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Sensitivity analysis and density estimation for finite-time ruin probabilities

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Abstract

The goal of this paper is to obtain probabilistic representation formulas that are suitable for the numerical computation of the (possibly non-continuous) density functions of infima of reserve processes commonly used in insurance. In particular we show, using Monte Carlo simulations, that these representation formulas perform better than standard finite difference methods. Our approach differs from standard Malliavin probabilistic representation techniques which generally require more smoothness on random variables, entailing the continuity of their functions.

Key words: Ruin probability, Malliavin calculus, insurance, integration by parts.
MSC Classification codes: 60J75, 60H07, 91B30.

1 Introduction

In ruin theory, computational methods for finite-time ruin probabilities have received considerable attention in the last decade. The reader is referred to the books by Gerber [8], Grandell [10], Panjer and Willmot [17], Asmussen [2], and Kaas et al. [12] for general results on ruin-related issues; see also, e.g., Gerber and Shiu [9], Albrecher et al. [1] for more recent results.

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Consider the classical compound Poisson risk model, in which the surplus process $(R_x(t))_{t \geq 0}$ is defined as

$$R_x(t) = x + f(t) - S(t), \quad t \geq 0, \quad (1.1)$$

where $x \geq 0$ is the amount of initial reserves and $f(t)$ is the premium income received between time 0 and time $t > 0$. Here, the aggregate claim amount up to time t is described by the compound Poisson process

$$S(t) = \sum_{k=1}^{N_t} W_k,$$

where the claim amounts W_k , $k \geq 1$, are non-negative independent, identically-distributed random variables, with the convention $S(t) = 0$ if $N_t = 0$. The number of claims N_t until $t \geq 0$ is modeled by an homogeneous Poisson process $(N_t)_{t \geq 0}$ with intensity $\lambda > 0$, and claim amounts are assumed to be independent of arrival times. However we do not make any assumption on the claim amount distribution. Our results are general and operational for light or heavy-tailed, discrete or continuous claim amount distributions, as illustrated in Section 5.

Given $T > 0$ a finite time horizon, a formula for the finite-time ruin probability

$$\psi(x, T) = \mathbb{P}(\exists t \in [0, T] : R_x(t) < 0)$$

has been proposed by Picard and Lefèvre [18], discussed by De Vylder [24] and Ignatov et al. [11], and compared to a Prabhu or Seal-type formula by Rullière and Loisel [23]. Its influence function and estimation risk (related to the jump size distribution) have been studied by Loisel, Mazza and Rullière in [14] and [13].

Another important practical problem is to obtain numerical values of the sensitivity of the finite-time ruin probability with respect to the initial reserve

$$\frac{\partial \psi}{\partial x}(x, T),$$

in particular due to new solvency regulations in Europe. In [22] Privault and Wei used the Malliavin calculus to compute the sensitivity of the probability

$$\mathbb{P}(R_x(T) < 0)$$

that the terminal surplus is negative with respect to parameters such as the initial reserve x or the interest rate of the model.

However the problem of computing the corresponding sensitivity for the finite-time ruin probability $\psi(x, T)$ has not been covered in [22] because $R_x(T)$ does not satisfy the smoothness conditions imposed therein. For this reason, in this paper we develop an alternative solution to this problem, based on direct integration by parts techniques.

This paper is organized as follows. We start in Section 2 by a review of the principles of Malliavin's calculus applied to density estimation. In Section 3 we present the particular setting of Malliavin's calculus on the Poisson space and we show in particular why the conditions imposed in this calculus are too stringent for the random variables (infima) considered here. Indeed, as noted in Privault and Wei [22], Remark 5.2, the differentiability conditions imposed above are not satisfied e.g. by the infimum

$$\inf_{0 \leq t \leq T} R_x(t),$$

corresponding to a finite time ruin probability, whose density may not be continuous.

In Section 4 we propose to use a direct integration by parts on the Poisson space to compute the density of the infimum

$$\mathcal{M}_{[0, T]} = \inf\{f(t) - S(t), \quad t \in [0, T]\}$$

of the process

$$(f(t) - S(t))_{t \in [0, T]}$$

between time 0 and some fixed time horizon $T > 0$. This provides an explicit probabilistic representation formula in the case of a compound Poisson process (which corresponds to the classical Crámer-Lundberg risk model) which is suitable for simulation purposes. In the case of a jump-diffusion process (with an independent Brownian component that models investment of the surplus into a risky asset), we obtain a formula involving the density of the Brownian bridge, which suggests how our method could be extended to such models after further research. In the compound Poisson risk model, the ruin probability $\psi(x, T)$ equals

$$\psi(x, T) = \mathbb{P}(\mathcal{M}_{[0, T]} < -x), \quad x \geq 0,$$

and the density of $\mathcal{M}_{[0, T]}$ at $-x < 0$ equals

$$-\frac{\partial \psi}{\partial x}(x, T).$$

In Section 5, we show that our method is much more stable than classical finite-difference methods on several examples (for unit valued, exponential, and Pareto distributed claim amounts).

2 Malliavin calculus for density estimation

We start by a summary of Malliavin's calculus and integration by parts formulas applied to the existence and smoothness of probability densities of random variables. In particular, our goal is to determine more precisely the range of application of these techniques to the suprema of compensated jump processes.

Existence of densities

Here we state conditional versions of classical results on the existence of probability densities, see e.g. § 3.1 of Nualart [16] or Corollary 5.2.3 of Bouleau and Hirsch [3].

In the sequel we work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Proposition 1 *Let $A \in \mathcal{F}$ such that $\mathbb{P}(A) > 0$ and assume that F satisfies an integration by parts formula of the form*

$$\mathbb{E}[Gf'(F)|A] = \mathbb{E}[\Lambda_{F,G}f(F)|A] \quad f \in \mathcal{C}_b^1(\mathbb{R}), \quad (2.1)$$

where G is a.s. positive on A and $\Lambda_{F,G}$ is an integrable weight depending on F and G , and independent of $f \in \mathcal{C}_b^1(\mathbb{R})$.

Then:

i) the law of F has a conditional density $\varphi_{F|A}$ given A with respect to the Lebesgue measure.

ii) if in addition $G = \mathbf{1}$ a.s. on A then this density is given by

$$\varphi_{F|A}(y) = \mathbb{E}[\Lambda_{F,\mathbf{1}}\mathbf{1}_{\{y \leq F\}}|A], \quad y \in \mathbb{R}. \quad (2.2)$$

Proof. We have

$$\mathbb{E}[Gf'(F)|A] = \mathbb{E}[f(F)\Lambda_{F,G}|A] \leq \|f\|_\infty \mathbb{E}[|\Lambda_{F,G}||A],$$

and taking $f' = \mathbf{1}_B$ yields

$$\mathbb{E}[G\mathbf{1}_B(F)] \leq m(B)\mathbb{E}[|\Lambda_{F,G}|],$$

where $m(B)$ denotes the Lebesgue measure of any bounded Borel subset B of \mathbb{R} , hence the law of F is absolutely continuous with respect to the Lebesgue measure since $G > 0$ a.s. If $G = \mathbf{1}$ a.s. on A , we have

$$\begin{aligned}
\mathbb{E}[f(F)|A] &= \mathbb{E}\left[\int_{-\infty}^F f'(y)dy \middle| A\right] \\
&= \int_{-\infty}^0 \mathbb{E}\left[f'(y+F) \middle| A\right] dy \\
&= \int_{-\infty}^0 \mathbb{E}\left[\Lambda_{F,1} f(y+F) \middle| A\right] dy \\
&= \int_{-\infty}^{\infty} f(y) \mathbb{E}\left[\mathbf{1}_{\{F \geq y\}} \Lambda_{F,1} \middle| A\right] dy.
\end{aligned}$$

□

Formulas of the type (2.1) can be used to show that

$$\frac{\partial}{\partial \xi} \mathbb{E}[f(F_\xi)|A] = \mathbb{E}[\partial_\xi F_\xi f'(F_\xi)|A] = \mathbb{E}[\Lambda_{F_\xi, \partial_\xi F_\xi} f(F_\xi)|A], \quad (2.3)$$

where F_ξ is a random variable depending on a real parameter ξ , and can be applied to sensitivity analysis in finance, see e.g. Fournié et al. [7] and in insurance, Privault and Wei [22].

Continuity of densities

Proposition 1 ensures the existence of the density $\varphi_{F|A}$ but not its smoothness. The next proposition provides a more precise statement.

