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# Higher loop renormalization of a supersymmetric field theory II

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## Abstract

We obtain the contributions to the renormalization group functions of all the diagrams containing the unique one-loop primitive divergence of a simple supersymmetric Wess–Zumino model, up to more than 200 loops. The asymptotic behavior of the coefficients in the expansion of the anomalous dimension is described.

## 1 Introduction

Perturbative Quantum Field Theory is known for its tremendous successes, with its ability to obtain highly precise values in Quantum Electrodynamics, or to test the Standard Model with radiative corrections to weak interactions. However, actual calculations become rapidly cumbersome and display a conjunction of analytical and combinatorial difficulties. The fast growth of the number of relevant terms gets compounded by the need to subtract a growing number of subdivergences.

In the quest of organizing principles for taming the combinatorial problem, a major progress has been the recognition of a Hopf algebra structure in the renormalization of Quantum Field Theories [1, 2, 3, 4]. The cohomology of the introduced Hopf algebras has been related to Schwinger–Dyson equations (see [7] and the references therein). An early application to the summation of a category of diagrams in a simple Yukawa field theory or a  $\phi^3$  theory in 6 dimensions has been obtained by Broadhurst and Kreimer in [5]. It was however later shown that in this simple case, the Schwinger–Dyson equation, which is linear, could be solved exactly, without any reference to the Hopf algebra of diagrams, since it gives rise to a non-linear differential equation for the anomalous dimension [6].

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In a preceding work [8], we adapted this work to a supersymmetric model with a simple Wess–Zumino type interaction. As it is well-known, in view of its ability to limit renormalization phenomena, supersymmetry has allowed to resolve a number of theoretical problems. Although its relevance as a property of the physical world is up to now as elusive as ever, it is still a leading candidate for physics beyond the standard model. Of course, if supersymmetry plays a role in particle physics, it has to be broken through some (preferably spontaneous or dynamical) mechanism. In this respect, theories with metastable supersymmetry breaking have been recently very actively investigated, particularly some generalized Wess–Zumino models [9, 10].

The method employed in [6], if really powerful, does not generalize to more complex situations. A strategy has been proposed in [11] to deal with non-linear Schwinger–Dyson, in which the formulation of the renormalization group in terms of the Hopf algebra of diagrams becomes essential. We here apply this proposal in a supersymmetric setting. With respect to the cases envisaged in [11], the supersymmetric case is simpler because we need only consider a unique anomalous dimension, which is common to all the member of the supermultiplet. We further simplify the procedure by a direct computation of the renormalized propagator, so that we can obtain explicit results up to high orders.

We will first review the bases of this procedure, which are the following two results: (i) the renormalization group is a one-parameter subgroup of the group of evaluation functions on the Hopf algebra of graphs [4] and (ii) some combinations of graphs form sub-Hopf algebras. We will show that these results allow to recover the full propagators from the renormalization group functions.

We then apply this calculation scheme to the Wess–Zumino supersymmetric model, building on our preceding work [8]. As we shall see, the efficiency of the method allows to reach high degrees of perturbation theory, being essentially limited by the size of the expression in terms of values of the Riemann  $\zeta$  function. Switching to numerical approximations allows to go further and reach the asymptotic behavior of the coefficients of the development of the anomalous dimension  $\gamma$  in powers of the “fine coupling constant”  $a$ . This behavior allows for the definition of a convergent Borel transform of this serie. A singularity is however present on the positive axis but we discuss the options for nevertheless obtaining a sensible resummation. This work suggests further developments that we explore as a conclusion.

## 2 Exponentiation of the renormalization group

Our purpose is to reach high order of perturbation theory in the evaluation of renormalization group function. Along the lines suggested in [11], we use Schwinger–Dyson equations to produce a family of diagrams for which analytical evaluation is very simple. However, a naïve approach to this evaluation involves a triple serie in the coupling constant  $g$ , the logarithm of the momentum  $L = \log(q^2/\mu^2)$  and the dimensional regularization parameter  $\epsilon$ . It was shown in [6] how to reduce the problem to a single non-linear differential equation for the

renormalization group  $\gamma$ -function, but the proposal relied on the linearity of the Schwinger–Dyson equation.

