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Claude Viterbo

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# Symplectic Homogenization

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## Abstract

Let  $H(q, p)$  be a Hamiltonian on  $T^*T^n$ . We show that the sequence  $H_k(q, p) = H(kq, p)$  converges for the  $\gamma$  topology defined by the author, to  $\overline{H}(p)$ . This is extended to the case where only some of the variables are homogenized, that is the sequence  $H(kx, y, q, p)$  where the limit is of the type  $\overline{H}(y, q, p)$  and thus yields an “effective Hamiltonian”. We give here the proof of the convergence, and the first properties of the homogenization operator, and give some immediate consequences for solutions of Hamilton-Jacobi equations, construction of quasi-states, etc....

We also prove that the function  $\overline{H}$  coincides with Mather’s  $\alpha$  function defined in [Ma] which gives a new proof of its symplectic invariance proved by P. Bernard [Bern 2].

Keywords: Homogenization, symplectic topology, Hamiltonian flow, Hamilton-Jacobi equation, variational solutions.

AMS Classification: primary: 37J05, 53D35 secondary: 35F20, 49L25, 37J40, 37J50.

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# Contents

<b>1</b>	<b>Introduction</b>	<b>3</b>
<b>2</b>	<b>A crash course on generating function metric</b>	<b>5</b>
2.1	Variational solutions of Hamilton-Jacobi equations . . . . .	9
<b>3</b>	<b>Statement of the main results</b>	<b>9</b>
3.1	Standard homogenization. . . . .	9
3.2	Partial Homogenization . . . . .	11
3.3	Homogenized Hamilton-Jacobi equations . . . . .	12
<b>4</b>	<b>Proof of the main theorem.</b>	<b>13</b>
4.1	Reformulating the problem and finding the homogenized Hamiltonian. . . . .	14
4.2	Concluding the proof of the main theorem. . . . .	21
<b>5</b>	<b>Proof of theorem 3.2.</b>	<b>30</b>
<b>6</b>	<b>Proof of the partial homogenization case</b>	<b>32</b>
6.1	Resolution in the $(q_2, P_1)$ variables . . . . .	35
6.2	Generating functions for $H(kx, y, q, p)$ . . . . .	36
6.3	The case of $\ell$ terms . . . . .	36
<b>7</b>	<b>Proof of proposition 3.6</b>	<b>39</b>
<b>8</b>	<b>Non compact-supported Hamiltonians and the time dependent case</b>	<b>40</b>
8.1	The coercive case . . . . .	40
8.2	The non coercive case . . . . .	42
<b>9</b>	<b>Homogenization in the <math>p</math> variable</b>	<b>43</b>
9.1	Partial Legendre transform . . . . .	43
9.2	Connection with Mather $\alpha$ function . . . . .	46
<b>10</b>	<b>Some examples and applications</b>	<b>47</b>
10.1	Homogenization of $H(t, q, p)$ in the variable $t$ . . . . .	47
10.2	The one dimensional case: $\frac{1}{2} p ^2 - V(x)$ . . . . .	48
10.3	Some computations . . . . .	49
10.4	Homogenized metric and the Thurston-Gromov norm . . . . .	50
<b>11</b>	<b>Further questions</b>	<b>51</b>

Appendix: Capacity of completely integrable systems. 52

References 53

## 1 Introduction

The aim of this paper is to define the notion of homogenization for a Hamiltonian diffeomorphism of  $T^*T^n$ . In other words, given  $H(t, q, p)$  defined for  $(q, p)$  in  $T^*T^n$ ,  $t$  in  $\mathbb{R}$ , 1-periodic in  $t$  we shall study whether the sequence  $H_k$  defined by  $H_k(t, q, p) = H(kt, kq, p)$  converges for the symplectic metric  $\gamma$  defined in [V1] to some Hamiltonian  $\overline{H}$ , necessarily of the form  $\overline{H}(q, p) = \overline{h}(p)$ .

This convergence of  $H_k$  to  $H$  should be understood as the convergence of the time one flows of  $H_k$ ,  $\varphi_k^t$  to the time one flow of  $\overline{H}$ ,  $\overline{\varphi}^t$ -again for the symplectic metric  $\gamma$ . This metric is necessarily rather weak, since for example there cannot be any  $C^0$  convergence for the flows.

However such convergence implies convergence of the variational solution (see [O-V] for the definition) of Hamilton-Jacobi equations.

$$(HJ) \quad \begin{cases} \frac{\partial}{\partial t} u(t, q) + H(k \cdot t, k \cdot q, \frac{\partial}{\partial x} u(t, x)) = 0 \\ u(0, q) = f(q) . \end{cases}$$

to the variational solutions of

$$(\overline{HJ}) \quad \begin{cases} \frac{\partial}{\partial t} u(t, q) + \overline{H}(t, q, \frac{\partial}{\partial x} u(t, x)) = 0 \\ u(0, q) = f(q) . \end{cases}$$

It is important to notice that none of these convergences implies any kind of pointwise or almost everywhere convergence (however  $C^0$  convergence of the flows implies  $\gamma$ -convergence as we proved in [V1], we refer to Humilière's work in [Hu] for stronger statements), but rather to some variational notions of convergence, like  $\Gamma$ -convergence (see [de G], [Dal M]). This or similar notions are used in homogenization theory, the theory of viscosity solutions for Hamilton-Jacobi equations (see [L-P-V]), or the rescaling of metrics on  $T^n$  (see [Gr]).

All the above-mentioned papers can be considered as anticipating the theory of "symplectic homogenization" that is presented here. We believe some of the advantages of this unified treatment are

- (1) the disposal of any convexity or even coercivity (as in [L-P-V]) assumption on  $H$  in the  $p$  direction, usually needed to define  $\overline{H}$  because of the

use of minimization techniques for the Lagrangian. In fact our homogenization is defined on compact supported objects, and then showed to extend to a number of non compact supported situations.

- (2) the natural extension of homogenization to cases where  $H$  has very little regularity (less than continuity is needed).
- (3) a well defined and common definition of the convergence of  $H_k$  to  $\overline{H}$  or  $\varphi_k$  to  $\overline{\varphi}$  that applies to flows, Hamilton-Jacobi equations, etc.
- (4) The symplectic invariance of the homogenized Hamiltonian extends the invariance results defined for example in [Bern 1] for Mather's  $\alpha$  function, making his constructions slightly less mysterious.
- (5) geometric properties of the function  $\overline{H}$  (see proposition 3.2, (5)). yielding computational methods extending those obtained in the one-dimensional case in [L-P-V] or in other cases (see for example [Conc]).

This paper will address these fundamental questions, some other applications will be dealt with in subsequent papers.

## Acknowledgements

This paper originated in the attempt to understand the physical phenomenon of self-adaptative oscillators described in [B-C-B], that the author learned from a popularization CNRS journal "Images de la physique". We hope to be able to explain in a future publication the connection between Symplectic homogenization and self-adaptative phenomena in vibrating structures. We not only wish to thank the authors of [B-C-B] for writing the paper and its popularized version[Bou], but also CNRS for maintaining journals like "Images de la physique".

I wish to thank S. Kuksin and P. Pansu for useful discussions and the University of Beijing for hospitality during the preparation of this work. I am also grateful to the organizers of the Yashafest in Stanford, and the Conference in symplectic topology in Kyoto for giving me the opportunity to present preliminary versions of this work. I also warmly thank Claudine Harmide for her efficient and speedy typing.

## 2 A crash course on generating function metric

This section is devoted to defining the metric  $\gamma$  for which we shall later prove that the sequence  $H_k(q, p) = H(kq, p)$  is convergent. The reader may well skip this section and jump directly to section 3, possibly returning here for reference.

Let  $M$  be an  $n$ -dimensional closed manifold,  $L$  be a Lagrangian submanifold in  $T^*M$  Hamiltonianly isotopic to the zero section  $O_M$  (i.e. there is a Hamiltonian isotopy  $\varphi_t$  such that  $\varphi_1(O_M) = L$ ).

**Definition 2.1.** The smooth function  $S : M \times \mathbb{R}^k \rightarrow \mathbb{R}$  is a generating function quadratic at infinity for  $L$  if

i) there is a non degenerate quadratic form  $q$  on  $\mathbb{R}^q$  such that

$$|\nabla_\xi S(q; \xi) - \nabla B(\xi)| \leq C$$

ii) the map

$$(q; \xi) \rightarrow \frac{\partial S}{\partial \xi}(q; \xi)$$

has zero as a regular value

iii) by i) and ii),  $\Sigma_s = \{(q; \xi) \mid \frac{\partial S}{\partial \xi}(q; \xi) = 0\}$  is a compact submanifold in  $M \times \mathbb{R}^k$ . The map

$$\begin{aligned} i_\delta : \Sigma_s &\rightarrow T^*M \\ (q; \xi) &\rightarrow (q, \frac{\partial S}{\partial q}(q; \xi)) \end{aligned}$$

has image  $i_s(\Sigma_s) = L$ .

*Remarks 2.2.* **(1)** In this paper, we shall always use a semicolon to separate the “base variables”  $q$  from the “fibre variables”,  $\xi$ .

**(2)** We still speak of generating function when there are no fibre variables. In this case,  $L$  is the graph of the differential  $dS(q)$ .

**(3)** In the sequel we shall abbreviate “generating function quadratic at infinity” by “G.F.Q.I.”.

Now when  $L$  is Hamiltonianly isotopic to the zero section, we know according to [V1] (prop 1.5, page 688) that the generating function is unique up to some elementary operations (see [V1], loc.cit.). Moreover, an elementary computation shows that denoting by  $S^\lambda$  the set

$$\{(q; \xi) \in M \times \mathbb{R}^k \mid S(q; \xi) \leq \lambda\}$$

we have for  $C$  large enough that

$$H^*(S^c, S^{-c}) \simeq H^*(M) \otimes H^*(D^-, \partial D^-)$$

where  $D^-$  is the unit disc of the negative eigenspace of  $B$ . Therefore, to each cohomology class  $\alpha$  in  $H^*(M) - \{0\}$  we may associate the image of  $\alpha \otimes T$  ( $T$  is a chosen generator of  $H^*(D^-, \partial D^-) \simeq \mathbb{Z}$ ), and by minmax a critical level  $c(\alpha, S)$  (see [V1] section 2, p.690-693).

**Definition 2.3.** Let  $L$  be Hamiltonianly isotopic to the zero section. We set

$$\begin{aligned} c_-(L) &= c(1, S) & 1 \in H^0(M) \\ c_+(L) &= c(\mu, S) & \mu \in H^n(M) \setminus \{0\} \\ \gamma(L) &= & c(\mu, S) - c(1, S) \end{aligned}$$

*Remark 2.4.* 1) Note that we may always add a constant to  $S$ . This shifts  $c_-(L)$  and  $c_+(L)$  by the same constant, so that, unless we normalized in some way  $S$ ,  $c_-(L), c_+(L)$  are not well defined, but their difference  $\gamma(L)$  is well-defined. However, if we specify the Hamiltonian  $H$  yielding the isotopy between the zero section and  $L$ , we may normalize  $S$  by requiring that its critical values coincide with the actions

$$\int_{0^1} [p(t)\dot{q}(t) - H(t, q(t), p(t))] dt$$

where  $(q(t), p(t)) = \varphi^t(q(0), 0)$  satisfies  $p(1) = 0$ .

Thus  $c_{\pm}(H)$  is well defined. Since if  $\varphi^1$  is generated by some compact supported Hamiltonian, such a Hamiltonian is unique, we may define  $c_{\pm}(\varphi)$  for  $\varphi \in \mathcal{H}D_c(T^*T^n)$

2) We shall sometimes deal with the case  $M = \mathbb{R}^n$ . Then we need quadraticity of  $S$  both in the  $\xi$  and  $x$  variable, so that

i) in Definition 2.1 should be replaced by

i') there exists a quadratic form  $q(x, \xi)$  on  $M \times \mathbb{R}^k (= \mathbb{R}^n \times \mathbb{R}^k)$  such that

$$|\nabla S(q; \xi) - \nabla B(q; \xi)| \leq C .$$

The map  $\gamma$  is well defined on the set  $\mathcal{L}$  of Lagrangian submanifolds Hamiltonianly isotopic to the zero section, where the Hamiltonian is assumed to be compact supported.

According to [V1], the metric  $\gamma$  defines a metric on  $\mathcal{L}$  by setting

**Definition 2.5.**  $\gamma(L, L') = c(\mu, S \ominus S') - c(1, S \ominus S')$  where  $(S \ominus S')(q; \xi, \xi') = S(q; \xi) - S(q; \xi')$ .

That this is indeed a metric is a consequence of Lusternik-Shnirelman's theory, as we proved in [V1].

Our goal however is to define a metric on  $\mathcal{H}_c(T^*M) = C_c^\infty([0, 1] \times T^*T^n, \mathbb{R})$  the set of compact supported, time dependent Hamiltonian isotopies of  $T^*M$ , and on  $\mathcal{HD}_c(T^*M)$  the group of time one maps of Hamiltonians in  $\mathcal{H}_c(T^*M)$ . In general, we may set

**Definition 2.6.** We set

$$\widehat{\gamma}(\varphi) = \sup\{\gamma(\varphi(L), L) \mid L \in \mathcal{L}\} .$$

$$\widehat{\gamma}(\varphi, \psi) = \widehat{\gamma}(\varphi\psi^{-1}).$$

*Remark 2.7.* For  $M = T^n$ , the graph of  $\varphi$ ,

$$\Gamma(\varphi) = \{(z, \varphi(z)) \mid z \in T^*T^n\}$$

is a Lagrangian submanifold of  $T^*T^n \times \overline{T^*T^n}$  (where  $\overline{T^*M}$  is  $T^*M$  with the symplectic form of opposite sign :  $-dp \wedge dq$ ).

But  $T^*T^n \times \overline{T^*T^n}$  is covered by  $T^*(\Delta_{T^*T^n})$  where  $\Delta_{T^*T^n}$  is the diagonal, and we may lift  $\Gamma(\varphi)$  to  $\widetilde{\Gamma}(\varphi)$ , which is now a Lagrangian submanifold in  $T^*(\Delta_{T^*T^n})$ .

When  $\varphi$  has compact support, we may compactify both  $\widetilde{\Gamma}(\varphi)$  and  $\Delta_{T^*T^n}$  and we get a Lagrangian submanifold  $\Gamma(\varphi)$  in  $T^*(S^n \times T^n)$ . We then defined in [V1] (page 679)

$$\gamma(\varphi) = c(\mu_{T^n} \otimes \mu_{S^n}, \overline{\Gamma}(\varphi)) - c(1 \otimes 1, \overline{\Gamma}(\varphi)) .$$

We proved in [V3] that  $\widehat{\gamma}(\varphi) \leq \gamma(\varphi)$ .

**Proposition 2.8.** (see [V3])

The map  $\widehat{\gamma}$  defines a bi-invariant metric on  $\mathcal{HD}_c(T^*M)$  since

i) it is nondegenerate  $\widehat{\gamma}(\varphi) = 0 \iff \varphi = id$

ii) it is invariant by conjugation  $\widehat{\gamma}(\psi\varphi\psi^{-1}) = \widehat{\gamma}(\varphi)$  for any  $\psi$  in  $\mathcal{HD}_c(T^*M)$ .

iii) it satisfies the triangle inequality

$$\widehat{\gamma}(\varphi\psi) \leq \widehat{\gamma}(\varphi) + \widehat{\gamma}(\psi)$$

for any  $\varphi, \psi$  in  $\mathcal{HD}_c(T^*M)$ .

The above properties also hold for  $\gamma$  instead of  $\widehat{\gamma}$  when  $M = T^n$ .



Let  $\lambda = pdq$  be the Liouville form on  $T^*M$ . A vector field  $Z$  is called a Liouville vector field, if  $Z = Z_\lambda + X_H$  where  $X_H$  is Hamiltonian, while  $i_{Z_\lambda}\omega = \lambda$  (hence  $i_Z\omega = \lambda + dH$ ). In particular  $Z$  is conformal (i.e. the flow  $\psi_t$  of  $Z$  satisfies  $\psi_t^*\omega = e^t\omega$ ) and we have

$$(1) \quad \gamma(\psi_t\varphi\psi_t^{-1}) = e^t\gamma(\varphi) .$$

In the set  $\mathcal{H}_c(T^*M)$  the metric  $\widehat{\gamma}$  is defined as follows

**Definition 2.9.** Let  $H$  be a Hamiltonian, with flow  $\psi^t$ . We denote by

$$\widehat{\gamma}(H) = \sup \{ \widehat{\gamma}(\psi^t) \mid t \in [0, 1] \}$$

and similarly

$$\widehat{\gamma}(H) = \sup \{ \widehat{\gamma}(\psi^t) \mid t \in [0, 1] \}$$

Finally we state some convergence criterion for the  $\widehat{\gamma}$  metric.

**Proposition 2.10.** Let  $M = \mathbb{R}^n$  or  $T^n$ .

1) Assume the sequence  $H_k$  of Hamiltonians on  $T^*M$  with fixed support converges  $C^0$  to  $H$ . Then  $H_k$  converges for the  $\gamma$  metric to  $H$ .

2) There is a constant  $C$  such that if  $\varphi$  is supported in  $W_R = \{(q, p) \in T^*M \mid |p| \leq R\}$  and for any  $z$  in  $T^*M$   $d(z, \varphi(z)) \leq \varepsilon$ , we have

$$\gamma(\varphi) \leq C\varepsilon R$$

*Proof.* Part 1) is proved as in [V1] Proposition 4.6 (page 699).

