2\} \) and \( \{s_1, u_1, u_2, \ldots, u_n, s_2\} \) are totally ordered and the edge paths \( \phi(c_1) = s_1', t_1', t_2', \ldots, t'_n, s_2' \) and \( \phi(c_2) = s_1', t_1', t_2', \ldots, t'_n, s_2' \) are both equivalent to \( s_1', s_2' \).

The third relation is immediate because the equivalence relation on the edge paths set is compatible with the product of the group.

Let \( \phi_* : \Pi_1(P) \to \Pi_1(Sim(P)) \) denote the application induced by \( \phi \). Note that the image of a loop is also a loop.

First of all, \( \phi_* \) is a morphism. Indeed:

\[
\begin{align*}
\phi(p) = x_0v_n \ldots v_1x_0 & \quad \text{then} \quad \phi(p).\phi(q) = x_0v_n \ldots v_1x_0w_n \ldots w_1x_0 \\
\phi(q) = x_0w_n \ldots w_1x_0 & \quad \text{then} \quad \phi(p.q) = x_0v_n \ldots v_1x_0w_n \ldots w_1x_0
\end{align*}
\]

These two edge-paths are equivalent so \( \phi_*(p) \phi_*(q) = \phi_*(p.q) \).

To prove that \( \phi_* \) is bijective, we are going to construct its inverse \( \psi_* \).

Let \( s_i' \ldots s_1' \) be an edge-path. We fix \( i \in \{1, \ldots, n\} \). The set \( \{s_i', s_{i+1}'\} \) being a simplex, the set \( \{s_i, s_{i+1}\} \) is totally ordered. Then, there exists a maximal (for the inclusion) totally ordered set containing it and having \( s_i \) and \( s_{i+1} \) as extremities. The choice of this is not important because all paths corresponding to these sets have the same origin and the same end and therefore are parallel. This maximal set corresponds to a path or to an inverse path \( w_i^s \) of the associated quiver with origin \( s_i \) and end \( s_{i+1} \). So we can define a morphism \( \psi \) from edge paths group to \( \Pi_1(C) \) by \( \psi(s_{i+1}' \ldots s_i') = w_i^s \ldots w_1^s \) if \( n \geq 1 \) and \( \psi(s_1') = s_1 \).

We will prove now that this application is constant on the equivalence class. In deed \( \psi(s's') \) is a path from \( s \) to \( s \), so \( \psi(s's') = s = \psi(s') \). Moreover, let take \( s', t', u' \) such that \( \{s', t', u'\} \) is a simplex, so \( \psi(s't'u') = \psi(s't').\psi(t'u') \) and \( \psi(s'u') \) are paths from \( u \) to \( s \). Therefore they are parallel.

Finally, we will verify that \( \phi_* \circ \psi_* = \psi_* \circ \phi_* = Id \). Let \( f \) be an arrow of the ordered quiver associated to \( P \) from \( s_1 \) to \( s_2 \) then \( \phi_*(f) = s_1's_2' \) and \( \psi_*(f) \) is a path from \( s_1 \) to \( s_2 \). Since the quiver is ordered, it does not exist any other path than \( f \), so \( \psi_*(f) = \psi_* \circ f = \psi_* \circ \phi_* = Id \).

We first show that in a particular case of incidence algebra the result obtained by I. Assem and J.A. De La Peña ([1], p.200) is a consequence of a classic fact in algebraic topology; so this proof has the advantage to link this result to algebraic topology.

3. Applications
Theorem 3.1. Let $P$ be a poset and $A = kQ/I$ its incidence algebra over a field $k$. Then

$$\text{Hom}(\Pi_1(P), k^+) \simeq HH^1(A)$$

where $k^+$ is the additive group of the field $k$ and $HH^1(A)$ is the first Hochschild cohomology group of $A$.

Proof. Let $C$ be a simplicial complex and $C_\bullet(A)$ be the chain complex over a ring $\Lambda$ associated to the simplicial homology of $C$; the latter homology will be denoted by $H_\bullet(C, \Lambda)$. Moreover, the cohomology of the complex $\text{Hom}_\Lambda(C_\bullet(A), A)$, where $A$ is a $\Lambda$-module, is denoted by $H^\bullet(C, \Lambda, A)$.