Now, if we work only with renormalized quantities, we can avoid the  $\epsilon$ -parameter. The momentum dependence can then be recovered from the renormalization group. Consider the ratio of the full propagator with the free propagator which, in a Lorentz invariant scheme and for a massless theory, is a single number. It can be fixed to 1 at the scale  $\mu^2$  in a “physical” renormalization scheme and the value for any momentum  $p$  can be inferred from the action of the renormalization group with parameter  $t = \log(p^2/\mu^2)$ . Other renormalization schemes like the minimal subtraction one or its variants would introduce finite contributions at the reference momentum. Similarly, vertex functions, which generically depend on multiple scales, would involve finite contributions at the reference scale. However, here we are interested only in the propagator of a massless theory where these complications are not present and the full propagator can be recovered from the renormalization group.

In [4], it was shown that the renormalization group is a one parameter subgroup  $\{F_t\}$  of the group of characters of the Hopf algebra of graphs, which means:

$$\begin{aligned} F_t(XY) &= F_t(X)F_t(Y) \\ F_{s+t}(X) &= (F_s \otimes F_t)(\Delta X) \end{aligned} \tag{1}$$

Multiple derivations of this identity allows to recursively obtain the derivatives of  $F_t$  at the origin. Indeed:

$$\partial_t^{k+1} F_t|_{t=0} = \partial_s \partial_t^k F_{s+t}|_{s=0, t=0} = \partial_s F_s|_{s=0} \partial_t^k F_t|_{t=0} \tag{2}$$

Applied on a group-like element of the Hopf algebra, that is, on an element  $X$  such that  $\Delta X = X \otimes X$ , this identity shows that the successive derivatives are powers of the first one, so that  $F_t(X) = e^{\alpha t}$ . We will deal however with more general elements of the Hopf algebra of diagrams, so that the results will be more complicated.

Finally since  $F_t$  is a character, its evaluation on a product is the product of the evaluations, so that  $\beta = \partial_t F_t|_{t=0}$ , the generator of the renormalization group, is a derivation. This will be important for the applications of eq. (2).

### 3 Hopf algebra of Green functions

In order to make the best use of the Hopf algebra technique, it is convenient to work not at the level of the individual diagrams, but on some sums of diagrams. If the sums of interest can be shown to form sub Hopf algebras, this allows to overcome in part the combinatorial complexities of calculations.

Our starting point is that the effective coupling is a map from the Hopf algebra of the graphs to the Hopf algebra of formal diffeomorphism tangent to the identity in 0. For couplings associated to a three-point vertex, it will be convenient to express this fact not in terms of the coupling  $g$ , but of a “fine

structure constant"  $a = Cg^2$ , in order to avoid square roots. The proportionality constant  $C$  can be chosen as in QED to simplify further results. If the vertex couples particle species  $j$ ,  $k$  and  $l$ , the effective constant is defined as:

$$a_{\text{eff}} = \frac{C\Gamma_v^2}{\Gamma_{2,j}\Gamma_{2,k}\Gamma_{2,l}} \quad (3)$$

Here  $\Gamma_{2,j}$  is the two-point function for the  $j$  specie while  $\Gamma_v$  is the effective interaction vertex.

Writing

$$a_{\text{eff}} = \sum_n a_{\text{eff},n} a^n \quad (4)$$

the action of the coproduct on  $a_{\text{eff}}$  is given by [4, 12]

$$\Delta a_{\text{eff}} = \sum_n a_{\text{eff}}^n \otimes a_{\text{eff},n} \quad (5)$$

This corresponds to the fact that  $a_{\text{eff}}$  is a map to the Hopf algebra of (coordinates for) formal diffeomorphisms.

In fact, two refinements are easy to obtain. First, the same formula holds whenever we do not take for the  $\Gamma$ 's the full perturbative expansion, but only the terms which contain a certain set of primitive divergences. This holds in particular if we take for the  $\Gamma$ 's the approximation resulting from a given truncation of the Schwinger–Dyson equations.

An examination of the proof of the previous identities in [12] shows that stronger results hold for the two- or three-point functions which are involved,

$$\Gamma_{2,m} = \sum_n \Gamma_{m,n} a^n, \quad (6)$$

$$\Gamma_v = g \sum_{n=0} \Gamma_{v,n} a^n, \quad (7)$$

$$\Delta \Gamma_{2,m} = \sum_n \Gamma_{2,m} a_{\text{eff}}^n \otimes \Gamma_{m,n} \quad (8)$$

$$\Delta \Gamma_v = \sum_n \Gamma_v a_{\text{eff}}^n \otimes \Gamma_{v,n} \quad (9)$$

In the cases in which the coupling constant is not renormalized,  $a_{\text{eff}} = a$ , these equations indicate that the  $\Gamma$ 's are group like elements of the Hopf algebra. Now, when deriving relation (5) from relations (8,9), one uses the fact that formulas of this kind are multiplicative. Indeed, suppose that  $A = \sum A_n a^n$  and  $B = \sum B_n a^n$  are two series satisfying a property like (8):

$$\Delta A = \sum_n A a_{\text{eff}}^n \otimes A_n. \quad (10)$$

Then the product  $AB$  satisfies the same property. We can further remark that the coproducts of the propagators satisfy similar equations. This easily follows from the fact that the identity element of the Hopf algebra trivially satisfies (10). This allows to prove (5) from eqs. (8,9).

