



**HAL**  
open science

## Minimization of $\lambda_2(\Omega)$ with a perimeter constraint

Dorin Bucur, Antoine Henrot

► **To cite this version:**

Dorin Bucur, Antoine Henrot. Minimization of  $\lambda_2(\Omega)$  with a perimeter constraint. 2008. hal-00201946v1

**HAL Id: hal-00201946**

**<https://hal.science/hal-00201946v1>**

Preprint submitted on 3 Jan 2008 (v1), last revised 2 Jun 2009 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Minimization of $\lambda_2(\Omega)$ with a perimeter constraint

Dorin Bucur

Laboratoire de Mathématiques (LAMA) UMR 5127, Université de Savoie  
Campus Scientifique 73 376 Le-Bourget-Du-Lac, France  
email: dorin.bucur@univ-savoie.fr

Antoine Henrot

Institut Élie Cartan, UMR 7502, Nancy Université - CNRS - INRIA  
B.P. 239 54506 Vandoeuvre les Nancy Cedex, France  
email: henrot@iecn.u-nancy.fr

January 3, 2008

## Abstract

We study the problem of minimizing the second Dirichlet eigenvalue for the Laplacian operator among sets of given perimeter. In two dimension, we prove that the optimum exists, is convex, regular, symmetric and its boundary contains exactly two points where the curvature vanishes. In  $N$  dimension, we prove existence of a minimizer in a slightly different (relaxed) class and we prove, assuming enough regularity, that this minimizer has cylindrical symmetry.

## 1 Introduction

Let  $\Omega$  be a bounded open set in  $\mathbb{R}^N$  and let us denote by  $0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \lambda_3(\Omega) \dots$  its eigenvalue for the Laplacian operator with homogeneous Dirichlet boundary condition. Problems linking the shape of a domain to the sequence of its eigenvalues, or some of them, are among the most fascinating of mathematical analysis or differential geometry. In particular, problems of minimization of eigenvalues, or combination of eigenvalues, brought about many deep works since the early part of the twentieth century. Actually, this question appears first in the famous book of Lord Rayleigh "The theory of sound". Thanks to some explicit computations and "physical evidence", Lord Rayleigh conjectured that the disk should minimize the first Dirichlet eigenvalue  $\lambda_1$  of the Laplacian among plane open sets of given area. This result has been proved later by Faber and Krahn using a rearrangement technique. Then, many other similar "isoperimetric problems" have been considered. For a survey on these questions, we refer

to the papers [3], [5], [23], [26] or the recent book [17]. Usually, in these minimization problems, one works in the class of sets with a given measure. In this paper, we choose to look at a similar problem but with a constraint on the perimeter of the sets. Apart the mathematical own interest of this question, the reason which led us to consider this problem is the following. Studying the famous gap problem (see [25]), we were interested in minimizing  $\lambda_2(\Omega) - \lambda_1(\Omega)$ , and more generally  $\lambda_2(\Omega) - k\lambda_1(\Omega)$ , with  $0 \leq k \leq 1$ , among (convex) open sets of given diameter. Looking at the limiting case  $k = 0$ , we realized that the optimal set (which does exist) is a body of constant width. Since any body of constant width have the same perimeter in dimension two, we were naturally led to consider the problem of minimizing  $\lambda_2(\Omega)$  among sets of given perimeter. In particular, if the solution was a ball (or more generally a body of constant width), it would give the answer to the previous problem. Unfortunately, as it is shown in Theorem 2.6, it is not the case! The minimizer that we are able to identify and characterize here (at least in two dimension) is a particular regular convex body, with (at least) one axis of symmetry and two points on its boundary where the curvature vanishes. It is worth observing that the four following minimization problems for the second eigenvalue have different solutions:

- with a volume constraint: two identical disks (see [20] or [17]),
- with a volume and a convexity constraint: a stadium-like set (see [18]),
- with a perimeter constraint: the convex set described in this paper,
- with a diameter constraint: we conjecture that the solution is a disk.

Let us finally remark that the same problems for the first eigenvalue all have the disk as the solution thanks to Faber-Krahn inequality and classical isoperimetric inequality.

This paper is organized as follows: section 2 is devoted to the complete study of the two-dimensional problem. We first prove existence of a minimizer and its  $C^\infty$  regularity. Then, we prove that it has (at least) one axis of symmetry. Note that the first author with E. Oudet was unable to prove such a symmetry result for the same problem with convexity and area constraint in [18]! In the end of this section, we give some other qualitative and geometric properties of the minimizer. In section 3, we consider the problem in higher dimension. This one is much more complicated since we cannot use here the trick of convexification. We first give an equivalent relaxed formulation of the problem, then we prove existence of a minimizer for the relaxed formulation. At last, we prove that the optimum (assuming some regularity) has cylindrical symmetry.

## 2 The two-dimensional case

### 2.1 Existence, regularity

We want to solve the minimization problem

$$\min\{\lambda_2(\Omega), \Omega \subset \mathbb{R}^2, P(\Omega) \leq c\} \quad (1)$$

where  $\lambda_2(\Omega)$  is the second eigenvalue of the Laplacian with Dirichlet boundary condition on the bounded open set  $\Omega$  and  $P(\Omega)$  denotes the perimeter (e.g. in the sense of De Giorgi) of  $\Omega$ . The monotonicity of the eigenvalues of the Dirichlet-Laplacian with respect to the inclusion has two easy consequences:

1. Let  $\Omega^*$  denotes the convex hull of  $\Omega$ , since **in two dimension**,  $P(\Omega^*) \leq P(\Omega)$ , it is clear that one can restrict ourselves to look for minimizers in the class of **convex sets**.
2. Obviously, it is equivalent to consider the constraint  $P(\Omega) \leq c$  or  $P(\Omega) = c$ .

Of course, point 1. easily implies existence, see below, but is no longer true in higher dimension which makes the existence proof much harder, see Theorem 3.2.

**Theorem 2.1** *There exists a minimizer  $\Omega$  for problem (1) and  $\Omega$  is  $C^\infty$ .*

**Proof :**

**Existence** We use the direct method of calculus of variations. Let  $\Omega_n$  be a minimizing sequence, according to point 1. above, we can assume that  $\Omega_n$  is a sequence of convex sets. Moreover, it is a bounded sequence because of the perimeter constraint. Therefore, there exists a convex domain  $\Omega$  and a subsequence still denoted by  $\Omega_n$  such that:

- $\Omega_n$  converges to  $\Omega$  for the Hausdorff metric or for the topology given by characteristic functions (see e.g. [19, Theorem 2.4.10]);
- $P(\Omega) \leq c$  (lower semi-continuity of the perimeter for the convergence of characteristic functions, see [19, Proposition 2.3.6]);
- $\lambda_2(\Omega_n) \rightarrow \lambda_2(\Omega)$  (continuity of the eigenvalues for the  $\gamma$ -convergence, see [9, Proposition 2.4.6] or [17, Theorem 2.3.17]).