As for part 2), we may follow the some pattern as in the proof of proposition 4.15 (loc.cit. page 703), provided we prove that there is a  $C^2$  small Hamiltonian supported in  $W_{2R}$  such that its time one map,  $\psi$ , satisfies

$$d(z, \psi(z)) \geq \varepsilon \quad \forall z \in W_R .$$

Given  $H$  without critical point in  $W_R$ , and  $C^2$  small, it is well known that  $\psi^t$  has no periodic orbit of period less than 1, hence  $d(z, \psi(z))$  is bounded from below by some  $\varepsilon_0 > 0$ . This concludes our proof.  $\square$

We may therefore define, as Humilière did in [Hu], the completion  $\widehat{\mathcal{H}}(T^*M)$  of  $\mathcal{H}_c(T^*M)$  for  $\gamma$ . By the above proposition we deduce that a  $C^0$ -converging sequence of Hamiltonians will be a Cauchy sequence for  $\gamma$ , hence defines an element in  $\widehat{\mathcal{H}}(T^*M)$ . We thus get

**Proposition 2.11.** There is an inclusion map

$$C_c^0([0, 1] \times T^*M, \mathbb{R}) \rightarrow \widehat{\mathcal{H}}(T^*M)$$

Similarly if  $\mathcal{HD}_0(T^*M)$  is the  $C^0$  closure of  $\mathcal{HD}_c(T^*M)$  we have an inclusion

$$\mathcal{HD}_0(T^*M) \rightarrow \widehat{\mathcal{HD}}(T^*M)$$

## 2.1 Variational solutions of Hamilton-Jacobi equations

Let us consider the symplectic covering of  $T^*T^n \times \overline{T^*T^n}$  by  $T^*(T^n \times \mathbb{R}^n)$  given by

$$(q, p, Q, P) \longrightarrow (Q, p, p - P, Q - q)$$

Let  $\varphi^t$  be the Hamiltonian flow of  $H(q, p)$ . Then, the graph of  $\varphi^t$  has image  $\tilde{\Gamma}(\varphi^t)$ . Let  $S_t(q, P, \xi)$  be a generating function for  $\tilde{\Gamma}(\varphi^t)$ . We denote by  $c(1(q) \otimes \mu, S_t)$  the number  $c(\mu, S_{t,q})$  where  $S_{t,q}(P, \xi) = S(q, P, \xi)$ . Then  $u_t(q, P) = c(1(q) \otimes 1(P), S_t)$  is a variational solution of

$$\begin{cases} \frac{\partial}{\partial t} u_t(q, P) + H(q, P + \frac{\partial}{\partial t} u_t(q)) = 0 \\ u_0(q, P) = 0 \end{cases}$$

We refer to [O-V] and [C-V] for more informations on variational solutions.

## 3 Statement of the main results

### 3.1 Standard homogenization.

Let  $H(t, q, p)$  be a  $C^2$  Hamiltonian on  $T^*T^n$ , 1-periodic in  $t$ , and compact supported

#### Theorem 3.1 (Main theorem).

Let  $H(t, q, p)$  be a  $C^2$  Hamiltonian on the torus  $T^n$ . Then the following holds:

- (1) The sequence  $H_k(t, q, p) = H(kt, kq, p)$   $\gamma$ -converges to  $\overline{H}(t, q, p) = \overline{h}(p)$ , where  $h$  is continuous.
- (2) The function  $\overline{H}$  only depends on  $\varphi^1$ , the time one map associated to  $H$  (i.e. it does not depend on the isotopy  $(\varphi^t)_{t \in [0,1]}$ ).
- (3) The map

$$\mathcal{A} : C^2([0, 1] \times T^*T^n, \mathbb{R}) \rightarrow C^0(\mathbb{R}^n, \mathbb{R})$$

given by  $\mathcal{A}(H) = \overline{H}$  extends to a nonlinear projector (i.e. it satisfies  $\mathcal{A}^2 = \mathcal{A}$ ) with Lipschitz constant 1

$$\mathcal{A} : \widehat{\mathcal{H}}(T^*T^n) \rightarrow C^0(\mathbb{R}^n, \mathbb{R})$$

where the metric on  $\widehat{\mathcal{H}}$  is given by  $\gamma$  and the metric on  $C^0(\mathbb{R}^n, \mathbb{R})$  is the  $C^0$  metric.

The next theorem states some properties of the map  $\mathcal{A}$

**Theorem 3.2 (Main properties of symplectic homogenization).**

Let  $\mathcal{A}$  be the map defined in the above theorem. Then it satisfies the following properties:

- (1) The map  $\mathcal{A}$  is monotone, i.e. if  $H_1 \leq H_2$  then  $\mathcal{A}(H_1) \leq \mathcal{A}(H_2)$ .
- (2) The map  $\mathcal{A}$  is invariant by Hamiltonian symplectomorphism:  
 $\mathcal{A}(H \circ \psi) = \mathcal{A}(H)$  for all  $\psi \in \mathcal{HD}(T^*\mathbb{T}^n)$
- (3) We have  $\mathcal{A}(-H) = -\mathcal{A}(H)$ .
- (4) The map  $\mathcal{A}$  extends to characteristic functions of subsets, hence induces a map (still denoted by  $\mathcal{A}$ ) between the set of subsets of  $T^*\mathbb{T}^n$ ,  $\mathcal{P}(T^*\mathbb{T}^n)$ , to the set of subsets of  $\mathbb{R}^n$ ,  $\mathcal{P}(\mathbb{R}^n)$ . This map is bounded by the symplectic shape of Sikorav (see [Be, Sik, El]), i.e.

$$\text{shape}(U) \subset \mathcal{A}(U)$$

- (5) If  $L$  is a Lagrangian Hamiltonianly isotopic to  $L_{p_0} = \{(q, p_0) \in T^*\mathbb{T}^n\}$  and  $H(L) \geq h$  (resp.  $\leq h$ ) we have  $\mathcal{A}(H)(p_0) \geq h$  (resp.  $\leq h$ ).

- (6) We have

$$\lim_{k \rightarrow \infty} \frac{1}{k} c_+(\varphi^k) = \sup_{p \in \mathbb{R}^n} \overline{H}(p)$$

$$\lim_{k \rightarrow \infty} \frac{1}{k} c_-(\varphi^k) = \inf_{p \in \mathbb{R}^n} \overline{H}(p)$$

- (7) Given any measure  $\mu$  on  $\mathbb{R}^n$  the map

$$\zeta(H) = \int_{\mathbb{R}^n} \mathcal{A}(H)(p) d\mu(p)$$

is a **symplectic quasi-state** (cf. [E-P] for the definition and properties of this notion, based on earlier work by [Aarnes]). In particular we have  $\mathcal{A}(H + K) = \mathcal{A}(H) + \mathcal{A}(K)$  whenever  $H$  and  $K$  Poisson-commute (i.e.  $\{H, K\} = 0$ ).

*Remarks 3.3.* • In (1) the assumption could be replaced by the property that  $H_1 \preceq H_2$  in the sense of [V1].

- As a result of (5) if  $u$  is a smooth subsolution of the stationary Hamilton-Jacobi equation, that is  $H(x, p+du(x)) \leq h$  then  $\overline{H}(p) \leq h$ . Similarly if  $u$  is a smooth supersolution, that is  $H(q, p+du(q)) \geq h$  then  $\overline{H}(p) \geq h$ .
- From (5), we get the following statement: let

$$E_c^+ = \{p_0 \in \mathbb{R}^n \mid \exists L \text{ Hamiltonianly isotopic to } L_{p_0}, H(L) \geq c\}$$

$$E_c^- = \{p_0 \in \mathbb{R}^n \mid \exists L \text{ Hamiltonianly isotopic to } L_{p_0}, H(L) \leq c\}$$

As a result, if  $p \in \overline{E}_c^+ \cap \overline{E}_c^-$ , we have  $H(p) = c$ .

### 3.2 Partial Homogenization

We here consider the case where the Hamiltonian is defined on  $T^*T^n \times M$ , where  $M$  is some symplectic manifold. We shall only consider here the case where  $M = T^*T^m$ , but the general case can be easily adapted.

**Theorem 3.4 (Main theorem, partial homogenization case).** *Let  $H(x, y, q, p)$  be a Hamiltonian on  $T^*T^{n+m}$ . Then*

- (1) *The sequence*

$$H_k(x, y, q, p) = H(kx, y, q, p)$$

*$\gamma$ -converges to  $\overline{H}(y, q, p)$*

- (2) *The map*

$$\mathcal{A}_x : C_c^2([0, 1] \times T^*T^{n+m}, \mathbb{R}) \rightarrow C^0(\mathbb{R}^n \times T^*T^m, \mathbb{R})$$

*given by  $\mathcal{A}_x(H) = \overline{H}$  extends to a projector (i.e. it satisfies  $\mathcal{A}_x^2 = \mathcal{A}_x$ ) with Lipschitz constant 1*

$$\mathcal{A}_x : \widehat{\mathcal{H}}(T^*T^{n+m}) \rightarrow \widehat{\mathcal{H}}(\mathbb{R}^n \times T^*T^m)$$

*where the metric on  $\widehat{\mathcal{H}}$  is  $\gamma$ .*

- (3) *If  $H_{(q,p)}(x, y) = H(x, y, q, p)$ , we have*

$$\mathcal{A}_x(H)(y, q, p) = \mathcal{A}(H_{(q,p)})(y)$$

*Remark 3.5.* (1) The Hamiltonian  $\overline{H}(y, q, p)$  is called the effective Hamiltonian. In case it is smooth, its flow is given by  $\overline{\Phi}(x_0, y_0, q_0, p_0) = (x(t), y(t), q(t), p(t))$

$$y(t) = y_0, \quad x(t) = \int_0^t \frac{\partial \overline{H}}{\partial y}(y_0, q(t), p(t)) dt,$$

$$\dot{q}(t) = \frac{\partial \overline{H}}{\partial p}(y_0, q(t), p(t)), \quad \dot{p}(t) = -\frac{\partial \overline{H}}{\partial q}(y_0, q(t), p(t))$$

- (2) It is not true anymore that  $\overline{H}$  depends only on the time one map of  $H$ . It however only depends on the family of time one maps of  $H_{(q,p)}$ .
- (3) More generally, using (3), we may translate in our situation the properties of  $\mathcal{A}$  stated in the first proposition. The projector  $\mathcal{A}_x$  is not invariant by symplectic maps. It is however invariant by fiber-preserving hamiltonian symplectic maps: if  $\psi = \psi_1 \times \psi_2$ . in other words

$$\mathcal{A}_x(H \circ \psi)(y, q, p) = \mathcal{A}_x(H)(y, \psi_2(q, p))$$

### 3.3 Homogenized Hamilton-Jacobi equations

Our theorem has some interesting applications to generalized solutions of evolution Hamilton-Jacobi equations

$$(HJ) \begin{cases} \frac{\partial}{\partial t} u(t, q) + H(t, q, \frac{\partial}{\partial x} u(t, x)) = 0 \\ u(0, q) = f(q) . \end{cases}$$

where  $t \in \mathbb{R}, q \in T^n$ .

Smooth solutions to such equations are only defined for  $t$  less than some  $T_0$  : solutions exhibit shocks, that is  $|u|_{C^1([0,T] \times T^n, \mathbb{R})}$  blows up as  $T$  goes to  $T_0$ .

There are essentially two types of generalized solutions for such equations : viscosity solutions (cf. [C-L]) and variational solutions (cf. [O-V]). These two solutions do not coincide in general, with one notable exception: when the Hamiltonian is convex in  $p$ .

From [L-P-V] it follows that if  $H$  is convex in  $p$ , and  $u_k$  is the solution of

$$(HJ_k) \begin{cases} \frac{\partial}{\partial t} u_k(t, q) + H(kt, kq, \frac{\partial}{\partial x} u_k(t, q)) = 0 \\ u_k(0, q) = f(q) \end{cases}$$

the sequence  $(u_k)_{k \geq 1}$  converges to  $\bar{u}$ , the solution of

$$(\overline{HJ}) \begin{cases} \frac{\partial}{\partial t} \bar{u}(t, q) + \overline{H}(\frac{\partial}{\partial x} \bar{u}(t, q)) = 0 \\ \bar{u}(0, q) = f(q) \end{cases}$$

Our theorem, together with results by Humilière (cf. [Hu]) implies that this extends to the non convex case, provided  $u_k$  is the variational solution and  $\overline{H}$  is given by our main theorem. We now state the more general proposition, yielding the analog of [L-P-V] when  $n = 0$ :

**Proposition 3.6.** *Let  $H \in C^0(T^*T^{n+m})$ ,  $f \in C^0(T^{n+m})$  and  $u_k$  be the variational solution of*

$$(HJP_k) \begin{cases} \frac{\partial}{\partial t} u_k(t, x, q) + H(kx, q, \frac{\partial}{\partial x} u_k(t, x, q)) = 0 \\ u_k(0, x, q) = f(x, q) \end{cases}$$

*Then  $\lim_{k \rightarrow +\infty} u_k(t, q) = \bar{u}(t, q)$  where convergence is uniform on compact time intervals and  $\bar{u}$  is the variational solution of  $(\overline{HJP})$ .*

$$(HJP) \begin{cases} \frac{\partial}{\partial t} \bar{u}(t, x, q) + \overline{H}(kx, q, \frac{\partial}{\partial x} \bar{u}(t, x, q)) = 0 \\ \bar{u}(0, x, q) = f(x, q) \end{cases}$$

*More precisely, there is a sequence  $\varepsilon_k$  going to zero, such that*

$$|u_k(t, x, q) - \bar{u}(t, x, q)| \leq \varepsilon_k t$$

The next three sections will be devoted to the proof of our main theorem, first in the “standard case”, then in the “partial homogenization” setting.

## 4 Proof of the main theorem.

Let us give the reader the main steps of the proof. We denote by  $\varphi_k^t$  be the flow of  $H(k \cdot q, p)$ . Starting from a G.F.Q.I. of the flow  $\varphi^t = \varphi_1^t$ , we shall in the first part of subsection 4.1, construct a G.F.Q.I. of  $\varphi_k^t$ .

our proof will then be split in two steps

- Finding a candidate  $\bar{\varphi}^t$  for the limit of  $\varphi_k^t$
- Showing that the limit of  $\varphi_k^t$  is indeed  $\bar{\varphi}^t$

The first step goes along the following lines: if  $H$  is independent from  $q$ , then  $c(\mu_q \otimes 1(p), \varphi_k^t) = H(p)$ , so we have to prove that

$$\lim_{k \rightarrow \infty} c(\mu_q \otimes 1(p), \varphi_k^t) = \overline{H}(p)$$

exists. This is the second part of subsection 4.1 and is proved in proposition 4.7.

The second step is more delicate, and is dealt with in subsection 4.2. In fact, the formula obtained for the G.F.Q.I. of  $\varphi_k^t$  yields an inequality valid for any Hamiltonian map  $\alpha$

$$\liminf_{k \rightarrow \infty} c(\mu, \varphi_k \alpha) \leq c(\mu, \overline{\varphi} \alpha)$$

proved in proposition 4.11.

We must then prove the reverse inequality, This relies in a crucial way on the main result of [V5], which implies that a  $1/k$ -periodic Lagrangian in  $T^*T^n$  contained in the unit disc bundle, has  $\gamma$  norm converging to zero.

We now give the details of the proof.

## 4.1 Reformulating the problem and finding the homogenized Hamiltonian.

First of all, we shall assume we are dealing with an autonomous Hamiltonian. We shall see in the next section that the general case reduces to this one.

Let  $\varphi^t$  be the flow associated to  $H$ , and  $\varphi = \varphi^1$ .

Similarly let  $\varphi_k^t$  be the flow associated to  $H_k(q, p) = H(kq, p)$ , and  $\varphi_k = \varphi_k^1$ . We first compute  $\varphi_k$  as a function of  $\varphi$ .

**Lemma 4.1.** *Let  $\rho_k(q, p) = (kq, p)$ , then  $\varphi_k = \rho_k^{-1} \varphi \rho_k$ .*

*Proof.* The map  $\rho_k$  is conformally symplectic, hence

$$\begin{aligned} dH_k(z)\xi &= dH(\rho_k(z))d\rho_k(z)\xi = \omega(X_H(\rho_k(z)), d\rho_k(z)\xi) \\ &= (\rho_k^* \omega)(d\rho_k^{-1}(z)X_H(\rho_k(z)), \xi) \end{aligned}$$

Since  $\rho_k^* \omega = k\omega$ , we get

$$\begin{aligned} X_{H_k}(z) &= k((\rho_k)_* X_H)(z) \\ &= (\rho_k)_*(kX_H)(z) \end{aligned}$$

The flow of  $kX_H$  is  $\varphi^{kt}$ , hence the flow of  $(\rho_k)_*(kX_H)$  is  $\rho_k^{-1} \varphi^{kt} \rho_k$ .  $\square$

We are thus looking for the  $\gamma$ -limit of  $\rho_k^{-1} \varphi^k \rho_k$ .