Proposition 2 *Under the hypothesis of Corollary 2 with $G = \mathbf{1}$ on A , if in addition $\Lambda_{F,1} \in L^p(A)$ for some $p > 1$, the conditional probability density $\varphi_{F|A}$ is continuous on \mathbb{R} .*

Proof. Use the bound

$$|\varphi_{F|A}(y) - \varphi_{F|A}(z)| \leq \frac{1}{\mathbb{P}(A)} \|\Lambda_F\|_{L^p(A)} (\mathbb{E}[\mathbf{1}_{[z,y]}(F)])^{1/q}, \quad y, z \in \mathbb{R}, \quad (2.4)$$

that follows from (2.2), with $1/p + 1/q = 1$. □

The integrability of Λ_F in $L^p(A)$ for $p > 1$ can essentially be obtained under strong (second order) differentiability conditions in the Malliavin sense, cf. Section 3.

Integration by parts

Proposition 1 can be implemented by assuming the existence of a gradient operator $D : L^2(\Omega) \rightarrow L^2(\Omega \times \mathbb{R}_+)$ admitting an adjoint $\delta : L^2(\Omega \times \mathbb{R}_+) \rightarrow$

$L^2(\Omega)$, with respective domains $\text{Dom}(D|A) \subset L^2(\Omega)$ and $\text{Dom}(\delta|A) \subset L^2(\Omega \times \mathbb{R}_+)$, such that

$$\mathbb{E}[\langle DF, u \rangle | A] = \mathbb{E}[F\delta(u)|A], \quad F \in \text{Dom}(D|A), \quad u \in \text{Dom}(\delta|A), \quad (2.5)$$

as recalled in the Corollaries 1 and 2 below, where $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{L^2([0, T])}$ denotes the scalar product in $L^2([0, T])$.

First, we treat the existence of densities in a corollary of Proposition 1, using the duality (2.5) between D and δ .

Corollary 1 *Let $w \in \text{Dom}(\delta|A)$ and $F \in \text{Dom}(D|A)$ such that*

$$\langle DF, w \rangle > 0, \quad \text{a.s. on } A \quad (2.6)$$

Then the law of F has a conditional density $\varphi_{F|A}$ given A with respect to the Lebesgue measure.

Proof. Letting $G = \langle DF, w \rangle$ we get

$$\mathbb{E}[\langle DF, w \rangle f'(F) | A] = \mathbb{E}[\langle Df(F), w \rangle | A] = \mathbb{E}[f(F)\delta(w) | A],$$

hence it suffices to apply Proposition 1 with $\Lambda_{F,G} = \delta(w)$. \square

As a consequence, the existence of density for F can be obtained under first order Malliavin differentiability conditions, cf. Section 3 for an implementation in the setting of jump processes. In particular when $w = DF$ in Corollary 1, Condition (2.6) becomes that of Theorem 6 of Privault [20], cf. also Proposition 4.2.4 of Decreusefond [5] on the Poisson space.

Next we recall how the operators D and δ can be applied to the representation and continuity of densities.

Corollary 2 *Assume that (2.5) holds, that $F \in \text{Dom}(D|A)$, and that there exists $w \in L^2(\Omega \times \mathbb{R}_+)$ such that $\frac{wG}{\langle DF, w \rangle} \in \text{Dom}(\delta|A)$. Then the conclusion of Proposition 1 holds with the weight*

$$\Lambda_{F,G} = \delta \left(G \frac{w}{\langle DF, w \rangle} \right). \quad (2.7)$$

Moreover, the density $\varphi_{F|A}$ is continuous on \mathbb{R} in case $G = \mathbf{1}$ a.s. on A .

Proof. Using the relation

$$f'(F) = \frac{\langle Df(F), w \rangle}{\langle DF, w \rangle}, \quad f \in \mathcal{C}_b^1(\mathbb{R}),$$

we get

$$\mathbb{E}[Gf'(F)|A] = \mathbb{E}\left[G\frac{\langle Df(F), w \rangle}{\langle DF, w \rangle} \Big| A\right] = \mathbb{E}\left[f(F)\delta\left(\frac{wG}{\langle DF, w \rangle}\right) \Big| A\right],$$

hence the existence of a conditional density follows from Proposition 1. The continuity of $\varphi_{F|A}$ in case $G = \mathbf{1}$ a.s. on A follows from Proposition 2 and the fact that δ is $L^2(\Omega)$ -valued on $\text{Dom}(\delta|A)$. \square

Note that we may take $w = DF$ in Corollary 2, in which case Relation (2.7) becomes

$$\Lambda_{F,G} = \delta\left(\frac{GDF}{\langle DF, DF \rangle}\right),$$

cf. e.g. Nualart [15], Proposition 2.1.1.

Non-continuous densities

To close this section, note that (2.1) can be replaced by an expression of the form

$$\mathbb{E}[f'(F)|A] = \mathbb{E}\left[\sum_{j=1}^Z \Lambda_j f(F_j) \Big| A\right]$$

where $Z, F_j, \Lambda_j, j \geq 1$, are random variables, which also implies the existence of a conditional density of F given A as

$$\varphi_{F|A}(y) = \mathbb{E}\left[\sum_{j=1}^Z \Lambda_j \mathbf{1}_{\{y \leq F_j\}} \Big| A\right]. \quad (2.8)$$

However, Relation (2.8) no longer ensures the continuity of $\varphi_{F|A}$ as the bound (2.4) is no longer valid. Such expressions will be obtained in Section 4, Proposition 4, for the infimum $\mathcal{M}_{[0,T]}$ of $R_\xi(t), t \in [0, T]$.

3 Malliavin calculus on the Poisson space

In order to apply the results of the preceding section to functionals of jump processes, we now turn to a specific implementation of the Malliavin calculus on Poisson space, cf. Carlen and Pardoux [4], Privault [19]. Here, $(\Omega, \mathcal{F}, \mathbb{P})$ denotes the canonical probability space of the standard the Poisson process $(N_t)_{t \in \mathbb{R}_+}$ whose jumps are denoted by $(T_k)_{k \geq 1}$, with $T_0 = 0$.

Definition 1 *Given $N \in \mathbb{N}$ we denote by \mathcal{S}_N the space of Poisson functionals of the form*

$$F = h(T_1 \wedge T, \dots, T_n \wedge T), \quad h \in \mathcal{C}^1([0, T]^n), \quad n \geq N, \quad (3.1)$$

with the boundary condition $F = 0$ on $\{N_T < N\}$, i.e.

$$h(t_1, \dots, t_{N-1}, T) = 0, \quad t_1, \dots, t_{N-1} \in [0, T]. \quad (3.2)$$

Every $F \in \mathcal{S}_N$ can be written as

$$F = \sum_{k=N}^{\infty} \mathbf{1}_{\{N_T=k\}} f_k(T_1, \dots, T_k), \quad (3.3)$$

where $f_0 \in \mathbb{R}$ and $f_k \in \mathcal{C}^1([0, T]^k)$ satisfies

$$f_k(T_1, \dots, T_k) = h(T_1, \dots, T_{n \wedge k}, T, \dots, T), \quad k \geq N, \text{ on } \{N_T = k\}. \quad (3.4)$$

Note that Condition (3.2) is void when $N = 0$.

Definition 2 Let $D_t F$, $t \in \mathbb{R}_+$, denote the gradient of $F \in \mathcal{S}_N$, defined as

$$D_t F = - \sum_{k=1}^n \mathbf{1}_{[0, T_k]}(t) \partial_k h(T_1 \wedge T, \dots, T_n \wedge T),$$

for $F \in \mathcal{S}_N$ of the form (3.1), where $\partial_k h$ denotes the partial derivative of h with respect to its k -th variable.

For F of the form (3.3) we have:

$$D_t F = - \sum_{n=N \vee 1}^{\infty} \mathbf{1}_{\{N_T=n\}} \sum_{k=1}^n \mathbf{1}_{[0, T_k]}(t) \partial_k f_n(T_1, \dots, T_n).$$

From now on we consider A of the form $A = \{N_T \geq N\}$ for some $N \in \mathbb{N}$, and let $\text{Dom}_N(D)$, $\text{Dom}_N(\delta)$ respectively denote $\text{Dom}(D|_{N_T \geq N})$ and $\text{Dom}(\delta|_{N_T \geq N})$. Similarly we will denote $\mathbb{E}[F|N_T \geq N]$ by $\mathbb{E}_N[F]$ for simplicity of notation.