For any simplicial complex, the abelianisation $\Pi_1^{ab}$ of $\Pi_1$ is isomorphic to the first homology group (see for example [11], 6.4.7). If we denote by $C$ the simplicial complex associated to $Q$, we have

$$\text{Hom}(\Pi_1(C))^{ab} \simeq H_1(C, \mathbb{Z})$$

and

$$\text{Hom}(\text{Hom}(\Pi_1(C))^{ab}, k^+) \simeq \text{Hom}(H_1(C, \mathbb{Z}), k^+).$$

Note that since $k^+$ is abelian we have:

$$\text{Hom}(\Pi_1(C), k^+) \simeq \text{Hom}(H_1(C, \mathbb{Z}), k^+).$$

Making use of theorem we have:

$$\text{Hom}(\Pi_1(Q, I_Q), k^+) \simeq \text{Hom}(H_1(C, \mathbb{Z}), k^+) \quad (\ast)$$

Adjunction, for two rings $R$ and $S$ and bimodules $A_R$, $rBS$, $C_S$, provides an isomorphism, (see [17], p.37, for example):

$$\text{Hom}_S(A \otimes_B C) \simeq \text{Hom}_R(A, \text{Hom}_S(B, C))$$

In our case, we consider $B = C = S = k$, $R = \mathbb{Z}$ and $A = H_1(C, \mathbb{Z})$. Identifying commutative groups and $\mathbb{Z}$-modules, this isomorphism becomes

$$\text{Hom}_k(H_1(C, \mathbb{Z}) \otimes \mathbb{Z} k, k) \simeq \text{Hom}(H_1(C, \mathbb{Z}), k^+).$$

Then $(\ast)$ becomes

$$\text{Hom}(\Pi_1(Q, I_Q), k^+) \simeq \text{Hom}_k(H_1(C, \mathbb{Z}) \otimes \mathbb{Z} k, k) \quad (\ast)$$

Moreover, the universal coefficients theorem in homology and in cohomology (see for example [10], p.176-179) are as follows. Let $\Lambda$ be a principal ring and $C$ be a flat chain complex over $\Lambda$, and let $A$ be a $\Lambda$-module, then

$$\begin{cases} 0 \to H_n(C) \otimes \Lambda A \to H_n(C \otimes \Lambda A) \to \text{Tor}^1(\Lambda, H_{n-1}(C), A) \to 0 \\ 0 \to \text{Ext}^1(\Lambda, H_{n-1}(C), A) \to H^n(\text{Hom}_\Lambda(C, A)) \to \text{Hom}_\Lambda(H_n(C), A) \to 0 \end{cases}$$

are exact.

As $C_\bullet(\mathbb{Z})$ is free and therefore flat, the hypotheses are verified with $C = C_\bullet(\mathbb{Z})$, $\Lambda = \mathbb{Z}$, $A = k$ and $n = 1$ in the first theorem and $C = C_\bullet(k)$, $\Lambda = k$, $A = k$ et $n = 1$ in the second.
Moreover, $\text{Ext}^1_k(H_{n-1}(C), A) = 0$ because $k$ is a field, and $\text{Tor}^1(H_0(C), k) = 0$ because $H_0(Z)$ is free (see for example [10], p.63, cor.2.4.7).

Then we have
\[
\begin{cases}
    H_1(C, Z) \otimes Z \simeq H_1(C \otimes Z, Z) \\
    H^1(C, k, k) \simeq \text{Hom}_k(H_1(C, k), k)
\end{cases}
\]

Since the complexes $C_\bullet(Z) \otimes Z$ and $C_\bullet(k)$ are isomorphic, the equation (*) becomes
\[
\text{Hom}(\Pi_1(Q, I), k^+) \simeq \text{Hom}_k(H_1(C \otimes Z, Z), k) \simeq \text{Hom}_k(H_1(C, k), k) \simeq H^1(C, k, k)
\]

We conclude using the Gerstenhaber-Schack theorem ([4] 1.4, or [8]) which shows that the cohomologies $H^i(C, k, k)$ and $HH^i(kQ/I)$ are isomorphic.