**Remark :** The map  $\rho_k^{-1}$  is not well defined on  $T^*T^n$ , so that *a priori* the lemma only makes sense on  $T^*\mathbb{R}^n$ . However given a continuous path  $z(t)$  from  $z_0$  to  $z_1$  in  $T^*T^n$ , and  $u_0$  such that  $ku_0 = z_0$ , we may find a unique continuous path  $u(t)$  such that  $u(0) = u_0$  and  $k.u(t) = z(t)$ . Therefore given an isotopy  $\psi^t$  starting at the identity, there is a unique isotopy  $\tilde{\psi}^t$  such that  $\rho_k \tilde{\psi}^t = \psi^t$ . Moreover the map  $\tilde{\psi}^1$  only depends on  $\psi^1$  and not on the choice of the Hamiltonian isotopy.

Note also that we may replace  $\varphi = \varphi^1$  by  $\varphi^{1/r}$  for some fixed integer  $r$ . Indeed, if  $\rho_k^{-1} \varphi^{k/r} \rho_k$   $c$ -converges to  $\psi$ , we have that

$$\rho_k^{-1} \varphi^k \rho_k = \rho_r \left( \rho_{kr}^{-1} \varphi^{kr/r} \rho_{kr} \right) \rho_r^{-1}$$

$c$ -converges to  $\rho_r \psi \rho_r^{-1}$ . If our theorem is proved for  $\varphi^{1/r}$ ,  $\psi$  will be generated by a Hamiltonian depending only on the  $p$  variable. We easily check that in this case

$$\rho_r \psi \rho_r^{-1} = \psi^r.$$

In other words,  $\rho_k^{-1} \varphi^k \rho_k$  converges to  $\tilde{\psi}^r$ .

We assume in the sequel that  $\varphi$  is  $C^1$  close to the identity, so that it lifts to a Hamiltonian diffeomorphism of  $T^*\mathbb{R}^n$   $C^1$  close to the identity,  $\tilde{\varphi}$ . The map  $\tilde{\varphi}$  has a generating function

$$\widehat{S}(Q, p) = \langle p, Q \rangle + S(Q, p)$$

where  $S$  is defined on  $T^*T^n$ , and  $S$  defines  $\tilde{\varphi}$  by the relation

$$\tilde{\varphi} \left( Q + \frac{\partial S}{\partial p}(Q, p), p \right) = \left( Q, p + \frac{\partial S}{\partial Q}(Q, p) \right)$$

or else

$$\tilde{\varphi} \left( \frac{\partial \widehat{S}}{\partial p}(Q, p), p \right) = \left( Q, \frac{\partial \widehat{S}}{\partial Q}(Q, p) \right)$$

This means that the graph of  $\tilde{\varphi}$ ,  $\overline{\Gamma}(\varphi)$ , in  $T^*\mathbb{R}^n \times \overline{T^*\mathbb{R}^n} \simeq T^*\Delta_{\mathbb{R}^{2n}}$ , has the compact supported generating function  $S(q, p)$  defined on  $T^*T^n$ .

In other words, if  $\tilde{\varphi}(q, p) = (Q, P)$  we have

$$\begin{cases} P - p = \frac{\partial S}{\partial Q}(Q, p) \\ q - Q = \frac{\partial S}{\partial p}(Q, p) \end{cases}$$

We now give the composition law for generating functions, due to Chekanov (cf. [Che])



**Lemma 4.2.** *Let  $\varphi_1, \varphi_2$  be Hamiltonian maps having  $S_1, S_2$  as generating functions. Then  $\varphi_1 \circ \varphi_2$  has the generating function*

$$\begin{aligned} S(q_1, p_2; q_2, p_1) &= S_1(q_1, p_1) + S_2(q_2, p_2) + \langle p_1, q_1 \rangle + \langle p_2, q_2 \rangle - \langle p_1, q_2 \rangle - \langle p_2, q_1 \rangle. \\ &= S_1(q_1, p_1) + S_2(q_2, p_2) + \langle p_1 - p_2, q_1 - q_2 \rangle. \end{aligned}$$

Note that if we set  $p_2 = p_1 - v$ ,  $q_2 = q_1 + u$ , we have

$$S(q_1, p_2; q_1 + u, p_2 + v) =$$

$$S_1(q_1, p_2 + v) + S_2(q_1 + u, p_2) - \langle v, u \rangle$$

so that  $S$  is a generating function quadratic at infinity.

*Proof.* It is a simple computation. The lemma claims that

$$\widehat{S}(q_1, p_2; q_2, p_1) = \widehat{S}_1(q_1, p_1) + \widehat{S}_2(q_2, p_2) - \langle p_1, q_2 \rangle$$

so we must prove that  $\varphi_1 \circ \varphi_2$  maps

$$\left( \frac{\partial \widehat{S}}{\partial p_2}(q_1, p_2; q_2, p_1), p_2 \right) = \left( \frac{\partial \widehat{S}_2}{\partial p_2}(q_2, p_2), p_2 \right)$$

to

$$\left( q_1, \frac{\partial \widehat{S}}{\partial q_1}(q_1, p_2; q_2, p_1) \right) = \left( q_1, \frac{\partial \widehat{S}_1}{\partial q_1}(q_1, p_1) \right)$$

where  $(q_1, p_1, q_2, p_2)$  are constrained by

$$\begin{cases} \frac{\partial \widehat{S}}{\partial p_1}(q_1, p_2; q_2, p_1) = 0 & \text{i.e. } q_2 = \frac{\partial \widehat{S}_1}{\partial p_1}(q_1, p_1) \\ \frac{\partial \widehat{S}}{\partial q_2}(q_1, p_2; q_2, p_1) = 0 & \text{i.e. } p_1 = \frac{\partial \widehat{S}_2}{\partial q_2}(q_2, p_2) \end{cases}$$

These last two equations are equivalent to

$$\begin{aligned} \varphi_2 \left( \frac{\partial \widehat{S}_2}{\partial p_2}(q_2, p_2), p_2 \right) &= \left( q_2, \frac{\partial \widehat{S}_2}{\partial q_2}(q_2, p_2) \right) \\ &= (q_2, p_1) \end{aligned}$$

and

$$\varphi_1(q_2, p_1) = \varphi_1 \left( \frac{\partial \widehat{S}_1}{\partial p_1}(q_1, p_1), p_1 \right) = \left( q_1, \frac{\partial \widehat{S}_1}{\partial q_1}(q_1, p_1) \right)$$

and thus

$$\varphi_1 \circ \varphi_2 \left( \frac{\partial \widehat{S}}{\partial p_2}(q_1, p_2; q_2, p_1), p_2 \right) = \left( q_1, \frac{\partial \widehat{S}}{\partial q_1}(q_1, p_2; q_2, p_1) \right)$$

where  $(q_1, p_2; q_2, p_1)$  are constrained by

$$\frac{\partial \widehat{S}}{\partial q_2}(q_1, p_2; q_2, p_1) = \frac{\partial \widehat{S}}{\partial p_1}(q_1, p_2; q_2, p_1) = 0$$

□

More generally, we get

**Lemma 4.3.** *The map  $\widehat{\varphi}^k$  is generated by*

$$\widehat{S}_k(q_1, p_k; p_1, q_2, p_2, \dots, q_{k-1}, p_{k-1}, q_k) = \Sigma_k(q_1, p_k; p_1, q_2, p_2, \dots, q_{k-1}, p_{k-1}, q_k) + Q_k(q_1, p_k; p_1, q_2, p_2, \dots, q_{k-1}, p_{k-1}, q_k)$$

where

$$\Sigma_k(q_1, p_k; p_1, q_2, p_2, \dots, q_{k-1}, p_{k-1}, q_k) = \sum_{j=1}^k S(q_j, p_j)$$

and

$$Q_k(q_1, p_k, p_1, q_2, \dots, q_{k-1}, p_{k-1}, q_k) = \sum_{j=1}^{k-1} \langle p_j - p_{j+1}, q_j - q_{j+1} \rangle + \langle p_k, q_1 \rangle$$

*Proof.* The proof follows immediately by induction from the previous lemma. □

Note that  $Q_k$  could be rewritten as

$$Q_k(q_1, p_k; p_1, q_2, \dots, q_{k-1}, p_{k-1}, q_k) = \sum_{j=1}^{k-1} \langle p_j, q_j - q_{j+1} \rangle + \langle p_k, q_k \rangle.$$

Again,  $\Sigma_k$  is defined on  $(T^*T^n)^k$ , while  $Q_k$  is defined on  $(T^*\mathbb{R}^n)^k$ . Finally we have

**Lemma 4.4.** *Let  $\varphi$  be generated by  $S(q, p)$ , then  $\varphi_k = \rho_k^{-1}\varphi^k\rho_k$  is generated by  $\widehat{F}_k$  given by*

$$\begin{aligned} \widehat{F}_k(q_1, p_k; p_1, \dots, q_{k-1}, p_{k-1}, q_k) = \\ \frac{1}{k}\widehat{\Sigma}_k(kq_1, p_k; p_1, \dots, kq_{k-1}, p_{k-1}, kq_k) + Q_k(q_1, p_k; p_1, \dots, q_{k-1}, p_{k-1}, q_k) \end{aligned}$$

*Proof.* Indeed if  $S(q, p; \xi)$  is a generating function for  $\psi$ , we have that  $\rho_k^{-1}\psi\rho_k$  is generated by  $\frac{1}{k}S(kq, p; \xi)$ .

Thus in our case, we expect the generating function

$$\begin{aligned} \widehat{G}_k(q_1, p_k; p_1, q_2, \dots, q_{k-1}, p_{k-1}, q_k) = \\ \frac{1}{k}\widehat{\Sigma}_k(kq_1, p_k; p_1, q_2, \dots, q_{k-1}, p_{k-1}, q_k) \\ + \frac{1}{k}Q_k(kq_1, p_k; p_1, q_2, \dots, q_{k-1}, p_{k-1}, q_k) . \end{aligned}$$

But the fiber preserving change of variable  $q_j \rightarrow kq_j (j \geq 2)$  sends  $\widehat{G}_k$  to

$$\begin{aligned} F_k(q_1, p_k; p_1, q_2, \dots, q_{k-1}, p_{k-1}, q_k) = \\ \frac{1}{k}\widehat{\Sigma}_k(kq_1, p_k; p_1, kq_2, \dots, kq_{k-1}, p_{k-1}, q_k) \\ + \frac{1}{k}Q_k(kq_1, p_k; p_1, kq_2, \dots, kq_{k-1}, p_{k-1}, q_k) . \end{aligned}$$

It is easy to check that the last term is equal to

$$Q_k(q_1, p_k; p_1, q_2, \dots, q_{k-1}, p_{k-1}, p_k) .$$

□

We now set

$$\begin{aligned} F_k(q_1, p_k; p_1, q_2, \dots, q_{k-1}, p_{k-1}, p_k) = \\ \widehat{F}_k(q_1, p_k; p_1, q_2, \dots, q_{k-1}, p_{k-1}, p_k) - \langle p_k, q_1 \rangle \end{aligned}$$

and to simplify our notations

$$x = q_1, y = p_k, \xi = (p_1, q_2, \dots, q_{k-1}, p_{k-1}, p_k) .$$

thus

$$F_k(x, y; \xi) = \sum_{j=1}^k S(q_j, p_j) + Q_k(x, y, \xi) - \langle y, x \rangle$$

**Definition 4.5.** We set  $h_k(y) = c(\mu_x, F_{k,y})$  where  $F_{k,y} = F_k(x, y; \xi)$ . We will also denote this function as  $c(\mu_x \otimes 1(y), F_k)$ .

*Remark 4.6.* As long as we write  $c(\mu_x \otimes 1(y), S)$  for a generating function  $S$ , there is no ambiguity. However, if  $\Lambda$  is the Lagrangian associated to  $S$ , and we write an expression like  $c(\mu, (\Lambda)_y)$ , one should be careful since  $S$  is only defined up to a constant, and this constant yields a coherent choice of a G.F.Q.I. for  $(\Lambda)_y$  for each  $y$ , so that the  $c(\mu, \Lambda_y)$  are well-defined up to the same constant for all values of the parameter  $y$ , and not up to a function of  $y$  as one could expect.

Clearly we take a G.F.Q.I.  $S$  for  $L$  and then  $(L)_y$  has G.F.Q.I.  $S_y = S(y, \bullet)$ .

**Proposition 4.7.** *The sequence  $(h_k)$  is a precompact sequence for the  $C^0$  topology.*

The proposition will follow from Ascoli-Arzelà's theorem once we prove the following

**Lemma 4.8.** *The sequence of  $(h_k)$  is equicontinuous.*

*Proof.* Indeed let  $\tilde{\varphi}$  be the lift of  $\varphi_k = \rho_k^{-1} \varphi^k \rho_k$  to  $T^*\mathbb{R}^n$ . It has support in some tube

$$T_A^*\mathbb{R}^n = \{(q, p) \in T^*\mathbb{R}^n \mid |p| \leq A\} .$$

Now for each  $p$  there exists  $q(p), \xi(p)$  such that

$$\frac{\partial F_k}{\partial x}(q(p), p; \xi(p)) = 0, \quad \frac{\partial F_k}{\partial \xi}(q(p), p, \xi(p)) = 0$$

$$F_k(q(p), p, \xi(p)) = h_k(p) .$$

Moreover, for generic  $\varphi$ , the map  $p \rightarrow (q(p), \xi(p))$  is smooth on the complement of some codimension 1 set. Thus for  $p$  in this complement,

$$\begin{aligned} dh_k(p) &= \frac{\partial}{\partial y} F_k(x(p), p, \xi(p)) \\ &= q(p) - Q_k(q(p), p) \end{aligned}$$

where  $Q_k$  is defined by

$$\tilde{\varphi}_k(q, p) = (Q_k(q, p), P_k(q, p)) .$$

The quantity  $q(p) - Q_k(q(p), p)$  can be estimated as follows : the first coordinate of the flow  $\tilde{\varphi}_k^t$  satisfies

$$\dot{q}_k(t) = \frac{\partial H}{\partial p}(kq_k(t), p_k(t))$$

hence  $|\dot{q}_k(t)|$  is bounded by  $C = \sup \left\{ \left| \frac{\partial H}{\partial p}(q, p) \right| \mid (q, p) \in T^*T^n \right\}$  hence  $|q - Q_k(q, p)| \leq C$ . From this we get the inequality

$$|dh_k(p)| = \left| \frac{\partial}{\partial p} F_k(q(p), p, \xi(p)) \right| \leq C$$

and since  $h_k$  is continuous, it is  $C$ -Lipschitz.  $\square$

From Ascoli-Arzelà and the above lemma, we see that the sequence  $h_k$  is relatively compact.

We will then need to prove that

**Lemma 4.9.** *If a subsequence of  $\rho_k^{-1} \varphi^k \rho_k$  has a limit  $\bar{\varphi}$ , then the sequence itself converges to this limit.*

*Proof.* We first claim that

$$(1) \quad \gamma(\varphi^k \psi^k) \leq k \gamma(\varphi \psi)$$

$$(2) \quad \gamma(\rho_k^{-1} \varphi \rho_k) = \frac{1}{k} \gamma(\varphi) .$$

Indeed, we may write

$$\varphi^k \psi^k = \varphi \psi (\psi^{-1}(\varphi \psi) \psi) (\psi^{-2}(\varphi \psi) \psi^2) \dots \psi^{-(k-1)}(\varphi \psi) \psi^{k-1}$$

Since each factor is conjugate to  $\varphi \psi$ , and we have  $k$  factors, property (1) follows immediately. Property (2) follows from the scaling property of  $\gamma$  by conformal conjugation (see 1).

Now let  $k$  large enough, so that  $\gamma(\rho_k^{-1} \varphi^k \rho_k, \bar{\varphi})$  is less than  $\varepsilon$ . Let  $n = kq + r$ . Note first that if  $r = 0$ ,

$$\gamma(\rho_{kd}^{-1} \varphi^{kd} \rho_{kd}, \bar{\varphi}) = \gamma(\rho_d^{-1} (\rho_k^{-1} \varphi^k \rho_k)^d \rho_d (\rho_d^{-1} \bar{\varphi}^{-d} \rho_d))$$

since

$$\bar{\varphi} = \rho_d^{-1} \bar{\varphi}^d \rho_d$$

But using (2), we get

$$\begin{aligned} \gamma(\rho_d^{-1} (\rho_k^{-1} \varphi^k \rho_k)^d \rho_d (\rho_d^{-1} \bar{\varphi}^{-d} \rho_d)) &= \gamma(\rho_d^{-1} (\rho_k^{-1} \varphi^k \rho_k)^d \bar{\varphi}^{-d} \rho_d) \leq \frac{1}{d} \gamma((\rho_k^{-1} \varphi^k \rho_k)^d \bar{\varphi}^{-d}) \leq \\ &\frac{1}{d} d \gamma(\rho_k^{-1} \varphi^k \rho_k \bar{\varphi}^{-1}) \leq \varepsilon \end{aligned}$$

Then we claim that

$$\lim_{q \rightarrow \infty} \gamma(\rho_{kq+r}^{-1} \varphi^{kq+r} \rho_{kq+r}, \rho_{kq}^{-1} \varphi^{kq} \rho_{kq}) = 0$$

Indeed, working in  $T^*\mathbb{R}^n$ , we may write  $\rho_{kq+r} \rho_{kq}^{-1} = \rho_{1+(r/kq)}$  and  $\varphi^{kq+r} \varphi^{-kq} = \varphi^r$  thus

$$\begin{aligned} \gamma(\rho_{kq+r}^{-1} \varphi^{kq+r} \rho_{kq+r}, \rho_{kq}^{-1} \varphi^{kq} \rho_{kq}) &= \gamma(\rho_{1+(r/kq)}^{-1} (\rho_{kq}^{-1} \varphi^{kq} \rho_{kq}) (\rho_{kq}^{-1} \varphi^r \rho_{kq}) \rho_{1+(r/kq)}, \rho_{kq}^{-1} \varphi^{kq} \rho_{kq}) = \\ &= \gamma(\rho_{1+(r/kq)}^{-1} (\rho_{kq}^{-1} \varphi^{kq} \rho_{kq}) (\rho_{kq}^{-1} \varphi^r \rho_{kq}) \rho_{1+(r/kq)} \rho_{kq}^{-1} \varphi^{-kq} \rho_{kq}) \end{aligned}$$

Now we use the fact we proved earlier, that

$$\gamma(\rho_{kq}^{-1} \varphi^{kq} \rho_{kq}, \bar{\varphi}) \leq \varepsilon$$

and that

$$\gamma(\rho_{kq}^{-1} \varphi^r \rho_{kq}) \leq \frac{1}{kq} \gamma(\varphi^r) \leq \varepsilon$$

for  $q$  large enough.