Therefore,  $\Omega$  is a solution of problem (1).

**Regularity** The proof of regularity is classical, see e.g. [10]. We refer also to [6] and [7] for similar results in a more complicated context. First of all, we need the simplicity of the second eigenvalue:

**Lemma 2.2** *Let  $\Omega$  be a minimizer of problem (1), then  $\lambda_2(\Omega)$  is simple.*

For the proof of this lemma, see the end of this section. We go on with the proof of regularity. Let us consider (locally) the boundary of  $\partial\Omega$  as the graph of a (concave) function  $h(x)$ , with  $x \in (-a, a)$ . We make a perturbation of  $\partial\Omega$  using a regular function  $\psi$  compactly supported in  $(-a, a)$ , i.e. we look at  $\Omega_t$  whose boundary is  $h(x) + t\psi(x)$ . The function  $t \mapsto P(\Omega_t)$  is differentiable at  $t = 0$  (see [15] or [19]) and its derivative  $dP(\Omega, \psi)$  at  $t = 0$  is given by:

$$dP(\Omega, \psi) := \int_{-a}^{+a} \frac{h'(x)\psi'(x) dx}{\sqrt{1 + h'(x)^2}}. \quad (2)$$

In the same way, the function  $t \mapsto \lambda_2(\Omega_t)$  is differentiable (see [19, Theorem 5.7.1]) and since the second (normalized) eigenfunction  $u_2$  belongs to the Sobolev space  $H^2(\Omega)$  (due to the convexity of  $\Omega$ , see [16, Theorem 3.2.1.2]), its derivative  $d\lambda_2(\Omega, \psi)$  at  $t = 0$  is

$$d\lambda_2(\Omega, \psi) := - \int_{-a}^{+a} |\nabla u_2(x, h(x))|^2 \psi(x) dx. \quad (3)$$

The optimality of  $\Omega$  implies that there exists a Lagrange multiplier  $\mu$  such that, for any  $\psi \in C_0^\infty(-a, a)$

$$\mu d\lambda_2(\Omega, \psi) + dP(\Omega, \psi) = 0$$

what implies, thanks to (2), (3) that  $h$  is a solution of the following o.d.e. (in the sense of distribution):

$$- \left( \frac{h'(x)}{\sqrt{1 + h'(x)^2}} \right)' = \mu |\nabla u_2(x, h(x))|^2. \quad (4)$$

Since  $u_2$  belongs to  $H^2(\Omega)$ , its first derivatives  $\frac{\partial u_2}{\partial x}$  and  $\frac{\partial u_2}{\partial y}$  have a trace on  $\partial\Omega$  which belong to  $H^{1/2}(\partial\Omega)$ . Now, the Sobolev embedding in one dimension  $H^{1/2}(\partial\Omega) \hookrightarrow L^p(\partial\Omega)$  for any  $p > 1$  shows that  $x \mapsto |\nabla u_2(x, h(x))|^2$  is in  $L^p(-a, a)$  for any  $p > 1$ . Therefore, according to (4), the function  $h'/\sqrt{1 + h'^2}$  is in  $W^{1,p}(-a, a)$  for any  $p > 1$  (we recall that  $h'$  is bounded because  $\Omega$  is convex), so it belongs to some Hölder space  $C^{0,\alpha}([-a, a])$  (for any  $\alpha < 1$ , according to Morrey-Sobolev embedding). Since  $h'$  is bounded, it follows immediately that  $h$  belongs to  $C^{1,\alpha}([-a, a])$ . Now, we come back to the partial differential equation and use an intermediate Schauder regularity result (see [13] or the remark after Lemma 6.18 in [14]) to claim that if  $\partial\Omega$  is  $C^{1,\alpha}$ , then the eigenfunction  $u_2$  is  $C^{1,\alpha}(\bar{\Omega})$  and  $|\nabla u_2|^2$  is  $C^{0,\alpha}$ . Then, looking again to the o.d.e. (4) and using the same kind of Schauder's regularity result yields that  $h \in C^{2,\alpha}$ . We iterate the process, thanks to a classical bootstrap argument to conclude that  $h$  is  $C^\infty$ .  $\square$

**Proof of Lemma 2.2** This result is similar to [17, Theorem 2.5.10]. The new difficulties here are the perimeter constraint (instead of the volume) and the fact that the domain  $\Omega$  is convex, but not necessarily regular. Nevertheless, we know that any eigenfunction of a convex domain is in the Sobolev space  $H^2(\Omega)$ , see [16]. Let us assume, for a contradiction, that  $\lambda_2(\Omega)$  is not simple, then it is double because  $\Omega$  is a convex domain in the plane, see [21]. Let us recall the result of derivability of eigenvalues in the multiple case (see [11] or [24]). Assume that the domain  $\Omega$  is modified by a regular vector field  $x \mapsto x + tV(x)$  (and we will denote by  $\Omega_t$  the image of  $\Omega$  by this transformation) and let us denote by  $u_2, u_3$  two orthonormal eigenfunctions associated to  $\lambda_2, \lambda_3$ . Then, the first variation of  $\lambda_2(\Omega_t), \lambda_3(\Omega_t)$  are the repeated eigenvalues of the  $2 \times 2$  matrix

$$\mathcal{M} = \begin{pmatrix} -\int_{\partial\Omega} \left(\frac{\partial u_2}{\partial n}\right)^2 V.n \, d\sigma & -\int_{\partial\Omega} \left(\frac{\partial u_2}{\partial n} \frac{\partial u_3}{\partial n}\right) V.n \, d\sigma \\ -\int_{\partial\Omega} \left(\frac{\partial u_2}{\partial n} \frac{\partial u_3}{\partial n}\right) V.n \, d\sigma & -\int_{\partial\Omega} \left(\frac{\partial u_3}{\partial n}\right)^2 V.n \, d\sigma \end{pmatrix} \quad (5)$$