We introduce now the notion of completed quiver which is interesting in view of the following results, and also in order to adapt Van Kampen’s theorem to an algebraic setting. A quiver is said to be a completed quiver if

1. it does not contain cycles.
2. each path of length at least two has a parallel arrow,
3. there are no couples of parallel arrows.

**Theorem 3.2.** Let $Q$ be a quiver without cycles and without parallel arrows.

1. There exists an ordered quiver $Q^o$ obtained by considering the set of paths of length at least 2 and deleting all arrows parallel to these paths.
2. There exists an completed quiver $Q^c$ obtained from $Q$ by considering the set of paths of length at least 2 and adding a parallel arrow for each of these paths, unless there already is one, either added or in $Q$.
3. The three fundamental groups of the quivers $Q$, $Q^o$ and $Q^c$ with their own parallel ideals which are not necessary admissible, are isomorphic:
\[
\Pi_1(Q, I_Q) \simeq \Pi_1(Q^o, I_{Q^o}) \simeq \Pi_1(Q^c, I_{Q^c})
\]

**Proof.**

First, we give more details on the construction of $Q^c$ and $Q^o$. The vertices of $Q^c$ and $Q^o$ are the same as those of $Q$. Moreover, there exists an arrow from $a$ to $b$, $a \neq b$, in $Q^c$ if and only if there exists a path (possibly an arrow) from $a$ to $b$ in $Q$, and there exists an arrow in $Q^o$ if and only if there exists an arrow in $Q$ from $a$ to $b$ which is not parallel to another path in $Q$.

We prove that for any path in $Q^c$ (resp. in $Q$) there exists a parallel path constructed with arrows that come from $Q$ (resp. $Q^o$). Let $q = \alpha_1 \ldots \alpha_n$ be a path of $Q^c$. For any $\alpha_i$ that does not come from $Q$, there exists by construction a path $\omega_i$ in $Q^c$, already present in $Q$, parallel to $\alpha_i$. If $\alpha_i$ is an arrow which comes from $Q$, let us set $\omega_i = \alpha_i$. Then the path $q$ is parallel to the path $q' = \omega_1 \ldots \omega_n$ which is issued from $Q$.

The same process is applicable to the quivers $Q$ and $Q^o$. 
The fundamental groups $\Pi_1(Q, I_Q)$, $\Pi_1(Q^o, I_{Q^o})$ and $\Pi_1(Q^c, I_{Q^c})$ are isomorphic. Since any path in $Q^c$ is parallel to a path that comes from $Q$, every walk of $\Pi_1(Q^c, I_{Q^c})$ is equivalent to a walk that contains only arrows issuing from $Q$; then, the equivalent classes will not be changed by adding the parallel arrows of $Q^c$.

The proof of the isomorphism between $\Pi_1(Q^o, I_{Q^o})$ and $\Pi_1(Q, I_Q)$ is the same.

$Q^o$ is ordered. The quiver $Q^o$ does not contain cycles, and there is no arrow parallel to a path by construction.

$Q^c$ is completed. The quiver $Q^o$ does not contain cycles either, and if there exists a path $p$ from $a$ to $b$ and no arrow from $a$ to $b$, then there is a path in $Q$, parallel to $p$. This is in contradiction with the construction of $Q^o$.