Lemma 1 The operator D can be extended to its closed domain $\text{Dom}_N(D)$ and admits an adjoint δ with domain $\text{Dom}_N(\delta)$ such that

$$\mathbb{E}_N[\langle DF, u \rangle] = \mathbb{E}_N[F \delta(u)], \quad F \in \text{Dom}_N(D), \quad u \in \text{Dom}_N(\delta). \quad (3.5)$$

Moreover for all $u \in L^2([0, T])$ and $F \in \text{Dom}_N(D)$ we have

$$\delta(Fu) = F \int_0^T u(t) d(N_t - \lambda dt) - \int_0^\infty u(t) D_t F dt. \quad (3.6)$$

Proof. The proof of this proposition is a conditional version of classical integration by parts results on the Poisson space, and is given for completeness in the Appendix Section 6. \square

In case the continuity condition (6.1) is not satisfied, then assuming in addition $\int_0^T u(s)ds = 0$ we still get

$$\mathbb{E}_N[\langle DF, u \rangle] = \mathbb{E}_N \left[F \sum_{k=1}^{N_T} u(T_k) \right] = \mathbb{E}_N \left[F \int_0^T u(t) dN(t) \right]$$

for F of the form (3.3).

As an example of application of Propositions 1 and 2 in this context, consider a constant premium income rate equal to 1, with deterministic claim amounts equal to 1 as well. The infimum

$$\mathcal{M}_{[0,T]} = \inf_{0 \leq t \leq T} (t - N_t) = \inf_{T_k \leq T, k \geq 0} (T_k - k) = \mathbf{1}_{\{N_T \geq 1\}} \inf_{T_k \leq T, k \geq 1} (T_k - k), \quad (3.7)$$

satisfies (3.3) with $f_0 = 0$ and

$$f_n(t_1, \dots, t_n) = \inf_{1 \leq k \leq n} (t_k - k), \quad n \geq 1,$$

and we have $\{\mathcal{M}_{[0,T]} < 0\} = \{N_T \geq 1\}$. Hence taking $N = 1$,

$$\mathcal{M}_{[0,T]} = \sum_{k=1}^{\infty} (T_k - k) \mathbf{1}_{\{\mathcal{M}_{[0,T]} = T_k - k\}}$$

belongs to $\text{Dom}_1(D)$ with $\mathcal{M}_{[0,T]} = 0$ on $A = \{N_T \geq 1\}$, and

$$\langle D\mathcal{M}_{[0,T]}, w \rangle = -\mathbf{1}_{\{\mathcal{M}_{[0,T]} < 0\}} \sum_{k=1}^{\infty} w(T_k) \mathbf{1}_{\{\mathcal{M}_{[0,T]} = T_k - k\}}.$$

Moreover the gradient norm

$$\begin{aligned} \langle D\mathcal{M}_{[0,T]}, D\mathcal{M}_{[0,T]} \rangle &= \mathbf{1}_{\{\mathcal{M}_{[0,T]} < 0\}} \sum_{k=1}^{\infty} T_k \mathbf{1}_{\{\mathcal{M}_{[0,T]} = T_k - k\}} \\ &= \mathbf{1}_{\{\mathcal{M}_{[0,T]} < 0\}} \sum_{k=1}^{\infty} (\mathcal{M}_{[0,T]} + k) \mathbf{1}_{\{\mathcal{M}_{[0,T]} = T_k - k\}} \end{aligned}$$

is a.e. positive on $A = \{\mathcal{M}_{[0,T]} < 0\} = \{N_T \geq 1\}$, thus ensuring the existence of the density of $\mathcal{M}_{[0,T]}$ conditionally to $\{\mathcal{M}_{[0,T]} < 0\}$ from Proposition 1 and Corollary 1.

The continuity of the density of F can be obtained from Proposition 2 provided $\Lambda_{F,1} \in L^p$ for some $p > 1$, and from Corollary 2 the divergence formula (3.6) can be used to satisfy this condition. However, applying the divergence formula (3.6) to $G = 1/\langle DF, w \rangle$ requires F to be twice differentiable for D .

In order to check that $\Lambda_{F,G}$ defined in (2.7) belongs to L^p , we can proceed as follows.

Let \mathcal{U} denote the space of processes of the form

$$u = \sum_{k=1}^n F_k h_k, \quad h_1, \dots, h_n \in \mathcal{C}_c^1((0, T)), \quad F_1, \dots, F_n \in \mathcal{S}_N, \quad n \geq 1, \quad (3.8)$$

and let the covariant derivative ∇ be defined as

$$\nabla_s u_t = D_s u_t - \mathbf{1}_{[0,t]}(s) \dot{u}_t, \quad s, t \in \mathbb{R}_+, \quad u \in \mathcal{U}.$$

Proposition 3 *For all $u \in \mathcal{U}$ we have the Skorohod isometry*

$$\mathbb{E}_N[|\delta(u)|^2] = \mathbb{E}_N[\|u\|_{L^2([0,T])}^2] + \mathbb{E}_N \left[\int_0^T \int_0^T \nabla_s u_t \nabla_t u_s ds dt \right]. \quad (3.9)$$

Proof. cf. the Appendix Section 6 . □

The isometry (3.9) implies the bound

$$\mathbb{E}_N[|\delta(u)|^2] \leq \mathbb{E}_N[\|u\|_{L^2([0,T])}^2] + \mathbb{E}_N[\|\nabla u\|_{L^2([0,T]^2)}^2], \quad (3.10)$$

which provides sufficient conditions for a process $u \in \mathcal{U}$ to belong to $\text{Dom}(\delta)$.

Again, applying this bound to $u = w / \langle DF, w \rangle$ as in Corollary 2, $w \in \mathcal{C}_c^1((0, T))$, $F \in \mathcal{S}_N$, requires a second order D -differentiability of F which is not satisfied by $F = \mathcal{M}_{[0,T]}$.

In [22], the computation of the probability density of $R_x(T)$ defined in (1.1) relied on strong (second order) differentiability conditions in the Malliavin sense, which are not satisfied by the infimum $\mathcal{M}_{[0,T]}$ defined in (3.7). It is natural that such differentiability conditions do not hold here since they would ensure the continuity of the probability density of $\mathcal{M}_{[0,T]}$, a property which is not satisfied here, cf. Relation (5.1) and Figure 1 below.

4 Calculation of densities by integration by parts

Our goal in this section is to develop a direct integration by parts method in order to obtain an algorithm for the numerical computation of the non-continuous density functions of the infima of jump processes such as $\mathcal{M}_{[0,T]}$. We consider both the cases of deterministic and random drifts.

Monotone deterministic drift

Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be an increasing function mapping $t > 0$ to the premium income $f(t)$ received between time 0 and time t , such that $f(0) = 0$, and assume that $(S(t))_{t \in \mathbb{R}_+}$ has the form

$$S(t) = Y_{N_t}, \quad t \in \mathbb{R}_+,$$

where $Y_0 = 0$ and $(Y_k)_{k \geq 1}$ is a sequence of random variables, independent of $(N_t)_{t \in \mathbb{R}_+}$, i.e. in the compound Poisson risk model, $S(t)$ represents the aggregate claim amount and

$$Y_k = \sum_{j=1}^k W_j, \quad k \in \mathbb{N}.$$

Consider the infimum

$$\mathcal{M}_{[0,T]} = \inf_{0 \leq t \leq T} (f(t) - S(t)).$$

Clearly we have $\mathcal{M}_{[0,T]} \leq 0 = f(0)$ hence the law of $\mathcal{M}_{[0,T]}$ is carried by $(-\infty, 0]$. On the other hand, we have $\mathcal{M}_{[0,T]} = 0$ if and only if $N_T = 0$ or $f(T_k) - Y_k > 0$ for all $k = 1, \dots, N_T$. Hence the law of $\mathcal{M}_{[0,T]}$ has a Dirac mass at 0 with weight

$$\begin{aligned} \mathbb{P}(\mathcal{M}_{[0,T]} = 0) &= \mathbb{P}(N_T = 0) + \mathbb{P}(\{\mathcal{M}_{[0,T]} \geq 0\} \cap \{N_T \geq 1\}) \\ &= e^{-\lambda T} + e^{-\lambda T} \mathbb{E} \left[\sum_{k=1}^{\infty} \lambda^k \int_0^T \int_0^{t_k} \cdots \int_0^{t_2} \mathbf{1}_{\{f(t_1) > Y_1\}} \cdots \mathbf{1}_{\{f(t_k) > Y_k\}} dt_1 \cdots dt_k \right]. \end{aligned}$$

In the next proposition we not only compute the density of $\mathcal{M}_{[0,T]}$, we also provide a probabilistic representation which is suitable for simulation purposes.

