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# Convergence of multi-dimensional quantized $SDE$ 's

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## Abstract

We quantize a multidimensional  $SDE$  (in the Stratanovich sense) by solving the related  $ODE$ 's in which the Brownian motion has been replaced by the components of stationary quantizers. We make a connection with rough path theory to show that such quantizations converge toward the solution of the  $SDE$ . In some particular cases, we show that this procedure provide some rate optimal quantizations of the equation.

**Key words:** Functional quantization, Stochastic Differential Equations, Stratanovich stochastic integral, stationary quantizers, rough path theory,  $p$ -variation, Hölder norm.

## 1 Introduction and preliminaries

Quantization is a way to discretize the path space of a random phenomenon: a random vector in finite dimension, a stochastic process in infinite dimension. Optimal Vector Quantization theory (finite-dimensional) random vectors finds its origin in the early 1950's in order to discretize some emitted signal (see [6]). It was further developed by specialists in Signal processing and later in Information Theory. The infinite dimensional case started to be extensively investigated in the early 2000's by several authors (see *e.g.* [11], [4], [12], [13], [12], [3], [8], etc).

In a recent paper ([13]), the functional quantization of a class of Brownian diffusions has been investigated from a constructive viewpoint. The main feature of this class of diffusions was that the diffusion coefficient was the inverse of the gradient of a diffeomorphism (both coefficients being smooth). This class contains most (non degenerate) scalar diffusions. Starting from a sequence of rate optimal quantizers some sequences of quantizers of the diffusion as the solutions of (non coupled)  $ODE$ 's are produced. This approach relied on the Lamperti transform and was closely related to the Doss-Sussman representation formula of the flow of a diffusion as a functional of the Brownian motion. In many situations these

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quantizers are rate optimal (or almost rate optimal) *i.e.* that they quantize the diffusion at the same rate  $O((\log N)^{-\frac{1}{2}})$  as the Brownian motion itself where  $N$  denotes the generic size of the quantizer. In a companion paper ([18]), some cubature formulas based on some of these quantizers were implemented, namely those obtained from some optimal product quantizers based on the Karhunen-Loève expansion of the Brownian motion, to price some Asian options in a Heston stochastic volatility model. Rather unexpectedly in view of the theoretical rate of convergence, the numerical experiments provided quite good numerical results for some “small” sizes of quantizers. Note however that these numerical implementations included some further speeding up procedures based on the stationarity of the quantizers and the Romberg extrapolation leading to a  $O((\log N)^{-\frac{3}{2}})$  rate.

Recently a sharp quantization rate has been established for a class of 1-dimensional diffusions in a non constructive way (see [3]). Finally, the standard rate  $O((\log N)^{-\frac{1}{2}})$  has been extended in [15] to general  $d$ -dimensional diffusions (regardless of their ellipticity properties) and even to Itô processes. This latter approach is constructive as well, although the resulting quantizers are no longer stationary.

Our aim in this paper is to extend the constructive natural approach initiated in [13] to general  $d$ -dimensional diffusions. To this end, we will need to call upon the rough path theory to replace the “Doss-Sussman environment”. Namely, we consider a diffusion process

$$dX_t = b(t, X_t) dt + \sigma(t, X_t) \circ dW_t, \quad X_0 = x \in \mathbb{R}^d, \quad t \in [0, T]$$

in the Stratanovich sense where  $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  and  $\sigma : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times q}$  are continuously differentiable with linear growth (uniformly with respect to  $t$ ) and  $W = (W_t)_{t \in [0, T]}$  is a  $q$ -dimensional Brownian motion defined on a filtered probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ .

Such an *SDE* admits a unique strong solution denoted  $X^x = (X_t^x)_{t \in [0, T]}$  (the dependency in  $x$  will be dropped for convenience). The  $\mathbb{R}^d$ -valued process  $X$  is pathwise continuous and  $\sup_{t \in [0, T]} |X_t| \in L^r(\mathbb{P})$ ,  $r > 0$  (where  $|\cdot|$  denotes the canonical Euclidean norm on  $\mathbb{R}^d$ ). In particular  $X$  is bi-measurable and can be seen as an  $L^r(\mathbb{P})$ -Radon random variable taking values in the Banach spaces  $(L^p_{T, \mathbb{R}^d}, |\cdot|_{L^p_T})$  where  $L^p_{T, \mathbb{R}^d} = L^p_{\mathbb{R}^d}([0, T], dt)$  and  $|g|_{L^p_T} = \left( \int_0^T |g(t)|^p dt \right)^{\frac{1}{p}}$  denotes the usual  $L^p$ -norm when  $p \in [1, \infty)$ .

For every integer  $N \geq 1$ , we can investigate for  $X$  the level  $N$  ( $L^r, L^p_T$ )-quantization problem for this process  $X$ , namely solving the minimization of the  $L^r(\mathbb{P})$ -mean  $L^p_{T, \mathbb{R}^d}$ -quantization error

$$e_{N,r}(X, L^p) := \min \left\{ e_{N,r}(\alpha, X, L^p), \alpha \subset L^p_{T, \mathbb{R}^d}, |\alpha| \leq N \right\} \quad (1.1)$$

where  $e_{N,r}(\alpha, X, L^p)$  denotes the  $L^r$ -man quantization error induced by  $\alpha$ , namely

$$e_{N,r}(\alpha, X, L^p) := \left( \mathbb{E} \min_{a \in \alpha} |X - a|_p^r \right)^{\frac{1}{r}} = \left\| \min_{a \in \alpha} |X - a|_{L^p_{T, \mathbb{R}^d}} \right\|_{L^r(\mathbb{P})}.$$

The use of “min” in (1.1) is justified by the existence of an optimal quantizer to that problem as shown in [2, 9] in this infinite dimensional setting.

In finite dimension (when considering  $\mathbb{R}^d$ -valued random vectors instead of  $L^p_{T, \mathbb{R}^d}$ -valued processes) the answer is provided by the so-called Zador Theorem which says (see [6]) that

if  $\mathbb{E}|X|^{r+\delta} < +\infty$  for some  $\delta > 0$  and if  $g$  denotes the absolutely continuous part of its distribution then

$$N^{\frac{1}{d}} e_{N,r}(X, \mathbb{R}^d) \rightarrow \tilde{J}_{r,d} \|g\|_{\frac{r}{d+r}}^{\frac{1}{r}} \quad \text{as} \quad N \rightarrow \infty \quad (1.2)$$

where  $\tilde{J}_{r,d}$  is finite positive real constant obtained as the limit of the normalized quantization error when  $X \stackrel{d}{=} U([0, 1])$ . This constant is unknown except when  $d = 1$  or  $d = 2$ .

A non asymptotic version of Zador's Theorem can be found *e.g.* in [15]: for every  $r, \delta > 0$  there exists a universal constant  $C_{r,\delta} > 0$  and an integer  $N_{r,\delta} \geq$  such that

$$\forall N \geq N_{r,\delta}, \quad N^{\frac{1}{d}} e_{N,r}(X, \mathbb{R}^d) \leq C_{r,\delta} \|X\|_{r+\delta}.$$

In infinite dimension, the case of Gaussian processes was the first to have been extensively investigated, first in the purely quadratic case ( $r = p = 2$ ): sharp rates have been established for a wide family of Gaussian processes including the Brownian motion, the fractional Brownian motions (see [11, 12]). For these two processes sharp rates are also known for  $p \in [1, \infty]$  and  $r \in (0, \infty)$  (see [3]). More recently, a connection between mean regularity of  $t \mapsto X_t$  (from  $[0, T]$  into  $L^r(\mathbb{P})$ ) and the quantization rate has been established (see [15]): if the above mapping is  $\mu$ -Hölder for an index  $\mu \in (0, 1]$ , then

$$e_{N,r}(X, L^p) = O((\log N)^{-\mu}), \quad p \in (0, r).$$

Based on this result, some universal quantization rates have been obtained for general Lévy processes with or without Brownian component some of them turning out to be optimal, once compared with the lower bound estimates derived from small deviation theory (see *e.g.* [7] or [4]). One important feature of interest of the purely quadratic case is that it is possible to construct from the Karhunen-Loève expansion of the process two families of rate optimal (stationary) quantizers, relying on

- Some sequences  $(\alpha^{(N,prod)})_{N \geq 1}$  of optimal *product quantizers* which are rate optimal *i.e.* such that  $e_{N,r}(\alpha^{(N)}, X, L^2) = O(e_{N,2}(X, L^2))$  (although not with a sharp optimal rate).
- Some sequences of *true* optimal quantizers (or at least some good numerical approximations)  $(\alpha^{(N,*)})_{N \geq 1}$  *i.e.* such that  $e_{N,r}(\alpha^{(N,*)}, X, L^2) = e_{N,2}(X, L^2)$ .

We refer to Section 1.1 below for further insight on these objects (both being available on the website [www.quantize.math-fi.com](http://www.quantize.math-fi.com)).

The main objective of this paper is the following: let  $(\alpha^N)_{N \geq 1}$  denote a sequence of rate optimal stationary quadratic quantizers of a  $q$ -dimensional standard Brownian motion  $W = (W^1, \dots, W^d)$ . Define the sequence  $x^N = (x_n^N)_{n=1, \dots, N}$ ,  $N \geq 1$ , of solutions of the ODE's

$$x_n^N = x + \int_0^t b(x_n^N(s)) ds + \int_0^t \sigma(x_n^N(s)) d\alpha_n^N(s), \quad n = 1, \dots, N.$$

Then, the finitely valued-process defined by

$$\tilde{X}^N = \sum_{n=1}^N x_n^N \mathbf{1}_{\{W \in C_n(\alpha^N)\}}$$

converges in  $p$ -variation toward  $X$  (and consequently for the sup norm) as  $N \rightarrow \infty$  *i.e.*

$$\text{Var}_{p,[0,T]}(X - \hat{X}) \rightarrow 0$$

in probability where

$$\text{Var}_{p,[0,T]}(x) := \sup \left\{ \left( \sum_{\ell=0}^{k-1} |x(t_{\ell+1}) - x(t_{\ell})|^p \right)^{\frac{1}{p}}, 0 \leq t_0 \leq t_1 \leq \dots \leq t_k \leq T, k \geq 1 \right\}.$$

We also provide some results with respect to the Hölder norm.