## 4 Practical formulas

Joining the results of the two preceding sections allows to construct the full renormalized propagator at any order in perturbation theory from the first derivative with respect to  $L$ , that is, the renormalization group function  $\gamma$ . In fact, we can get at will the propagator or its inverse, the 1PI two-point function.

Let us consider the evaluation of the renormalization group on the propagator  $\Gamma_{2,j}^{-1}$ , normalized by the free propagator,

$$F_t(\Gamma_{2,j}^{-1}) = 1 + \sum_{n=1}^{\infty} \gamma_n t^n \quad (11)$$

where, as before,  $t = \log(p^2/\mu^2)$ . Notice that the coefficient  $\gamma_1$  in expansion (11) is related to the anomalous dimension  $\gamma$ ,

$$\gamma_1 = -2\gamma \quad (12)$$

The successive terms of the development of this expression in powers of  $t$  can be computed from eq. (2), using the coproduct in eq. (8),

$$(k+1)\gamma_{k+1} = \sum_n (\gamma_1 + n\beta) a^n \gamma_{k,n} = (\gamma_1 + \beta a \partial_a) \gamma_k \quad (13)$$

To obtain the preceding equation we made use of the simple differentiation rule given by (2). This allowed to apply the derivation property of the generator of the group  $\partial_t F_t|_{t=0}$  to further evaluate its effect on the left factor of  $\Delta A$ .

## 5 Application to the Wess–Zumino model

The model we will consider is the simplest Wess–Zumino model, with only one type of superfields (and not two as in our preceding work [8]). The general non-renormalization properties of such a model ensures that the vertex is not renormalized and in fact, in a massless theory, it cannot get any correction. The only contribution to the renormalization group comes from the correction to the propagator and the supersymmetry has here the effect that all components of the superfield get the same renormalization factor<sup>1</sup>. We therefore have the simple relation

$$\beta = 3\gamma_1. \quad (14)$$

so that eq.(13) becomes

$$(k+1)\gamma_{k+1} = \gamma_1(1 + 3a\partial_a)\gamma_k \quad (15)$$

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<sup>1</sup>In a supersymmetric gauge theory, the situation would be subtler, since the gauge symmetry and the supersymmetry are not easy to combine: in the component formalism, supersymmetry is not fully explicit and in a superfield formalism, the unconstrained superfield describing the gauge supermultiplet is of dimension zero, so any power of this field can appear in the renormalized Lagrangian. This can be overcome by an appropriate choice of the gauge fixing condition and therefore it does not affect physical properties [13].

The first step in our calculation is to compute the effect of the one-loop primitive divergence. There are no primitive divergences at two loops and the three-loop one is non-planar and therefore subleading in a large  $N$  study. For this we will insert the full renormalized propagator at order  $n$  in the one loop diagram to derive  $\gamma_1$  at the order  $n + 1$ . Since  $\gamma_k$  is at least of order  $k$ , we only have to consider a finite expansion in  $t$  of the correction factor of the propagator.

We therefore need to compute a simple one loop integral, but with propagators having logarithmic corrections. All of them can be obtained multiplying the propagators  $\Gamma_2^{-1}(p^2)$  by a factor  $(p^2/\mu^2)^x$ . Indeed, differentiating with respect to  $x$  at  $x = 0$  will give the logarithm factors in each of the propagators. This procedure is essentially the same as the Mellin transform method used in [11].