We use the fact that  $\bar{\varphi}$  and  $\rho_s$  satisfy  $\rho_s^{-1} \bar{\varphi} = \bar{\varphi}^s \rho_s$  to infer the next estimate of the above quantity

$$\gamma(\rho_{1+(r/kq)}^{-1} \bar{\varphi} \psi_1 \rho_{1+(r/kq)} \psi_2 \bar{\varphi}^{-1}) = \gamma(\bar{\varphi}^{r/kq} \rho_{1+(r/kq)}^{-1} \psi_1 \rho_{1+(r/kq)} \psi_2)$$

where  $\gamma(\psi_j) \leq \varepsilon$ . The above is then small as soon as  $kq$  is large enough.

We thus proved that for  $q$  large enough,

$$\gamma(\rho_{kq+r}^{-1} \varphi^{kq+r} \rho_{kq+r}, \rho_{kq}^{-1} \varphi^{kq} \rho_{kq})$$

is close to zero. Since we proved earlier that  $\gamma(\rho_{kq}^{-1} \varphi^{kq} \rho_{kq}) \leq \varepsilon$  this proves that the limit of  $\rho_k^{-1} \varphi^k \rho_k$  is indeed  $\bar{\varphi}$ . □

## 4.2 Concluding the proof of the main theorem.

The last section gave us a function  $h_\infty(p)$  limit of some subsequence  $h_{k_\nu}(p)$ . Since  $h_\infty$  is continuous, according to Humilière (cf. prop.2.11) it has a flow in  $\widehat{\mathcal{H}}(T^*T^n)$ . We denote by  $\bar{\varphi}^t$  this flow.

**Proposition 4.10.** *The map  $\bar{\varphi}$  is the limit of  $\varphi_k = \rho_k^{-1} \varphi^k \rho_k$  : we have*

$$\lim_{k \rightarrow +\infty} \gamma(\varphi_k, \bar{\varphi}) = 0 .$$

This will be based on the following two propositions

**Proposition 4.11.** *For any  $\alpha$  in  $\widehat{\mathcal{H}}(T^*T^n)$ , there exists a sequence  $\ell_\nu$  such that*

$$\lim_{\nu \rightarrow \infty} c(\mu, \varphi_{\ell_\nu} \alpha) \leq c(\mu, \bar{\varphi} \alpha)$$

**Proposition 4.12.** *Consider a subsequence of  $(\varphi_{k_\nu})$  such that*

$$\lim_{\nu} c(\mu \otimes 1(p), \varphi_{k_\nu}) = \lim_{\nu} h_{k_\nu}(p) = h_\infty(y)$$

Then we have

$$\lim_{\nu} c(\mu \otimes 1(p), \varphi_{k_\nu}^{-1}) = -h_\infty(p)$$

*Remark 4.13.* Note that this means that if we define  $\overline{H}$  as in proposition 4.7, the operator  $\mathcal{A}$  satisfies  $\mathcal{A}(-H) = -\mathcal{A}(H)$ . This is typically a statement that does not hold in the case of viscosity solutions, since if  $u(t, x)$  is a viscosity solution associated to  $H$ ,  $u(-t, x)$  is not in general a viscosity solution associated to  $-H$ .

*Proof that Proposition 4.11 and 4.12 imply proposition 4.10.* Indeed take  $\alpha = \bar{\varphi}^{-1}$ , where  $\bar{\varphi}$  is the limit associated by the previous subsection to some subsequence  $(k_\nu)_{\nu \geq 1}$ . We get

$$\lim_{\nu} c(\mu, \varphi_{\ell_\nu} \bar{\varphi}^{-1}) \leq c(\mu, \text{Id}) = 0.$$

and since for any  $\psi$ ,  $c(\mu, \psi) \geq 0$  we get,

$$\lim_{\nu} c(\mu, \varphi_{\ell_\nu} \bar{\varphi}^{-1}) = 0$$

Now we must prove  $\lim_{\nu} c(1, \varphi_{\nu \cdot k_\nu} \bar{\varphi}^{-1}) = 0$ , and it is enough to show that

$$\lim_{\nu} c(1, \varphi_{\ell_\nu} \alpha) \geq c(1, \bar{\varphi} \alpha)$$

for any  $\alpha$  in  $\widehat{\mathcal{H}}(T^*T^n)$ .

But

$$c(1, \varphi_{\nu \cdot k_\nu} \alpha) = -c(\mu, \alpha^{-1} \varphi_{\ell_\nu}^{-1}) = -c(\mu, \varphi_{\ell_\nu}^{-1} \alpha^{-1}).$$

We may then apply proposition 4.11 to the sequence  $(\varphi_{k_\nu}^{-1})$  since according to proposition 4.12

$$\lim_k c(\mu \otimes 1(p), \varphi_{\ell_\nu}^{-1}) = -h_\infty(p)$$

and  $-h_\infty(p)$  has flow  $\bar{\varphi}^{-1}$  in the completion  $\widehat{\mathcal{H}}(T^*T^n)$ .

As a result

$$\begin{aligned} \lim_k c(1, \varphi_{\ell_\nu} \alpha) &= - \lim_k c(\mu, \varphi_{\ell_\nu}^{-1} \alpha^{-1}) \\ &\geq -c(\mu, \bar{\varphi}^{-1} \alpha^{-1}) = c(1, \bar{\varphi} \alpha) . \end{aligned}$$

We thus proved the following statement: if  $c(\mu_x \otimes 1(p), \varphi_{k_\nu})$  converges to  $h_\infty$ , then  $\varphi_{\ell_\nu}$  converges to  $\bar{\varphi}$ . Now assume there are two subsequences,  $\varphi_{k_\nu}, \varphi_{l_\nu}$  such that  $c(\mu_x \otimes 1(p), \varphi_{k_\nu})$  converges to  $h_\infty$ , while  $c(\mu_x \otimes 1(p), \varphi_{l_\nu})$  converges to  $k_\infty$ . Then we find subsequences of  $(\varphi_k)$  converging to  $\bar{\varphi}$  and  $\bar{\psi}$  (where  $\bar{\varphi}$  is the flow of  $h_\infty$  while  $\bar{\psi}$  is the flow of  $\psi$ ).

But according to lemma 4.9, two converging subsequences of  $(\varphi_k)$  must have the same limit, thus  $h_\infty = k_\infty$ .

This concludes our proof of 4.10, modulo the proof of 4.11 and 4.12.  $\square$

*Proof of proposition 4.11.* First of all, if  $S(x, y, \eta)$  is a G.F.Q.I. of  $\alpha$ ,  $\varphi_k \alpha$  has the G.F.Q.I.

$$\begin{aligned} \Phi_k(u, v; x, y, \eta, \xi) &= \\ S(x, v; \eta) + F_k(u, y; \xi) + \langle y - x, v - u \rangle \end{aligned}$$

Note that for each  $y$ , there is a cycle  $C(y)$  homologous to  $T^n \times E_k^-$  ( $x$  lives in  $T^n$ ,  $\xi$  in  $E_k, E_k^-$  is the negative eigenspace for  $F_k$ ) such that

$$F_k(y, C(y)) \leq h_k(y) + \varepsilon .$$

(we denote by  $(y, C(y))$  the set of  $(x, y, \xi)$  such that  $(x, \xi) \in C(y)$ ). Unfortunately we may not get such an estimate if we simultaneously require that  $C(y)$  is to depend continuously on  $y$ . However, we may assume the above estimate holds outside of  $U_{2\delta}$  where  $U_\delta$  is a  $\delta$ -neighbourhood of some grid in  $(\mathbb{R}^n)^*$  (see figure 1), while inside  $U_\delta$ ,  $F_k(y, C(y)) \leq a$  for some constant  $a$ .

Indeed we have

**Lemma 4.14.** *Let  $F(u, x)$  be a smooth function on  $V \times X$  such that there exists  $C(u) \in H_*(X)$  with  $F(u, C(u)) \leq f(u)$ . Then for any subset  $U$  in  $V$ , such that each connected component of  $V - U$  has diameter less than  $\varepsilon$ , there exists a continuous map  $u \rightarrow \tilde{C}(u)$  and a constant  $a$ , such that*

$$F(u, \tilde{C}(u)) \leq f(u) + a\chi_U(u)$$

Thus  $F_k(y, C(y)) \leq h_k(y) + a\chi^\delta(y) + \varepsilon$  where  $\chi^\delta$  is a smooth function equal to one on  $U_\delta$  and to zero outside  $U_{2\delta}$ ,  $a$  is some constant, and  $\varepsilon$  is arbitrarily small.

Assume first  $U_\delta$  is empty (i.e.  $\chi^\delta = 0$ ).

Set

$$\bar{\Phi}_k(u, v; x, y, \eta) = S(x, v; \eta) + h_k(y) + \langle y - v, u - x \rangle$$



defined on  $T_{(u,v)}^n \times \mathbb{R}_x^n \times (\mathbb{R}_y^n)^* \times N$  ( $N$  is the vector spaces where  $\eta$  lives).

Let  $\Gamma$  be a cycle in  $T_{(u,v)}^n \times (\mathbb{R}_x^n) \times (\mathbb{R}_y^n)^* \times N$  in the homology class of  $T_{(u,v)}^n \times \Delta_{x,y} \times N^-$  ( $\Delta_{x,y}$  is the diagonal in  $\mathbb{R}_x^n \times (\mathbb{R}_y^n)^*$  and the negative eigenspace of  $\langle y, -x \rangle$ ).

We choose  $\Gamma$  such that  $\bar{\Phi}_k(\Gamma) \leq c(\mu, \bar{\Phi}_k) + \varepsilon = c(\mu, \bar{\varphi}_k \alpha) + \varepsilon$ , which is possible by definition.

Let now  $\Gamma \times_Y C$  be the cycle

$$\Gamma \times_Y C = \{(u, v, x, y, \xi, \eta) \mid (u, v, x, y, \eta) \in \Gamma, (u, \xi) \in C(y)\} .$$

We claim that

$$\Phi_k(\Gamma \times_Y C) \leq \bar{\Phi}_k(\Gamma)$$

and  $\Gamma \times_Y C$  is a cycle in the homology class of

$$T_{(u,v)}^n \times \Delta_{x,y} \times E_k^- \times N_1^- \times N_2^-$$

so that

$$c(\mu, \varphi_k \alpha) = c(\mu, \Phi_k) \leq \bar{\Phi}_k(\Gamma) \leq c(\mu, \bar{\Phi}_k) + \varepsilon \leq c(\mu, \bar{\varphi}_k \alpha) + \varepsilon$$

Let us now try to establish the inequality in the general case,

Let  $k_\nu$  be a sequence such that  $c(\mu_x \otimes 1(y), \varphi_{k_\nu})$  converges to  $h_\infty(y)$ . We replace  $\Phi_k$  by  $\Phi_{\ell,k}$  where  $F_k$  is replaced by the explicit formula for  $F_{\ell k}$  given by lemma 4.4.

$$\Phi_{\ell,k}(u, v; \bar{x}, \bar{y}, \bar{\xi}, \eta) = S(x_1, v, \eta) + \frac{1}{\ell} \sum_{j=1}^{\ell} F_k(\ell x_j, y_j, \xi_j) + Q_\ell(\bar{x}, \bar{y}) + \langle y_\ell - v, u - x_1 \rangle$$

We may now “spread our error”  $a\chi^\delta(y)$  by translating it. More precisely, let us choose different  $\chi_j^\delta(y)$  so that

$$F_k(y, C_j(y)) \leq h_k(y) + a_k \chi_j^\delta(y) + \varepsilon$$

and such that the supports,  $U_j^\delta$ , of the  $\chi_j^\delta$  are so chosen that the intersection of  $(n+2)$  distinct  $U_j^\delta$  is empty.

Consider

$$\begin{aligned} \bar{\Phi}_{\ell,k}(u, v; \bar{x}, \bar{y}, \eta) = \\ S(x_1, v; \eta) + \frac{1}{\ell} \sum_{j=1}^{\ell} (h_k(y_j) + a_k \chi_j^\delta(y_j)) + Q_\ell(\bar{x}, \bar{y}) + \langle y_\ell - v, u - x_1 \rangle \end{aligned}$$

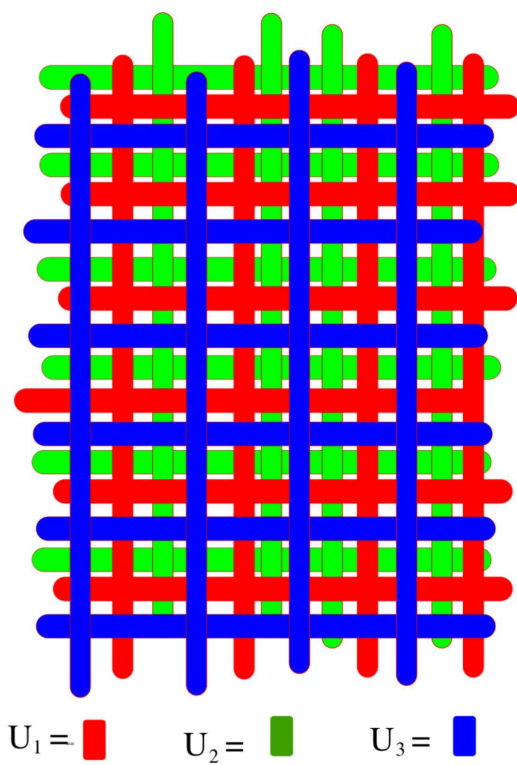


Figure 1: The sets  $U_\delta^j$

Then, let  $\Gamma$  be a cycle in the same homology class as above, such that

$$\bar{\Phi}_{\ell,k}(\Gamma) \leq c(\mu, \bar{\Phi}_{\ell,k}) + \varepsilon$$

and

$$\Gamma \times_Y C = \{(u, v; \bar{x}, \bar{y}, \bar{\xi}, \eta) \mid (u, v, \bar{x}, \bar{y}, \eta) \in \Gamma, (\ell x_j, \xi_j) \in C_j(y_j)\} .$$

Now  $\Gamma \times_Y C$  belongs to the suitable homology class, and we may thus infer that

$$c(\mu, \varphi_k \alpha) = c(\mu, \Phi_{\ell,k}) \leq \bar{\Phi}_{\ell,k}(\Gamma \times_Y C)$$

and  $\bar{\Phi}_{\ell,k}(\Gamma \times_Y C) \leq \bar{\Phi}_{\ell,k}(\Gamma)$ . We may conclude that

$$c(\mu, \Phi_{\ell,k}) \leq c(\mu, \bar{\Phi}_{\ell,k}) + 2\varepsilon$$

Finally, we must show

**Lemma 4.15.** *We have*

$$c(\mu, \bar{\Phi}_{\ell,k}) \leq c(\mu, \bar{\Phi}_k) + \frac{A_k}{\ell}$$

*Proof.* Indeed  $\bar{\Phi}_{\ell,k}$  is the generating function of  $\psi_{k,\delta,\ell} \alpha$  where

$$\psi_{k,\delta,\ell} = \rho_\ell^{-1} (\psi_{k,\delta}^1 \circ \cdots \circ \psi_{k,\delta}^\ell) \rho_\ell$$

where  $\psi_{k,\delta}^j$  is the time one flow of  $h_k(y) + a_k \chi_j^\delta(y)$ .