Now, let us introduce the Lagrangian  $L(\Omega) = \lambda_2(\Omega) + \mu P(\Omega)$ . As we have seen in the proof of Theorem 2.1, the perimeter is differentiable and the derivative is a linear form in  $V.n$  supported on  $\partial\Omega$  (see e.g. [19, Corollary 5.4.16]). We will denote by  $\langle dP_{\partial\Omega}, V.n \rangle$  this derivative. So the Lagrangian  $L(\Omega_t)$  has a derivative which is the smallest eigenvalue of the matrix  $\mathcal{M} + \mu \langle dP_{\partial\Omega}, V.n \rangle I$  where  $I$  is the identity matrix. Therefore, to reach a contradiction (with the optimality of  $\Omega$ ), it suffices to prove that one can always find a deformation field  $V$  such that the smallest eigenvalue of this matrix is negative. Let us consider two points  $A$  and  $B$  on  $\partial\Omega$  and two small neighborhoods  $\gamma_A$  and  $\gamma_B$  of these two points of same length, say  $2\delta$ . Let us choose any regular function  $\varphi(s)$  defined on  $(-\delta, +\delta)$  (vanishing at the extremities of the interval) and a deformation field  $V$  such that

$$V.n = +\varphi \text{ on } \gamma_A, \quad V.n = -\varphi \text{ on } \gamma_B, \quad V.n = 0 \text{ elsewhere.}$$

Then, the matrix  $\mathcal{M} + \mu \langle dP_{\partial\Omega}, V.n \rangle I$  splits into two matrices  $\mathcal{M}_A - \mathcal{M}_B$  where  $\mathcal{M}_A$  is defined by (and a similar formulae for  $\mathcal{M}_B$ ):

$$\mathcal{M}_A = \begin{pmatrix} \langle dP_{\gamma_A}, \varphi \rangle - \int_{\gamma_A} \left(\frac{\partial u_2}{\partial n}\right)^2 \varphi \, d\sigma & -\int_{\gamma_A} \left(\frac{\partial u_2}{\partial n} \frac{\partial u_3}{\partial n}\right) \varphi \, d\sigma \\ -\int_{\gamma_A} \left(\frac{\partial u_2}{\partial n} \frac{\partial u_3}{\partial n}\right) \varphi \, d\sigma & \langle dP_{\gamma_A}, \varphi \rangle - \int_{\gamma_A} \left(\frac{\partial u_3}{\partial n}\right)^2 \varphi \, d\sigma \end{pmatrix}. \quad (6)$$

In particular, it is clear that the exchange of  $A$  and  $B$  replace the matrix  $\mathcal{M}_A - \mathcal{M}_B$  by its opposite. Therefore, the only case where one would be unable to choose two points  $A, B$  and a deformation  $\varphi$  such that the matrix has a negative eigenvalue is if  $\mathcal{M}_A - \mathcal{M}_B$  is identically zero for any  $\varphi$ . But this implies, in particular

$$\int_{\gamma_A} \left(\frac{\partial u_2}{\partial n}\right) \left(\frac{\partial u_3}{\partial n}\right) \varphi \, d\sigma = \int_{\gamma_B} \left(\frac{\partial u_2}{\partial n}\right) \left(\frac{\partial u_3}{\partial n}\right) \varphi \, d\sigma \quad (7)$$

and

$$\int_{\gamma_A} \left[ \left( \frac{\partial u_2}{\partial n} \right)^2 - \left( \frac{\partial u_3}{\partial n} \right)^2 \right] \varphi d\sigma = \int_{\gamma_B} \left[ \left( \frac{\partial u_2}{\partial n} \right)^2 - \left( \frac{\partial u_3}{\partial n} \right)^2 \right] \varphi d\sigma \quad (8)$$

for any regular  $\varphi$  and any points  $A$  and  $B$  on  $\partial\Omega$ . This implies that the product  $\left( \frac{\partial u_2}{\partial n} \right) \left( \frac{\partial u_3}{\partial n} \right)$  and the difference  $\left( \frac{\partial u_2}{\partial n} \right)^2 - \left( \frac{\partial u_3}{\partial n} \right)^2$  should be constant a.e. on  $\partial\Omega$ . As a consequence  $\left( \frac{\partial u_2}{\partial n} \right)^2$  has to be constant. Since the nodal line of the second eigenfunction touches the boundary in two points (see [22] or [1]),  $\frac{\partial u_2}{\partial n}$  has to change sign. So we get a function belonging to  $H^{1/2}(\partial\Omega)$  taking values  $c$  and  $-c$  on sets of positive measure, which is absurd, unless  $c = 0$ . This last issue is impossible by the Holmgren uniqueness theorem.  $\square$

## 2.2 Symmetry

**Theorem 2.3** *The minimizer  $\Omega$  has (at least) one axis of symmetry.*

**Proof :**

Let  $u_1, u_2$  denote the two first normalized eigenfunctions of  $\Omega$ . For any angle  $\theta \in [0, 2\pi)$ , we divide  $\Omega$  in two parts  $\Omega_l(\theta)$  and  $\Omega_r(\theta)$  by a line  $D(\theta)$  (see

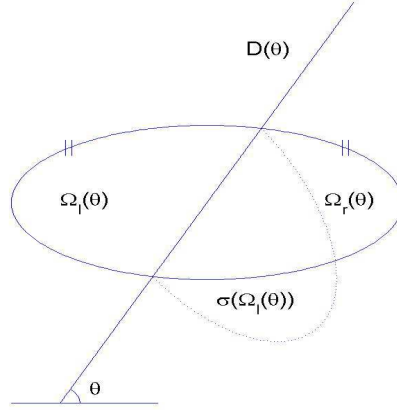


Figure 1: The line  $D(\theta)$  cuts the boundary in two parts of same perimeter

Figure 1) such that the two parts have the same perimeter. Let us denote by  $\Gamma_+$  the right half of the unit circle:

$$\Gamma_+ := \{(x, y) \in \mathbb{R}^2, x^2 + y^2 = 1, x \geq 0, y \neq -1\}.$$

For any  $\theta$ , we look at the following elementary variational problem:

$$\max_{(\alpha, \beta) \in \Gamma_+} \frac{\int_{\Omega_l(\theta)} |\alpha \nabla u_1 + \beta \nabla u_2|^2}{\int_{\Omega_l(\theta)} (\alpha u_1 + \beta u_2)^2} \quad (9)$$

and the same problem with  $\Omega_l(\theta)$  replaced by  $\Omega_r(\theta)$ . Since we are looking at the ratio of two positive quadratic forms in dimension two, this maximization problems have a unique solution in  $\Gamma_+$  unless the ratio is constant (in which case we can take any point in  $\Gamma_+$ ). Let us denote by  $(\alpha_l(\theta), \beta_l(\theta))$  and  $(\alpha_r(\theta), \beta_r(\theta))$  these solutions. Since  $(\alpha_l(\theta + \pi), \beta_l(\theta + \pi)) = (\alpha_r(\theta), \beta_r(\theta))$ , for continuity reasons, there exists an angle  $\theta^*$  such that  $(\alpha_l(\theta^*), \beta_l(\theta^*)) = (\alpha_r(\theta^*), \beta_r(\theta^*))$ . Let us denote by  $(\alpha^*, \beta^*)$  this common value. Now, we have

$$\lambda_2(\Omega) = \max_{(\alpha, \beta) \in \mathbb{R}^2} \frac{\int_{\Omega} |\alpha \nabla u_1 + \beta \nabla u_2|^2}{\int_{\Omega} (\alpha u_1 + \beta u_2)^2} \geq \frac{\int_{\Omega} |\alpha^* \nabla u_1 + \beta^* \nabla u_2|^2}{\int_{\Omega} (\alpha^* u_1 + \beta^* u_2)^2}. \quad (10)$$