NOTATION: • For every  $d \geq 1$ , one denotes  $\xi = (\xi^1, \dots, \xi^d)$  a row vector of  $\mathbb{R}^d$ .

•  $|\cdot|$  denotes the canonical Euclidean norm on  $\mathbb{R}^d$ .

• One denotes by  $(\mathcal{F}_t^X)_{t \geq 0}$  the augmented natural filtration of a process  $X = (X_t)_{t \geq 0}$  (so that it satisfies the usual conditions).

• For a bounded function  $f : [0, T] \rightarrow \mathbb{R}^d$ ,  $\|f\|_{\text{sup}} := \sup_{t \in [0, T]} |f(t)|$ . If  $f$  is a Borel function  $\|f\|_{L^p_{T, \mathbb{R}^d}} := \left( \int_0^T |f(t)|^p dt \right)^{\frac{1}{p}}$ .

• For an  $\mathbb{R}^d$ -valued bi-measurable process, we denote  $\|X\|_p := \| |X|_{L^p_{T, \mathbb{R}^d}} \|_p = \left( \mathbb{E} \int_0^T |X_t|^p dt \right)^{1/p}$ .

## 1.1 Some background on functional quantization

Functional quantization of stochastic processes can be seen as a discretization of the path-space of a process and the approximation (or coding) of a process by finitely many deterministic functions from its path-space. In a Hilbert space setting this reads as follows.

Let  $(H, \langle \cdot, \cdot \rangle)$  be a separable Hilbert space with norm  $|\cdot|$  and let  $X : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow H$  be a random vector taking its values in  $H$  with distribution  $\mathbb{P}_X$ . Assume the integrability condition

$$\mathbb{E} |X|^2 < +\infty. \quad (1.3)$$

For  $N \geq 1$ , the  $L^2$ -optimal  $N$ -quantization problem for  $X$  consists in minimizing

$$\left( \mathbb{E} \min_{a \in \alpha} |X - a|^2 \right)^{1/2} = \left\| \min_{a \in \alpha} |X - a| \right\|_{L^2(\mathbb{P})}$$

over all subsets  $\alpha \subset H$  with  $\text{card}(\alpha) \leq N$ . Such a set  $\alpha$  is called  $N$ -codebook or  $N$ -quantizer. The minimal  $N^{\text{th}}$  quantization error of  $X$  is then defined by

$$e_N(X, H) := \inf \left\{ \left( \mathbb{E} \min_{a \in \alpha} |X - a|^2 \right)^{1/2} : \alpha \subset H, \text{card}(\alpha) \leq N \right\}. \quad (1.4)$$

For a given  $N$ -quantizer  $\alpha$  one defines an associated nearest neighbour projection

$$\pi_\alpha := \sum_{a \in \alpha} a \mathbf{1}_{C_a(\alpha)}$$

and the induced  $\alpha$ -(Voronoi) quantization of  $X$  by setting

$$\hat{X}^\alpha := \pi_\alpha(X), \quad (1.5)$$

where  $\{C_a(\alpha) : a \in \alpha\}$  is a Voronoi partition induced by  $\alpha$ , that is a Borel partition of  $H$  satisfying

$$C_a(\alpha) \subset \{x \in H : |x - a| = \min_{b \in \alpha} |x - b|\} \quad (1.6)$$

for every  $a \in \alpha$ . Then one easily checks that, for any random vector  $X' : \Omega \rightarrow \alpha \subset H$ ,

$$\mathbb{E} |X - X'|^2 \geq \mathbb{E} |X - \hat{X}^\alpha|^2 = \mathbb{E} \min_{a \in \alpha} |X - a|^2$$

so that finally

$$\begin{aligned} e_n(X) &= \inf \left\{ (\mathbb{E} |X - \hat{X}|^2)^{1/2} : \hat{X} = f(X), f : H \rightarrow H \text{ Borel measurable, } \text{card}(f(H)) \leq N \right\} \\ &= \inf \left\{ (\mathbb{E} |X - \hat{X}|^2)^{1/2} : \hat{X} : \Omega \rightarrow H \text{ random vector, } \text{card}(\hat{X}(\Omega)) \leq N \right\}. \end{aligned} \quad (1.7)$$

A typical setting for functional quantization is  $H = L_T^2 := L^2([0, 1], dt)$  (equipped with  $\langle f, g \rangle_2 := \int_0^T fg(t)dt$  and  $\|f\|_{L_T^2} := \sqrt{\langle f, f \rangle}$ ). Thus any (bi-measurable) process  $X = (X_t)_{t \in [0, T]}$  defined on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  such that

$$\int_0^T \mathbb{E}(X_t^2) dt < +\infty$$

is a random variable  $X : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow L_T^2$ . But this Hilbert setting is not the only possible one for functional quantization (see *e.g.* [14], [8], [4], etc) since natural Banach spaces like  $L^p([0, T], dt)$  or  $\mathcal{C}([0, T])$  are natural path-spaces.

In the purely Hilbert setting the existence of (at least) one *optimal N-quantizer* for every integer  $N \geq 1$  is established so that the infimum in (1.4) holds as a minimum. This existence property holds true in any reflexive Banach space and  $L^1$ -spaces (see [8] for details).

## 1.2 Constructive aspects of functional quantization of the Brownian motion

One considers a scalar Brownian motion  $(W_t)_{t \in [0, T]}$  on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ . The two main classes of rate optimal quantizers of the Brownian motion are the product optimal quantizers and the true optimal quantizers. Both are based on the Karhunen-Loève expansion of the Brownian motion given by

$$W_t = \sum_{k \geq 1} \sqrt{\lambda_k} \xi_k e_k^W(t) \quad (1.8)$$

where, for every  $k \geq 1$ ,

$$\lambda_k = \left( \frac{T}{\pi(k - 1/2)} \right)^2 \quad \text{and} \quad e_k^W(t) = \sqrt{\frac{2}{T}} \sin \left( \frac{t}{\sqrt{\lambda_k}} \right) \quad (1.9)$$

and

$$\xi_k = \frac{(W | e_k^W)_2}{\sqrt{\lambda_k}} = \sqrt{\frac{2}{T}} \int_0^T W_t \sin(t/\sqrt{\lambda_k}) \frac{dt}{\sqrt{\lambda_k}}.$$

The sequence  $(e_k^W)_{k \geq 1}$  is an orthonormal basis of  $L_T^2$ . The system  $(\lambda_k, e_k^W)_{k \geq 1}$  can be characterized as the eigensystem of the symmetric positive trace covariance operator of  $f \mapsto (t \mapsto \int_0^T (s \wedge t) f(s) ds) \equiv (t \mapsto \mathbb{E}(\langle f | W \rangle_2 W_t))$ . In particular this implies that the Gaussian sequence  $(\xi_k)_{k \geq 1}$  is pairwise uncorrelated hence i.i.d.,  $\mathcal{N}(0; 1)$ -distributed. The Karhunen-Loève expansion of  $W$  plays the role of *PCA* of the process: it is the fastest way to exhaust the variance of  $W$  among all expansions on an ortho-normal basis.

The convergence of the series in the right hand side of (1.8) holds in  $L_T^2$  for every  $\omega \in \Omega$  and  $\mathbb{P}(d\omega)$ -a.s. for every  $t \in [0, T]$ . In fact this convergence also holds in  $L^2(\mathbb{P})$  and  $\mathbb{P}(d\omega)$ -a.s. for the sup norm over  $[0, T]$ . The first convergence follows from Theorem 2.1(a) further on applied with  $X = W$  and  $\mathcal{G}_N = \sigma(\xi_1, \dots, \xi_N)$  and the second one follows e.g. from [14]  $\mathbb{P}(d\omega)$ -a.s.. In particular the convergence holds in  $L^2(d\mathbb{P} \otimes dt)$  or equivalently in  $L_{L_T^2}^2(\mathbb{P})$ .

### 1.2.1 Optimal product quantization

A product  $N$ -quantization of  $W$  is defined by

$$\widehat{W}_t^{(N_1, \dots, N_L)} := \sqrt{\frac{2}{T}} \sum_{k=1}^L \sqrt{\lambda_k} \widehat{\xi}_k^{N_k} \sin\left(\frac{t}{\sqrt{\lambda_k}}\right) \quad (1.10)$$

where  $N_1, \dots, N_L$  are non zero integers satisfying  $N_1 \cdots N_L \leq N$  and  $\widehat{\xi}_1^{N_1}, \dots, \widehat{\xi}_L^{N_L}$  are optimal quadratic quantizations of  $\xi_1, \dots, \xi_L$ . An *optimal product  $N$ -quantization*  $\widehat{W}^{N, prod}$  is obtained as a solution to the following *integral bit allocation optimization* problem for the sequence  $(N_k)_{k \geq 1}$ :

$$\min \left\{ \|W - \widehat{W}^{(N_1, \dots, N_L)}\|_2, N_1, \dots, N_L \geq 1, N_1 \cdots N_L \leq N, L \geq 1 \right\} \quad (1.11)$$

(see [11] for further details and [18] for the numerical aspects). It is established in [11] (as a special case of a more general result on Gaussian processes) that

$$\frac{1}{T} \|W - \widehat{W}^{N, prod}\|_2 \asymp (\log N)^{-\frac{1}{2}} \quad (1.12)$$

Furthermore, the critical dimension  $L = L_W(N)$  satisfies  $L_W(N) \sim \log N$ . Numerical experiments carried out in [18] show that  $\frac{1}{T} \|W - \widehat{W}^{N, prod}\|_2 \approx c_W (\log N)^{-\frac{1}{2}}$  with  $c_W \approx 0.5$  ( $N \leq 10000$ ).