**Example 3.3.**

\[
Q = \begin{array}{c}
\downarrow \\
\downarrow \\
\end{array} \\
Q^o = \begin{array}{c}
\downarrow \\
\downarrow \\
\end{array} \\
Q^c = \begin{array}{c}
\downarrow \\
\downarrow \\
\end{array}
\]

**Theorem 3.4. (Van Kampen)** Let $Q$ be a connected ordered quiver and let $Q_1$, $Q_2$ be two connected subquivers such that $Q_1^c \cup Q_2^c = Q^c$ and such that $Q_0$, the ordered quiver associated to the completed quiver $Q_1^c \cap Q_2^c$, is also connected. The inclusions $i_1$ (resp. $i_2$) from $Q_0^c$ to $Q_1^c$ (resp. $Q_2^c$) induce $i_{1*}$ (resp $i_{2*}$) from $\Pi_1(Q_0, I_{Q_0})$ to $\Pi_1(Q_1, I_{Q_1})$ (resp. $\Pi_1(Q_2, I_{Q_2})$).

Therefore $\Pi_1(Q, I_Q)$ is the free product of $\Pi_1(Q_1, I_{Q_1})$ and $\Pi_1(Q_2, I_{Q_2})$ by adding the relations $i_{1*}(\alpha) = i_{2*}(\alpha)$ for all $\alpha$ in $\Pi(Q_0, I_{Q_0})$.

**Proof.** We use Van Kampen’s theorem for a simplicial complex (see [11], p243, for example) :

$K$ is a connected complex and $K_0$, $K_1$, $K_2$ are connected subcomplexes, such that $K_1 \cup K_2 = K$ and $K_1 \cap K_2 = K_0$; $a_0$ is a vertex of $K_0$; the injection maps $j_r : K_0 \to K_r$, $r = 1, 2$, induce

\[
j_{r*} : \Pi(K_0, a_0) \to \Pi(K_r, a_0)
\]

Then $\Pi_1(K)$ is the free product of $\Pi_1(K_1)$ and $\Pi_1(K_2)$ by means of the mapping $j_{1*}(\alpha) \mapsto j_{2*}(\alpha)$ for all $\alpha$ in $\Pi(Q_0, I_{Q_0})$.

To $Q, Q_0, Q_1, Q_2$, there correspond simplicial complexes that we denote $K, K_0, K_1, K_2$. By hypothesis, these simplicial complexes are connected. Moreover, as there is a bijection between the paths of $Q_i^c$ of length $n$ and the $n$-dimensional simplexes of $K_i$, we have $K = K_1 \cup K_2$ and $K_0 = K_1 \cap K_2$.

Let us denote by $\phi_*$ the isomorphism defined in theorem between $\Pi_1(Q, I_Q)$ and $\Pi_1(K)$. Then it is easy to see that the following diagrams are commutative (n=1,2):

\[
\begin{array}{ccc}
\Pi_1(Q_0, I_{Q_0}) & \xrightarrow{i_{k*}} & \Pi_1(Q_n, I_{Q_n}) \\
\phi_* & & \phi_* \\
\Pi_1(K_0) & \xrightarrow{j_{k*}} & \Pi_1(K_n)
\end{array}
\]
Therefore $\Pi_1(Q, I_Q)$ is the free product of $\Pi_1(Q_1, I_{Q_1})$ and $\Pi_1(Q_2, I_{Q_2})$ by adding the relations $i_{1*}(\alpha) = i_{2*}(\alpha)$ for all $\alpha$ in $\Pi(Q_0, I_{Q_0})$

**Corollary 3.5.** If $\Pi_1(Q_0, I_{Q_0}) = 0$, then $\Pi_1(Q, I_Q)$ is the free product of $\Pi_1(Q_1, I_{Q_1})$ and $\Pi_1(Q_2, I_{Q_2})$.

**Example 3.6.** We begin with an easy example to show how the theorem works

\[
\begin{align*}
Q &= \begin{array}{c}
\begin{array}{c}
\bullet \\
\end{array}
\end{array} \quad Q_1 &= \begin{array}{c}
\begin{array}{c}
\bullet \\
\end{array}
\end{array} \quad Q_2 &= \begin{array}{c}
\begin{array}{c}
\bullet \\
\end{array}
\end{array} \quad Q_0 &= \begin{array}{c}
\begin{array}{c}
\bullet \\
\end{array}
\end{array}
\end{align*}
\]

As the fundamental group of a tree is zero, we deduce from the corollary that $\Pi_1(Q, I_Q) = 0$.