We decompose the numerator and the denominator in the right-hand side of (10) in  $\int_{\Omega_l(\theta^*)} + \int_{\Omega_r(\theta^*)}$  and we take the smallest ratio. Let us assume without loss of generality, that this smallest ratio is achieved with  $\Omega_l(\theta^*)$ . We get,

$$\lambda_2(\Omega) \geq \frac{\int_{\Omega_l(\theta^*)} |\alpha^* \nabla u_1 + \beta^* \nabla u_2|^2}{\int_{\Omega_l(\theta^*)} (\alpha^* u_1 + \beta^* u_2)^2} \geq \frac{\int_{\Omega_l(\theta^*)} |\alpha \nabla u_1 + \beta \nabla u_2|^2}{\int_{\Omega_l(\theta^*)} (\alpha u_1 + \beta u_2)^2} \quad (11)$$

the second inequality being valid for any  $(\alpha, \beta)$  by definition of  $(\alpha^*, \beta^*)$ . Now, let us introduce  $\sigma$  the symmetry with respect to the line  $D(\theta^*)$  and the new set  $\widehat{\Omega} := \Omega_l(\theta^*) \cup \sigma(\Omega_l(\theta^*))$ , see Figure 1. Let us also introduce the test functions (which belong to the Sobolev space  $H_0^1(\widehat{\Omega})$ ):

$$\widehat{u}_1 = \begin{cases} u_1 & \text{on } \Omega_l(\theta^*) \\ u_1 \circ \sigma & \text{on } \sigma(\Omega_l(\theta^*)) \end{cases} \quad \widehat{u}_2 = \begin{cases} u_2 & \text{on } \Omega_l(\theta^*) \\ u_2 \circ \sigma & \text{on } \sigma(\Omega_l(\theta^*)). \end{cases}$$

Finally, we have, for any  $(\alpha, \beta)$ :

$$\frac{\int_{\Omega_l(\theta^*)} |\alpha \nabla u_1 + \beta \nabla u_2|^2}{\int_{\Omega_l(\theta^*)} (\alpha u_1 + \beta u_2)^2} = \frac{\int_{\widehat{\Omega}} |\alpha \nabla \widehat{u}_1 + \beta \nabla \widehat{u}_2|^2}{\int_{\widehat{\Omega}} (\alpha \widehat{u}_1 + \beta \widehat{u}_2)^2} \geq \lambda_2(\widehat{\Omega})$$

the last inequality coming from the min-max formula for  $\lambda_2$ . Now, since  $\Omega$  is a minimizer, we must have equality in all the above inequalities. In particular,  $\widehat{u}_1$  and  $\widehat{u}_2$  are the two first eigenfunctions of  $\widehat{\Omega}$ . But, since  $\widehat{u}_1 = u_1$  on  $\Omega_l(\theta^*)$ , analyticity of the eigenfunctions implies that  $\widehat{\Omega} = \Omega$  what proves the Theorem.  $\square$



**Remark 2.4** A more careful analysis of the equalities in the chain of inequalities shows that the nodal line of  $u_2$  cannot be on the line of symmetry  $D(\theta^*)$ . Indeed, if the nodal line was on the axis of symmetry  $D$ , we would have a first eigenvalue  $u_1$  (resp. a second eigenvalue  $u_2$ ) even (resp. odd) with respect to  $D$ . In particular, denoting by  $\Omega_l = \Omega_l(\theta^*)$  we would have

$$\int_{\Omega_l} |\nabla u_i|^2 = \lambda_i/2, \quad \int_{\Omega_l} u_i^2 = 1/2 \quad i = 1, 2$$

and (without loss of generality)  $2 \int_{\Omega_l} u_1 u_2 := d < 0$ . Moreover,  $d < 1$  by Cauchy-Schwarz inequality. So, the variational problem (9) would consist in maximizing

$$J(\alpha, \beta) = \frac{\lambda_1 \alpha^2 + 2b\lambda_1 \alpha \beta + \lambda_2 \beta^2}{\alpha^2 + 2b\alpha\beta + \beta^2}.$$

But it is straightforward to check that the maximum of  $J(\alpha, \beta)$  over  $\Gamma_+$  is obtained for  $\alpha, \beta$  such that  $\alpha = -d\beta$  and not for  $(\alpha, \beta) = (0, 1)$  which should be the case since we have equality in (10). A consequence is the following alternative:

- either we have only one axis of symmetry, with the nodal line of  $u_2$  crossing this axis of symmetry,
- or we have a second axis of symmetry which, necessarily, contains the nodal line of  $u_2$ .

Numerical simulations seems to show that the second part of the previous alternative occurs, see Figure 2.