It is possible to get a closed form for the underlying optimal product quantizers  $\alpha^N$ . First, note that the normal distribution on the real line being log-concave, there is exactly one stationary quadratic quantizer of full size  $M$  for every  $M \geq 1$  (hence it is the optimal one). So, let  $N \geq 1$  and let  $(N_k)_{k \geq 1}$  denote its optimal integral bit allocation for the Brownian motion  $W$ . For every  $N_k \geq 1$ , we denote by  $\beta^{(N_k)} := \{\beta_{i_k}^{(N_k)}, 1 \leq i_k \leq N_k\}$  the unique optimal quantizer of the normal distribution: thus  $\alpha(0) = \{0\}$  by symmetry of the normal distribution. Then, the optimal quadratic product  $N$ -quantizer  $\alpha^{N, prod}$  (of “true size”  $N_1 \times \cdots \times N_{L_W(N)} \leq N$ ) can be described using a multi-indexation as follows:

$$\alpha_{(n_1, \dots, n_k, \dots)}^{N, prod}(t) = \sum_{k \geq 1} \beta_{n_k}^{(N_k)} \sqrt{\lambda_k} e_k^W(t), \quad n_k \in \{1, \dots, N_k\}, \quad k \geq 1.$$

These sums are in fact all finite so that the functions  $\alpha_{(i_1, \dots, i_n, \dots)}^{N, prod}$  all are  $\mathcal{C}^\infty$  with finite variation on every interval of  $\mathbb{R}_+$ .

Explicit optimal integral bit allocations as well as optimal quadratic quantizations (quantizers and their weights) of the scalar normal distribution are available on the website [19]. Note for practical applications that this optimal product quantization is based on 1-dimensional quantizations of small size of the scalar normal distribution  $\mathcal{N}(0; 1)$ . This kind of functional quantization has been applied in [18] to price Asian options in a Heston stochastic volatility model.

### 1.2.2 Optimal quantization

It is established in [11] (Theorem 3.2) that the quadratic optimal quantization of the Brownian motion reads

$$\widehat{W}_t^{N, opt} = \sqrt{\frac{2}{T}} \sum_{k=1}^{d_W(N)} \sqrt{\lambda_k} (\widehat{\zeta}_{d_W(N)}^N)^k \sin\left(\frac{t}{\sqrt{\lambda_k}}\right) \quad (1.13)$$

where  $\zeta_{d_W(N)} = \text{Proj}_{E_{d(N)}}^\perp(W) \sim \mathcal{N}(0; \text{Diag}(\lambda_1, \dots, \lambda_{d_W(N)}))$  with  $E_{d_W(N)} := \mathbb{R}\text{-span}\{\sin(\cdot/\sqrt{\lambda_1}), \dots, \sin(\cdot/\sqrt{\lambda_{d_W(N)}})\}$  and  $\widehat{\zeta}_{d_W(N)}^N$  is an optimal quadratic quantization of  $\zeta_{d_W(N)}$  of size  $N$ .

If one considers an optimal quadratic  $N$ -quantizer  $\beta^N = \{\beta_n^N, n = 1, \dots, N\} \subset \mathbb{R}^{d_W(N)}$  of the distribution  $\mathcal{N}(0; \text{Diag}(\lambda_1, \dots, \lambda_{d_W(N)}))$  (*a priori* not unique)

$$\alpha_n^{N, opt}(t) = \sum_{k=1}^{d_W(N)} (\beta_n^N)^k \sqrt{\lambda_k} e_k^W(t), \quad n = 1, \dots, N.$$

Once again this defines a  $\mathcal{C}^\infty$  function with finite variation on every interval of  $\mathbb{R}_+$ .

The resulting optimal quantization error satisfies

$$\frac{1}{T} \|W - \widehat{W}^{N, opt}\|_2 \sim c_W^{opt} (\log N)^{-\frac{1}{2}} \quad \text{as } N \rightarrow \infty \quad (1.14)$$

where  $c_W^{opt} = \frac{\sqrt{2}}{\pi} \approx 0.4502$ .

The true value for  $d_W(N)$  is unknown. A conjecture supported by numerical evidences is that  $d_W(N) \sim \log N$  (what is proved is that  $d_W(N) = \Omega(\log(N))$ ).

Large scale computations of optimal quadratic quantizers of the Brownian motion have been carried out (up to  $N = 10\,000$  and  $d = 10$ ). They are available on the website [19].

### 1.2.3 Stationary functional quantizations, Wiener like integral

Both types of quantizations defined above share an important property of quantizers: self-stationarity.

**Definition 1.1** *Let  $\alpha \subset L_T^2$ . The quantizer  $\alpha$  is stationary for the Brownian motion if there is a Voronoi quantization  $\widehat{W} := \widehat{W}^\alpha$  induced by  $\alpha$  such that*

$$\widehat{W} = \mathbb{E}(W | \sigma(\widehat{W})) \quad \text{a.s.} \quad (1.15)$$



where  $\mathbb{E}(\cdot | \mathcal{G})$  denotes the functional conditional expectation given the  $\sigma$ -field  $\mathcal{G}$  on  $L^2_{L^2_T}(\mathbb{P})$  (see Annex).

In the case of product quantization  $\widehat{W}^{N,prod}$ , this follows from the stationarity property of the optimal quadratic quantization of the marginals  $\xi_n$  (see [11] or [18]). In the case of optimal quadratic quantization  $\widehat{W}^{N,opt}$  this follows from the optimality of the quantization of  $\zeta_{d_W(N)}$ .

We will see that it is possible to define a kind of Wiener integral with respect to a stationary quantization.

First, we must have in mind that if  $W$  is a  $(\mathcal{F}_t)$ -Brownian motion, then one can define the Wiener stochastic integral (on  $[0, T]$ ) of any process  $\varphi \in L^2([0, T] \times \Omega, \mathcal{W}([0, T]) \otimes \mathcal{F}_0, dt \otimes d\mathbb{P})$  with respect to  $W$ . One can see it as a special case of Itô stochastic integral or as an extended Wiener integral: If  $\varphi(t, \omega)$  denotes an elementary process of the form

$$\varphi(t, \cdot) := \sum_{k=1}^n \xi_k \mathbf{1}_{(s_k, s_{k+1}]}, \quad 0 = s_0 < s_1 < \dots < s_{n-1} < s_n = T.$$

where  $\xi_i$  are  $\mathcal{F}_0$ -measurable (hence independent of  $W$ ). Set  $I_T(\varphi) := \sum_{k=1}^n \xi_k (W_{s_{k+1}} - W_{s_k})$ .

Then,  $I_T$  is an isometry from  $L^2_{L^2_T}(\mathbb{P})$  into  $L^2(\mathcal{F}_T, \mathbb{P})$ . Furthermore, one checks that

$$\mathbb{E} \left( \int_0^T \varphi(s, \cdot) dW_s \mid \mathcal{F}_T^W \right) = \int_0^T \mathbb{E}(\varphi(s, \cdot) \mid \mathcal{F}_T^W) dW_s$$

where  $\mathcal{F}_T^W$  denotes the augmented filtration of  $W$  at time  $T$ . We follow the same lines to define the stochastic integral with respect to a stationary quantizer. Set for an elementary process  $\varphi$

$$\widehat{I}_T(\varphi) = \sum_{k=1}^n \xi_k (\widehat{W}_{s_{k+1}} - \widehat{W}_{s_k})$$

so that

$$\begin{aligned} \widehat{I}_T(\varphi) &= \sum_{k=1}^n \xi_k \mathbb{E}(W_{s_{k+1}} - W_{s_k} \mid \widehat{W}) \\ &= \sum_{k=1}^n \mathbb{E} \xi_k (W_{s_{k+1}} - W_{s_k}) \mid \mathcal{F}_0 \vee \sigma(\widehat{W}) \\ &= \mathbb{E} \left( \int_0^T \varphi(t, \cdot) dW_t \mid \mathcal{F}_0 \vee \sigma(\widehat{W}) \right) \end{aligned}$$

where we used that the  $\sigma$ -fields  $\sigma(\widehat{W})$  and  $\mathcal{F}_0$  are independent. It follows that

$$\|\widehat{I}_T(\varphi)\|_2^2 \leq \|I_T(\varphi)\|_2^2 = \|\varphi\|_{L^2_T}^2.$$

Hence the linear transformation  $\widehat{I}_T$  extends to a linear continuous mapping on  $L^2_{L^2_T}(\mathcal{F}_0, \mathbb{P})$ . Furthermore, one checks, first on elementary processes, then on the whole  $L^2_{L^2_T}(\mathcal{F}_0, \mathbb{P})$  by continuity of the (functional) conditional expectation that

$$\mathbb{E} \left( I_T(\varphi) \mid \mathcal{F}_0 \vee \sigma(\widehat{W}) \right) = \widehat{I}_T(\varphi).$$

We will denote from now on

$$\widehat{I}_T(\varphi)(\omega) := \int_0^T \varphi(t, \omega) d\widehat{W}_t(\omega).$$

Now set for every  $t \in [0, T]$ ,  $\int_0^t \varphi(s, \omega) d\widehat{W}_s(\omega) := \int_0^t \mathbf{1}_{[0,t]}(s) \varphi(s, \omega) d\widehat{W}_s(\omega)$ .

One checks using Jensen and Doob Inequality that,

$$\begin{aligned} \mathbb{E} \sup_{t \in [0, T]} \left| \int_0^t \varphi(s, \cdot) d\widehat{W}_s \right|^2 &\leq \mathbb{E} \sup_{t \in [0, T]} \left| \int_0^t \varphi(s, \cdot) dW_s \right|^2 \\ &\leq C \mathbb{E} \int_0^T \varphi^2(s, \cdot) ds. \end{aligned} \quad (1.16)$$

On the other hand, as soon as the stationary quantizer  $\alpha$  (such that  $\widehat{W} = \widehat{W}^\alpha$ ) has pathwise continuous elements, it follows that if  $\varphi$  is an elementary process,

$$\int_0^t \varphi(s, \cdot) dW_s = \sum_{k=1}^n \xi_k (\widehat{W}_{s_{k+1} \wedge t} - \widehat{W}_{s_k \wedge t})$$

admits a continuous modification in  $t \in [0, T]$ . One classically derives by combining this result with (1.16) and the everywhere density of elementary processes that, for every  $\varphi \in L^2_{L^2_T}(\mathcal{F}_0, \mathbb{P})$ , the process

$$\left( \int_0^t \varphi(s, \cdot) d\widehat{W}_s \right)_{t \in [0, T]} \quad \text{admits a continuous modification.}$$

This is always this modification that will be considered from now on. Furthermore, using the characterization of functional conditional expectation (see Annex), it follows that

$$\mathbb{E} \left( \int_0^\cdot \varphi(s, \cdot) d\widehat{W}_s \mid \mathcal{F}_0 \vee \sigma(\widehat{W}) \right) = \int_0^\cdot \varphi(s, \cdot) d\widehat{W}_s. \quad (1.17)$$

#### 1.2.4 Extension to multi-dimensional brownian motions

To quantize a standard  $d$ -dimensional Brownian motion one uses a kind of product-quantization of each of its components with marginal rate optimal quantizers of size  $[N^{\frac{1}{d}}]$ . Such quantizers share the same properties as the above ones defined in the scalar case.