But because these flows commute, we have that  $\psi_{k,\delta,\ell}$  is the time one flow of

$$K_{k,\delta,\ell}(y) = \frac{1}{\ell} \left( \sum_{j=1}^{\ell} h_k(y) + a_k \chi_j^\delta(y) \right) .$$

Now since  $(n+2)$  sets  $U_\delta^j$  have empty intersection, we have that  $\left| \sum_{j=1}^{\ell} \chi_j^\delta(y) \right| \leq (n+2)$  hence

$$|K_{k,\delta,\ell}(y) - h_k(y)| \leq \frac{A_k}{\ell}$$

As a result

$$\gamma(\psi_{k,\delta,\ell}, \psi_k) \leq \frac{A_k}{\ell}$$

where  $\psi_k$  is the time one flow of  $h_k(y)$  hence

$$\gamma(\psi_{k,\delta,\ell} \alpha, \psi_k \alpha)$$

$$\leq \frac{A_k}{\ell}$$

and

$$\begin{aligned} c(\mu, \bar{\Phi}_{\ell,k}) &\leq c(\mu, \psi_{k,\delta,\ell}\alpha) \\ &\leq c(\mu, \psi_k\alpha) + \frac{a_k}{\ell} \\ &\leq c(\mu, \bar{\varphi}_k\alpha) + \frac{A_k}{\ell} \end{aligned}$$

□

Since by assumption, as  $\nu$  goes to infinity along a subsequence  $\psi_{k_\nu}$  goes to  $\bar{\varphi}$  we get

$$\begin{aligned} c(\mu, \varphi_{k_\nu}\alpha) &= c(\mu, \bar{\Phi}_{\ell,k}) \leq \\ c(\mu, \bar{\Phi}_{\ell,k}) + \varepsilon &\leq c(\mu, \bar{\Phi}_k) + \varepsilon + \frac{a_k}{\ell} \\ &\leq c(\mu, \bar{\varphi}\alpha) + \frac{A_k}{\ell} \end{aligned}$$

Taking  $\ell$  large enough, we see that

$$\begin{aligned} \lim_{\nu} c(\mu, \varphi_{\ell_\nu, k_\nu}\alpha) \\ \leq c(\mu, \bar{\varphi}\alpha) \end{aligned}$$

as announced. This concludes the proof of Proposition 4.11. □

*Remark 4.16.* It is important to notice that here

$$\bar{\Phi}_{\ell,k} = S(x_1, v; \eta) + \frac{1}{\ell} \sum_{j=1}^{\ell} (h_k(y_j) + a\chi_j^\delta(y_j)) + Q_\ell(\bar{x}, \bar{y}) + \langle y_\ell - v, u - x_1 \rangle$$

cannot be bounded from above by

$$S(x_1, v; \eta) + \frac{1}{\ell} \sum_{j=1}^{\ell} h_k(y_j) + Q_\ell(\bar{x}, \bar{y}) + \langle y_\ell - v, u - x_1 \rangle + \frac{a}{\ell}$$

as it is obvious by choosing  $(y_1, \dots, y_\ell)$  such that each  $y_j$  is in  $U_j^\delta$ . the above proof would not hold if we replace  $\chi_j^\delta(y)$  by an analogous function  $\chi_j^\delta(q, p)$ .

*Proof of proposition 5.3.* Let  $\Gamma_\varphi$  be the graph of  $\varphi$  in coordinates

$$\Gamma_\varphi = \{(X(x, y), x, y - X(x, y), X(x, y) - x) \mid \varphi(x, y) = (X(x, y), Y(x, y))\}$$

Then the reduction of  $\Gamma_\varphi$  at  $y = y_0$  is  $(\Gamma_\varphi)_{y_0} = \{(X(x, y), y_0 - Y(x, y_0))\}$  that is  $L_{y_0} - \varphi(L_{y_0})$  where

$$L_{y_0} = \{(x, y_0) \mid x \in T^n\}$$

Now  $c(\mu_x \otimes 1(y), \Gamma_\varphi) = c(\mu, (\Gamma_\varphi)_y) = c(\mu, L_y - \varphi(L_y))$ .

*Remark 4.17.* We refer to remark 4.6 and remind once again the reader of the extra care that has to be taken when using such expressions as  $c(\mu, \Lambda_y)$ . Here  $c(\mu, (\Gamma_\varphi)_y)$  is well defined if we choose a G.F.Q.I. for  $\Gamma_\varphi$ .

If  $\varphi = \varphi_k = \rho_k^{-1} \varphi^k \rho_k$ , then  $\varphi_k$  is  $1/k$ -periodic, hence so is  $L_y - \varphi_k(L_y)$ .

**Lemma 4.18.** *The quantity  $\gamma(L_y - \varphi_k(L_y))$  goes to 0 as  $k$  goes to infinity.*

*Proof.* Note that  $L_y$  is not exact, but if  $\tau_y$  is the translation by the vector  $y$  in the  $p$  direction

$$\begin{aligned} L_y &= \tau_y(0_{T^n}) \\ \varphi_k(L_y) &= \varphi_k \tau_k(0_{T^n}) \end{aligned}$$

and we identify

$$\gamma(L_y - \varphi_k(L_y))$$

to

$$\gamma(0_{T^n} - (\tau_y^{-1} \varphi_k \circ \tau_y)(0_{T^n})) = \gamma(\tau_y^{-1} \varphi_k \tau_y(0_{T^n}))$$

Moreover  $(\tau_y^{-1} \varphi_k \tau_y)(0_{T^n})$  is Hamiltonianly isotopic to the zero section and invariant by any translation in the  $x$  direction by a vector in  $\frac{1}{k}\mathbb{Z}^n$ . More precisely the Hamiltonian isotopy  $(\tau_y^{-1} \varphi_k \tau_y)(0_{T^n})$  is invariant by such translation.

Thus  $\gamma(\tau_y^{-1} \varphi_k \tau_y(0_{T^n}))$  is the  $\gamma$  invariant of some Lagrangian contained in the product  $T_1^* T_k^n$  (where  $T_k^n = (\mathbb{R}/\frac{1}{k}\mathbb{Z})^n$ ), Hamiltonianly isotopic (in  $T_1^* T_k^n$ ) to the zero section. Since

$$T_1^* T_k^n \simeq T_{1/k}^* T^n = \{(q, p) \mid x \in T^n, |p| \leq \frac{1}{k}\}$$

we get, according to the main result in [V5], that

$$\gamma(\tau_y^{-1} \varphi_k \tau_y(0_{T^n})) \leq 0(1/k)$$

□

As a result,  $c(\mu, L_y - \varphi_k(L_y)) = -c(1, \varphi_k(L_y) - L_y) = -c(1, L_y - \varphi_k^{-1}(L_y))$

This last quantity differs from

$$-c(\mu, L_y - \varphi_k^{-1}(L_y))$$

by

$$\gamma(L_y - \varphi_k^{-1}(L_y)) = \gamma(\varphi_k(L_y) - L_y)$$

which goes to zero as  $k$  goes to infinity.

Hence

$$c(\mu, L_y - \varphi_k^{-1}(L_y)) = c(\mu, \varphi_k(L_y) - L_y) = -c(1, L_y - \varphi_k(L_y)) \simeq -c(\mu, L_y - \varphi_k(L_y))$$

In other words, denoting  $\bar{h}_k$  the number  $c(\mu_x \otimes 1(y), \varphi_k^{-1})$ , we proved

$$\begin{aligned} \bar{h}_k(y) &= c(\mu_x \otimes 1(y), \varphi_k^{-1}) = c(\mu \otimes 1(y), L_y - \varphi_k^{-1}(L_y)) = \\ &= -c(\mu \otimes 1(y), L_y - \varphi_k^{-1}(L_y)) + 0\left(\frac{1}{k}\right) = -h_k(y) + 0\left(\frac{1}{k}\right) \end{aligned}$$

□

We thus showed that some subsequence of  $\varphi_k$  converges to  $\bar{\varphi}$ . On the other hand, we proved in lemma 4.9 that this implies convergence of the sequence itself to  $\bar{\varphi}$ . We may finally conclude that  $\varphi_k$  converges to  $\bar{\varphi}$ , and thus that the sequence  $H(kq, p)$  converges to  $\bar{H}$  for the  $\gamma$  metric, which proves the first statement of the main theorem.

*End of the proof of theorem 3.1.* Assertion (2) follows from the fact that  $\bar{\varphi}^1$  determines  $\bar{H}$ , and that

$$\bar{\varphi}^1 = \lim_{k \rightarrow \infty} \rho_k^{-1} \varphi^k \rho_k$$

which only depends on  $\varphi^1$ .

We finally prove assertion (3).

$$\begin{aligned} |h_{k,1}(y) - h_{k,2}(y)| &\leq |c(\mu_x \otimes 1(y), \rho_k^{-1} \varphi_1^k \rho_k) - c(\mu_x \otimes 1(y), \rho_k^{-1} \varphi_2^k \rho_k)| \leq \\ \gamma((\rho_k^{-1} \varphi_1^k \rho_k)^{-1} \circ \rho_k^{-1} \varphi_2^k \rho_k) &\leq \gamma(\rho_k^{-1} \varphi_1^{-k} \varphi_2^k \rho_k) \leq \frac{1}{k} \gamma(\varphi_1^{-k} \varphi_2^k) \leq \gamma(\varphi_1^{-1} \varphi_2) \end{aligned}$$

Therefore  $\mathcal{A}$  is Lipschitz, with Lipschitz constant one, for the norms  $\gamma$  and  $C^0$  and thus extends to a Lipschitz map from  $\widehat{\mathcal{H}}(T^*T^n)$  to  $C^0(\mathbb{R}^n, \mathbb{R})$ .

Since  $H$  only depends on  $p$ , then  $\bar{H} = H$ , we get that  $\mathcal{A}$  is a projector.

This concludes our proof of theorem 3.1. □

## 5 Proof of theorem 3.2.

In order to prove (1) of theorem 3.2, we need to prove that if  $H_1 \leq H_2$  then  $h_{\infty,1} \leq h_{\infty,2}$ . But this follows immediately from the fact that  $S_1(q,p) \leq S_2(q,p)$  hence,  $F_{k,1} \leq F_{k,2}$  and therefore

$$h_{k,1}(y) = c(\mu \otimes 1(y), F_{k,1}) \leq c(\mu \otimes 1(y), F_{k,2}) = h_{k,2}(y)$$

As a result,  $h_{\infty,1}(y) \leq h_{\infty,2}(y)$ .

If we wanted to assume only  $H_1 \preceq H_2$  so that  $\varphi_1^1 \preceq \varphi_2^1$  and

$$\rho_k^{-1} \varphi_1^1 \rho_k \preceq \rho_k^{-1} \varphi_2^1 \rho_k$$

and by going to the limit,  $\overline{\varphi}_1 \preceq \overline{\varphi}_2$ . Now  $\overline{\varphi}_1$  and  $\overline{\varphi}_2$  are the flows of  $\overline{H}_1$  and  $\overline{H}_2$  which depend only on  $p$ .

Therefore they commute, and our assertion follows from the

**Lemma 5.1.** *If  $\varphi^1$ , the time one flow of  $H(p)$ , satisfies  $\text{Id} \preceq \varphi^1$  then  $H$  is positive.*

*Proof.* In Appendix A we prove that  $c_-(\varphi^1) = \inf_{p \in \mathbb{R}^n} H(p)$ . Therefore if  $c_-(\varphi^1)$  is positive,  $H$  must be non-negative.  $\square$

To prove (2), we have to compare  $\mathcal{A}(H \circ \psi)$  to  $\mathcal{A}(H)$ . Note that the flow associated to  $H \circ \psi$  is  $\psi^{-1} \varphi^t \circ \psi$ . Thus  $\mathcal{A}(H \circ \psi)$  is associated to the  $\gamma$ -limit of

$$\rho_k^{-1} \psi^{-1} \varphi^k \psi \rho_k = (\rho_k^{-1} \psi^{-1} \rho_k) (\rho_k^{-1} \varphi^k \rho_k) (\rho_k^{-1} \psi \rho_k)$$

But  $\lim_{k \rightarrow \infty} \gamma(\rho_k^{-1} \psi^{-1} \rho_k) = 0$ , hence

$$\lim_{k \rightarrow \infty} \rho_k^{-1} \psi^{-1} \varphi^k \psi \rho_k = \lim_{k \rightarrow \infty} \rho_k^{-1} \varphi^k \rho_k^{-1}$$

Similarly for property (3), we have to compare  $\lim_{k \rightarrow \infty} \rho_k^{-1} \varphi^k \rho_k$  and  $\lim_{k \rightarrow \infty} \rho_k^{-1} \varphi^{-k} \rho_k$ . Clearly, if the limit exists, they must be inverse from each other, that is they are given by  $\overline{\varphi}$  and  $(\overline{\varphi})^{-1}$ . Now it follows from [Hu] that two continuous compact supported Hamiltonians  $H, K$  in  $\widehat{\mathcal{H}}$ , such that their flows satisfy  $\varphi \psi = \text{Id}$  in  $\widehat{H}(T^*T^n)$  must satisfy  $H + K = 0$ .

We now prove property (4). Now, we may consider a decreasing sequence of smooth functions  $H_\nu$  such that  $\lim_\nu H_\nu = \chi_U$ , the limit being here a pointwise limit. Then  $\overline{H}_\nu$  is also a decreasing sequence, and therefore has a limit  $\overline{H}_\infty$ , and we denote by  $\mathcal{A}(U)$  the support of  $\overline{H}_\infty$ . Since for any sequence  $K_\nu$  decreasing to  $\chi_U$ , there is for each  $\nu$  a  $\mu$  such that  $K_\mu \leq H_\nu$ , we have  $\overline{K}_\infty \leq \overline{H}_\infty$ . By symmetry, we get  $\overline{K}_\infty = \overline{H}_\infty$  hence the support of  $\overline{K}_\infty$  coincides with the support of  $\overline{H}_\infty$ . By symmetry, we see that the supports of

$\overline{K}_\infty$  and  $\overline{H}_\infty$  must coincide (in fact the functions will coincide). This support defines  $\mathcal{A}(U)$ .

Assume now  $L$  is a Lagrangian submanifold Hamiltonianly isotopic to  $L_{y_0}$ . Then, by the Hamiltonian invariance we just proved,  $\mathcal{A}(L) = \mathcal{A}(L_{y_0})$ . Now it is easy to show that

$$\mathcal{A}(L_{y_0}) = \{y_0\}$$

Since  $shape(U)$  contains  $p$  if and only if  $U$  contains a Lagrangian  $L$ , Hamiltonianly isotopic to  $L_p$ , we get that for  $p \in shape(U)$ , we must have  $p_0 \in shape(U)$ . This concludes the proof of (4).

As for property (5), it is an easy consequence of the above. Indeed, assume first  $H(L) \geq h$  where  $L$  is Hamiltonianly isotopic to  $L_{p_0}$ . Let  $\kappa_{p_0}$ , be a function on  $(\mathbb{R}^*)^n$  equal to 1 near  $p_0$ , very negative in a tube containing the  $p$ -projection of the support of  $H$ , and compact supported. Then if  $\psi(L_{p_0}) = L$ , we have

$$H \geq h \cdot \kappa_{p_0} \circ \psi$$

hence

$$\overline{H} \geq h \cdot \overline{\kappa_{p_0} \circ \psi} = h \cdot \overline{\kappa_{p_0}} = h \cdot \kappa_{p_0}$$

As a result,

$$\overline{H}(p_0) \geq h \kappa_{p_0}(p_0) = h$$

Changing  $H$  to  $-H$ , and using (3) we get the second statement.

Finally, to show that  $\zeta$  is a quasi-state, it is enough to deal with the case where  $\mu$  is a Dirac mass at  $p$ . We must then prove

- (1) (Monotonicity)  $H_1 \leq H_2$  implies  $\overline{H}_1 \leq \overline{H}_2$ . This follows from (1).
- (2) (Quasi-linearity) If  $H, K$  Poisson commute, then  $\overline{(H + K)}(p) = \overline{H}(p) + \overline{K}(p)$ . this follows from the fact that if  $H, K$  commute, with respective flows  $\varphi^t, \psi^t$ , then  $H + K$  has flow  $\psi^t \varphi^t$  and then  $\mathcal{A}(H + K)$  corresponds to

$$\lim_{k \rightarrow \infty} \rho_k^{-1} \varphi^{kt} \psi^{kt} \rho_k^{-1} = \left( \lim_{k \rightarrow \infty} \rho_k^{-1} \varphi^{kt} \rho_k \right) \lim_{k \rightarrow \infty} \rho_k^{-1} \psi^{kt} \rho_k$$

and this corresponds to  $\mathcal{A}(H)(p) + \mathcal{A}(K)(p)$ .

- (3) (Normalization)  $\zeta(1) = 1$ . We see that  $H_R(q, p) = \chi_{[-R, R]}(p)$  so

$$\mathcal{A}(1) = \mathcal{A}\left(\lim_{R \rightarrow \infty} H_R\right) = 1 = \lim_{R \rightarrow \infty} \mathcal{A}(H_R) = \lim_{R \rightarrow \infty} H_R = 1$$

Note that the inversion of limits will only be justified when we deal with non compact supported Hamiltonians in section 8.

This concludes the proof of theorem 3.2.



## 6 Proof of the partial homogenization case

We here consider the case of the sequence  $H(kx, y, q, p)$  and prove that it converges to  $\overline{H}(y, q, p)$  obtained by performing the above homogenization, on the variables  $(x, y)$  and freezing the  $(q, p)$  variable.