### 2.3 Other qualitative properties

Since we know that the minimizer is  $C^\infty$ , we can now write rigorously the optimality condition. Under variations of the boundary (replace  $\Omega$  by  $\Omega_t = (I + tV)(\Omega)$ ), the shape derivative of the perimeter is given by (see section 2.1 and [19, Corollary 5.4.16])

$$dP(\Omega; V) = \int_{\partial\Omega} \mathcal{C} V \cdot n d\sigma$$

where  $\mathcal{C}$  is the curvature of the boundary and  $n$  the exterior normal vector. Using the expression of the derivative of the eigenvalue given in (3) (see also [19, Theorem 5.7.1]), the proportionality of these two derivatives through some Lagrange multiplier yields existence of a constant  $\mu$  such that

$$|\nabla u_2|^2 = \mu \mathcal{C} \quad \text{on } \partial\Omega. \quad (12)$$

Multiplying the equality in (12) by  $X \cdot n$  and integrating on  $\partial\Omega$  yields, thanks to Gauss formulae ( $\int_{\partial\Omega} \mathcal{C} X \cdot n d\sigma = P(\Omega)$ ) and a classical application of the Rellich formulae ( $\int_{\partial\Omega} |\nabla u_2|^2 X \cdot n d\sigma = 2\lambda_2(\Omega)$ ), the value of the Lagrange multiplier. So, we have proved:

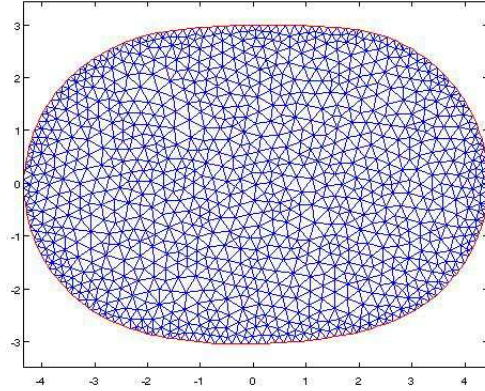


Figure 2: A possible minimizer obtained numerically (by courtesy of Edouard Oudet)

**Proposition 2.5** *Any minimizer satisfies*

$$|\nabla u_2(x)|^2 = \frac{2\lambda_2(\Omega)}{P(\Omega)} \mathcal{C}(x), \quad x \in \partial\Omega \quad (13)$$

where  $\mathcal{C}(x)$  is the curvature at point  $x$ .

As a consequence, we can state some qualitative properties of the optimal domain(s):

**Theorem 2.6** *An optimal domain satisfies:*

1. *Its boundary does not contain any segment.*
2. *Its boundary does not contain any arc of circle.*
3. *Its boundary contains exactly two points where the curvature vanishes.*

**Proof :** An easy consequence of Hopf's lemma (applied to each nodal domain) is that the normal derivative of  $u_2$  can only vanish on  $\partial\Omega$  at points where the nodal line hits the boundary. Now, we know (see [22] or [1]) that there are exactly two such points. Then, the first and third item follow immediately from the "over-determined" condition (13). The second item has already been proved in a similar situation in [18]. We repeat the proof here for sake of completeness. Let us assume that  $\partial\Omega$  contains a piece of circle  $\gamma$ . According to (13),  $\Omega$  satisfies the optimality condition

$$\frac{\partial u_2}{\partial n} = c \text{ (constant) } \quad \text{on } \gamma. \quad (14)$$

We put the origin at the center of the corresponding disk and we introduce the function

$$w(x, y) = x \frac{\partial u_2}{\partial y} - y \frac{\partial u_2}{\partial x}.$$

Then, we easily verify that

$$\begin{aligned} -\Delta w &= \lambda_2 w \text{ in } \Omega \\ w &= 0 \text{ on } \gamma \\ \frac{\partial w}{\partial n} &= 0 \text{ on } \gamma. \end{aligned}$$

Now we conclude, using Holmgren uniqueness theorem, that  $w$  must vanish in a neighborhood of  $\gamma$ , so in the whole domain by analyticity. Now, it is classical that  $w = 0$  imply that  $u_2$  is radially symmetric in  $\Omega$ . Indeed, in polar coordinates,  $w = 0$  implies  $\frac{\partial u}{\partial \theta} = 0$ . Therefore  $\Omega$  would be a disk which is impossible since it would contradict point 3.  $\square$

### 3 The N-dimensional case

We start by considering the smooth version of the minimization problem

$$\inf\{\lambda_2(\Omega) : \Omega \text{ bounded, open and smooth, } P(\Omega) \leq c\} := l_o. \quad (15)$$

Since in dimension  $N \geq 3$ , the perimeter  $P(\Omega)$  is not anymore decreasing upon convexification, we can not restrict the minimization problem only to the family of convex sets. This is the reason for which we have to consider in a first step a relaxed form of problem (15) and, in a second step, to prove the regularity of the relaxed solution and get in this way a solution to problem (15). In this section we prove only the existence of a relaxed solution.

Although it is classical to relax the problem on the family of quasi-open sets (see for instance [9] or [19]), the constraint on the perimeter leads to a natural relaxation of the optimization problem on the family of measurable sets. If  $M \subseteq \mathbb{R}^N$  is a measurable set of finite measure, we define

$$\widehat{H}_0^1(M) = \{u \in H^1(\mathbb{R}^N) : u = 0 \text{ a.e. on } \mathbb{R}^N \setminus M\}. \quad (16)$$

With this definition, one can define  $\lambda_1(M), \lambda_2(M), \dots$  thanks to the classical min-max formulae. For an arbitrary open set  $M$ , this definition does not coincide with the usual definition of  $H_0^1(M)$ . Nevertheless, we point out that it is not restrictive to consider this definition since:

1. For every measurable set  $M \subseteq \mathbb{R}^N$  there exists a quasi open set  $\omega$  (see for instance [4]) such that

$$H_0^1(\omega) = \widehat{H}_0^1(M);$$

2. Problem (15) is equivalent to problem

$$\min\{\lambda_2(M) : M \text{ measurable}, P(M) \leq c\} := l_m, \quad (17)$$

in the sense that  $l_o = l_m$ .

We shall prove this equivalence in the sequel, but we point out first the main reason for the equivalence. For a smooth set  $M$ , (e.g. Lipschitz, see [19, Lemma 3.2.15]) the definition of the spaces  $H_0^1(M)$  and  $\widehat{H}_0^1(M)$  coincide. For a non-smooth open set  $M$  (say with a crack), we have that  $\widehat{H}_0^1(M)$  contains strictly  $H_0^1(M)$ , which may lead to the idea that  $l_m < l_o$ . In practice, when solving (15), the minimizing sequence will not develop cracks, precisely because by erasing a crack the generalized perimeter is unchanged and the eigenvalue decreases. This hint, is effective, and we can give the following.

**Theorem 3.1** *Problems (15) and (17) are equivalent, in the sense that  $l_o = l_m$ .*

**Proof :**

Clearly,  $l_m \leq l_o$ , since for every smooth open set  $M$  we have the identity  $H_0^1(M) = \widehat{H}_0^1(M)$ .

In order to prove the converse inequality, let  $M$  be measurable, of finite measure, such that  $P(M) \leq c$ . There exists a quasi open set  $\omega$ ,  $\omega \subseteq M$  a.e. and

$$H_0^1(\omega) = \widehat{H}_0^1(M).$$

We point out that the measure of  $M \setminus \omega$  may be strictly positive, and that  $P(\omega)$  may be greater than  $c$ .