Proposition 4 *The density at $y \in \mathbb{R}$ of $\mathcal{M}_{[0,T]}$ given that $N_T \geq 1$ is equal to*

$$\begin{aligned} \varphi_{\mathcal{M}_{[0,T]} | N_T \geq 1}(y) &= \\ &= \frac{\lambda}{1 - e^{-\lambda T}} \mathbb{E} \left[\sum_{j=1}^{N_T} \mathbf{1}_{\{y \leq \inf_{1 \leq l \leq j} (f(T_l) - Y_l)\}} \mathbf{1}_{\{f(T_{j-1}) - Y_j < y\}} \mathbf{1}_{\{y \leq \inf_{j \leq l \leq N_T} (f(T_l) - Y_{l+1})\}} \right] \\ &+ \frac{\lambda}{1 - e^{-\lambda T}} \mathbb{E} \left[\mathbf{1}_{\{0 < Y_{N_T+1} + y < f(T)\}} \mathbf{1}_{\{f(T_{N_T}) < Y_{N_T+1} + y\}} \mathbf{1}_{\{\inf_{1 \leq l \leq N_T} (f(T_l) - Y_l) > y\}} \right], \end{aligned}$$

where we use the convention $\inf_{\emptyset} = +\infty$.

Proof. Since f is increasing we have

$$\mathcal{M}_{[0,T]} = \inf_{T_k \leq T, k \geq 0} (f(T_k) - Y_k) = \mathbf{1}_{\{N_T \geq 1\}} \inf_{T_k \leq T, k \geq 1} (f(T_k) - Y_k),$$

with $T_0 = 0$. Hence

$$\begin{aligned}
& \mathbb{P}(\{\mathcal{M}_{[0,T]} \geq y\} \cap \{N_T \geq 1\}) \tag{4.1} \\
&= e^{-\lambda T} \mathbb{E} \left[\sum_{k=1}^{\infty} \lambda^k \int_0^T \int_0^{t_k} \cdots \int_0^{t_2} \mathbf{1}_{\{y < \inf_{1 \leq l \leq k} (f(t_l) - Y_l)\}} dt_1 \cdots dt_k \right] \\
&= \lambda e^{-\lambda T} \mathbb{E} \left[\sum_{k=0}^{\infty} \lambda^k \int_0^T \int_0^{t_{k+1}} \cdots \int_0^{t_2} \mathbf{1}_{\{f(t_1) > Y_1 + y\}} \cdots \mathbf{1}_{\{f(t_{k+1}) > Y_{k+1} + y\}} dt_1 \cdots dt_{k+1} \right]
\end{aligned}$$

In order to determine the density of $\mathcal{M}_{[0,T]}$ given that $N_T \geq 1$, it suffices to compute the derivative

$$\frac{\partial}{\partial y} \mathbb{P}(\{\mathcal{M}_{[0,T]} \geq y\} \cap \{N_T \geq 1\}).$$

Now, (4.1) implies:

$$\begin{aligned}
& -\frac{\partial}{\partial y} \mathbb{P}(\{\mathcal{M}_{[0,T]} \geq y\} \cap \{N_T \geq 1\}) \\
&= \lambda e^{-\lambda T} \mathbb{E} \left[\sum_{k=0}^{\infty} \lambda^k \sum_{j=1}^k \int_0^T \int_0^{t_{k+1}} \cdots \int_0^{t_{j+1}} \int_0^{t_{j-1}} \cdots \int_0^{t_2} \right. \\
& \quad \left. \mathbf{1}_{\{y < \inf_{1 \leq l < j} (f(t_l) - Y_l)\}} \mathbf{1}_{\{f(t_{j-1}) < Y_j + y < f(t_{j+1})\}} \mathbf{1}_{\{y < \inf_{j < l \leq k+1} (f(t_l) - Y_l)\}} dt_1 \cdots dt_{k+1} \right] \\
&+ \lambda e^{-\lambda T} \mathbb{E} \left[\sum_{k=0}^{\infty} \lambda^k \mathbf{1}_{\{0 < Y_{k+1} + y < f(T)\}} \right. \\
& \quad \left. \int_0^T \mathbf{1}_{\{f(t_k) < Y_{k+1} + y\}} \int_0^{t_k} \cdots \int_0^{t_2} \mathbf{1}_{\{y < \inf_{1 \leq l \leq k} (f(t_l) - Y_l)\}} dt_1 \cdots dt_k \right] \\
&= \lambda e^{-\lambda T} \mathbb{E} \left[\sum_{k=0}^{\infty} \lambda^k \sum_{j=1}^k \int_0^T \int_0^{t_{k-1}} \cdots \int_0^{t_2} \right. \\
& \quad \left. \mathbf{1}_{\{y \leq \inf_{1 \leq l < j} (f(t_l) - Y_l)\}} \mathbf{1}_{\{f(t_{j-1}) < y + Y_j < f(t_j)\}} \mathbf{1}_{\{y \leq \inf_{j \leq l \leq k} (f(t_l) - Y_{l+1})\}} dt_1 \cdots dt_k \right. \\
& \quad \left. + \sum_{k=0}^{\infty} \lambda^k \mathbf{1}_{\{0 < Y_{k+1} + y < f(T)\}} \int_0^T \mathbf{1}_{\{f(t_k) < Y_{k+1} + y\}} \int_0^{t_k} \cdots \int_0^{t_2} \mathbf{1}_{\{y < \inf_{1 \leq l \leq k} (f(t_l) - Y_l)\}} dt_1 \cdots dt_k \right] \\
&= \lambda e^{-\lambda T} \mathbb{E} \left[\sum_{k=0}^{\infty} \lambda^k \sum_{j=1}^k \int_0^T \int_0^{t_k} \cdots \int_0^{t_2} \right. \\
& \quad \left. \mathbf{1}_{\{y \leq \inf_{1 \leq l \leq j} (f(t_l) - Y_l)\}} \mathbf{1}_{\{f(t_{j-1}) - Y_j < y\}} \mathbf{1}_{\{y \leq \inf_{j \leq l \leq k} (f(t_l) - Y_{l+1})\}} dt_1 \cdots dt_k \right. \\
& \quad \left. + \sum_{k=0}^{\infty} \lambda^k \mathbf{1}_{\{Y_{k+1} + y < f(T)\}} \int_0^T \int_0^{t_k} \cdots \int_0^{t_2} \mathbf{1}_{\{f(t_k) < Y_{k+1} + y\}} \mathbf{1}_{\{y < \inf_{1 \leq l \leq k} (f(t_l) - Y_l)\}} dt_1 \cdots dt_k \right] \\
&= \lambda \mathbb{E} \left[\sum_{j=1}^{N_T} \mathbf{1}_{\{y \leq \inf_{1 \leq l \leq j} (f(T_l) - Y_l)\}} \mathbf{1}_{\{f(T_{j-1}) - Y_j < y\}} \mathbf{1}_{\{y \leq \inf_{j \leq l \leq N_T} (f(T_l) - Y_{l+1})\}} \right] \\
&+ \lambda \mathbb{E} \left[\mathbf{1}_{\{0 < Y_{N_T+1} + y < f(T)\}} \mathbf{1}_{\{f(T_{N_T}) < Y_{N_T+1} + y\}} \mathbf{1}_{\{\inf_{1 \leq l \leq N_T} (f(T_l) - Y_l) > y\}} \right].
\end{aligned}$$

□

Note that in some cases, other analytic expressions for the density of $\mathcal{M}_{[0,T]}$ can be obtained. For example, when $(Y_k)_{k \geq 1}$ are independent, exponentially distributed random variables with parameter $\mu > 0$ and $f(t) = \alpha t$ is linear, $\alpha \geq 0$, Theorem 4.1 and Relation (4.6) of Dozzi and Vallois [6] show that

$$\begin{aligned} & \mathbb{P}(\mathcal{M}_{[0,T]} < x) \\ &= \lambda \int_0^T \left(x \sum_{n=0}^{\infty} \frac{(\lambda \mu t(x + \alpha t))^n}{(n!)^2} + \alpha t \sum_{n=0}^{\infty} \frac{(\lambda \mu t(x + \alpha t))^n}{n!(n+1)!} \right) \frac{e^{-\mu(x+\alpha t)-\lambda t}}{x + \alpha t} dt, \end{aligned}$$

which provides another expression for the density of $\mathcal{M}_{[0,T]}$ by differentiation with respect to x .