As in [8], we only need to compute the graph for the correction to the auxiliary field propagator. Indeed, the fact that we have corrections on the two propagators does not change the observation already made there: the corrections to the free propagators of all the members of the supermultiplet are equal. We are interested in the quantity

$$\begin{aligned}\Gamma(q^2, x, y) &= +\frac{g^2}{8\pi^4} \int d^4p \frac{1}{(p^2)^{1-x} [(q-p)^2]^{1-y}} \\ &= +\frac{g^2}{8\pi^2 \Gamma(1-x)\Gamma(1-y)} \int du dv \frac{u^{-x} v^{-y}}{(u+v)^2} \exp\left(-\frac{uv}{u+v} q^2\right) \\ &= +\frac{g^2}{8\pi^2} (q^2)^{x+y} \frac{\Gamma(-x-y)\Gamma(1+x)\Gamma(1+y)}{\Gamma(2+x+y)\Gamma(1-x)\Gamma(1-y)}\end{aligned}\quad (16)$$

The overall plus sign comes from the product of the minus sign in every correction to the 1PI two-point function and the free propagator which is  $-1$ . It will be convenient to introduce the “fine structure constant”  $a = g^2/8\pi^2$ . Now, we can get an expression similar (but simpler) to the ones in [11] using the expansion of the logarithm of the  $\Gamma$  function

$$\Gamma(q^2, x, y) = -\frac{a(q^2/\mu^2)^z}{z(1+z)} \exp\left(2 \sum_{l=1}^{\infty} \frac{\zeta(2l+1)}{2l+1} (z^{2l+1} - x^{2l+1} - y^{2l+1})\right) \quad (17)$$

where  $z = x + y$  and  $\zeta(2l+1)$  is the Riemann zeta function. From (17) we can obtain the two-point functions calculated with modified propagators by applying differential operator in  $x$  and  $y$ . Differentiation with respect to  $\log(q^2)$  produces a factor  $z$  which exactly compensates the term which is singular for  $z = 0$ . Let us define

$$H(x, y) = -\left. \frac{\partial \Gamma(q^2, x, y)}{\partial \log(q^2)} \right|_{q^2=\mu^2} \quad (18)$$

Then, one has

$$H(x, y) = \frac{a}{1+x+y} \exp\left(2 \sum_{l=1}^{\infty} \frac{\zeta(2l+1)}{2l+1} ((x+y)^{2l+1} - x^{2l+1} - y^{2l+1})\right) \quad (19)$$

and

$$\gamma_1 = (1 + \sum \gamma_n \frac{d^n}{dx^n})(1 + \sum \gamma_n \frac{d^n}{dy^n})H(x, y)|_{x=y=0} \quad (20)$$

The evaluation of the  $\gamma$  function does not require any further regularization. After the order  $a$  term, which is obtained making  $x = y = 0$ , the following terms involve corrections to the propagators, which are obtained by taking derivatives with respect to  $x$  and  $y$  evaluated at the origin, this corresponding to diagrams with the propagators multiplied by given powers of  $\log(p^2)$ .

We could suppose that all the formalism we developed is superfluous and that we could simply derive further eq. (17) with respect to  $\log(q^2)$  to obtain directly the higher terms in the expansion. Although this is possible, the computation are really much more cumbersome, even for the leading term of the derivatives of  $\Gamma_2$ . Furthermore, one would obtain the development of the 2-point function in the effective action, so that an inversion of the expansion would be needed to obtain the propagator.

## 6 Calculation and results

Once the Mellin transform (19) is known, it is straightforward to obtain the first orders in the development of the  $\gamma$  function from eqs. (15) and (20). We give in table 1 the 12 first terms of the expansion of  $\gamma$ . The principal limitation for calculating this “exact” form for each term stems from the rapid growth, which behaves as  $\exp(\alpha n^{2/3})$ , of the number of contribution one has to take into account. Already in the terms presented in table 1, one can see the rapid growth of their size. At order 25, there are already 122 possible products of  $\zeta$  values contributing to the result and this number reaches 409174 at order 100. Such a number, as well as the size of the corresponding rational coefficients, limit to less than 30 the number of terms which can be computed with a straightforward implementation on today’s average computer, mainly due to the memory footprint. The rapid growth of the required resources limit to a few units the number of additional terms which can be computed using more efficient coding schemes and bigger memories. The highest orders of the development cannot therefore be reached in this way, all the more so that at the end we will need to convert these results to some concrete numerical approximation.

It is therefore convenient to work from the start with numerical values. A much higher number of terms is now easily within reach, since for a fixed precision, the computation work grows only quartically with the desired number of terms. In Table 2, we give the development of  $\gamma_1$  up to 16 loops and the 7 last terms we calculated. The full table, with all results up to 201 loops, is available upon request from the authors.