Following the density result of smooth sets into the family of measurable sets [2, Theorem 3.42], there exists a sequence of smooth sets  $\Omega_n$ , such that

$$1_{\Omega_n} \xrightarrow{L^1(\mathbb{R}^N)} 1_M \quad (18)$$

$$\mathcal{H}^{N-1}(\partial\Omega_n) \longrightarrow P(M). \quad (19)$$

Unfortunately, it is immediate to observe that this implies only

$$\lambda_2(M) \leq \liminf_{n \rightarrow \infty} \lambda_2(\Omega_n),$$

while we are seeking precisely the opposite inequality.

We consider the function  $w = R_\omega(1)$  and a convolution kernel  $(\rho_k)_k$ . Here, and in the sequel, for every quasi open set of finite measure we denote  $R_\omega : L^2(\mathbb{R}^N) \longrightarrow L^2(\mathbb{R}^N)$  the operator defined by  $R_\omega(f) = u$ , where  $u$  solves in the variational sense (see for details [9, Chapter 4])

$$-\Delta u = f \text{ in } \omega, \quad u \in H_0^1(\omega).$$

In the same way, for a positive Borel measure  $\mu$ , absolutely continuous with respect to the capacity and with a regular set of finite Lebesgue measure, we define  $R_\mu(f)$  as being the solution of

$$-\Delta u + \mu u = f \text{ in } \mathbb{R}^N, u \in H^1(\mathbb{R}^N) \cap L^2(\mu).$$

As in the proof of [2, Theorem 3.42], by the definition of the perimeter and the coarea formula we have

$$\begin{aligned} P(M) &= \|\nabla 1_M\|_{meas} = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla(1_M * \rho_k)| dx \\ &= \lim_{k \rightarrow \infty} \int_0^1 P(\{1_M * \rho_k > t\}) dt \geq \int_0^1 \liminf_{k \rightarrow \infty} P(\{1_M * \rho_k > t\}) dt. \end{aligned}$$

For every  $t \in (0, 1)$  we have

$$\begin{aligned} |\{1_M * \rho_k > t\} \setminus M| &\leq \frac{1}{t} \int_{\mathbb{R}^N} |1_M * \rho_k - 1_M| dx \\ |M \setminus \{1_M * \rho_k > t\}| &\leq \frac{1}{1-t} \int_{\mathbb{R}^N} |1_M * \rho_k - 1_M| dx, \end{aligned}$$

so that  $1_{\{1_M * \rho_k > t\}}$  converges in  $L^1(\mathbb{R}^N)$  to  $1_M$  and

$$\liminf_{k \rightarrow \infty} P(\{1_M * \rho_k > t\}) \geq P(M).$$

The above inequalities imply that for almost every  $t \in (0, 1)$

$$\liminf_{k \rightarrow \infty} P(\{1_M * \rho_k > t\}) = P(M).$$

For a subsequence (still denoted using the index  $k$ ) we have

$$1_{\{1_M * \rho_k > t\}} \xrightarrow{L^1(\mathbb{R}^N)} 1_M$$

and

$$\lim_{k \rightarrow \infty} P(\{1_M * \rho_k > t\}) = P(M).$$

We notice that up to a set of zero measure

$$\{w > 0\} \subseteq M.$$

We may assume that  $w \leq 1$ . This is always true for a constant  $c$  small enough, and the isoperimetric problem being independent on  $c$ , we can choose it as small as needed. So, we get that

$$w * \rho_k \leq 1_M * \rho_k$$

so

$$\{w * \rho_k > t\} \subseteq \{1_M * \rho_k > t\}$$

and

$$\lambda_2(\{1_M * \rho_k > t\}) \leq \lambda_2(\{w * \rho_k > t\}).$$

Let us prove that

$$\limsup_{k \rightarrow \infty} \lambda_2(\{w * \rho_k > t\}) \leq \lambda_2(\{w > t\}).$$

It is enough to show (see for instance [9, Chapters 4,5]) that one of the Mosco conditions hold, namely for every  $\varphi \in H_0^1(\{w > t\})$  there exists a sequence  $\varphi_k \in H_0^1(\{w * \rho_k > t\})$  such that  $\varphi_k \rightarrow \varphi$  strongly in  $H_0^1$ .

Using the density result of [12], it is enough to choose  $\varphi \geq 0$  such that  $\varphi \leq (w - t)^+$  and take

$$\varphi_k = \min\{\varphi, ((w * \rho_k) - t)^+\}.$$

This being true for every  $t$ , from the convergence  $\lambda_2(\{w > t\}) \rightarrow \lambda_2(\{w > 0\})$  for  $t \rightarrow 0$ , by a diagonal procedure we can choose  $t_k \rightarrow 0$  such that

$$\limsup_{k \rightarrow \infty} \lambda_2(\{w * \rho_k > t_k\}) \leq \lambda_2(\{w > 0\}) = \lambda_2(\omega) = \lambda_2(M),$$

and

$$\begin{aligned} 1_{\{1_M * \rho_k > t_k\}} &\xrightarrow{L^1} 1_M, \\ \lim_{k \rightarrow \infty} P(\{1_M * \rho_k > t_k\}) &= P(M). \end{aligned}$$

This proves that  $l_o \leq l_m$ . □

In the sequel, we prove the existence of a relaxed solution.

**Theorem 3.2** *Problem (17) has a solution.*

**Proof :**

Let  $(M_n)$  be a minimizing sequence for problem (17). We shall use a concentration compactness argument for the resolvent operators (see [8, Theorem 2.2]). Let  $\omega_n$  be the quasi open sets such that  $\omega_n \subseteq M_n$  a.e. and  $\lambda_2(\omega_n) = \lambda_2(M_n)$ . From the classical isoperimetric inequality, the measures of  $\omega_n$  are uniformly bounded. Consequently, for a subsequence (still denoted using the same index) two situations may occur: compactness and dichotomy.

If the compactness issue holds, there exists a measure  $\mu$  and a sequence of vectors  $y_n \in \mathbb{R}^N$  such that the resolvent operators

$$R_{\omega_n + y_n} \rightarrow R_\mu$$

converge strongly in  $\mathcal{L}(L^2(\mathbb{R}^N))$ . Since the perimeters of  $M_n$  are uniformly bounded, we can define (up to subsequences) sets  $M^k$  as the limits of  $M_n \cap B(0, k)$ , and set  $M = \cup_k M^k$ .