The following corollary is useful to derive sensitivity analysis formulas such as Relation (2.3) above.

Corollary 3 *We have for all $g \in \mathcal{C}_b^1(\mathbb{R})$:*

$$\begin{aligned} & \mathbb{E}_1[g'(\mathcal{M}_{[0,T]})] \\ &= \lambda \mathbb{E}_1 \left[\sum_{j=1}^{N_T} g \left(\min \left(\inf_{1 \leq l \leq j} (f(T_l) - Y_l), \inf_{j \leq l \leq N_T} (f(T_l) - Y_{l+1}) \right) \right) - g(f(T_{j-1}) - Y_j) \right] \\ &+ \lambda \mathbb{E}_1 \left[g \left(\min \left(\inf_{1 \leq l \leq N_T} (f(T_l) - Y_l), f(T) - Y_{N_T+1} \right) \right) - g(f(T_{N_T}) - Y_{N_T+1}) \right]. \end{aligned}$$

Proof. We apply Proposition 4 and the relation

$$\mathbb{E}_1[g'(\mathcal{M}_{[0,T]})] = \int_{-\infty}^{\infty} g'(z) \varphi_{\mathcal{M}_{[0,T]} | N_T \geq 1}(z) dz.$$

□

Note that the above formula has the same form as (2.8) which, as noted in Section 2, does not ensure the continuity of the associated probability density.

Random drift

Consider now the infimum

$$\mathcal{M}_{[0,T]} = \inf_{0 \leq t \leq T} (Z_t - S(t))$$

where $(Z_t)_{t \in \mathbb{R}_+}$ is a stochastic process with independent increments and $Z_0 = 0$, independent of $(S(t))_{t \in \mathbb{R}_+}$, and such that

$$\inf_{t \in [a, b]} Z_t, \quad 0 \leq a < b,$$

has a density denoted by $\varphi_{a,b}(x)$. For example, if $(Z_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion then $\varphi_{a,b}(x)$ is given by

$$\begin{aligned} \int_x^\infty \varphi_{a,b}(z) dz &= \mathbb{P} \left(\inf_{t \in [a, b]} Z_t \geq x \right) \\ &= \mathbb{E} \left[\mathbf{1}_{\{Z_a < x\}} \mathbb{P} \left(\inf_{t \in [a, b]} Z_t \geq x \mid Z_a \right) \right] + \mathbb{E} \left[\mathbf{1}_{\{Z_a \geq x\}} \mathbb{P} \left(\inf_{t \in [a, b]} Z_t \geq x \mid Z_a \right) \right] \\ &= \mathbb{E} \left[\mathbf{1}_{\{Z_a < x\}} \mathbb{P} \left(\inf_{t \in [0, b-a]} B_t \geq x - Z_a \mid Z_a \right) \right] + \mathbb{P}(Z_a \geq x) \\ &= 2\mathbb{E} \left[\mathbf{1}_{\{Z_a < x\}} \mathbb{P}(B_{b-a} \geq x - Z_a \mid Z_a) \right] + \mathbb{P}(Z_a \geq x) \\ &= \frac{1}{\pi \sqrt{a(b-a)}} \int_0^\infty e^{-(x-y)^2/(2a)} \int_y^\infty e^{-z^2/(2(b-a))} dz dy + \frac{1}{\sqrt{2\pi a}} \int_x^\infty e^{-z^2/(2a)} dz. \end{aligned}$$

We have $\mathcal{M}_{[0, T]} \leq Z_0 = 0$ a.s., hence the law of $\mathcal{M}_{[0, T]}$ is carried by $(-\infty, 0]$.

Proposition 5 *The probability density at $y \in \mathbb{R}$ of $\mathcal{M}_{[0, T]}$ is equal to*

$$\begin{aligned} \varphi_{\mathcal{M}_{[0, T]}}(y) &= -\lambda e^{-\lambda T} \mathbb{E} \left[\sum_{k=0}^\infty \lambda^k \sum_{j=1}^{k+1} \int_0^T \int_0^{t_{k+1}} \cdots \int_0^{t_2} \varphi_{t_{j-1}, t_j}(y + S(t_{j-1})) \right. \\ &\quad \left. \mathbb{P} \left(y + S(t_{k+1}) < \inf_{t \in [t_{k+1}, T]} Z_t \mid S \right) \prod_{\substack{l=1 \\ l \neq j}}^{k+1} \mathbb{P} \left(y + S(t_{l-1}) < \inf_{t \in [t_{l-1}, t_l]} Z_t \mid S \right) dt_1 \cdots dt_{k+1} \right] \\ &= -\lambda e^{-\lambda T} \mathbb{E} \left[\sum_{k=0}^\infty \lambda^k \int_0^T \int_0^{t_{k+1}} \cdots \int_0^{t_2} \right. \\ &\quad \left. \varphi_{t_{k+1}, T}(y + S(t_{k+1})) \prod_{l=1}^{k+1} \mathbb{P} \left(y + S(t_{l-1}) < \inf_{t \in [t_{l-1}, t_l]} Z_t \mid S \right) dt_1 \cdots dt_{k+1} \right]. \end{aligned}$$

Proof. We have

$$\mathcal{M}_{[0, T]} = \min \left(\min_{T_k \leq T, k \geq 1} \inf_{t \in [T_{k-1}, T_k]} (Z_t - S(T_{k-1})), \inf_{t \in [T_{N_T}, T]} (Z_t - S(T_{N_T})) \right).$$

Hence

$$\begin{aligned}
\mathbb{P}(\mathcal{M}_{[0,T]} \geq y) &= e^{-\lambda T} \mathbb{E} \left[\sum_{k=1}^{\infty} \lambda^k \int_0^T \int_0^{t_k} \cdots \int_0^{t_2} \right. \\
&\quad \left. \mathbf{1}_{\{y+S(t_k) < \inf_{t \in [t_k, T]} Z_t\}} \prod_{l=1}^k \mathbf{1}_{\{y+S(t_{l-1}) < \inf_{t \in [t_{l-1}, t_l]} Z_t\}} dt_1 \cdots dt_k \right] \\
&= \lambda e^{-\lambda T} \mathbb{E} \left[\sum_{k=0}^{\infty} \lambda^k \int_0^T \int_0^{t_{k+1}} \cdots \int_0^{t_2} \right. \\
&\quad \left. \mathbf{1}_{\{y+S(t_{k+1}) < \inf_{t \in [t_{k+1}, T]} Z_t\}} \prod_{l=1}^{k+1} \mathbf{1}_{\{y+S(t_{l-1}) < \inf_{t \in [t_{l-1}, t_l]} Z_t\}} dt_1 \cdots dt_{k+1} \right] \\
&= \lambda e^{-\lambda T} \mathbb{E} \left[\sum_{k=0}^{\infty} \lambda^k \int_0^T \int_0^{t_{k+1}} \cdots \int_0^{t_2} \right. \\
&\quad \left. \mathbb{P} \left(y + S(t_{k+1}) < \inf_{t \in [t_{k+1}, T]} Z_t \middle| S \right) \prod_{l=1}^{k+1} \mathbb{P} \left(y + S(t_{l-1}) < \inf_{t \in [t_{l-1}, t_l]} Z_t \middle| S \right) dt_1 \cdots dt_{k+1} \right],
\end{aligned} \tag{4.2}$$

and in order to determine the density $\varphi_{\mathcal{M}_{[0,T]}}$ of $\mathcal{M}_{[0,T]}$ it remains to compute the derivative $-\frac{\partial}{\partial y} \mathbb{P}(\mathcal{M}_{[0,T]} \geq y)$. \square

By a simple change of variable this allows one to treat exponential jump diffusion processes of the form $e^{Z_t - S(t)}$.

The limitation of this result is that it involves the density of a stochastic (e.g. Brownian) bridge. A more detailed analysis could lead to a more operational formula, which would be useful for applications not only in ruin theory, but also in finance and in credit risk to obtain greek letters for some barrier options and CDO's in jump-diffusion models.