**Proposition 1.1** *Let  $W =: (W^1, \dots, W^d)$  denote a  $d$ -dimensional standard Brownian motion and  $\widehat{W} =: (\widehat{W}^1, \dots, \widehat{W}^d)$  a pathwise continuous stationary quantization of  $W$ . Then, for every  $i \neq j$ ,  $i, j \in \{1, \dots, d\}$ ,*

$$\mathbb{E} \left( \int_0^\cdot W_s^i dW_s^j \mid \sigma(\widehat{W}^i, \widehat{W}^j) \right) = \int_0^\cdot \widehat{W}_s^i d\widehat{W}_s^j.$$

**Proof.** The stochastic integral  $\int_0^\cdot W_s^i dW_s^j$  coincides with the (extended) Wiener integral defined with respect to the filtration  $\mathcal{G}_{i,t}^j := \sigma(\mathcal{F}_T^{W^i}, \mathcal{F}_t^{W^j})$  (it is clear that  $W^j$  is a  $\mathcal{G}_{i,t}^j$ -standard Brownian motion). The result is then a straightforward consequence of (1.17).  $\diamond$

**Remark.** The above result still holds true if one considers an additional “0” component to the Brownian motion  $W_t^0 = t$  (with  $\widehat{W}_t^0 = t$ ).

## 2 Kolmogorov criterion and convergence in $p$ -variation

The aim of this section is to investigate the connections between the celebrated Kolmogorov criterion and the tightness of some classes of sequences of processes for the topology of  $p$ -variation convergence. In fact this connection is somehow the starting point of the rough path theory, but we will look at it in a slightly different way. Namely if  $X$  satisfies the Kolmogorov criterion and  $(\mathcal{G}_N)_{N \geq 1}$  denotes a sequence of sub- $\sigma$ -algebras of  $\mathcal{A}$  then a sequence of processes defined by

$$X^N := \mathbb{E}(X | \mathcal{G}^N), \quad N \geq 1,$$

is ( $C$ -tight and) tight for a whole family of topologies induced by convergence in  $p$ -variation (the conditional expectation is considered in the functional sense).

First let us define the  $p$ -variation (semi-)norm (and Hölder semi-norm).

**Definition 2.1** (a) Let  $x : [0, T] \rightarrow \mathbb{R}^d$  be a continuous function and let  $p \in [1, \infty)$ . One defines the  $p$ -variation semi-norm by

$$Var_{p,[0,T]}(x) := \sup \left\{ \left( \sum_{\ell=0}^{k-1} |x(t_{\ell+1}) - x(t_{\ell})|^p \right)^{\frac{1}{p}}, \quad 0 \leq t_0 \leq t_1 \leq \dots \leq t_k \leq T, \quad k \geq 1 \right\}.$$

(b) The  $\frac{1}{p}$ -Hölder semi-norm is closely connected to the  $p$ -variation semi-norm defined for every continuous function  $x : [0, T] \rightarrow \mathbb{R}^d$  by

$$\|x\|_{p,Hol} = \sup_{s,t \in [0,T]} \frac{|x(t) - x(s)|}{|t - s|^{\frac{1}{p}}}.$$

**Remark.** Indeed, it is clear that

$$Var_{p,[0,T]}(x) \leq \|x\|_{p,Hol}. \quad (2.1)$$

Although already appearing in one of the seminal papers of rough path theory (see [16]), its interest has been emphasized in [5].

When  $p \geq 1$ , one defines a norm on  $\mathcal{V}^p := \{x \in \mathcal{C}([0, T]), \quad Var_{p,[0,T]}(x) < +\infty\}$  by setting

$$Var_{p,[0,T]}^0(x) := |x(0)| + Var_{p,[0,T]}(x).$$

One proceeds likewise with the Hölder semi-norm.

**Definition 2.2** Let  $p \geq 1$ ,  $\theta > 0$ . A process  $X = (X_t)_{t \in [0, T]}$  satisfies the Kolmogorov criterion  $(K_{p,\theta})$  if there is a real constant  $C > 0$  such that

$$\forall s, t \in [0, T], \quad \mathbb{E}|X_t - X_s|^p \leq C |t - s|^{1+\theta} \quad \text{and} \quad X_0 \in L^p(\mathbb{P}).$$

**Theorem 2.1** Let  $X := (X_t)_{t \in [0, T]}$  be a pathwise continuous process defined on  $(\Omega, \mathcal{A}, \mathbb{P})$  satisfying the Kolmogorov criterion  $(K_{p,\theta})$ . Let  $(\mathcal{G}_N)_{N \geq 1}$  be a sequence of sub- $\sigma$ -fields of  $\mathcal{A}$ . For every  $N \geq 1$  set

$$X^N := \mathbb{E}(X | \mathcal{G}_N).$$

For every  $N \geq 1$ ,  $X^N$  has a pathwise continuous version. Furthermore, if one of the following conditions is satisfied

(a)  $\mathcal{G}_N \subset \mathcal{G}_{N+1}$ ,

(b) There exists a subset  $D \subset [0, T]$  such that  $X_t^N \xrightarrow{\mathbb{P}} X_t$ ,  $t \in D$ ,

(c)  $|X^N - X|_{L_T^r} \xrightarrow{\mathbb{P}} 0$  for some  $r \geq 1$ ,

then

$$\forall q > \frac{p}{\theta}, \quad \|X - X^N\|_{\text{sup}} + \|X - X^N\|_{q, \text{Hol}} \xrightarrow{L^p} 0$$

so that, in particular,

$$\text{Var}_{q, [0, T]}(X - X^N) \xrightarrow{L^p} 0$$

The proof of the theorem is a variant of the proof of the Kolmogorov criterion for functional tightness of processes. It consists in a string of several lemmas. For the following easy lemma, we refer to [10].

**Lemma 2.1** *Let  $x, y \in \mathcal{C}([0, T], \mathbb{R}^d)$  and let  $p > 0$ . Then*

(a)  $\|x - x(0)\|_{\text{sup}} \leq \text{Var}_{p, [0, T]}(x)$ .

(b) If  $p \geq 1$ ,  $\text{Var}_{p, [0, T]}(x + y) \leq \text{Var}_{p, [0, T]}(x) + \text{Var}_{p, [0, T]}(y)$  and if  $0 \leq p \leq 1$ ,

$(\text{Var}_{p, [0, T]}(x + y))^p \leq (\text{Var}_{p, [0, T]}(x))^p + (\text{Var}_{p, [0, T]}(y))^p$

(c) For every  $p' > p$ ,  $(\text{Var}_{p', [0, T]}(x))^{p'} \leq 2^{p'-p} \|x\|_{\text{sup}}^{p'-p} (\text{Var}_{p', [0, T]}(x))^p$ .

(d) Claims (a)-(b)-(c) remain true with the  $\frac{1}{p}$ -Hölder semi-norm  $\|\cdot\|_{p, \text{Hol}}$  instead of the  $p$ -variation semi-norm.

**Lemma 2.2** *If  $X$  satisfies the Kolmogorov criterion  $(K_{p, \theta})$  and  $X^N = \mathbb{E}(X | \mathcal{G}_N)$ ,  $N \geq 1$  then, the sequence  $(X^N)_{N \geq 1}$  is  $C$ -tight and for every  $\theta' \in (0, \theta)$ , there exists a random variable  $Z \in L_{\mathbb{R}}^{\theta'}(\mathbb{P})$  such that*

$$\mathbb{P}(d\omega)\text{-a.s. } \forall s, t \in [0, T], \quad |X_t^N(\omega) - X_s^N(\omega)|^p = \mathbb{E}(Z^{\theta'} | \mathcal{G}_N)(\omega) |t - s|^{\theta'}. \quad (2.2)$$

In particular

$$\text{Var}_{\frac{p}{\theta'}}(X^N) \leq \|X^N\|_{p, \text{Hol}} \leq \left( \mathbb{E}((TZ)^{\theta'} | \mathcal{G}_N) \right)^{\frac{1}{p}}.$$

**Proof.** Let  $s, t \in [0, T]$ . Let  $m, n \geq 1$  be two fixed integers.

$$\begin{aligned} \sup_{s, t \in [0, T], t \leq s \leq t + \frac{T}{2^n}} |X_t - X_s|^p &\leq 3^p \max_{0 \leq k \leq 2^n - 1} \sup_{s \in [\frac{kT}{2^n}, \frac{(k+1)T}{2^n}]} |X_s - X_{\frac{kT}{2^n}}|^p \\ &\leq 3^p \max_{0 \leq k \leq 2^n - 1} \sup_{m \geq 0} \sum_{\ell=1}^{2^m} |X_{\frac{kT}{2^n} + \ell \frac{T}{2^{n+m}}} - X_{\frac{kT}{2^n} + (\ell-1) \frac{T}{2^{n+m}}}|^p \\ &\leq \sum_{k=0}^{2^n - 1} \sum_{m \geq 0} \sum_{\ell=1}^{2^m} |X_{\frac{kT}{2^n} + \ell \frac{T}{2^{n+m}}} - X_{\frac{kT}{2^n} + (\ell-1) \frac{T}{2^{n+m}}}|^p. \end{aligned} \quad (2.3)$$