Since  $w_{\omega_n+y_n}$  converges strongly in  $L^2(\mathbb{R}^N)$  to  $w_\mu$ , we get that  $\{w_\mu > 0\} \subseteq M$  a.e. so that  $\lambda_2(M) \leq \lambda_2(\{w_\mu > 0\}) \leq \lambda_2(\mu) = \lim_{n \rightarrow \infty} \lambda_2(\omega_n)$ .

On the other hand

$$P(M) = \lim_{k \rightarrow \infty} P(M \cap B(0, k), B(0, k)) \leq \liminf_{k \rightarrow \infty} P(M_k),$$

so that  $M$  is a solution to problem (17) .

Let us assume that we are in the dichotomy issue. There exists two sequences of quasi open sets such that

$$\begin{aligned} \omega_n^1 \cup \omega_n^2 &\subseteq \omega \\ \omega_n^1 &= \omega_n \cap B(0, R_n^1), \omega_n^2 = \omega_n \cap (\mathbb{R}^N \setminus B(0, R_n^2)), R_n^2 - R_n^1 \rightarrow +\infty \\ \liminf_{n \rightarrow \infty} |\omega_n^i| &> 0, \quad i = 1, 2 \\ R_{\omega_n} - R_{\omega_n^1} - R_{\omega_n^2} &\rightarrow 0, \text{ in } \mathcal{L}(L^2(\mathbb{R}^N)) \end{aligned}$$

Let us denote

$$M_n^1 = M_n \cap B(0, R_n^1), M_n^2 = M_n \cap (\mathbb{R}^N \setminus B(0, R_n^2)).$$

Since the measures of  $M_n$  are uniformly bounded, one can suitably increase  $R_n^1$  and decrease  $R_n^2$  such that

$$\limsup_{n \rightarrow \infty} P(M_n^1 \cup M_n^2) \leq c$$

and all other properties of the dichotomy issue remain valid.

We have that  $|\lambda_2(\omega) - \lambda_2(\omega_n^1 \cup \omega_n^2)| \rightarrow 0$ . Since  $\omega_n^1$  and  $\omega_n^2$  are disconnected, either  $\lambda_2(\omega_n)$  equals  $\lambda_2(\omega_n^1)$  or  $\lambda_1(\omega_n^1)$ . The first situation is to be excluded since this implies that  $M_n^1$  is a minimizing sequence with perimeter less than or equal to some constant  $\alpha < c$ , which is absurd. The second situation leads to an optimum consisting on two disjoint ball (this is a consequence of the Faber-Krahn isoperimetric inequality for the first eigenvalue).  $\square$

We now extend the symmetry result proved in Theorem 2.3. For that purpose, we need to assume some (weak) regularity for the optimal domain  $\Omega$ . We recall that an open set is said to be Caratheodory if it coincides with the interior of its closure.

**Theorem 3.3** *Let  $\Omega \subset \mathbb{R}^N$  be a minimizer of  $\lambda_2$  with a perimeter constraint and assume that  $\Omega$  is a bounded Caratheodory open set. Then,  $\Omega$  has a cylindrical symmetry (in the sense that it is invariant under all rotations around some  $N - 2$  vector space).*

**Proof :**

In order to keep an intuitive image, we make the proof in the case  $N = 3$ , but it could be handled exactly in the same way in the general case.

First, let us fix a plane  $T$  and consider all the planes orthogonal to  $T$  dividing the set  $\Omega$  in two parts of equal perimeter. This is always possible in any direction of  $T$  provided that  $\Omega$  does not have flat parts on its boundary. Provided the second eigenvalue is simple, the same arguments as in Theorem 2.6 are valid. Otherwise, the flat parts with positive  $\mathcal{H}^2$  measure are contained in a countable family of planes, which we may ignore without loosing the generality (as seen from the proof below).

As in Theorem 2.3, there exists a position such that one side of  $\Omega$  together with its reflection with respect to the plane (call it  $\tilde{T}$ ) is at least as good as the optimum. By analyticity, the optimum is symmetric with respect to this plane. Moreover,  $\tilde{T}$  has to contain the center of mass of  $\Omega$ , which is assumed to be the origin of the space.

Let us assume that  $P_0 = [x_1 0 x_3]$  is plane of symmetry. For every unit vector  $e$  there exists a plane  $\Pi_e$  such that  $e \in \Pi_e$  and  $\Pi_e$  is plane of symmetry (take  $T$  orthogonal to  $e$ ). Let  $D_e = \Pi_e \cap P_0$ . If we fix a sense on  $D_e$  and denote  $\alpha_e$  the angle between  $\Pi_e$  and  $P_0$ , the set  $\Omega$  is invariant for the rotation  $R(D_e, 2\alpha_e)$  of angle  $2\alpha_e$  and axis  $D_e$ .

**Step 1.** Assume that  $\alpha_e \in (\mathbb{R} \setminus \mathbb{Q})\pi$ . There exists a dense set  $M \subseteq \mathbb{R}$  such that  $\forall \alpha \in M$

$$R(D_e, \alpha)(\Omega) = \Omega.$$

Using the fact that  $\Omega$  is open, we get for every  $\theta \in \mathbb{R}$

$$\Omega \subseteq R(D_e, \theta)(\Omega) \subseteq \overline{\Omega}.$$

Since  $\Omega$  is assumed to be Caratheodory, we find that  $D_e$  is an axis of cylindrical symmetry.

**Step 2.** Following Step 1. every angle of any rotation should be a rational number, otherwise the problem is solved. Assume that for a sequence of unit vectors  $(e_n)_n$  not belonging to  $P_0$ , the angles  $\alpha_n \neq 0$  obtained in Step 1 (which are rational numbers) converge to zero. By compactness, we may assume that  $D_{e_n}$  converge to some line  $D$ . Since the set  $\{k \cdot \alpha_n : n \in \mathbb{N}, k \in \mathbb{Z}\}$  is dense in  $\mathbb{R}$ , we get that for every compact set  $K \subseteq \Omega$ , for every  $\varepsilon > 0$  and for every  $\theta \in \mathbb{R}$

$$K \subseteq R(D, \theta)(\Omega) \subseteq \bigcup_{x \in \Omega} B(x, \varepsilon),$$

so

$$\Omega \subseteq R(D, \theta)(\Omega) \subseteq \overline{\Omega},$$

and we conclude as in Step 1.

**Step 3.** If  $R(D, \theta)$  is a rotation leaving invariant  $\Omega$  and  $\theta$  is a rational number of the form  $p/q\pi$ , with  $(p, q) = 1$ , then obviously  $R(D, 1/q\pi)$  leaves



invariant  $\Omega$ . Following this remark and Step 2, one can find some  $q_0 \in \mathbb{N}$  such that any angle of rotation about any line leaving invariant  $\Omega$  is of the form  $k/q\pi$  with  $k \in \mathbb{N}$  and  $q \leq q_0$  (i.e. the family of possible angles is finite).