5 Numerical simulations

We present an example of simulation when $f(t) = t$ and $W_k = 1$, $k \in \mathbb{N}$, i.e. for the infimum

$$\mathcal{M}_{[0,T]} = \inf_{0 \leq t \leq T} (t - N_t) = \inf_{T_k \leq T, k \geq 0} (T_k - k) = \mathbf{1}_{\{N_T \geq 1\}} \inf_{T_k \leq T, k \geq 1} (T_k - k).$$

In this case the density function found in Proposition 4 rewrites as

$$\begin{aligned}
&\frac{\partial}{\partial y} \mathbb{P}(\{\mathcal{M}_{[0,T]} \geq y\} \cap \{N_T \geq 1\}) \\
&= \lambda \mathbb{E} \left[\sum_{j=1}^{N_T} \mathbf{1}_{\{y \leq \inf_{1 \leq l \leq j-1} (T_l - l)\}} \mathbf{1}_{\{T_{j-1} - j < y\}} \mathbf{1}_{\{y \leq \inf_{j \leq l \leq N_T} (T_l - l)\}} \right] \\
&\quad + \lambda \mathbb{E} \left[\mathbf{1}_{\{T_{N_T} < N_T + 1 + y < T\}} \mathbf{1}_{\{y < \inf_{1 \leq l \leq N_T} (T_l - l)\}} \right]
\end{aligned}$$

$$\begin{aligned}
&= \lambda \mathbb{E} \left[\sum_{j=1}^{N_T} \mathbf{1}_{\{y \leq \inf_{1 \leq l \leq j-1} (T_l - l)\}} \mathbf{1}_{\{T_{j-1} - j < y\}} \mathbf{1}_{\{y \leq \inf_{j \leq l \leq N_T} (T_l - l - 1)\}} \right] \\
&\quad + \lambda \sum_{k=0}^{\infty} \mathbb{E} \left[\mathbf{1}_{\{N_T = k\}} \mathbf{1}_{\{T_k < k+1+y < T\}} \mathbf{1}_{\{y < \inf_{1 \leq l \leq k} (T_l - l)\}} \right] \\
&= \lambda \mathbb{E} \left[\sum_{j=1}^{N_T} \mathbf{1}_{\{y \leq \inf_{1 \leq l \leq j-1} (T_l - l)\}} \mathbf{1}_{\{T_{j-1} - j < y\}} \mathbf{1}_{\{y \leq \inf_{j \leq l \leq N_T} (T_l - l - 1)\}} \right] \\
&\quad + \lambda \sum_{i=0}^{\infty} \mathbf{1}_{[T-i-2, T-i-1]}(y) \sum_{k=0}^i \mathbb{E} \left[\mathbf{1}_{\{N_T = k\}} \mathbf{1}_{\{T_k < k+1+y\}} \mathbf{1}_{\{y < \inf_{1 \leq l \leq k} (T_l - l)\}} \right] \quad (5.1) \\
&= \lambda \mathbb{E} \left[\sum_{j=1}^{N_T+1} \mathbf{1}_{\{T_{j-1} - j < y \leq \min(\inf_{1 \leq l \leq j} (T \wedge T_l - l), \inf_{j \leq l \leq N_T} (T_l - l - 1))\}} \right].
\end{aligned}$$

Note that the non-continuous component of the density appears explicitly in (5.1) of the above expression. For the purpose of sensitivity analysis, the result of Corollary 3 becomes:

$$\begin{aligned}
&\mathbb{E}[g'(y + \mathcal{M}_{[0, T]})] \\
&= \lambda \mathbb{E} \left[\sum_{j=1}^{N_T} g \left(\min \left(\inf_{1 \leq l \leq j-1} (T_l - l), \inf_{j \leq l \leq N_T} (T_l - l - 1) \right) \right) - g(T_{j-1} - j) \right] \\
&\quad + \lambda \mathbb{E} \left[g \left(\min \left(\inf_{1 \leq l \leq N_T} (T_l - l), T - N_T - 1 \right) \right) - g(T_{N_T} - N_T - 1) \right] \\
&= \lambda \mathbb{E} \left[\sum_{j=1}^{N_T+1} g \left(\min \left(\inf_{1 \leq l \leq j} (T \wedge T_l - l), \inf_{j \leq l \leq N_T} (T_l - l - 1) \right) \right) - g(T_{j-1} - j) \right].
\end{aligned}$$

For a same number of iteration, the integration by parts algorithm is not significantly slower than the finite differences method, because it only involves the computation of two infima instead of one. However it yields a much greater level of precision: one can check in Figure 1 that our results are much less noisy than the ones of finite-difference method. Besides, the density at each point is obtained independently from other points, which is not the case with finite difference or kernel estimation methods. This is especially important for non-continuous densities, for which kernel estimators will introduce some form of unwanted smoothing.

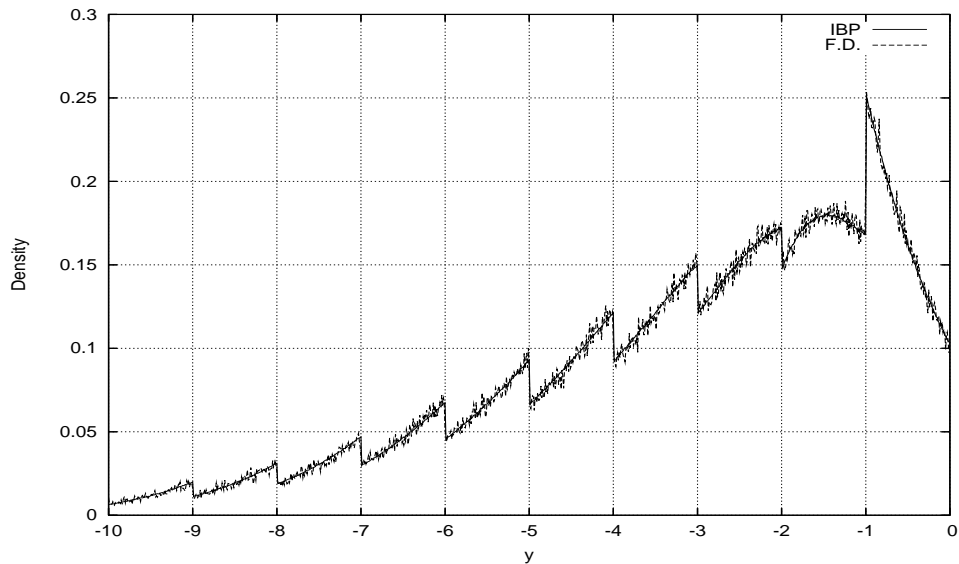


Figure 1. Estimation of the probability density of $\mathcal{M}_{[0,T]}$ by our method (IBP) and by finite differences (FD) with $N = 100000$ trials.

In Figure 2 we illustrate the fact that our method requires much fewer trials to accurately estimate the target value.

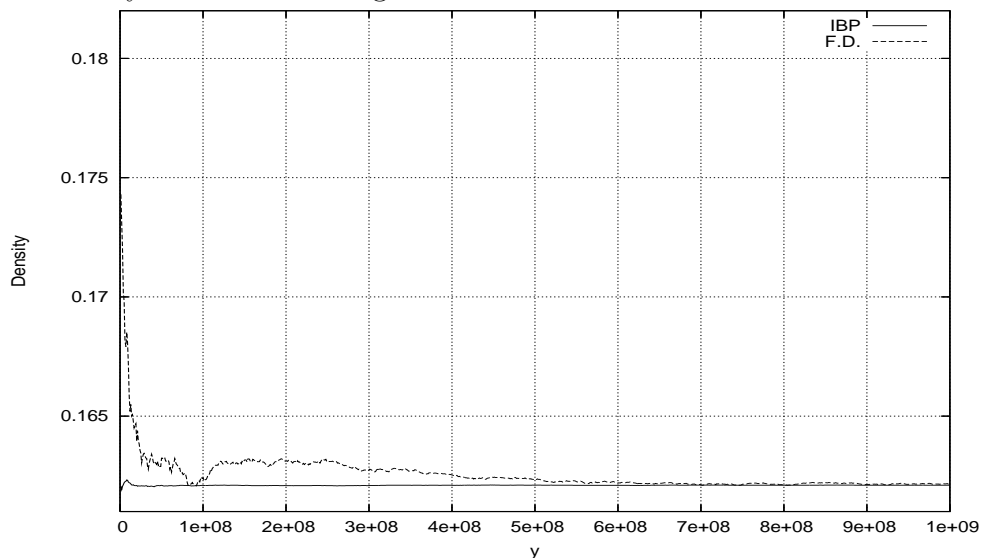


Figure 2. Estimation of the probability density of $\mathcal{M}_{[0,T]}$ at $y = -0.5$ vs number of trials by our method (IBP) and by finite differences (FD).

After this simple example, we also illustrate the case of exponentially and Pareto distributed claim amounts in Figures 3, 4 and 5 below, to show that our method is operational for typical light and heavy-tailed insurance models.