The flow of  $H(kx, y, q, p)$  is given by

$$\left\{ \begin{array}{l} \dot{x} = \frac{\partial}{\partial y} H(k \cdot x, y, q, p) \\ \dot{y} = -k \frac{\partial}{\partial x} H(k \cdot x, y, q, p) \\ \dot{q} = \frac{\partial}{\partial p} H(k \cdot x, y, q, p) \\ \dot{p} = -\frac{\partial}{\partial q} H(k \cdot x, y, q, p) \end{array} \right.$$

Set

$$x_k(t) = k \cdot x\left(\frac{t}{k}\right), y_k(t) = y\left(\frac{t}{k}\right), q_k(t) = q\left(\frac{t}{k}\right), p_k(t) = p_2\left(\frac{t}{k}\right)$$

We shall consider the flow  $\varphi_k^t$  associated to the Hamiltonian.

$$\left\{ \begin{array}{l} \dot{x}_k = \frac{\partial}{\partial y} H(x_k, y_k, q_k, p_k) \\ \dot{y}_k = -\frac{\partial}{\partial x} H(x_k, y_k, q_k, p_k) \\ \dot{q}_k = \frac{1}{k} \frac{\partial}{\partial p} H(x_k, y_k, q_k, p_k) \\ \dot{p}_k = -\frac{1}{k} \frac{\partial}{\partial q} H(x_k, y_k, q_k, p_k) \end{array} \right.$$

Then our map is generated by  $\rho_k^{-1} \varphi_k^{kt} \rho_k$  where

$$\rho_k(x_k, y_k, q_k, p_k) = (k \cdot x_k, y_k, q_k, p_k)$$

Let  $S_k(x_k, y_k, q_k, p_k, \xi)$  be a generating function for the flow above. The candidate for the homogenization is again given by  $\lim_{k \rightarrow \infty} \overline{H}_k$  where

$$\overline{H}_k(y, q, p) = c(\mu_x \otimes 1(y) \otimes 1(q, p), S_k)$$

is obtained by freezing the  $(q, p)$  variables and performing homogenization as in the previous section. The precompactness of the sequence is proved as in proposition 4.7.

Let us reformulate the problem by considering the symplectic form  $\sigma_k$  on  $T^*T^{m+n}$  given by  $dy \wedge dx + kdp \wedge dq$ . For a Hamiltonian  $H(x, y, q, p)$  its flow for  $\sigma_k$  is defined by the equations

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial y}(x, y, q, p), & \dot{y} = -\frac{\partial H}{\partial x}(x, y, q, p) \\ \dot{q} = \frac{1}{k} \frac{\partial H}{\partial P}(x, y, q, p), & \dot{p} = -\frac{1}{k} \frac{\partial H}{\partial q}(x, y, q, p) \end{cases}$$

Now to a function  $S(x, Y, q, P)$  on  $T^*(T^{n+m})$  we may associate the Hamiltonian map given by

$$\begin{cases} X - x = \frac{\partial S}{\partial Y}(x, Y, q, P), & y - Y = \frac{\partial S}{\partial x}(x, Y, q, P) \\ Q - q = \frac{1}{k} \frac{\partial S}{\partial P}(x, Y, q, P), & (p - P) = \frac{1}{k} \frac{\partial S}{\partial q}(x, Y, q, P) \end{cases}$$

Indeed this amounts to the identification of  $T^*(T^{m+n}) \times T^*(T^{m+n})$  endowed with  $\sigma_k \ominus \sigma_k$ , to  $T^*(T^{n+m} \times \mathbb{R}^{n+m})$  endowed with the standard form by

$$(x, y, q, p, X, Y, Q, P) \longrightarrow (x, Y, q, P, y - Y, X - x, k(p - P), k(Q - q))$$

Two such transformation are composed by the following formula: If  $S_1(x_1, Y_1, q_1, P_1), S_2(x_2, Y_2, q_2, P_2)$  are the generating functions for  $\varphi_1, \varphi_2$ , we will have

$$S(x, Y, q, P; x_2, Y_1, q_2, P_1) = S_1(x, Y_1, q, P_1) + S_2(x_2, Y, q_2, P) - \langle x - x_2, Y_1 - Y \rangle - k \langle P_1 - P, q - q_2 \rangle$$

Indeed, the constraining equations are

$$\begin{cases} \frac{\partial S}{\partial x_2} = 0 \iff \frac{\partial S_2}{\partial x}(x_2, Y, q_2, P) - Y + Y_1 = 0 \\ \frac{\partial S}{\partial Y_1} = 0 \iff \frac{\partial S_1}{\partial y}(x, Y_1, q, P_1) + -x + x_2 = 0 \\ \frac{\partial S}{\partial q_2} = 0 \iff \frac{\partial S_2}{\partial q}(x_2, Y, q_2, P) + k(P - P_1) = 0 \\ \frac{\partial S}{\partial P_1} = 0 \iff \frac{\partial S_1}{\partial p}(x, Y_1, q, P_1) + k(q - q_2) = 0 \end{cases}$$

and the map  $\varphi$  is given by

$$\begin{aligned} & \left( x, Y + \frac{\partial S}{\partial x}(x, Y, q, P; x_2, Y_1, q_2, P_1), q, P + \frac{1}{k} \frac{\partial S}{\partial q}(x, Y, q, P; x_2, Y_1, q_2, P_1) \right) \longrightarrow \\ & \left( x + \frac{\partial S}{\partial Y}(x, Y, q, P; x_2, Y_1, q_2, P_1), Y, q + \frac{1}{k} \frac{\partial S}{\partial P}(x, Y, q, P; x_2, Y_1, q_2, P_1), P \right) \end{aligned}$$

that is

$$\begin{aligned} & \left( x, Y + \frac{\partial S_1}{\partial x}(x, Y_1, q, P_1), q, P + \frac{1}{k} \frac{\partial S_1}{\partial q}(x, Y_1, q, P_1) \right) \longrightarrow \\ & \left( x + \frac{\partial S_2}{\partial Y}(x_2, Y, q_2, P), Y, q + \frac{1}{k} \frac{\partial S_2}{\partial P}(x_2, Y, q_2, P) \right) \end{aligned}$$

Now the map  $\varphi_1$  sends

$$\begin{aligned} & \left( x, Y_1 + \frac{\partial S_1}{\partial x}(x, Y_1, q, P_1), q, P_1 + \frac{1}{k} \frac{\partial S_1}{\partial q}(x, Y_1, q, P_1) \right) \longrightarrow \\ & \left( x + \frac{\partial S_1}{\partial Y_1}(x, Y_1, q, P_1), Y_1, q + \frac{1}{k} \frac{\partial S_1}{\partial P_1}(x, Y_1, q, P_1), P_1 \right) \end{aligned}$$

and the map  $\varphi_2$  sends

$$\begin{aligned} & \left( x_2, Y + \frac{\partial S_2}{\partial x_2}(x_2, Y, q_2, P), q, P + \frac{1}{k} \frac{\partial S_2}{\partial q_2}(x_2, Y, q_2, P) \right) \longrightarrow \\ & \left( x_2 + \frac{\partial S_2}{\partial Y}(x_2, Y, q_2, P), Y, q_2 + \frac{1}{k} \frac{\partial S_2}{\partial P}(x_2, Y, q_2, P), P \right) \end{aligned}$$

Since

$$\left\{ \begin{array}{l} Y = Y_1 + \frac{\partial S_2}{\partial x}(x_2, Y, q_2, P) \\ x = x_2 + \frac{\partial S_2}{\partial y}(x_2, Y, q_2, P) \\ P = P_1 + \frac{1}{k} \frac{\partial S_2}{\partial q}(x_2, Y, q_2, P) \\ q = q_2 + \frac{1}{k} \frac{\partial S_2}{\partial p}(x_2, Y, q_2, P) \end{array} \right.$$

we may infer

$$\varphi = \varphi_1 \circ \varphi_2$$

## 6.1 Resolution in the $(q_2, P_1)$ variables

For  $j = 1, 2$ , let the functions

$$\frac{\partial S_j}{\partial P_1}(x_2, Y, q_2, P), \quad \frac{\partial S_j}{\partial q_2}(x_2, Y, q_2, P)$$

be  $C^1$  bounded, and assume  $k$  is large. We may then solve

$$\begin{cases} \frac{\partial S_2}{\partial q}(x_2, Y, q_2, P) + k(P - P_1) = 0 \\ \frac{\partial S_1}{\partial p}(x, Y_1, q, P_1) + k(q - q_2) = 0 \end{cases}$$

in

$$(q_2, P_1) = (q_2(x, Y, q, P; x_2, Y_1), P_1(x, Y, q, P; x_2, Y_1))$$

This requires the following matrix to be invertible:

$$\begin{pmatrix} I - \frac{1}{k} \frac{\partial^2 S_2}{\partial q \partial p}(x_2, Y, q_2, P) & \frac{1}{k} \frac{\partial^2 S_2}{\partial q^2}(x_2, Y, q_2, P) \\ \frac{1}{k} \frac{\partial^2 S_1}{\partial p \partial q}(x_2, Y, q_2, P) & I - \frac{1}{k} \frac{\partial^2 S_1}{\partial p^2}(x, Y_1, q, P_1) \end{pmatrix}$$

We thus get a new generating function

$$\widehat{S}(x, Y, q, P; x_2, Y_1) = S(x, Y, q, P; x_2, Y_1, q_2(x, Y, q, P; x_2, Y_1), P_1(x, Y, q, P; x_2, Y_1))$$

Note that

$$\|q_2(x, Y, q, P; x_2, Y_1) - q\|_{C^1} = O(1/k), \quad \|P_1(x, Y, q, P; x_2, Y_1) - P\|_{C^1} = O(1/k)$$

hence

$$\|\widehat{S}(x, Y, q, P; x_2, Y_1) - S(x, Y, q, P; x_2, Y_1, q, P)\|_{C^1} = O(1/k)$$

where

$$S(x, Y, q, P; x_2, Y_1, q, P) = S_1(x, Y_1, q, P) + S_2(x_2, Y, q, P) - \langle x - x_2, Y_1 - Y \rangle$$

## 6.2 Generating functions for $H(kx, y, q, p)$

Suppose now that  $S_1$  is a function of  $(x_1, Y_1, q_1, P_1, \xi_1)$  and  $S_2$  of  $(x_2, Y_2, q_2, P_2, \xi_2)$

Then

$$S(x, Y, q, P; x_2, Y_1, q_2, P_1, \xi_1, \xi_2) = S_1(x, Y_1, q, P_1, \xi_1) + S_2(x_2, Y, q_2, P, \xi_2) - \langle x - x_2, Y_1 - Y \rangle - k \langle P_1 - P, q - q_2 \rangle$$

The conditions are then given by

$$\begin{cases} \frac{\partial S_2}{\partial q_2}(x_2, Y, q_2, P, \xi_2) + k(P - P_1) = 0 \\ \frac{\partial S_1}{\partial P_1}(x, Y_1, q, P_1, \xi_1) + k(q - q_2) = 0 \end{cases}$$

and for  $k$  large enough we may write, as in the previous section

$$(q_2, P_1) = (q_2(x, Y, q, P; x_2, Y_1, \xi_1, \xi_2), P_1(x, Y, q, P; x_2, Y_1, \xi_1, \xi_2))$$

hence we set

$$\widehat{S}(x, Y, q, P; x_2, Y_1, \xi_1, \xi_2) = S_1(x, Y_1, q, P_1(x, Y, q, P; x_2, Y_1, \xi_1, \xi_2), \xi_1) + S_2(x_2, Y, q_2(x, Y, q, P; x_2, Y_1, \xi_1, \xi_2), P, \xi_2) - \langle x - x_2, Y_1 - Y \rangle - k \langle P_1 - P, q - q_2 \rangle$$

Again, we have, as above

$$\|\widehat{S}(x, Y, q, P; x_2, Y_1, \xi_1, \xi_2) - S(x, Y, q, P; x_2, Y_1, q, P, \xi_1, \xi_2)\|_{C^1} = O(1/k)$$

## 6.3 The case of $\ell$ terms

Let  $\varphi_k^t$  be the flow associated to  $H(x, y, q, p)$  for the symplectic form  $\sigma_k$ , the flow  $\Psi_k^t$  associated to  $H(kx, y, q, p)$  for  $\sigma_1$  is given by

$$\Psi_k^t = \varphi_k^{kt}$$

Indeed  $\Psi_k^t$  is defined by the equations

$$\begin{cases} \dot{x}(t) = \frac{\partial H}{\partial y}(kx, y, q, p), & \dot{y}(t) = -k \frac{\partial H}{\partial x}(kx, y, q, p) \\ \dot{q}(t) = \frac{\partial H}{\partial p}(kx, y, q, p), & \dot{p}(t) = -\frac{\partial H}{\partial q}(kx, y, q, p) \end{cases}$$

and setting  $kx = u$ , we get

$$\begin{cases} \dot{u}(t) = k \frac{\partial H}{\partial y}(u, y, q, p), & \dot{y}(t) = -k \frac{\partial H}{\partial x}(u, y, q, p) \\ \dot{q}(t) = \frac{\partial H}{\partial p}(u, y, q, p), & \dot{p}(t) = -\frac{\partial H}{\partial q}(u, y, q, p) \end{cases}$$

Defining  $\rho_k(x, y, q, p) = (kx, y, q, p)$ , we get

$$\begin{aligned} \Psi_k^t(x(0), y(0), q(0), p(0)) &= (x(t), y(t), q(t), p(t)) = \left(\frac{1}{k}u(t), y(t), q(t), p(t)\right) = \\ &= \rho_k^{-1}(u(t), y(t), q(t), p(t)) = \rho_k^{-1} \varphi_k^{kt}(u(0), y(0), q(0), p(0)) \end{aligned}$$

hence

$$\Psi_k^t = \rho_k^{-1} \varphi_k^{kt} \rho_k$$

This is not surprising, since  $\Psi_k^t$  is Hamiltonian for  $\sigma_1$  (hence it is  $\sigma_1$ -symplectic) while  $\varphi_k^t$  is Hamiltonian for  $\sigma_k$  (hence is  $\sigma_k$ -symplectic) and  $\rho_k^* \sigma_k = k \sigma_1$ .

Let  $F_k(x, Y, q, P, \xi)$  be a generating function associated to the time one flow of  $H(x, y, q, p)$ , for  $\sigma_k$ , we shall have

$$\begin{aligned} F_\ell(x, Y, q, P; \bar{x}, \bar{Y}, \bar{q}, \bar{P}, \bar{\xi}) = \\ \sum_{j=1}^{\ell} F_k(x_j, Y_j, q_j, P_j, \xi_j) - \sum_{j=1}^{\ell} \langle x_j - x_{j+1}, Y_j - Y_{j+1} \rangle - k \langle q_j - q_{j+1}, P_j - P_{j+1} \rangle \end{aligned}$$

Here

$$\begin{aligned} x_1 &= x, q_1 = q, P_\ell = P, Y_\ell = Y, \\ \bar{x} &= (x_2, \dots, x_\ell), \bar{q} = (q_2, \dots, q_\ell), \bar{P} = (P_1, \dots, P_{\ell-1}), \bar{Y} = (Y_1, \dots, Y_{\ell-1}), \bar{\xi} = (\xi_1, \dots, \xi_\ell) \end{aligned}$$

The condition for solving the constrains in  $(\bar{q}, \bar{P})$  is the invertibility of the following matrix

$$\begin{pmatrix} I - \frac{1}{k} \frac{\partial^2 F_\ell}{\partial \bar{q} \partial \bar{P}}(\bar{x}, Y, \bar{q}, P) & \frac{1}{k} \frac{\partial^2 F_\ell}{\partial \bar{q}^2}(\bar{x}, Y, \bar{q}, P) \\ \frac{1}{k} \frac{\partial^2 F_\ell}{\partial \bar{P} \partial \bar{q}}(\bar{x}, Y, \bar{q}, P) & I - \frac{1}{k} \frac{\partial^2 F_\ell}{\partial \bar{P}^2}(x, \bar{Y}, q, \bar{P}) \end{pmatrix}$$

This amounts to the inequality

$$(\star) \quad \frac{1}{k} \left\| \begin{pmatrix} \frac{\partial^2 F}{\partial q_2 \partial P} (x, Y, q, P) & \frac{\partial^2 F}{\partial q^2} (x, Y, q, P) \\ \frac{\partial^2 F}{\partial P \partial q} (x, Y, q, P) & \frac{\partial^2 F}{\partial P^2} (x, Y, q, P) \end{pmatrix} \right\| \leq \varepsilon$$

since a matrix of the type .

$$\begin{pmatrix} I + A & C & 0 & \dots & \dots & 0 \\ B & I + A & C & 0 & \dots & 0 \\ 0 & 0 & I + A & C & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \\ 0 & \dots & 0 & B & I + A & C \\ 0 & \dots & 0 & 0 & B & I + A \end{pmatrix}$$

is invertible provided  $\|A\|, \|B\|, \|C\|$  are small enough (independently from the number of blocks: this follows from Gershgorin's theorem, stating that if  $R$  bounds the sum on any line of the off diagonal terms, the eigenvalues of the matrix are at distance less than  $R$  from the diagonal terms).

Under the above assumption  $(\star)$ , we have that

$$\|F_\ell(x, Y, q, P; \bar{x}, \bar{Y}, \bar{q}, \bar{P}, \bar{\xi}) - \widehat{F}_\ell(x, Y, q, P; \bar{x}, \bar{Y}, \bar{\xi})\|_{C^1} \leq C\ell\varepsilon$$

where

$$\widehat{F}_\ell(x, Y, q, P; \bar{x}, \bar{Y}, \bar{\xi}) = \sum_{j=1}^{\ell} F(x_j, Y_j, q, P, \xi_j) - \sum_{j=1}^{\ell} \langle x_j - x_{j+1}, Y_j - Y_{j+1} \rangle$$

Now let us for typographical convenience revert to  $(x, y, q, p)$  notation instead of  $(x, Y, q, P)$ . Let the generating function associated to  $\Psi_k^1$  be given by

$$F_k(x, y, q, p; \bar{\xi})$$

We thus have according to proposition 4.11 a function  $h_k(y, q, p)$  and a cycle  $\Gamma(y, q, p)$  with the proper homology class such that

$$F_k(y, q, p; \Gamma(y, q, p)) \leq h_k(q, y, p)$$

where  $\lim_{k \rightarrow \infty} h_k(y, q, p) = h_\infty(y, q, p)$ .