**Example 3.7.** Now, let’s consider the quiver $Q$ drawn on the next figure. The dotted line indicates the limits of quivers $Q_1$ and $Q_2$. Then the quivers $Q$, $Q_1$, $Q_2$ and $Q_0$ are :

\[
\begin{align*}
Q &= \begin{array}{c}
\begin{array}{c}
\bullet \\
\end{array}
\end{array} \quad Q_1 &= \begin{array}{c}
\begin{array}{c}
\bullet \\
\end{array}
\end{array} \quad Q_2 &= \begin{array}{c}
\begin{array}{c}
\bullet \\
\end{array}
\end{array} \quad Q_0 &= \begin{array}{c}
\begin{array}{c}
\bullet \\
\end{array}
\end{array}
\end{align*}
\]

Since the quiver $Q_1$ and $Q_2$ are the same, we only had to calculate the fundamental group of $Q_1$. Let’s use again the Van Kampen theorem, and decompose the quiver $Q_1$ in $Q_{11}$ and $Q_{12}$. The intersection quiver will be denoted by $Q_{10}$ :

\[
\begin{align*}
Q_1 &= \begin{array}{c}
\begin{array}{c}
\bullet \\
\end{array}
\end{array} \quad Q_{11} &= \begin{array}{c}
\begin{array}{c}
\bullet \\
\end{array}
\end{array} \quad Q_{12} &= \begin{array}{c}
\begin{array}{c}
\bullet \\
\end{array}
\end{array} \quad Q_{10} &= \begin{array}{c}
\begin{array}{c}
\bullet \\
\end{array}
\end{array}
\end{align*}
\]

The fundamental groups of quivers $Q_{10}$, $Q_{11}$ and $Q_{12}$ is 0. Then it is the same for the quiver $Q_1$ and therefore for $Q$ itself.

**Example 3.8.** The last example shows a situation where $Q_0$ is not simply connected :

\[
\begin{align*}
Q &= \begin{array}{c}
\begin{array}{c}
\bullet \\
\end{array}
\end{array} \quad Q_1 &= \begin{array}{c}
\begin{array}{c}
\bullet \\
\end{array}
\end{array} \quad Q_2 &= \begin{array}{c}
\begin{array}{c}
\bullet \\
\end{array}
\end{array} \quad Q_0 &= \begin{array}{c}
\begin{array}{c}
\bullet \\
\end{array}
\end{array}
\end{align*}
\]
Once again, the quivers $Q_1$ and $Q_2$ are the same and we use again the Van Kampen theorem to calculate it:

$$Q_1 = \begin{tikzpicture}
\draw[->, thick] (0,0) -- (1,0);
\draw[->, thick] (1,0) -- (1,1);
\draw[->, thick] (1,1) -- (0,0);
\end{tikzpicture}$$

$$Q_{11} = \begin{tikzpicture}
\draw[->, thick] (0,0) -- (1,0);
\draw[->, thick] (1,0) -- (1,1);
\draw[->, thick] (1,1) -- (0,0);
\end{tikzpicture}$$

$$Q_{12} = \begin{tikzpicture}
\draw[->, thick] (0,0) -- (1,0);
\draw[->, thick] (1,0) -- (1,1);
\end{tikzpicture}$$

$$Q_{10} = \begin{tikzpicture}
\draw[->, thick] (0,0) -- (1,0);
\end{tikzpicture}$$

Since the fundamental group $Q_{10}$ is 0, $Q_1$ is the free product of $Q_{11}$ and $Q_{12}$ which are isomorphic to $\mathbb{Z}$. Then, if $Q_{11}$ and $Q_{12}$ are generated by $a$ and $b$, $Q_1$ is the group generated by \{a, b\}. In the same way, $Q_2$ is generated by two elements $c$ and $d$. If we add the relation $i_{1*}(Q_0) - i_{2*}(Q_0) = 0$, we obtain that $d = b$ and therefore the fundamental group of $Q$ is the group generated by \{a, b, c\}.

References