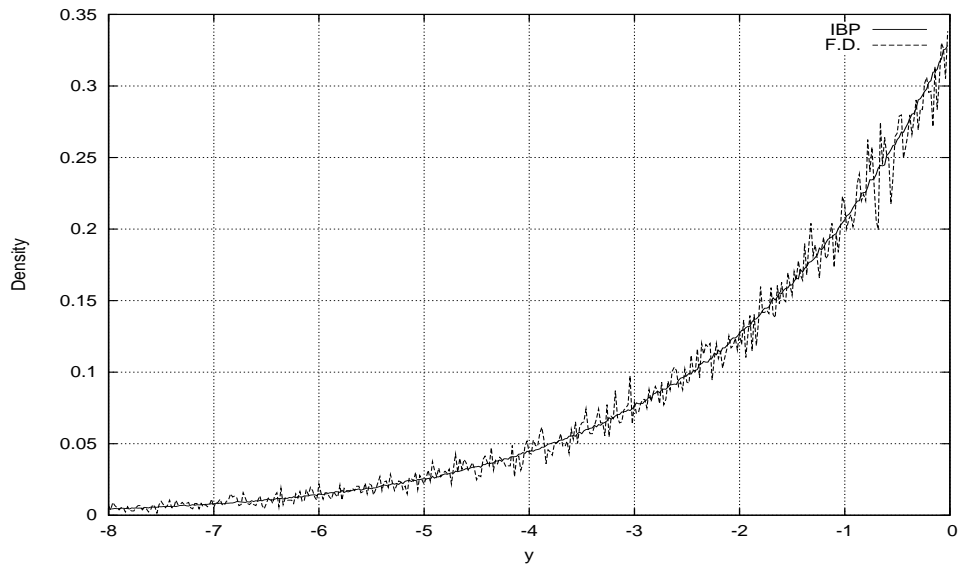


Figure 3. Probability density of $\mathcal{M}_{[0,T]}$ by finite differences and integration by parts for exponentially distributed claim amounts with $N = 100000$ trials.

The respective computation times to obtain the graph of Figure 3 above are 2m35s for the finite difference method and 4m5s for the integration by parts method.

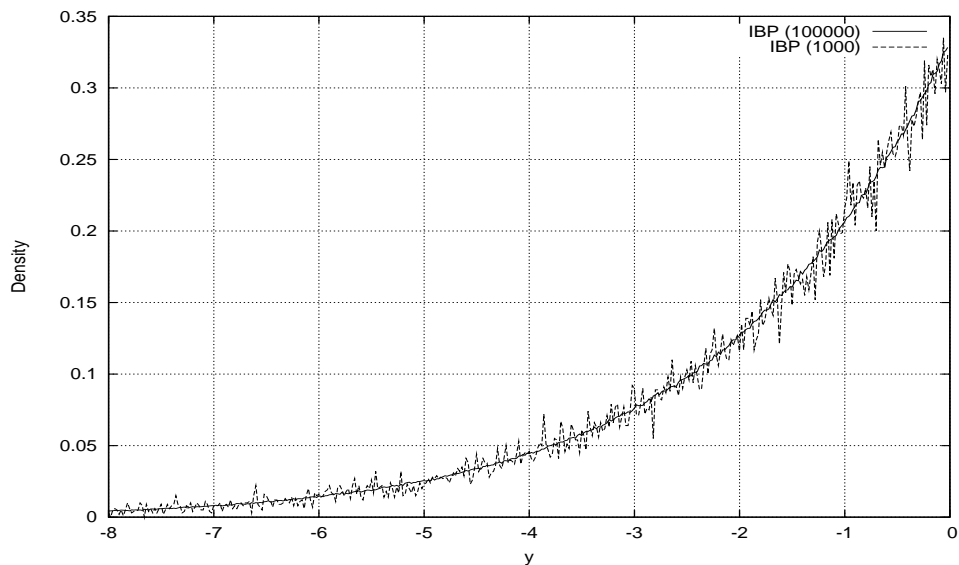


Figure 4. Comparison of density estimates of $\mathcal{M}_{[0,T]}$ for exponentially distributed claim amounts by integration by parts with $N = 1000$ and $N = 100000$ trials.

In Figure 4 above we present a density estimate obtained via the integration by parts method with $N = 1000$ samples and a computation time of 2.6s, to be compared with the similar level of precision reached in Figure 3 by a finite difference method with $N = 100000$ samples and a computation time of 4m5s. Finally, in Figure 5 below we consider the case of Pareto-distributed claims.

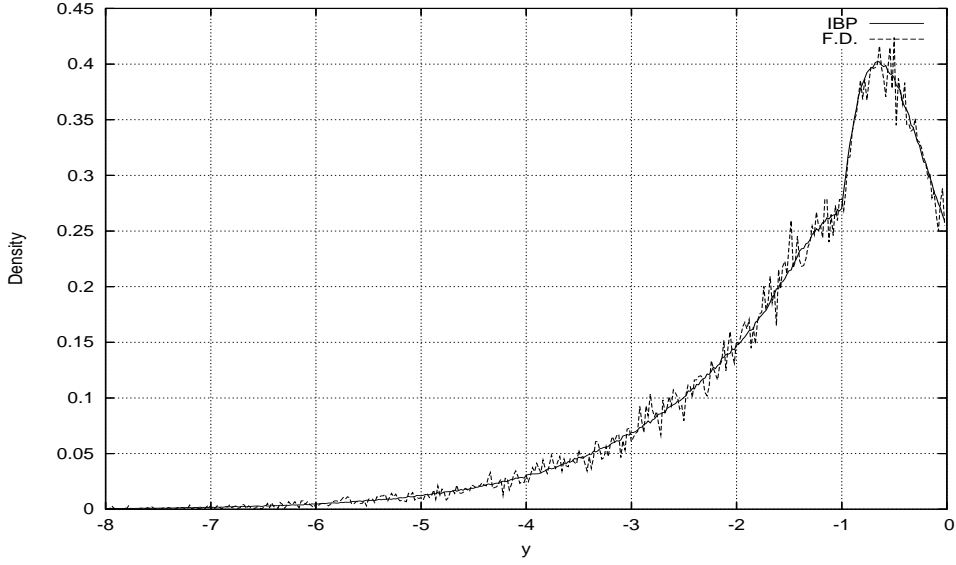


Figure 5. Probability density of $\mathcal{M}_{[0,T]}$ by finite differences and integration by parts for Pareto-distributed claim amounts with $N = 100000$ trials.

6 Appendix

For completeness, in this appendix we provide the proofs of Lemma 1 and Proposition 3, which are conditional versions of existing results, see e.g. [21], [22], and references therein.

Proof of Lemma 1. Recall that for all $F \in \mathcal{S}_N$ of the form (3.3) we have:

$$\mathbb{E}_N[F] = e^{-\lambda T} \sum_{n=N}^{\infty} \lambda^n \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1, \dots, t_n) dt_1 \cdots dt_n.$$

By standard integration by parts we first prove (3.5) when $u \in L^2([0, T])$:

$$\begin{aligned} & \mathbb{E}_N[\langle DF, u \rangle] \\ &= -e^{-\lambda T} \sum_{n=N \vee 1}^{\infty} \frac{\lambda^n}{n!} \sum_{k=1}^n \int_0^T \cdots \int_0^T \int_0^{t_k} u(s) ds \partial_k f_n(t_1, \dots, t_n) dt_1 \cdots dt_n \\ &= e^{-\lambda T} \sum_{n=N \vee 1}^{\infty} \frac{\lambda^n}{n!} \sum_{k=1}^n \int_0^T \cdots \int_0^T f_n(t_1, \dots, t_n) u(t_k) ds dt_1 \cdots dt_n \\ & - e^{-\lambda T} \sum_{n=N \vee 1}^{\infty} \frac{\lambda^n}{(n-1)!} \int_0^T u(s) ds \int_0^T \cdots \int_0^T f_n(t_1, \dots, t_{n-1}, T) dt_1 \cdots dt_{n-1}. \end{aligned}$$