For every  $\theta' \in (0, \theta)$ , set

$$Z := \left( \sum_{n \geq 0} 2^{n\theta'} \sup_{s, t \in [0, T], t \leq s \leq t + \frac{T}{2^n}} |X_t - X_s|^p \right)^{\frac{1}{\theta'}}.$$

Taking expectation in (2.3) yields

$$\mathbb{E}(Z^{\theta'}) \leq C 3^p \sum_n 2^{n\theta'} \sum_{k=0}^{2^n-1} \sum_{m \geq 0} \left( \frac{T}{2^{n+m}} \right)^{1+\theta} \leq C_{p, \theta} \sum_n 2^{-n(\theta-\theta')} < +\infty.$$

On the other hand, for every  $\delta \in [0, T]$ ,  $2^{-(1+n\delta)} \leq \delta/T \leq 2^{-n\delta}$ , so that

$$\delta^{-\theta'} \sup_{s, t \in [0, T], t \leq s \leq t + \delta} |X_t - X_s|^p \leq 2^{(1+n\delta)\theta'} T^{-\theta'} \sup_{s, t \in [0, T], t \leq s \leq t + \frac{T}{2^n}} |X_t - X_s|^p \leq \left( \frac{2}{T} \right)^{\theta'} Z^{\theta'}.$$

Consequently, for every  $s, t \in [0, T]$ , and every  $\omega \in \Omega$ ,

$$|X_t(\omega) - X_s(\omega)|^p \leq Z^{\theta'}(\omega) |t - s|^{\theta'}.$$

Finally, it follows from Jensen Inequality that

$$\begin{aligned} \mathbb{P}(d\omega)\text{-a.s. } \forall s, t \in [0, T], \quad |X_t^N(\omega) - X_s^N(\omega)|^p &= |\mathbb{E}(X_t - X_s | \mathcal{G}_N)(\omega)|^p \\ &\leq \mathbb{E}(|X_t - X_s|^p | \mathcal{G}_N)(\omega) \\ &\leq \mathbb{E}(Z^{\theta'} | \mathcal{G}_N)(\omega) |t - s|^{\theta'}. \end{aligned} \quad (2.4)$$

so that, on the one hand every  $X^N$ ,  $N \geq 1$  has a pathwise continuous version, and

$$\mathbb{P}(d\omega)\text{-a.s.} \quad \text{Var}_{\frac{p}{\theta'}, [0, T]}(X^N) \leq \|X^N(\omega)\|_{p, \text{Hol}} \leq T^{\frac{\theta'}{p}} \left( \mathbb{E}(Z^{\theta'} | \mathcal{G}_N)(\omega) \right)^{\frac{1}{p}}. \quad \diamond$$

**Proof of Theorem 2.1.** As concerns  $C$ -tightness (which is quite classical, see [1], since it is the original aim of Kolmogorov's criterion), first note that  $X_0^N = \mathbb{E}(X_0 | \mathcal{G}_N)$ ,  $N \geq 1$ , is uniformly integrable hence tight. On the other hand it follows from (2.4) that the sequence of processes  $(X^N - X_0^N)_{N \geq 1}$  is  $C$ -tight (this is but the original proof of Kolmogorov criterion). The space  $(\mathcal{C}([0, T], \mathbb{R}^d), \|\cdot\|_{\text{sup}})$  being a topological space, it is clear that the sequence  $(X^N, X)_{N \geq 1}$  is  $C$ -tight on  $(\mathcal{C}([0, T], \mathbb{R}^{2d}), \|\cdot\|_{\text{sup}})$ . Let  $\mathbb{Q} = w\text{-}\lim_N \mathbb{P}_{(X^N, X)}$  denote a weak functional limiting value of  $(X^N, X)_{N \geq 1}$ . If  $Z = (Z^1, Z^2)$  denotes the canonical process on  $(\mathcal{C}([0, T], \mathbb{R}^{2d}), \|\cdot\|_{\text{sup}})$ , it is clear that  $\mathbb{Q}_{Z^2} = \mathbb{P}_X$ .

▷ *Convergence of the sup-norm.* Assume that (c) holds: the functional  $x \mapsto |x^1(t) - x^2(t)|_{L_T^r}$  is continuous on  $(\mathcal{C}([0, T], \mathbb{R}^{2d}), \|\cdot\|_{\text{sup}})$ . Consequently  $|X^{N'} - X|_{L_T^r} \xrightarrow{\mathcal{L}} |Z^1 - Z^2|_{L_T^r}$ . Hence  $Z^1 = Z^2$   $\mathbb{Q}$ -a.s.. i.e.  $\mathbb{Q} = \mathbb{P}_{X, X}$  so that  $(X^N, X) \xrightarrow{\mathcal{L}(\|\cdot\|_{\text{sup}})} (X, X)$  as  $N \rightarrow \infty$ . This means that  $\|X^N - X\|_{\text{sup}} \xrightarrow{\mathbb{P}} 0$ . On the other hand,

$$\|X^N\|_{\text{sup}}^p \leq C_{p, T} \mathbb{E}(Z^{\theta'} + |X_0|^p | \mathcal{G}_N)$$

which implies that  $(\|X^N - X\|_{\text{sup}}^p)_{N \geq 1}$  is uniformly integrable and finally that

$$\mathbb{E}(\|X^N - X\|_{\text{sup}}^p) \longrightarrow 0 \quad \text{as} \quad N \rightarrow \infty.$$

Assume that (b) holds: It follows that for every  $t_1, \dots, t_k \in D$ ,  $(X_{t_1}^N, \dots, X_{t_k}^N) \longrightarrow (X_{t_1}, \dots, X_{t_k})$  in distribution so that  $\mathbb{Q}_{Z^1}$  and  $\mathbb{P}_X$  have the same marginals. Hence  $Z^1 \sim X$  in distribution which means that  $\|X^N - X\|_{\text{sup}} \rightarrow 0$  in probability. One concludes like in (c).

If (a) holds, for every  $t \in [0, T]$ ,  $X_t^N \rightarrow X_t$   $\mathbb{P}$ -a.s.. So that condition (b) is satisfied.

▷ *Convergence of the Hölder semi-norm.* As concerns the convergence of the  $\frac{1}{p}$ -Hölder semi-norm, one proceeds as follows. Let  $q' \in (\frac{p}{\theta}, q)$  and  $\theta' := \frac{p}{q'} \in (0, \theta)$ . Assume e.g. that  $q \geq 1$ . It follows from Lemma 2.1(b)-(c)-(d) that

$$\|X - X^N\|_{q, \text{Hol}} \leq 2^{1-\frac{q'}{q}} \|X - X^N\|_{\text{sup}}^{1-\frac{q'}{q}} \times (\|X\|_{q, \text{Hol}} + \|X^N\|_{q, \text{Hol}})^{\frac{q'}{q}}.$$

Consequently  $\|X - X^N\|_{q, \text{Hol}} \xrightarrow{\mathbb{P}} 0$ . On the other hand

$$\|X - X^N\|_{q, \text{Hol}} \leq \|X\|_{q, \text{Hol}} + \|X^N\|_{q, \text{Hol}} \leq \|X\|_{q, \text{Hol}} + (\mathbb{E}(Z^q | \mathcal{G}_N))^{\frac{1}{p}}$$

so that uniform integrability implies  $\|X - X^N\|_{q, \text{Hol}} \xrightarrow{L^p} 0$ . As concerns  $q$ -variation the result follows from the inequality (2.1).  $\diamond$

**Proposition 2.1** *Let  $Y^N : \Omega \times [0, T] \rightarrow \mathbb{R}^d$ ,  $N \in \mathbb{N} \cup \{\infty\}$ , be a sequence of bi-measurable processes. Assume  $Y^N \in L_T^2$  for every  $N \in \mathbb{N} \cup \{\infty\}$  and*

$$Y^N \xrightarrow{L^2(\mathbb{P})} Y \quad (\text{i.e. } \int_0^T \mathbb{E}|Y_t^N - Y_t|^2 dt \rightarrow 0).$$

*Let  $X : \Omega \times [0, T] \rightarrow \mathbb{R}^d$  be a (bi-measurable) process. Assume that  $X \in L_{L_T^2}^p(\mathbb{P})$  for some  $p \geq 1$  and that  $X$  is bi-measurable with respect to the  $\sigma$ -field  $\mathcal{T}_Y$ -measurable where  $\mathcal{T}_Y$  denotes the smallest  $\sigma$ -field on  $[0, T] \times Y$  that makes  $(\omega, t) \mapsto Y_t(\omega)$  measurable. Then,*

$$X^N := \mathbb{E}(X | Y^N) \xrightarrow{L^p(\mathbb{P})} X.$$

**Proof.** To establish this result, one may assume without loss of generality that  $X$  is a real-valued process. Then there exists a Borel function  $\Phi : \mathbb{R} \rightarrow \mathbb{R}$  such that  $X_t(\omega) = \Phi(Y_t(\omega))$ .