Let  $\varphi \in [0, 2\pi)$ . For  $N$  large enough, we consider the vectors

$$e_i = \cos\left(\frac{\pi}{4} + \frac{i}{2^N}\right), \sin\left(\frac{\pi}{4} + \frac{i}{2^N}\right), 0) \quad i = 0, 1, \dots, 2q.$$

We construct the symmetry planes  $\Pi_{e_i, \varphi}$  associated to  $R(0x_1, \varphi)e_i$ , and notice that at least two of them have the same angle with  $P_0$ . For  $N$  large, the angle between these two planes is less than  $\pi/q_0$ , so that is necessarily equal to zero. Consequently, the plane  $R(0x_1, \varphi)[x_1 0x_2]$  is plane of symmetry. Finally, the line  $[0x_1]$  is a line of cylindrical symmetry.  $\square$

**Remark 3.4** Assuming  $C^2$  regularity (which seems harder to prove here), one can find the same kind of optimality condition as in (13) with the mean curvature instead of the curvature. Since, the gradient of  $u_2$  still vanishes where the nodal surface hits the boundary, the mean curvature has to vanish and, according to the cylindrical symmetry, it is along a circle. In particular, this shows that the optimal domain is not convex in dimension  $N \geq 3$ .

## References

- [1] G. ALESSANDRINI, *Nodal lines of eigenfunctions of the fixed membrane problem in general convex domains*, Comment. Math. Helv., **69** (1994), no. 1, 142-154.
- [2] L. AMBROSIO, N. FUSCO, D. PALLARA *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.
- [3] M.S. ASHBAUGH, R.D. BENGURIA, *Isoperimetric inequalities for eigenvalues of the Laplacian*, Spectral theory and mathematical physics: a Festschrift in honor of Barry Simon's 60th birthday, 105–139, Proc. Sympos. Pure Math., 76, Part 1, Amer. Math. Soc., Providence, RI, 2007.
- [4] M.S. ASHBAUGH, D. BUCUR, *On the isoperimetric inequality for the buckling of a clamped plate*. Z. Angew. Math. Phys., **54** (2003), no. 5, 756–770.
- [5] Z. BELHACHMI, D. BUCUR, G. BUTTAZZO, J.M. SAC-EPÉE, *Shape optimization problems for eigenvalues of elliptic operators*, ZAMM Z. Angew. Math. Mech., **86** (2006), no. 3, 171-184.
- [6] T. BRIANÇON, *Regularity of optimal shapes for the Dirichlet's energy with volume constraint*, ESAIM: COCV, **10** (2004), 99-122.

- [7] T. BRIANÇON, M. HAYOUNI, M. PIERRE, *Lipschitz Continuity of State Functions in Some Optimal Shaping*, Calc. Var. Partial Differential Equations, **23** (2005), no. 1, 13-32.
- [8] D. BUCUR, *Uniform concentration-compactness for Sobolev spaces on variable domains*. J. Differential Equations, **162** (2000), no. 2, 427–450.
- [9] D. BUCUR, G. BUTTAZZO, *Variational Methods in Shape Optimization Problems*, Progress in Nonlinear Differential Equations and Their Applications, **65** Birkhäuser, Basel, Boston 2005.
- [10] A. CHAMBOLLE, C. LARSEN,  *$C^\infty$  regularity of the free boundary for a two-dimensional optimal compliance problem*, Calc. Var. Partial Differential Equations, **18** (2003), no. 1, 77-94.
- [11] S.J. COX, *Extremal eigenvalue problems for the Laplacian*, Recent advances in partial differential equations (El Escorial, 1992), 37-53, RAM Res. Appl. Math., 30, Masson, Paris, 1994.
- [12] G. DAL MASO, F. MURAT *Asymptotic behaviour and correctors for Dirichlet problems in perforated domains with homogeneous monotone operators*. Ann. Scuola Norm. Sup. Pisa Cl. Sci., (4) **24** (1997), no. 2, 239–290.
- [13] D. GILBARG, L. HÖRMANDER, *Intermediate Schauder estimates*, Arch. Rational Mech. Anal., **74** (1980), no. 4, 297-318.
- [14] D. GILBARG, N.S. TRUDINGER, *Elliptic partial differential equations of second order*. Reprint of the 1998 edition, Classics in Mathematics. Springer-Verlag, Berlin, 2001.
- [15] E. GIUSTI, *Minimal surfaces and functions of bounded variation*, Monographs in Mathematics, 80, Birkhuser Verlag, Basel, 1984.
- [16] P. GRISVARD, *Elliptic problems in non-smooth domains*, Pitman, London, 1985.
- [17] A. HENROT, *Extremum problems for eigenvalues of elliptic operators*, Birkhäuser, Basel, Boston 2006.
- [18] A. HENROT, E. OUDET, *Minimizing the second eigenvalue of the Laplace operator with Dirichlet boundary conditions*, Arch. Rational Mech. Anal., **169** (2003), 73-87.
- [19] A. HENROT, M. PIERRE, *Variation et optimisation de formes*, Mathématiques et Applications **48**, Springer, 2005.
- [20] E. KRAHN, *Über Minimaleigenschaften der Kugel in drei un mehr Dimensionen*, Acta Comm. Univ. Dorpat., **A9** (1926), 1-44.

- [21] C.S. LIN, *On the second eigenfunctions of the Laplacian in  $\mathbb{R}^2$* , Comm. Math. Phys., **111** no2 (1987), 161-166.
- [22] A. MELAS, *On the nodal line of the second eigenfunction of the Laplacian in  $\mathbb{R}^2$* , J. Diff. Geometry, **35** (1992), 255-263.
- [23] L.E. PAYNE, *Isoperimetric inequalities and their applications*, SIAM Rev. **9** (1967), 453-488.
- [24] B. ROUSSELET, *Shape Design Sensitivity of a Membrane*, J. Opt. Theory and Appl., **40** (1983), 595-623.
- [25] M. VAN DEN BERG, *On condensation in the free-boson gas and the spectrum of the Laplacian*, J. Statist. Phys., **31** (1983), no. 3, 623-637.
- [26] S.-T. YAU *Open problems in geometry. Differential geometry: partial differential equations on manifolds*, (Los Angeles, CA, 1990), 1-28, Proc. Sympos. Pure Math., 54, Part 1, Amer. Math. Soc., Providence, RI, 1993.