From (3.4) we have the continuity condition

$$f_{n-1}(t_1, \dots, t_{n-1}) = f_n(t_1, \dots, t_{n-1}, T), \quad n \geq N, \quad (6.1)$$

hence

$$\begin{aligned}
\mathbb{E}_N[\langle DF, u \rangle] &= e^{-\lambda T} \sum_{n=N \vee 1}^{\infty} \frac{\lambda^n}{n!} \int_0^T \cdots \int_0^T f_n(t_1, \dots, t_n) \sum_{k=1}^n u(t_k) dt_1 \cdots dt_n \\
&\quad - \lambda e^{-\lambda T} \int_0^T u(s) ds \sum_{n=N}^{\infty} \frac{\lambda^n}{n!} \int_0^T \cdots \int_0^T f_n(t_1, \dots, t_n) dt_1 \cdots dt_n \\
&= \mathbb{E}_N \left[F \left(\sum_{k=1}^{N_T} u(T_k) - \lambda \int_0^T u(s) ds \right) \right] \\
&= \mathbb{E}_N \left[F \int_0^T u(t) d(N_t - \lambda t) \right].
\end{aligned}$$

Next we define $\delta(uG)$, $G \in \mathcal{S}_N$, by (3.6), i.e.

$$\delta(uG) := G \int_0^T u(t) d(N_t - \lambda dt) - \langle u, DG \rangle,$$

with for all $F \in \mathcal{S}_N$:

$$\begin{aligned}
\mathbb{E}_N[G \langle DF, u \rangle] &= \mathbb{E}_N[\langle D(FG), u \rangle - F \langle DG, u \rangle] \\
&= \mathbb{E}_N \left[F \left(G \int_0^T u(t) dN(t) - \langle DG, u \rangle \right) \right] \\
&= \mathbb{E}_N[F \delta(uG)],
\end{aligned}$$

which proves (3.5). The closability of D then follows from the integration by parts formula (3.5): if $(F_n)_{n \in \mathbb{N}} \subset \mathcal{S}_N$ is such that $F_n \rightarrow 0$ in $L^2(\Omega)$ and $DF_n \rightarrow U$ in $L^2(\Omega)$, then (3.5) implies

$$\begin{aligned}
|\mathbb{E}_N[\langle U, Gu \rangle_{L^2([0, T])}]]| &\leq |\mathbb{E}_N[F_n \delta(uG)] - \mathbb{E}_N[UG]| + |\mathbb{E}_N[F_n \delta(uG)]| \\
&= |\mathbb{E}_N[\langle DF_n, u \rangle - U]G]| + |\mathbb{E}_N[F_n \delta(uG)]| \\
&\leq \|\langle DF_n, u \rangle - U\|_{L^2(\{N_T \geq N\})} \|G\|_{L^2(\{N_T \geq N\})} \\
&\quad + \|F_n\|_{L^2(\{N_T \geq N\})} \|\delta(uG)\|_{L^2(\{N_T \geq N\})}, \quad n \in \mathbb{N},
\end{aligned}$$

hence $\mathbb{E}_N[UG] = 0$, $G \in \mathcal{S}_N$, i.e. $U = 0$ in $L^2(\{N_T \geq N\})$, which implies $U = 0$ in $L^2(\Omega)$ by construction of \mathcal{S}_N .

As a consequence of (3.5) the operator D can be extended to its closed domain $\text{Dom}_N(D)$ of functionals $F \in L^2(\{N_T \geq N\})$ for which there exists a sequence $(F_n)_{n \in \mathbb{N}} \subset \mathcal{S}_N$ converging to F such that $(DF_n)_{n \in \mathbb{N}}$ converges in $L^2(\Omega \times \mathbb{R}_+)$, by letting

$$DF = \lim_{n \rightarrow \infty} DF_n,$$

for all such $F \in \text{Dom}_N(D)$, and DF is well-defined due to the closability of D . The argument is similar for δ . \square

Proof of Proposition 3. See also Proposition 2 of Privault [21] for unconditional versions of this result. For simplicity of notation, let

$$D_u F = \langle DF, u \rangle, \quad F \in \text{Dom}_N(D), \quad u \in L^2(\Omega \times [0, T]).$$

and

$$\nabla_u v(t) = \int_0^T \nabla_s v(t) ds, \quad u \in \mathcal{C}_c^1((0, T)).$$

For all $u, v \in \mathcal{C}_c^2((0, T))$ we have

$$\begin{aligned} (D_u D_v - D_v D_u) T_n &= -D_u \int_0^{T_n} v_s ds + D_v \int_0^{T_n} u_s ds \\ &= v_{T_n} \int_0^{T_n} u_s ds - u_{T_n} \int_0^{T_n} v_s ds \\ &= \int_0^{T_n} \left(\dot{v}(t) \int_0^t u_s ds - \dot{u}(t) \int_0^t v_s ds \right) dt \\ &= D_{\nabla_u v - \nabla_v u} T_n, \end{aligned}$$

hence

$$(D_u D_v - D_v D_u) F = D_{\nabla_u v - \nabla_v u} F, \quad F \in \mathcal{S}_N. \quad (6.2)$$

On the other hand we have

$$\begin{aligned} D_u \delta(v) &= - \sum_{k=1}^{\infty} \dot{v}(T_k) \int_0^{T_k} u_s ds \\ &= -\delta \left(v \cdot \int_0^\cdot u_s ds \right) - \int_0^\infty \dot{v}(t) \int_0^t u_s ds dt \\ &= \delta(\nabla_u v) + \langle u, v \rangle_{L^2(\mathbb{R}_+)}, \end{aligned}$$

hence the commutation relation

$$D_u \delta(v) = \delta(\nabla_u v) + \langle u, v \rangle_{L^2(\mathbb{R}_+)}, \quad u, v \in \mathcal{C}_c^2((0, T)), \quad (6.3)$$

between D and δ .

Next, note that for $u = \sum_{i=1}^n h_i F_i \in \mathcal{U}$ of the form (3.8) we have $\delta(u) \in \text{Dom}_N(D)$ and

$$\begin{aligned} \mathbb{E}_N [\delta(h_i F_i) \delta(h_j F_j)] &= \mathbb{E}_N [F_i D_{h_i} \delta(h_j F_j)] \\ &= \mathbb{E}_N [F_i D_{h_i} (F_j \delta(h_j) - D_{h_j} F_j)] \\ &= \mathbb{E}_N [(F_i F_j D_{h_i} \delta h_j + F_i \delta(h_j) D_{h_i} F_j - F_i D_{h_i} D_{h_j} F_j)] \\ &= \mathbb{E}_N [(F_i F_j \langle h_i, h_j \rangle_{L^2(\mathbb{R}_+)} + F_i F_j \delta(\nabla_{h_i} h_j) + F_i \delta(h_j) D_{h_i} F_j - F_i D_{h_i} D_{h_j} F_j)] \\ &= \mathbb{E}_N [(F_i F_j \langle h_i, h_j \rangle_{L^2(\mathbb{R}_+)} + D_{\nabla_{h_i} h_j} (F_i F_j) + D_{h_j} (F_i D_{h_i} F_j) - F_i D_{h_i} D_{h_j} F_j)] \end{aligned}$$

$$\begin{aligned}
&= \mathbb{E}_N \left[(F_i F_j \langle h_i, h_j \rangle_{L^2(\mathbb{R}_+)}) + D_{\nabla_{h_i} h_j} (F_i F_j) + D_{h_j} F_i D_{h_i} F_j + F_i (D_{h_j} D_{h_i} F_j - D_{h_i} D_{h_j} F_j) \right] \\
&= \mathbb{E}_N \left[(F_i F_j \langle h_i, h_j \rangle_{L^2(\mathbb{R}_+)}) + D_{\nabla_{h_i} h_j} (F_i F_j) + D_{h_j} F_i D_{h_i} F_j + F_i D_{\nabla_{h_j} h_i - \nabla_{h_i} h_j} F_j \right] \\
&= \mathbb{E}_N \left[(F_i F_j \langle h_i, h_j \rangle_{L^2(\mathbb{R}_+)}) + F_j D_{\nabla_{h_i} h_j} F_i + F_i D_{\nabla_{h_j} h_i} F_j + D_{h_j} F_i D_{h_i} F_j \right] \\
&= \mathbb{E}_N \left[F_i F_j \langle h_i, h_j \rangle_{L^2(\mathbb{R}_+)} + F_j \int_0^T D_s F_i \int_0^T \nabla_t h_j(s) h_i(t) dt ds \right. \\
&\quad \left. + F_i \int_0^T D_t F_j \int_0^T \nabla_s h_i(t) h_j(s) ds dt + \int_0^T h_i(t) D_t F_j \int_0^T h_j(s) D_s F_i ds dt \right],
\end{aligned}$$

where we used the commutation relations (6.2) and (6.3). \square

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