Set  $\mu := (\mathbb{P} \otimes \lambda_{[0, T]}) \circ X^{-1}$ . Let  $\Phi_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$  Lipschitz continuous such that

$$\int_{\Omega} \int_0^T |\Phi(X_t) - \Phi_\varepsilon(X_t)|^2 dt d\mathbb{P} = \int_{\mathbb{R}} |\Phi - \Phi_\varepsilon|^2 d\mu \leq \varepsilon^2.$$

It follows from the contraction property of the conditional expectation in  $L_{L_T^2}^p$  that

$$\begin{aligned} \|X - X^N\|_{L_{L_T^2}^p} &\leq \|X - \Phi(Y^N)\|_{L_{L_T^2}^p} + \|\mathbb{E}(\Phi(Y^N) - X | Y^N)\|_{L_{L_T^2}^p} \\ &\leq 2\|X - \Phi_\varepsilon(Y^N)\|_{L_{L_T^2}^p}. \end{aligned}$$

Now

$$\|X - \Phi_\varepsilon(Y^N)\|_{L^p_{L^2_T}} \leq \|X - \Phi_\varepsilon(Y)\|_{L^p_{L^2_T}} + [\Phi_\varepsilon]_{\text{Lip}} \|Y - Y^N\|_{L^p_{L^2_T}}$$

Consequently, for every  $\varepsilon > 0$ ,

$$\limsup_N \|X - X^N\|_{L^p_{L^2_T}} \leq \inf_{\varepsilon > 0} \|\Phi(Y) - \Phi_\varepsilon(Y)\|_{L^p_{L^2_T}} = 0. \quad \diamond$$

### 3 Application to the convergence of quantized *SDE*'s

#### 3.1 From Itô to Stratanovich

An *SDE*

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \quad X_0 \in L^p_{\mathbb{R}^d}(\mathbb{P})$$

where  $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  and  $\sigma : [0, T] \times \mathbb{R}^d \rightarrow \mathcal{M}(d \times q)$  are smooth enough functions (*e.g.* continuously differentiable with bounded differentials) and  $W = (W_t)_{t \in [0, T]}$  is a  $q$ -dimensional Brownian motion. First note that without loss of generality one may assume that  $q = d$  by increasing the dimension of  $W$  or adding some identically zero components to  $X$  (no ellipticity like assumption is needed here). This *SDE* can be written in the Stratanovich sense as follows

$$dX_t = f(X_t) \circ dW_t, \quad X_0 \in L^p_{\mathbb{R}^d}(\mathbb{P}), \quad (3.1)$$

where, for notational convenience  $W = (W^0, W^1, \dots, W^q)$  stands for  $(t, W_t)$ ,  $X_t = (X_t^0, X_t^1, \dots, X_t^d)$  stands for  $(t, X_t)$  and  $f : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}_+ \times \mathbb{R}^d$  (with  $f^0(t, x) = t$ ) is a differentiable function with bounded differentials.

Following rough paths theory developed by T. Lyons ([17]) and many co-authors (see [10] for an introduction), one can also solve this equation in the sense of rough paths with finite  $p$ -variation,  $p \geq 2$ , since we know (*e.g.* from the former Kolmogorov criterion) that  $W$  *a.s.* does have finite  $p$ -variation, for any  $p > 2$ . Namely this means solving an equation formally reading

$$dx_t = f(x_t)dy_t, \quad x_0 \in \mathbb{R}^d. \quad (3.2)$$

In this equation  $\mathbf{y}$  does not represent the path (null at 0)  $(y_t)$  associated to  $y_t = W_t(\omega)$  itself but the so-called *geometric multiplicative functional lying on  $y$*  with controlled  $p$ -variation, namely a couple  $\mathbf{y} = ((\mathbf{y}_{s,t}^1)_{0 \leq s \leq t \leq T}, (\mathbf{y}_{s,t}^2)_{0 \leq s \leq t \leq T})$  where  $\mathbf{y}_{s,t}^1 = y_t - y_s \in \mathbb{R}^{d+1}$ ,  $0 \leq s \leq t \leq T$ , can be identified with the path  $(y_t)$  and  $(\mathbf{y}_{s,t}^2)_{0 \leq s \leq t \leq T}$  satisfies,  $\mathbf{y}_{s,t}^2 \in \mathbb{R}^{(d+1)^2}$  for every  $0 \leq s \leq u \leq t \leq T$  and the following tensor multiplicative property

$$\mathbf{y}_{s,t}^2 = \mathbf{y}_{s,u}^2 + \mathbf{y}_{u,t}^2 + \mathbf{y}_{s,u}^1 \otimes \mathbf{y}_{u,t}^1.$$

Different choices for this functional are possible, leading to different solutions to the above Equation (3.2). Thus, the choice that makes coincide *a.s.* the solution of (3.1) and the pathwise solutions of (3.2) is given by

$$\mathbf{y}_{s,t}^2 := \left( \int_s^t (W_u^i - W_s^i) \circ dW_u^j \right)_{i,j=0,\dots,d} (\omega) \quad (3.3)$$

so that

$$\mathbf{y}_{s,u}^1 \otimes \mathbf{y}_{u,t}^1 = (\mathbf{y}_{s,u}^{1,i} \mathbf{y}_{s,u}^{1,j})_{i,j=0,\dots,d}.$$

When  $p \in [2, 3)$ , the *p-variation distance* between two such multiplicative functionals  $\mathbf{y}, \mathbf{z}$  is defined by

$$\delta_p(\mathbf{y}, \mathbf{z}) = \text{Var}_{p,[0,T]}(\mathbf{y}^1 - \mathbf{z}^1) + \text{Var}_{p/2,[0,T]}(\mathbf{y}^2 - \mathbf{z}^2)$$

where

$$\text{Var}_{q,[0,T]}(\mathbf{y}^2) := \sup \left\{ \left( \sum_{\ell=0}^{k-1} |\mathbf{y}_{t_\ell, t_{\ell+1}}^2|^q \right)^{\frac{1}{q}}, 0 \leq t_0 \leq t_1 \leq \dots \leq t_k \leq T, k \geq 1 \right\}.$$

**Remark.** Likewise one defines the Hölder distance by setting

$$\rho_p(\mathbf{y} - \mathbf{x}) = \|\mathbf{y}^1 - \mathbf{x}^1\|_{p, \text{Hol}} + \|\mathbf{y}^2 - \mathbf{x}^2\|_{p/2, \text{Hol}}$$

where

$$\|\mathbf{x}^2\|_{p/2, \text{Hol}} := \sup_{s,t \in [0,T]} \frac{|\mathbf{x}_t^2 - \mathbf{x}_s^2|}{|t - s|^{\frac{2}{p}}}$$

The following theorem describes the continuity of the so-called Itô map  $\mathbf{y} \mapsto x$  for the  $\delta_p$ -convergence.

**Theorem 3.1** (see [10, 17]) *Let  $f \in \mathcal{C}^{2+\alpha}(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R}_+ \times \mathbb{R}^d)$ ,  $\alpha > 0$ , bounded with bounded first two differentials, and let  $\mathbf{y}$  be a multiplicative functional satisfying  $\delta_p(\mathbf{y}, 0) < +\infty$  for every  $p \in (2, 2 + \alpha)$ . Then Equation (3.2) has a unique solution.*

*When  $\mathbf{y}$  is given by (3.3), the solution a.s. coincides with that of the SDE in the Stratonovich sense.*

*Furthermore, let  $(\mathbf{y}^N)_{N \geq 1}$  be a sequence of multiplicative functionals satisfying  $\delta_p(\mathbf{y}^N, \mathbf{y}) \rightarrow 0$  as  $N \rightarrow \infty$ , for every  $p \in (2, 2 + \alpha)$ . Then, the sequence  $(x^N)_{N \geq 1}$  of (unique) solutions to the differential equations  $dx_t^N = f(x_t^N) d\mathbf{y}_t^N$ ,  $x_0^N = x_0$ ,  $N \geq 1$ , satisfy*

$$\forall p \in (2, 2 + \alpha), \quad \text{Var}_{p,[0,T]}(x - x^N) \xrightarrow{N \rightarrow \infty} 0.$$

**Remark.** Following Theorem 4.1 in [5], one also has convergence result with respect to the Hölder norm  $\rho_p$ . To be precise, if  $f$  is three times differentiable on  $\mathbb{R}^d$  and globally Lipschitz, then the above theorem remains true by replacing *mutatis mutandis*,  $\delta_p$  by  $\rho_p$ .

### 3.2 Quantization of the SDE

Let  $(\alpha^N)_{N \geq 1}$  denote a sequence of quantizers of the Brownian motion. Each  $\alpha^N$  is made up of  $N$  functions (or elementary quantizer)  $\alpha_n^N : [0, T] \rightarrow \mathbb{R}^d$ ,  $n = 1, \dots, N$ . For convenience component “0” will be added accordingly to each elementary quantizer  $\alpha_n^N$  by setting  $\alpha_n^{N,0}(t) = t$  (which exactly quantizes the function  $W_t^0 = t$ ). We assume that every elementary quantizer  $\alpha_n^N$  is a continuous function with finite variation over  $[0, T]$ .

From now on, our aim is to rely on this theorem to establish that the solutions of the ODE’s

$$d\tilde{x}_n^N(t) = f(\tilde{x}_n^N(t)) d\alpha_n^N(t), \quad \tilde{x}_n^N(0) = x_0, \quad n = 1, \dots, N,$$



produce a sequence of (non Voronoi) converging quantization of the solution  $X$  of the SDE (3.1) when  $N \rightarrow \infty$  by setting

$$\tilde{X}^N = \sum_{n=1}^N \tilde{x}_n^N \mathbf{1}_{\{W \in C_n(\alpha^N)\}}$$

(keep in mind that  $\{W \in C_n(\alpha^N)\} = \{\widehat{W}^N = \alpha_n^N\}$ ,  $n = 1, \dots, N$ ). At this stage, it is quite classical background using the definition of the Stratanovich integral and the independence of the components  $W^i$ , to note that, for every  $i, j \in \{1, \dots, d\}$ ,

$$\int_s^t (W_u^i - W_s^i) \circ dW_u^j = \begin{cases} \int_0^t W_u^i dW_u^j - \int_0^s W_u^i dW_u^j - W_s^i(W_t^j - W_s^j), & \text{if } i \neq j \\ \int_0^t W_u^i dW_u^i - \int_0^s W_u^i dW_u^i + \frac{1}{2}(t-s) - W_s^i(W_t^i - W_s^i), & \text{if } i = j. \end{cases}$$

When  $i$  or  $j = 0$ , the computation is trivial. As concerns  $\widehat{W}^N$ , which is made of finitely many smooth paths  $\alpha_n^{(N)}$ , the multiplicative functional  $\widehat{\mathbf{y}}^N$  is defined formally by replacing the Stratanovich integral by a standard Stieltjes integral.