In the calculation of the expansion of  $\gamma_1$ , we first have to determine the Taylor expansion of the function  $H(x, y)$ . This is the part of the calculation which is numerically most demanding, since these coefficients are the sum of a great number of terms which nearly cancel, so that care must be taken that the true precision of the result is lesser than the number of digits used in the computation. However, something rather remarkable happens: the Taylor

$$\begin{aligned}
\gamma(a) = & a - 2a^2 + 14a^3 + \left(-160 + 16\zeta(3)\right)a^4 \\
& + \left(2444 - 328\zeta(3)\right)a^5 + \left(-45792 + 7056\zeta(3) + 2016\zeta(5)\right)a^6 \\
& + \left(1\,005\,480 - 169\,152\zeta(3) - 70\,896\zeta(5) + 8960\zeta^2(3)\right)a^7 \\
& + \left(-25\,169\,760 + 4\,509\,408\zeta(3) + 2\,199\,840\zeta(5) \right. \\
& \quad \left. + 564\,480\zeta(7) - 390\,400\zeta^2(3)\right)a^8 \\
& + \left(705\,321\,200 - 132\,548\,640\zeta(3) - 69\,922\,848\zeta(5) \right. \\
& \quad \left. - 29\,005\,632\zeta(7) + 14\,193\,504\zeta^2(3) + 6\,397\,056\zeta(3)\zeta(5)\right)a^9 \\
& + \left(-21\,841\,420\,384 + 4\,261\,047\,424\zeta(3) + 2\,354\,993\,856\zeta(5) \right. \\
& \quad + 1\,194\,909\,696\zeta(7) + \frac{858\,457\,600}{3}\zeta(9) - 512\,441\,536\zeta^2(3) \\
& \quad \left. - 383\,788\,416\zeta(3)\zeta(5) + \frac{49\,556\,480}{3}\zeta^3(3)\right)a^{10} \\
& + \left(740\,194\,188\,032 - 148\,784\,410\,432\zeta(3) - 84\,779\,661\,888\zeta(5) \right. \\
& \quad - 47\,818\,582\,272\zeta(7) - \frac{58\,999\,853\,440}{3}\zeta(9) + 19\,225\,297\,088\zeta^2(3) \\
& \quad + 17\,828\,697\,216\zeta(3)\zeta(5) + 1\,829\,076\,480\zeta^2(5) \\
& \quad \left. + 3\,838\,602\,240\zeta(3)\zeta(7) - \frac{3\,432\,237\,056}{3}\zeta^3(3)\right)a^{11} \\
& + \left(-27\,243\,674\,154\,368 + 5\,610\,375\,120\,768\zeta(3) + 3\,266\,192\,145\,024\zeta(5) \right. \\
& \quad + 1\,961\,976\,190\,464\zeta(7) + 1\,019\,076\,124\,160\zeta(9) + 230\,546\,534\,400\zeta(11) \\
& \quad - 760\,702\,109\,184\zeta^2(3) - 788\,057\,929\,728\zeta(3)\zeta(5) - 141\,297\,435\,648\zeta^2(5) \\
& \quad - 297\,887\,016\,960\zeta(3)\zeta(7) + 59\,550\,068\,736\zeta^3(3) \\
& \quad \left. + 34\,512\,334\,848\zeta^2(3)\zeta(5)\right)a^{12} + \dots
\end{aligned}$$

Table 1: The first 12 terms of the development of  $\gamma_1$

$$\begin{aligned}
\gamma(a) = & a - 2 a^2 + 14 a^3 - 140.767089549446491434 a^4 \\
& + 2049.72533576365307439 a^5 - 35219.8401369368689507 a^6 \\
& + 741582.310142069315875 a^7 - 1.74630317191742523615 \times 10^7 a^8 \\
& + 4.72719801334671229530 \times 10^8 a^9 - 1.39759545666280992694 \times 10^{10} a^{10} \\
& + 4.60146704077682933925 \times 10^{11} a^{11} - 1.63220296094286720854 \times 10^{13} a^{12} \\
& + 6.32651854893093835423 \times 10^{14} a^{13} - 2.61715263667021333524 \times 10^{16} a^{14} \\
& + 1.16791189443603376676 \times 10^{18} a^{15} - 5.52247245848724267096 \times 10^{19} a^{16} \\
& + \dots \\
& + 8.4053176185682527526 \times 10^{454} a^{195} - 4.9339110330514367678 \times 10^{457} a^{196} \\
& + 2.9112362346747106444 \times 10^{460} a^{197} - 1.7263592738217495952 \times 10^{463} a^{198} \\
& + 1.0289894774008300571 \times 10^{466} a^{199} - 6.1636327768018535021 \times 10^{468} a^{200} \\
& + 3.7107878544109289292 \times 10^{471} a^{201} + \dots
\end{aligned}$$