Moreover  $\Gamma(y, q, p)$  may be allowed to depend continuously on  $(y, q, p)$  provide we allow the weaker inequality

$$F_k(y, q, p; \Gamma_j(y, q, p)) \leq h_k(q, y, p) + a\chi_j(q, p)$$

where now  $\chi_j$  is supported on  $W_j^\delta$ , a  $\delta$  neighbourhood of a grid in the  $(q, p)$  variables. Note that we used proposition 4.11, in order to get rid of the  $y$  dependence of  $\chi$ .

Then

$$\widehat{F}_{k\ell}(x, y, q, p; \bar{x}, \bar{y}, \bar{\xi}) = \frac{1}{\ell} \sum_{j=1}^{\ell} F_k(x_j, y_j, q, p, \xi_j) - \langle y_j - y_{j+1}, x_j - x_{j+1} \rangle$$

will satisfy on

$$\widehat{\Gamma}_{k\ell} = \{(x_j, y_j, q, p, \xi_j) \mid (x_j, \xi_j) \in \Gamma_j(y_j, q, p)\}$$

the inequality

$$\begin{aligned} \widehat{F}_{k\ell}(x, y, q, p; \bar{x}, \bar{y}, \bar{\xi}) &\leq \\ \frac{1}{\ell} \sum_{j=1}^{\ell} h_k(y, q, p) + \frac{a}{\ell} \chi_j(q, p) - \langle x_j - x_{j+1}, y_j - y_{j+1} \rangle \end{aligned}$$

As before we choose the  $\chi_j$  so that the intersection of  $2m + 2$  supports  $\text{supp}(\chi_j)$  is empty.

Thus  $\widehat{F}_{k\ell}$  is bounded by the generating function of  $h_k(y, q, p)$  plus  $\frac{a}{\ell}$ . We therefore get for all  $\alpha$ , that

$$c(\mu, \Psi_k^1 \alpha) \leq c(\mu, \bar{\Psi}_k^1 \alpha) + \varepsilon$$

Finally, we may conclude the proof of theorem 3.4 as in the standard case.

## 7 Proof of proposition 3.6

We shall limit ourselves to the case where homogenization is done on all variables.

Now according to [Hu], if  $u_1, u_2$  are given by  $c(1(q), L_1)$  and  $c(1(q), L_2)$ , we have

$$|c(1(q), L_1) - c(1(q), L_2)| \leq \gamma(L_1, L_2)$$

Thus if  $L_1 = \varphi_1(\Lambda)$  and  $L_2 = \varphi_2(\Lambda)$ , we have

$$\gamma(L_1, L_2) \leq \gamma(\varphi_1 \varphi_2^{-1})$$

In our case,  $L = \varphi_k^t, \bar{L} = \bar{\varphi}^t$ , we get

$$|u_k(t, q) - \bar{u}(t, q)| \leq \gamma(\varphi_k^t \bar{\varphi}^{-t})$$

Now we have according to the computation of lemma 4.9, that

$$\gamma(\varphi_k^{kt} \bar{\varphi}^{-kt}) \leq k \gamma(\varphi_k^t \bar{\varphi}^{-t})$$



Taking the supremum over  $t$  in  $[0, 1]$ , we get

$$\sup_{t \in [0, k]} \gamma(\varphi_k^t \bar{\varphi}^{-t}) \leq k \sup_{t \in [0, 1]} \gamma(\varphi_k^t \bar{\varphi}^{-t})$$

thus implying our estimate.

## 8 Non compact-supported Hamiltonians and the time dependent case

### 8.1 The coercive case

Assume first that  $H(q, p)$  is not compact supported, and that  $H$  is coercive, that is

$$\lim_{|p| \rightarrow \infty} H(q, p) = +\infty$$

Then let  $\chi : \mathbb{R} \rightarrow \mathbb{R}$  be a truncation function, that is

- (1)  $\chi$  is supported in  $[-2A, 2A]$
- (2)  $\chi = 1$  on  $[-A, A]$

We then consider  $\chi(|p|)H(q, p) = K(q, p)$ , and denote by  $\varphi^t$  the flow of  $H$ ,  $\psi^t$  the flow of  $K$ . Since  $\varphi^t$  preserves  $H$ , we have that if  $a(c), b(c)$  are defined by

$$W^{a(\lambda)} = \{(q, p) \mid |p| \leq a(\lambda)\} \subset \{(q, p) \mid H(q, p) \leq \lambda\} \subset \{(q, p) \mid |p| \leq b(\lambda)\} = W^{b(\lambda)}$$

Then  $\varphi^t$  sends  $W^{a(\lambda)}$  into  $W^{b(\lambda)}$  thus, for  $A \geq b(\lambda)$ , we have  $\psi^t = \varphi^t$ . Since  $\rho_k$  preserves  $W^\lambda$ , we get that  $\varphi_k^t = \rho_k^{-1} \varphi^{kt} \rho_k$  sends also  $W^{a(\lambda)}$  into  $W^{b(\lambda)}$  and moreover coincides with  $\varphi_k^t$  on  $W^{a(\lambda)}$ .

The conclusion is given by the following result due to [Hu]

**Lemma 8.1.** *Let  $\varphi_k^t, \psi_k^t$  be two sequences of Hamiltonian flows Let  $U \subset V \subset Z$  such that for any  $t$ ,  $\varphi_k^t(U) \subset V$ ,  $\varphi_k^t(V) \subset Z$   $\psi_k^t(U) \subset V$  and  $\varphi_k^t = \psi_k^t$  on  $V$ . Then if  $\gamma - \lim_{k \rightarrow \infty} \varphi_k^t = \bar{\varphi}^t$  and  $\gamma - \lim_{k \rightarrow \infty} \psi_k^t = \bar{\psi}^t$  we have*

$$\bar{\psi}^t = \bar{\varphi}^t$$

on  $U$ .

*Proof.* Indeed  $\varphi_k^t \circ \psi_k^{-t}(U) \subset W$  hence  $(\varphi_k^t \circ \psi_k^{-t})|_U = \text{Id}_U$ . Now we claim that if a sequence  $\rho_k$  such that  $(\rho_k)|_U = \text{Id}_U$  having  $\gamma$ -limit  $\bar{\rho}$ , then  $\bar{\rho}|_U = \text{Id}_U$ . Indeed this follows from [Hu] section 4.2.  $\square$

As a result, taking  $c(\lambda) = a(b(\lambda))$ , we get that  $U = W^{a(\lambda)}$ ,  $V = W^{b(\lambda)}$ ,  $Z = W^{c(\lambda)}$ , the assumptions of the lemma are satisfied, hence the restriction of  $\bar{\psi}^t$  to  $U$  is independent from the choice of  $\chi$ .

Note that the condition that  $\bar{\psi}^t = \bar{\varphi}^t$  on  $U$  only implies that the generating Hamiltonians  $\bar{H}, \bar{K}$  differ by a constant.

Applying this to  $\chi_1(|p|)$  and  $\chi_2(|p|)$  we get that

**Definition 8.2.** Let  $H$  be a Hamiltonian such that

$$\lim_{|p| \rightarrow \infty} H(q, p) = +\infty$$

Then we denote by  $\bar{H}$  the Hamiltonian equal to the common value of the  $\bar{K}_\chi$  for  $A$  going to infinity.

Thus any autonomous proper Hamiltonian can be homogenized.

**Proposition 8.3.** *The map  $\mathcal{A}$  from  $\widehat{\text{Ham}}(T^*T^n)$  to  $C^0(\mathbb{R}^n)$  extends to a map defined on the set of coercive Hamiltonians, i.e. such that  $\lim_{|p| \rightarrow +\infty} H(q, p) = +\infty$*

Consider now a 1-periodic Hamiltonian  $H(t, q, p)$  on  $T^*T^n$  and consider the Hamiltonian  $K(t, \tau, q, p) = \tau + H(t, q, p)$ . This new Hamiltonian is not compact supported, but, considering the function  $\chi$  as defined above,  $K_\chi(t, \tau, q, p) = \chi(\tau)(\tau + H(t, q, p))$  is compact supported.

The same argument as above show that  $K_\chi$  can be homogenized, and the limit  $\bar{K}_\chi$  is of the form  $\tau + \bar{H}(p)$  on  $|\tau| \leq A$ .

*Remark 8.4.* Note that we may also use the distance  $\hat{\gamma}$  defined by

$$\hat{\gamma}(\varphi, id) = \sup\{\gamma(\varphi(L), L) \mid L \in \mathcal{L}\}$$

and we may also define the weak limit as  $\varphi = \lim_k \varphi_k$  if and only if for any  $L$  in  $\mathcal{L}$  we have

$$\lim_k \gamma(\varphi_k(L), \varphi(L)) = 0$$

We may now consider applications of the non compact situation to homogenization for Hamilton-Jacobi equations. Indeed, let us consider a Hamiltonian  $H(t, q, p)$  and a function  $f$  of class  $C^1$ , and  $\varphi^t$  its flow. Since the graph of  $df$  is bounded, we may truncate  $H$ , in such a way that  $\varphi^t(\Gamma_{df})$  is unchanged.

Thus  $H$  is now compact supported, and we get a function  $u_f(t, x)$ , and we get that the solution  $u_{f,k}(t, x)$  of

$$(HJ_k) \quad \begin{cases} \frac{\partial}{\partial t} u_k(t, q) + H(kt, kq, \frac{\partial}{\partial x} u_k(t, q)) = 0 \\ u_k(0, q) = f(q) \end{cases}$$

converges to the solution  $\bar{u}_f$  of

$$(\overline{HJ}) \quad \begin{cases} \frac{\partial}{\partial t} u(t, q) + \overline{H}(\frac{\partial}{\partial x} u(t, q)) = 0 \\ u(0, q) = f(q) \end{cases}$$

If  $f$  is only  $C^0$ , we need to consider a sequence  $f_\nu$  converging to  $f$ . Since

**Lemma 8.5.**

$$|\bar{u}_f - \bar{u}_g|_{C^0} \leq |f - g|_{C^0}$$

*Proof.* Indeed, let  $\Psi$  be a Hamiltonian diffeomorphism of  $T^*N$  such that  $\Psi(\Lambda_f) = \Lambda_g$ , where  $\Lambda_f = \{(x, df(x)) \mid x \in N\}$ , and such that  $\gamma(\Psi) \leq |f - g|_{C^0}$ .

Then the function  $u_f$  is obtained as  $c(1(x), \varphi^t(\Lambda_f))$ , and we have

$$\begin{aligned} |c(1(x), \varphi^t(\Lambda_f)) - c(1(x), \varphi^t(\Lambda_g))| &= |c(1(x), \varphi^t(\Lambda_f)) - c(1(x), \varphi^t \Psi(\Lambda_f))| \leq \\ &|c(1(x), \varphi^t(\Lambda_f) - \varphi^t \Psi(\Lambda_f))| \leq |c(1(x), \Lambda_f - \Psi(\Lambda_f))| \leq \gamma(\Lambda_f, \Psi(\Lambda_f)) \leq \\ &\gamma(\Psi) = |f - g|_{C^0} \end{aligned}$$

□

## 8.2 The non coercive case

Assume for example that

$$H(x_1, x_2, p_1, p_2) = h(p_1, p_2)$$

outside a compact set. Notice that the Poisson brackets,  $\{H, p_1\} = \{H, p_2\} = 0$  outside a compact set, therefore  $\{H, |p_1|^2 + |p_2|^2\} = 0$  outside a compact set. The flow  $\varphi^t$  of  $H$  will then remain inside a bounded domain  $W^\lambda$  for  $\lambda$  large enough. We may then use the same truncation method as above, and infer that we may homogenize  $H$ :

**Proposition 8.6.** *Let  $H(x_1, x_2, p_1, p_2) = h(p)$  outside a compact set. Then we have a homogenization operator  $\mathcal{A}$  with the same properties as in the compact supported case.*

**Corollary 8.7.** *Assume  $u_k$  is a variational solution of Hamilton-Jacobi equation  $(HJ_k)$ . Then the sequence  $u_k$  converges uniformly to  $\bar{u}$  solution of  $(\overline{HJ})$ .*

*Remark 8.8.* By an approximation method, this will work for any hamiltonian such that

$$\lim_{|p| \rightarrow \infty} |H(q, p) - h(p)| = 0$$

## 9 Homogenization in the $p$ variable

### 9.1 Partial Legendre transform

Let  $\Lambda$  be some Lagrangian submanifold Hamiltonianly isotopic to the diagonal in  $T^*(T^n) \times T^*T^n$ .

We may consider  $\Lambda$  as a graph over the diagonal: it has a G.F.Q.I. of the form  $S(q, P, \xi)$ , so that

$$\Lambda = \left\{ \left( q, P - \frac{\partial S}{\partial q}(q, P, \xi), q + \frac{\partial S}{\partial p}(q, P, \xi), P \right) \mid (q, P) \in T^*T^n \right\}$$

But we may consider  $\Lambda$  as a Lagrangian in  $T^*(T^n \times T^n)$  and as such, it may have a G.F.Q.I. in the  $(q, Q)$  variables, that is

$$L = \left\{ \left( q, -\frac{\partial F}{\partial q}(q, Q, \eta), Q, \frac{\partial F}{\partial Q}(q, Q, \eta) \right) \mid (q, Q) \in \widetilde{T^n \times T^n} \right\}$$

where  $\widetilde{T^n \times T^n}$  is the covering of  $T^n \times T^n$  such that  $q - Q$  is defined in  $\mathbb{R}^n$ .

We may say that  $S$  and  $F$  are Legendre dual to each other. Note that while in the case without fibre variables, Legendre duality is a map, here it is only a correspondence, since the G.F.Q.I. are not unique. If  $F$  is a convex function in  $Q$ , we have

$$S(q, P) = \sup \{ \langle p, Q \rangle - F(q, Q) \mid Q - q \in \mathbb{R}^n \}$$

Note that the use of  $F$  is more convenient when we must find a generating function of  $\varphi(L)$ . Indeed, if  $S_L(q, \xi)$  is a G.F.Q.I. for  $L$ , then

$$S_{\varphi(L)}(Q, q, \xi, \eta) = S_L(q, \xi) + F(q, Q, \eta)$$

is a G.F.Q.I. for  $\varphi(L)$ .

Let us consider the sequence  $H(x, k \cdot p)$ . Its flow is given by

$$\psi_k^t = \sigma_k^{-1} \varphi^{kt} \sigma_k$$

where  $\sigma_k(q, p) = (q, k \cdot p)$ .

Note that here  $\sigma_k$  is a bona fide map on  $T^*T^n$ , so that we do not have to invoke covering arguments. Since  $\sigma_k$  satisfies  $\sigma_k^*\omega = \frac{1}{k}\omega$ , we get, that

$$\gamma(\sigma_k^{-1}\varphi^{kt}\sigma_k) = \frac{1}{k}\gamma(\varphi^{kt})$$

There is *a priori* no limit for the sequence  $\sigma_k^{-1}\varphi^{kt}\sigma_k$ : indeed if  $\varphi^t$  is the flow of  $H(p)$ ,  $\sigma_k^{-1}\varphi^{kt}\sigma_k$  will be the flow of  $H(kp)$ . However let us write  $\tau_k(q, p) = (k \cdot x, \frac{p}{k})$ , then

$$\psi_k^t = \sigma_k^{-1}\varphi^{kt}\sigma_k = \tau_k^{-1}\rho_k^{-1}\varphi^{kt}\rho_k\tau_k = \tau_k^{-1}\varphi_k^t\tau_k$$

Now

$$\gamma(\varphi_k^t\overline{\varphi}^{-t}) \leq \varepsilon_k t$$

thus

$$\gamma((\tau_k^{-1}\varphi_k^t\tau_k)(\tau_k^{-1}\overline{\varphi}^{-t}\tau_k)) = \gamma(\tau_k^{-1}(\varphi_k^t\overline{\varphi}^{-t})\tau_k) = \gamma(\varphi_k^t\overline{\varphi}^{-t}) \leq \varepsilon_k t$$

Now since  $\tau_k^{-1}\varphi_k^t\tau_k$  is generated by  $\overline{H}(k \cdot p)$ , we do not get a limit for  $H(x, k \cdot p)$  but we get:

**Proposition 9.1.**

$$\lim_{k \rightarrow \infty} \gamma(H(x, k \cdot p), \overline{H}(k \cdot p)) = \lim_{k \rightarrow \infty} \gamma(\psi_k^t\overline{\varphi}^{-t}) = 0$$

In spite of the fact that  $\overline{H}(k \cdot p)$  has no limit as  $k$  goes to infinity, this has a number of applications.