Then first classical step is to prove that  $\delta_p$ -convergence of  $\widehat{W}^N$  to  $W$  reduces to that of approximate Lévy areas  $\int_0^t \widehat{W}_s^{N,i} d\widehat{W}_s^{N,j}$  toward the true ones  $\int_0^t W_s^i dW_s^j$ ,  $i \neq j$ . This follows from the following easy abstract lemma which proof, based on Lemma 2.1, is left to the reader

**Lemma 3.1** *Let  $p \geq 2$ . Let  $(W^N)_{N \geq 1}$  be a sequence of  $\mathbb{R}^d$ -valued processes such that  $\|\text{Var}_{p,[0,T]}(W^i - W^{N,i})\|_p \rightarrow 0$  as  $N \rightarrow \infty$ . Then (still using the convention  $W_t^0 = W_t^{N,0} = t$ ), for every  $i, j \in \{0, \dots, d\}$*

$$\begin{aligned} \|\text{Var}_{p,[0,T]}((W_s^i(W_t^j - W_s^j) - W_s^{N,i}(W_t^{N,j} - W_s^{N,j}))\|_p &\rightarrow 0 & \text{if } i \neq j, \\ \|\text{Var}_{p,[0,T]}((W_s^i(W_t^i - W_s^i) - W_s^{N,i}(W_t^{N,i} - W_s^{N,i}))\|_{p/2} &\rightarrow 0 & \text{if } i = j. \end{aligned}$$

It follows that, if the assumption of this lemma is satisfied by a sequence  $(W^N)_{N \geq 1}$ , the  $\delta_p$ -continuity of the Itô map for the  $p$ -variation reduces in this simple case to considering the approximation in  $p$ -variation of the Lévy areas  $\int_0^t W_s^i \circ dW_s^j = \int_0^t W_s^i dW_s^j$ ,  $i \neq j$ , and  $\int_0^t W_s^i \circ dW_s^i = \int_0^t W_s^i dW_s^i + \frac{1}{2}t$  by their counterpart defined using  $W^N$ .

**Remark.** The same holds true with respect to the  $p$ -Hölder semi-norm.

### 3.3 Rate optimal quantizers

**Proposition 3.1** *Let  $(\widehat{W}^N)_{N \geq 1}$  be a sequence of stationary rate optimal quadratic functional quantizers of a standard  $d$ -dimensional Brownian motion  $W$  defined by (1.10) or (1.13). Then*

$$\forall q > \frac{2p}{p-2}, \quad \text{Var}_{q,[0,T]}(W - \widehat{W}^N) \xrightarrow{L^p} 0 \quad \text{as } N \rightarrow \infty.$$

**Proof.** This follows from Theorem 2.1(c) since  $W$  satisfies the Kolmogorov criterion  $K_{p,p/2-1}$  for every  $p > 2$  and the stationarity property (1.15).  $\diamond$

**Remark.** Rate optimality of  $(\widehat{W}^N)_{N \geq 1}$  is not necessary to derive this convergence.

**Lemma 3.2** *If  $1 \leq i \neq j \leq d$ , the Lévy area  $\left(\int_0^t W_s^i dW_s^j\right)_{t \in [0, T]}$  satisfies the Kolmogorov criterion  $(K_{p, p/2-1})$  for every  $p > 2$ .*

**Proof.** This is a classical result that derives from a simple application of B.D.G. Inequality and Holder Inequality.  $\diamond$

**Lemma 3.3** *Let  $W$  be a (real-valued) standard Brownian motion.*

(a) *For every  $\varphi \in L^2_{L^2_T}(\mathcal{F}_0, \mathbb{P})$*

$$\int_0^t \varphi(s, \cdot) dW_s = \sqrt{\frac{2}{T}} \sum_{k \geq 1} \xi_k \int_0^t \varphi(s, \cdot) \cos(s/\sqrt{\lambda_k}) ds \quad (3.4)$$

where  $\xi_k := (W|e_k^W)_2/\sqrt{\lambda_k}$  are independent,  $\mathcal{N}(0, 1)$ -distributed (see (1.8) and (1.9)) and independent of  $\varphi$ .

(b) *For every  $\varphi \in L^2_{L^2_T}(\mathcal{F}_0, \mathbb{P})$*

$$\int_0^t \varphi(s, \cdot) d\widehat{W}_s = \sqrt{\frac{2}{T}} \sum_{k \geq 1} \frac{(\widehat{W}|e_k^W)_2}{\sqrt{\lambda_k}} \int_0^t \varphi(s, \cdot) \cos(s/\sqrt{\lambda_k}) ds. \quad (3.5)$$

**Proof.** (a) Set for every  $\varphi \in L^2_{L^2_T}(\mathcal{F}_0, \mathbb{P})$ ,

$$\begin{aligned} J_T(\varphi) &:= \sqrt{\frac{2}{T}} \sum_{k \geq 1} \xi_k \sqrt{\lambda_k} \int_0^T \varphi(s, \cdot) d \sin(s/\sqrt{\lambda_k}) \\ &= \sqrt{\frac{2}{T}} \sum_{k \geq 1} \xi_k \int_0^T \varphi(s, \cdot) \cos(s/\sqrt{\lambda_k}) ds. \end{aligned} \quad (3.6)$$

This defines clearly an isometry from  $L^2_{L^2_T}(\mathcal{F}_0, \mathbb{P})$  into the Gaussian space spanned by  $(\xi_n)_{n \geq 1}$  since

$$\mathbb{E}(J_T(\varphi)^2) = \frac{2}{T} \sum_{k \geq 1} \mathbb{E}(\xi_k^2) \mathbb{E} \left( \int_0^T g(s) \frac{1}{\sqrt{\lambda_k}} \cos(s/\sqrt{\lambda_k}) ds \right)^2 = \mathbb{E} \int_0^T g^2(t) dt.$$

The last equality uses that the sequence  $(\sqrt{\frac{2}{T}} \cos(\pi(k - \frac{1}{2})t))_{k \geq 1}$  is an orthonormal basis of  $L^2_T$ . Finally, note that for every  $t \in [0, T]$ ,  $J_T(\mathbf{1}_{[0, t]}) = \sqrt{\frac{2}{T}} \sum_{k \geq 1} \sqrt{\lambda_k} \xi_k \sin(t/\sqrt{\lambda_k}) = W_t$ . This proves that  $J_T = I_T$  i.e. is but the (extended) Wiener integral with respect to  $W$ .

(b) This follows by taking the (functional) conditional expectation of (3.4).  $\diamond$

Let  $\mathcal{C}_b^r([0, T] \times \mathbb{R}^d)$   $r > 0$ , denote the set of  $[r]$ -times differentiable bounded functions  $f : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  with bounded partial derivatives up to order  $[r]$  and whose partial derivatives of order  $[r]$  are  $(r - [r])$ -Hölder.

**Theorem 3.2** Let  $p \geq 2$ . Let  $(\widehat{W}^N)_{N \geq 1}$  be a sequence of stationary quadratic functional quantizers of the Brownian motion converging to  $W$  in  $L^2_T(\mathbb{P})$ .

(a) For every  $p \geq 2$  and every  $q > \frac{2p}{p-2}$ ,

$$\begin{aligned} \text{Var}_{q,[0,T]} \left( \int_0^\cdot W_s^i dW_s^j - \int_0^\cdot \widehat{W}_s^{N,i} d\widehat{W}_s^{N,j} \right) &\xrightarrow{L^p} 0 \quad \text{as } N \rightarrow \infty \\ \text{Var}_{q,[0,T]} \left( \int_0^\cdot W_s^i d \circ W_s^i - \int_0^\cdot \widehat{W}_s^{N,i} d\widehat{W}_s^{N,i} \right) &\xrightarrow{L^p} 0 \quad \text{as } N \rightarrow \infty. \end{aligned}$$

(b) Let  $f : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}$  be  $\mathcal{C}^{2+\alpha}$  ( $\alpha > 0$ ). Consider for every  $N \geq 1$ , the solutions of the quantized ODE's

$$d\widehat{X}_t^N = f(\widehat{X}_t^N) d\widehat{W}_t^N, \quad N \geq 1.$$

Then, for every  $q > 2$ ,

$$\text{Var}_{q,[0,T]}(\widehat{X}^N - X) \xrightarrow{\mathbb{P}} 0.$$

**Remarks.** • There is an abuse of notation in the above claim (b) since  $\widehat{X}^N$  is not a Voronoi quantizer of  $X$ : this quantization of  $X$  is defined on the Voronoi partition induced by the quantization of the Brownian motion  $W$ .

- Using the same approach based on Theorem 2.1(c) one can retrieve the classical results about “natural” rough paths of the Brownian motion defined as the linear interpolation of the Brownian motion at times  $t_k$ ,  $k = 0, 1, \dots, n$ .
- Other extensions to the Brownian bridge or the Ornstein-Uhlenbeck processes and more generally to Gaussian semi-martingales that satisfy the Kolmogorov criterion are possible by this method.

**Proof.** (a) When  $i \neq j$ , this is a straightforward consequence of Theorem 2.1(c). When  $i = j$ ,

$$\int_0^t \widehat{W}_s^i d\widehat{W}_s^i = \frac{1}{2}(\widehat{W}_t^i)^2 \quad \text{and} \quad \int_0^t W_s^i d \circ W_s^i = \frac{1}{2}(W_t^i)^2$$

and

$$\begin{aligned} \text{Var}_{q,[0,T]}((W^i)^2 - (\widehat{W}^i)^2) &\leq \left( \|W^i\|_{\text{sup}} + \|\widehat{W}^i\|_{\text{sup}} \right) \text{Var}_{q,[0,T]}(W^i - \widehat{W}^i) \\ &\leq \left( \|W^i\|_{\text{sup}} + \mathbb{E}(\|W^i\|_{\text{sup}} | \widehat{W}^i) \right) \text{Var}_{q,[0,T]}(W^i - \widehat{W}^i). \end{aligned}$$

This finally implies that for every  $q > \frac{2p}{p-2}$

$$\text{Var}_{q,[0,T]} \left( \int_0^\cdot W_s^i d \circ W_s^i - \int_0^\cdot \widehat{W}_s^{N,i} d\widehat{W}_s^{N,i} \right) \xrightarrow{L^p} 0. \quad \diamond$$

## 4 Toward some rate of convergence

We first consider two *independent* Brownian motions  $B$  and  $W$ . All the subsequent quantizations are designed from some pathwise continuous stationary quantizers.