Table 2: Approximations of the development of  $\gamma_1$

expansion of the rather complicated function  $H$  in (19) is, asymptotically, the same as the one of  $h$ , defined by:

$$h(x, y) = (1 + xy) \left( \frac{1}{1+x} + \frac{1}{1+y} - 1 \right) + \frac{1}{2} \frac{xy}{1-x-y} \quad (21)$$

More precisely, the terms of total degree  $n$  in the Taylor expansion of  $h$  and the original function  $H$  are the same up to a relative error which is smaller than  $2^{-n}$ . If it is relatively easy to understand the properties of the terms in  $xy^n$  from the serie with general term  $\zeta(2p+1) - 1$ , the interplay of different terms which yields such a simple result for the general terms appears quite miraculous. Even more, the error term proportional to  $2^{-n}$ ,  $3^{-n}$  and  $4^{-n}$  can be given similar explicit forms and the determination of error terms proportional to powers of bigger integer seem only to require more terms with higher precision. These asymptotic formulas have been used to obtain the higher degrees of the Taylor expansion for our numerical application.

The asymptotic behavior of the coefficients  $g_n$  in the expansion of  $\gamma_1$  in powers of  $a$  can be determined. With

$$\gamma_1(a) = \sum_{n=1}^{\infty} g_n a^n, \quad (22)$$

we have that

$$g_n \propto (-3)^n \Gamma(n + 2/3). \quad (23)$$

This implies that a suitably defined Borel transform will have a single pole at  $-1/3$  as dominant singularity. However the asymmetry between odd and even terms of the series implies that there is also a weaker singularity at  $+1/3$ . It is

not a simple pole but the singularity of the dilogarithm  $\text{Li}_2$ . This precludes the analytic continuation of the Borel transform on the positive real axis and its approximation by Padé methods. A brutal answer to this problem is simply to integrate the Borel transform only on the interval  $[0, \frac{1}{3}]$ . This gives an analytic function which has the desired asymptotic expansion at the origin. However this procedure is rather ad hoc and gives a far from unique solution.

There is also the possibility that higher order corrections to the Schwinger–Dyson equation stemming from the addition of multiloop primitive divergences would weaken the singularity on the positive axis, so that the singularity on the positive axis would disappear in a complete calculation.

Taken as an asymptotic formula, eq.(21) could be useful for a demonstration of the asymptotic properties of the expansion of  $\gamma_1$ . The second term in  $h$  has positive Taylor coefficients. Combined with the alternating signs of the main term in the development of  $\gamma_1$ , this introduces strong compensation between the terms with even total powers of derivatives in eq. (20) and those with odd total powers. As a result, in the calculation of the term of degree  $n$ , the terms with one of the propagator with the correction of degree  $n - 1$  and the other one equal to the free propagator are dominant.

## 7 Discussion

High orders of the perturbative development of the renormalization group function have been obtained, resumming all diagrams with the simplest primitive divergence. The number of terms calculated in the corresponding expansion allows to reach a clearly asymptotic regime. The Borel transform of the resulting series however presents singularities on the positive real axis. The question remains open whether this signals a fundamental limitation of the perturbative calculations or it is just an artifact of the present approximation. Indeed, in a large  $N$  approximation, other primitive divergences are possible, beginning at the fourth order.

However, one should first test if the additional contributions remain subdominant. Indeed, in the contribution from the four loop primitive divergence, the corrections to the propagators can be split between the eleven propagators, generating a rapidly growing number of terms and this could more than compensate for the fact that corrections to the propagator of order  $n - 3$  are of order  $1/n^3$  with respect to the propagator of order  $n$  which gives the main contribution in the term we have calculated. This would also be an important step in the way of conjecturing  $\zeta$ -function style functional equations generalizing the one obtained in [11] for the simple case where the  $\gamma$  function satisfies a differential equation.

One interesting feature of the approach we presented is that it provides a clear path for evaluating the correction stemming from additional terms in the Schwinger–Dyson equation. The evaluation of the primitive diagrams looks formally as the analytic regularization proposed long ago [14, 15, 16] but the fact that we need such an evaluation only for primitively divergent diagrams avoids much of the technicalities stemming from the complex multidimensional

pole structures appearing in general diagrams.

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