First, let us consider the standard parabolic Hamilton-Jacobi equations

$$\begin{cases} \frac{\partial}{\partial t}u(t, q) + H(q, \frac{\partial}{\partial x}u(t, q)) = 0 \\ u(0, q) = f(q) \end{cases}$$

Set  $u_k(t, q) = \frac{1}{k}u(k \cdot t, k \cdot x)$ , then  $u_k$  satisfies the equation

$$\begin{cases} \frac{\partial}{\partial t}u_k(t, q) + H(kq, \frac{\partial}{\partial x}u_k(t, q)) = 0 \\ u_k(0, q) = \frac{1}{k}f(kq) \end{cases}$$

and since  $H(kq, p)$  converges to  $\overline{H}$ , we get that

$$\lim_{k \rightarrow \infty} u_k(t, q) = \overline{u}(t, q)$$

that is

$$u(t, q) \simeq k\bar{u}\left(\frac{t}{k}, \frac{q}{k}\right)$$

where  $\bar{u}(t, q)$  is solution of

$$\begin{cases} \frac{\partial}{\partial t}\bar{u} + \bar{H}\left(\frac{\partial}{\partial q}\bar{u}(t, q)\right) \\ \bar{u}(0, q) = 0 \end{cases}$$

Since the solution is given by

$$\bar{u}(t, q) = -t\bar{H}(0)$$

we get

$$u(kt, kq) = -kt\bar{H}(0) + kv_k(t, q)$$

where  $v_k$  goes to zero with  $k$ , since so does  $\gamma(H(kq, p), \bar{H})$  and the initial condition is of the order  $\frac{1}{k}$ .

In other words,

$$u(t, q) = -t\bar{H}(0) + w(t, q)$$

where  $w$  is bounded.

The sequences of Hamilton-Jacobi equations

$$(HJ_k) \quad \frac{\partial}{\partial t}u(t, q) + H(kq, \frac{\partial}{\partial x}u(t, q)) = 0$$

$$(HJ'_k) \quad \frac{\partial}{\partial t}u(t, q) + H(q, k\frac{\partial}{\partial x}u(t, q)) = 0$$

We shall assume  $H$  is smooth for simplicity. Note that if we set  $u_k(t, x) = v_k(t, kx)$ , then if  $v_k$  is a solution of  $(HJ'_k)$ , we have that  $u_k$  is a solution of  $(HJ_k)$ . Indeed, let  $L_k$  be the geometric solution for  $(HJ_k)$ , then the image of  $L_k$  by the map

$$(t, \tau, q, p) \longrightarrow (t, \tau, \frac{x}{k}, kp)$$

is a geometric solution for  $(HJ'_k)$ . Note that if the initial condition for  $v_k$  is given by  $f(x)$ , then the initial condition for  $u_k$  is given by  $f(kx)$ . However if  $w_k$  is a variational solution for  $(HJ'_k)$  with initial condition  $g(x)$ , then, according to lemma 8.5,

$$|w_k(t, x) - u_k(t, x)| \leq \sup_{x \in T^n} |g(x) - f(kx)| \leq \|g - f_k\| \leq \|g\| + \|f\|$$

Now

$$|u_k(t, x) - \bar{u}(t, x)| \leq \varepsilon_k t$$

so that  $v_k(t, kx)$  is approximated by  $\bar{u}(t, x)$ . In other words,

$$v_k(t, x) = \bar{u}(t, \frac{x}{k}) + o(1)$$

Thus

$$|v_k(t, x) + t\bar{H}(0)| \leq |\bar{u}(t, \frac{x}{k}) + t\bar{H}(0)| + |u_k(t, \frac{x}{k}) - \bar{u}(t, \frac{x}{k})| \leq o(t) + \varepsilon_k t$$

$$\lim_{t \rightarrow \infty} \frac{1}{t} v_1(t, x) = -\bar{H}(0)$$

**Proposition 9.2.** *Let  $u$  be a variational solution of*

$$(HJ) \quad \begin{cases} \frac{\partial}{\partial t} u(t, q) + H(q, \frac{\partial}{\partial x} u(t, q)) = 0 \\ u(0, q) = f(q) \end{cases}$$

then

$$\lim_{t \rightarrow \infty} \frac{1}{t} u(t, q) = -\bar{H}(0)$$

## 9.2 Connection with Mather $\alpha$ function

The  $\alpha$  function has been defined by Mather for a Lagrangian  $L(x, \xi)$  as

$$\alpha(p) = \lim_{T \rightarrow \infty} \frac{1}{T} \inf \left\{ \int_0^T L(t, q(t), \dot{q}(t)) dt - \langle p, x_1 - x_0 \rangle \mid q(0) = x_0, q(T) = x_1 \right\}$$

As a special case, we may show

**Proposition 9.3.** *Let  $H$  be the Legendre dual of the Lagrangian  $L$ , i.e.  $H$  is strictly convex in  $p$  and*

$$L(t, x, \xi) = \sup \{ \langle p, \xi \rangle - H(t, q, p) \}.$$

Then

$$\bar{H}(p) = \lim_{T \rightarrow \infty} \inf \left\{ \frac{1}{T} \int_0^T L(t, q(t), \dot{q}(t)) dt - \langle p, x_1 - x_0 \rangle \mid q(0) = x_0, q(T) = x_1 \right\}$$

*Proof.* It is enough to consider the case  $p = 0$ . Then let

$$\mathcal{P}_t = \{q : [0, 1] \longrightarrow M \mid q(0) = x_0\}$$

and  $\pi : \mathcal{P}_t \longrightarrow M$  the map  $q \mapsto q(1)$ . Let

$$E_t(q) = \int_0^t L(q(ts), t\dot{q}(ts)) ds$$

defined on  $\mathcal{P}$ , and consider  $E_t$  as a G.F.Q.I. . We shall write  $(x_1, q)$  to remind the reader that  $\pi(q) = q(1) = x_1$ .

Now

$$DE_t(x_1, q) = \int_0^t \left[ \frac{\partial L}{\partial x}(q(ts), t\dot{q}(ts)) - \frac{d}{dt} \frac{\partial L}{\partial \xi}(q(ts), t\dot{q}(ts)) \right] \delta q(ts) ds + \frac{\partial L}{\partial \xi}(q(t), t\dot{q}(t)) \delta q(t) - \frac{\partial L}{\partial \xi}(q(0), t\dot{q}(0)) \delta q(0)$$

Setting

$$p(t) = \frac{\partial L}{\partial \xi}(q(t), t\dot{q}(t))$$

we get  $(x_1, \frac{\partial E_t}{\partial x_1}) = (x_1, p_1) = \varphi^t(x_0, 0)$ . Therefore  $E_t$  is a G.F.Q.I. of  $\varphi^t(0_N)$ . and since

$$\inf\{E_t(x, q) \mid q \in \mathcal{P}, q(1) = x\} = c(1(x), E_t) = u_{L,t}(x)$$

is a geometric solution of  $HJ$ , we have, as proved in

$$\lim_{t \rightarrow \infty} \frac{1}{t} u_{L,t}(x) = -\overline{H}(0)$$

□

## 10 Some examples and applications

### 10.1 Homogenization of $H(t, q, p)$ in the variable $t$

Applying the above in the case of partial homogenization, we see that we may associate to a time-periodic Hamiltonian  $H(t, q, p)$  an autonomous Hamiltonian  $\overline{H}(q, p)$ . However, this is nothing else than

$$\overline{H}(q, p) = \frac{1}{T} \int_0^T H(t, q, p) dt$$



Indeed, if  $H(kt, q, p)$  has flow  $\varphi_k^t$ , we have

$$\lim_{k \rightarrow \infty} \varphi_k^t = \overline{\varphi}^t$$

in the  $C^0$  topology, where  $\overline{\varphi}^t$  is the flow of

$$\frac{1}{T} \int_0^T X_H(t, q, p) dt = X_{\overline{H}}(q, p)$$

by the fundamental theorem of classical averaging. Since  $C^0$  limit implies  $\gamma$  convergence, we get our claim.

## 10.2 The one dimensional case: $\frac{1}{2}|p|^2 - V(x)$

In [L-P-V], the case  $H(q, p) = \frac{1}{2}|p|^2 - V(x)$  is explicitly dealt with, in the case  $V$  is bounded from below. We shall see here that this follows immediately from property (2) of theorem 3.2. Indeed, we have, assuming  $V \geq 0$  is one-periodic

$$\begin{cases} \overline{H}(p) = 0 & \text{if } |p| \leq \int_0^1 (V(x))^{1/2} dx \\ \overline{H}(p) = \lambda & \text{where } \lambda \text{ solves } |p| = \int_0^1 (V(x) + \lambda)^{1/2} dx \text{ if } |p| \geq \int_0^1 \sqrt{V(x)} dx \end{cases}$$

Indeed, this follows immediately from theorem 3.2 property (3).

### 10.3 Some computations

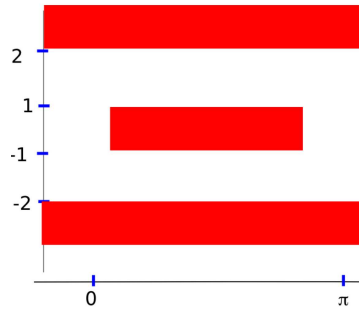


Figure 2: The red set is the complement of the support of the Hamiltonian  $H(q, p)$ .

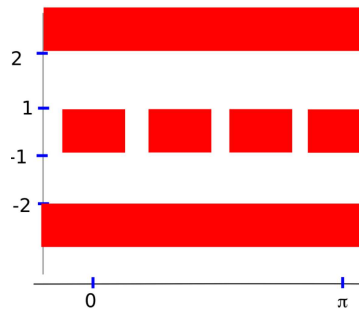


Figure 3: The red set is the complement of the support of the Hamiltonian  $H(kq, p)$ .

We let  $H$  be a Hamiltonian vanishing on the complement of the red set in figure 2, and equal to one in the complement. We claim that  $\overline{H}(p)$  for  $p$  in

$[-2, 2]$ .

Indeed, we see that a clever path in the labyrinth of figure 3 will have any possible Liouville form. Thus, we have  $L \in \mathcal{L}_p$  contained in  $H_k = 1$  for any  $p$  in  $[-1, 1]$ . As a result,  $\overline{H}(p) \geq 1$  for any  $p$  in  $[-1, 1]$ . Since obviously,  $\overline{H}(p) = 1$  for  $|p| \in [1, 2]$ , we get  $\overline{H}(p) = 1$  on  $[-2, 2]$  while  $\overline{H}(p) = 0$  for  $|p| \geq 2$ .

*Remark 10.1.* This example can be easily adapted to get Homogenized Hamiltonians taking more than two values.

## 10.4 Homogenized metric and the Thurston-Gromov norm

First consider the case where  $H$  generates the geodesic flow of  $g$ , even though, since  $H(q, p) = |p|_g^2$ ,  $H$  is not compact supported (but we shall prove that it may be extended to this setting). Then  $H_k$  generates the geodesic flow of the rescaled metric by the covering map

$$\begin{aligned} T^n &\longrightarrow T^n \\ q &\longrightarrow kq \end{aligned}$$

of degree  $k^n$ .

It is well known that if  $d$  is the distance defined by  $g$  (i.e.  $d(x, y)$  is the length of the shortest geodesic for  $g$  connecting  $x$  to  $y$ ) and  $d_k$  the one defined by  $g_k$  (corresponding to  $H_k$ ), we have

$$d_k(x, y) = \frac{1}{k} d(kx, ky)$$

and

$$\lim_{k \rightarrow +\infty} d_k(x, y) = \overline{d}(x, y)$$

where  $\overline{d}$  is the metric associated to some flat Finsler metric  $\overline{g}$ . It is well known that  $g_k$  does not converge to  $\overline{g}$  in any reasonable sense, except for the convergence of minimizers of the associated energy functional

$$E(\gamma) = \int_0^1 |\dot{\gamma}(t)|_g^2 dt$$

This is connected to the notion of  $\Gamma$ -convergence (cf. [Dal M], [Br]). In particular we easily see that the length for  $d_k$  of the shortest closed geodesic in the homotopy class  $\alpha$  (in  $\mathbb{Z}^n$ ),  $\ell_k(\alpha)$ , converges to the length for  $\overline{d}$  of the shortest closed geodesic in the homotopy class  $\alpha$ ,  $\overline{\ell}(\alpha)$ .

Indeed,

$$\ell_k(\alpha) = \inf\{d_k(x, x + \alpha) \mid x \in \mathbb{R}^n\}$$

and since  $d_k(x, x + \alpha)$  converges uniformly to  $d(x, x + \alpha)$ , and  $x$  needs only to vary in a fundamental domain of the covering of the torus by  $\mathbb{R}^n$ , we get that

$$\lim_{j \rightarrow +\infty} \ell_k(\alpha) = \bar{\ell}(\alpha) .$$

Now it is well known that in the class  $\alpha$  there is a second closed geodesic, obtained by a min max procedure. If we denote by

$$\ell_k(\alpha, \beta) = \inf \{ \sup \{ d_k(x(t), x(t) + \alpha) \mid t \in [0, 1] \} \mid [x(t)] = \beta \text{ in } \pi_1(T^n) \} .$$

and similarly for  $\bar{\ell}(\alpha, \beta)$ , do we have

$$\lim_{k \rightarrow +\infty} \ell_k(\alpha, \beta) = \bar{\ell}(\alpha, \beta) ?$$

The methods of our theorem imply a positive answer, since

$$\ell_k(\alpha, \beta) = c(\alpha \wedge \beta, E)$$

Note that the analogous statement cannot hold for the whole length spectrum of  $g_k$  (i.e. the set of lengths of closed geodesics), as it is easy to construct examples for which the length spectrum of  $g_k$  becomes dense as  $k$  goes to infinity i.e. for any  $\lambda \in \mathbb{R}_+$  and  $\delta > 0$  there is  $k_0$  in  $\mathbb{N}$  such that for all  $k \geq k_0$ ,  $\text{Spec}(g_k) \cap [\lambda - \delta, \lambda + \delta] \neq \emptyset$ .

## 11 Further questions

We could try to compare the homogenization point of view with the KAM point of view: consider the Hamiltonian  $H_0(p) + \varepsilon H_1(q, p) = H_\varepsilon(q, p)$ . The question is to compute the symplectic homogenization of  $H_\varepsilon$ .

As was pointed out by Sergei Kuksin, the type of homogenization or averaging described here is a kind of “dequantized averaging”, in the sense that usual homogenization is concerned with the limit of the “quantized Hamiltonian”,  $H(\frac{x}{\varepsilon}, D_x)$  as  $\varepsilon$  goes to zero. Here we deal directly with  $H(\frac{x}{\varepsilon}, p)$  the “classical Hamiltonian”. It is natural to ask whether in the framework of the above section, the laplacian associated to the homogenized metrics converges to some operator. This is true according to  $\Gamma$ -convergence classical results ([Br]), the limiting operator is  $\Delta_\infty$ . But this is not the Laplacian of the metric  $g_\infty$ . First of all  $g_\infty$  is not riemannian, but only Finslerian. Moreover, it seems that  $g_\infty$  detects changes in the metric on small sets: typically a

three torus with a metric made small along three lines in three orthogonal directions will have a much smaller  $g_\infty$  than one without such “short directions“. But the Laplacian does not detect this, since the Brownian motion will not see such lines. So the only reasonable question is whether the metric  $g_\infty$  determines the Laplacian  $\Delta_\infty$ .

One may ask a more general question, that is

*Question 11.1.* Assume  $H_\nu$  converges to  $H$  for the  $c$ -topology. Does the spectrum of the operators  $H_\nu(x, D_x)$  converge to the spectrum of  $H(x, D_x)$ ?

Finally in a publication in preparation, we shall explain the connection between the above homogenization and the self-tuning of oscillators.

## Appendix A: Capacity of completely integrable systems

Let  $\varphi^1$  be the time on flow associated to the Hamiltonian  $h(p)$  defined on  $T^*\mathbb{T}^n$ . Our goal is to prove the following

**Proposition A.1.**

$$c_+(\varphi_1) = \sup_p h \quad , \quad c_-(\varphi_1) = \inf_p h$$

$$\text{osc}_p h = \gamma(\varphi_1)$$

*Proof.* Set  $\varphi^t(q, p) = (Q_t(q, p), P_t(q, p))$ , then the graph of  $\varphi_t$  defines a Lagrangian submanifold  $\Gamma_t$  in  $T^*(T^n \times \mathbb{R}^n)$  as the image of  $(\theta, r) \rightarrow (\frac{q+Q_t}{2}, \frac{r+R_t}{2}, r - R_t, Q_t - q)$ . Note that even though  $Q_t$  is in  $T^n$ ,  $Q_t - q$  has a unique determination in  $\mathbb{R}^n$  which is continuous in  $t$  and equals 0 for  $t = 0$ . The same argument allows us to define  $\frac{q+Q_t}{2} = q + \frac{Q_t - q}{2}$ .

Moreover, if we set  $x = \frac{q+Q_t}{2}$ ,  $y = \frac{r+R_t}{2}$ ,  $\xi = p - P_t$ ,  $\eta = Q_t - q$ , the symplectic form is given by  $d\xi \wedge dx + d\eta \wedge dy$ . In our case, where  $\varphi_t$  is the flow of a “standard” integrable system, we have

$$x_t = q + \frac{t}{2}h'(p) \quad y_t = p \quad \xi_t = 0 \quad \eta_t = h'(p)$$

Thus if we set  $f_t(x, y) = th(y)$ , we have

$$\xi_t = \frac{\partial}{\partial x} f_t(x, y) \quad , \quad \eta_t = \frac{\partial}{\partial y} f_t(x, y)$$

that is  $f_t$  is a generating function of  $\Gamma_t$  with no “fibre variables”. It is then easy to see that

$$c_+(\varphi_t) = \sup f_t \quad , \quad c_-(\varphi_t) = \inf f_t$$

$$\gamma(\varphi_t) = \sup f_t - \inf f_t$$

Since  $f_1(x, y) = h(y)$  this proves our proposition. □

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