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  an  $\alpha$ -Holder function,  $\alpha \in (0, 1]$ . Then, for every  $p \geq 1$ , it follows from B.D.G. and Jensen Inequality that

$$\begin{aligned} \mathbb{E} \left( \sup_{t \in [0, T]} \left| \int_0^t (f(W_s) - f(\widehat{W}_s)) d\widehat{B}_s \right|^p \right) &\leq \mathbb{E} \left( \sup_{t \in [0, T]} \left| \int_0^t (f(W_s) - f(\widehat{W}_s)) dB_s \right|^p \right) \\ &\leq C_{p, T} \int_0^T \mathbb{E} |f(W_s) - f(\widehat{W}_s)|^p ds \\ &\leq C_{p, T} [f]_\alpha^p \int_0^T \mathbb{E} |W_s - \widehat{W}_s|^{p\alpha} ds \end{aligned}$$

so that

$$\left\| \sup_{t \in [0, T]} \left| \int_0^t (f(W_s) - f(\widehat{W}_s)) d\widehat{B}_s \right| \right\|_p \leq C_{p, T} \| |W - \widehat{W}|_{L_T^{p\alpha}} \|_p. \quad (4.1)$$

To obtain some global rates of convergence we focus on the second error term which writes in full generality

$$\int_0^T \varphi(s, \cdot) d(B_s - \widehat{B}_s) = \sum_{n \geq 1} (B - \widehat{B} | e_n^W)_2 \int_0^T \varphi(s, \cdot) de_n^W(s)$$

where  $\varphi(t, \omega) = f(W_t(\omega))$ .

From now on, we make the assumption that,

for every  $N \geq 1$ ,  $\widehat{B}^N$  is an optimal product  $N$ -quantization of  $B$

given by (1.10) where the sequence  $(N_n)$  of integral bit allocation satisfies (1.11).

Then,  $(\widehat{B}^N | e_n^W)_2$  is a quantization of  $(B | e_n^W)_2$ . To be precise, if  $(\xi_n)_{n \geq 1}$  denotes the  $\mathcal{N}(0; 1)$ -distributed i.i.d. sequence in the Karhunen-Loève expansion of  $B$ ,

$$(B - \widehat{B}^N | e_n^W)_2 = \sqrt{\lambda_n} (\xi_n - \widehat{\xi}_n^{N_n})$$

where  $\widehat{\xi}_n^{N_n}$  are optimal quantizations of  $\xi_n$

$$\mathbb{E} \left( \left| \int_0^T \varphi(s, \omega) d(B_s - \widehat{B}_s) \right|^2 \middle| \mathcal{F}_T^W \right) = \sum_{n \geq 1} \mathbb{E} (\xi_n - \widehat{\xi}_n^{N_n})^2 \lambda_n \left( \int_0^T \varphi(s, \cdot) de_n^W(s) \right)^2$$

so that, owing to Equation (1.2) (Zador's Theorem),

$$\mathbb{E} \left( \left| \int_0^T \varphi(s, \omega) d(B_s - \widehat{B}_s^N) \right|^2 \right) \leq C \sum_{n \geq 1} \frac{1}{n^2 N_n^2} \mathbb{E} \left( \int_0^T \varphi(s, \cdot) de_n^W(s) \right)^2.$$

Assume now that  $f$  is  $\mathcal{C}^2$ . Then,

$$\begin{aligned} \int_0^T f(W_s) de_n^W(s) &= \sqrt{\frac{2}{T}} \left[ f(0) \sin \left( \frac{T}{\sqrt{\lambda_n}} \right) + \int_0^T \left( \sin \left( \frac{T}{\sqrt{\lambda_n}} \right) - \sin \left( \frac{s}{\sqrt{\lambda_n}} \right) \right) f'(W_s) dW_s \right. \\ &\quad \left. + \frac{1}{2} \int_0^T \left( \sin \left( \frac{T}{\sqrt{\lambda_n}} \right) - \sin \left( \frac{s}{\sqrt{\lambda_n}} \right) \right) f''(W_s) ds \right]. \end{aligned}$$

Then, if  $\sup_{t \in [0, T]} \mathbb{E}(f'(W_t))^2 + \mathbb{E}(f''(W_t))^2 < +\infty$ ,

$$\mathbb{E} \left( \int_0^T f(W_s) de_n^W(s) \right)^2 = C_T \left( f(0)^2 + \sup_{t \in [0, T]} \mathbb{E}(f'(W_t))^2 + \mathbb{E}(f''(W_t))^2 \right) < +\infty$$

where  $C_T$  denotes a positive real constant. Consequently

$$\left\| \int_0^T f(W_s) d(B_s - \widehat{B}_s^N) \right\|_2^2 \leq C_T \sum_{n \geq 1} \frac{1}{n^2 N_n^2}. \quad (4.2)$$

It follows from the optimal integral bit allocation (1.11) of the  $N_n$ 's that

$$\left\| \int_0^T f(W_s) d(B_s - \widehat{B}_s) \right\|_2^2 \leq C_T \|B - \widehat{B}^N\|_2 \leq C_T (\log N)^{-1}. \quad (4.3)$$

Combining this estimate with (4.1) yields the following proposition.

**Proposition 4.1** *Suppose  $f$  is  $C^2$  and  $f'$  is bounded and  $\sup_{t \in [0, T]} \|f''(W_t)\|_2 < +\infty$ . Suppose that  $\widehat{W}^N$ ,  $N \geq 1$ , is a rate optimal sequence of  $N$ -quantizations of  $W$  and that  $\widehat{B}^N$ ,  $N \geq 1$ , is a sequence of optimal  $N$ -product quantizations of  $B$ . Then*

$$\left\| \int_0^T f(W_s) dB_s - \int_0^T f(\widehat{W}_s^N) d\widehat{B}_s^N \right\|_2 \leq C_T (\log N)^{-\frac{1}{2}}. \quad (4.4)$$

APPLICATION. The optimal product quantization of the Lévy area of  $(W^1, W^2)$  converges in  $L^2(\mathbb{P})$  toward the Lévy area at a  $O((\log N)^{-\frac{1}{2}})$ -rate.

This result can be slightly improved since it is established in [9] that the sequence of optimal  $N$ -product quantizers remains rate optimal in  $L^p(\mathbb{P})$  as long as  $0 < p < 3$ . This means that the above rate remains true in  $L^p(\mathbb{P})$ ,  $0 < p < 3$  at least if one considers a sequence of optimal  $N$ -product quantizers for  $W$  as well.

A careful reading and understanding of the proof of Theorem 3.1 should probably lead to some *a priori* error bounds in a much more general setting. A  $O((\log N)^{-\frac{1}{2}})$ -rate should hold for a wide class of multi-dimensional *SDE* (in the Stratonovich sense), extending the constructive results obtained in [13].

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## Annex: Functional conditional expectation

Let  $(X_t)_{t \in [0, T]}$  be a bi-measurable process defined on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  such that

$$\int_0^T \mathbb{E}(X_t^2) dt < +\infty.$$

One can consider  $X$  as a random variable  $X : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow L_T^2 := L^2([0, T], dt)$  and more precisely as an element of the Hilbert space

$$L_{L_T^2}^2(\Omega, \mathcal{A}, \mathbb{P}) := \left\{ Y : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow L_T^2, \mathbb{E}|Y|_{L_T^2}^2 < +\infty \right\}$$

where  $|f|_{L_T^2}^2 = \int_0^T f^2(t) dt$ . For the sake of simplicity, one denotes  $\|Y\|_2 := \sqrt{\mathbb{E}|Y|_{L_T^2}^2}$ . If  $\mathcal{B}$  denotes a sub- $\sigma$ -field of  $\mathcal{A}$  (containing all  $\mathbb{P}$ -negligible sets of  $\mathcal{A}$ ) then  $L_{L_T^2}^2(\Omega, \mathcal{B}, \mathbb{P})$  is a closed sub-space of  $L_{L_T^2}^2(\Omega, \mathcal{A}, \mathbb{P})$  and one can define the *functional conditional expectation* of  $X$  by

$$\mathbb{E}(X | \mathcal{B}) := \text{Proj}_{L_{L_T^2}^2(\Omega, \mathcal{B}, \mathbb{P})}^\perp(X).$$

Functional conditional expectation can be extended to bi-measurable processes  $X$  such that  $\|X\|_1 := \mathbb{E}|X|_{L_T^1} < +\infty$  following the approach used for  $\mathbb{R}^d$ -valued random vectors. Then,  $\mathbb{E}(X | \mathcal{B})$  is characterized by: for every  $\mathcal{B}([0, T]) \otimes \mathcal{B}$ -bi-measurable process  $Z = (Z_t)_{t \in [0, T]}$ , bounded by 1,

$$\mathbb{E} \int_0^T Z_t X_t dt = \mathbb{E} \int_0^T Z_t \mathbb{E}(X | \mathcal{B})_t dt.$$

Furthermore, as soon as the process  $(\mathbb{E}(X_t | \mathcal{B}))_{t \in [0, T]}$  has a  $\mathcal{B}([0, T]) \otimes \mathcal{B}$  bi-measurable version, one straightforwardly checks that one may define a modification of the functional conditional expectation by setting

$$\mathbb{E}(X | \mathcal{B})_t = \mathbb{E}(X_t | \mathcal{B}), \quad t \in [0, T].$$

EXAMPLES: (a) Let  $\mathcal{B} := \sigma(\mathcal{N}_A, B_i, i \in I)$  where  $(B_i)_{i \in I}$  is a finite measurable partition of  $\Omega$  such that  $\mathbb{P}(B_i) > 0, i \in I$ .

(b) Let  $X := (W_t)_{t \in [0, T]}$  a standard Brownian motion in  $\mathbb{R}^d$  and let  $\mathcal{B} := \sigma(W_{t_1}, \dots, W_{t_n})$  where  $0 = t_0 < t_1 < \dots < t_n = T$ . Then

$$\forall t \in [t_k, t_{k+1}), \quad \mathbb{E}(W | \mathcal{B})_t = W_{t_k} + \frac{t - t_k}{t_{k+1} - t_k} (W_{t_{k+1}} - W_{t_k}).$$