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# Nonparametric estimation for a stochastic volatility model.

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**Abstract** Consider discrete time observations  $(X_{\ell\delta})_{1 \leq \ell \leq n+1}$  of the process  $X$  satisfying  $dX_t = \sqrt{V_t}dB_t$ , with  $V_t$  a one-dimensional positive diffusion process independent of the Brownian motion  $B$ . For both the drift and the diffusion coefficient of the unobserved diffusion  $V$ , we propose nonparametric least square estimators, and provide bounds for their risk. Estimators are chosen among a collection of functions belonging to a finite dimensional space whose dimension is selected by a data driven procedure. Implementation on simulated data illustrates how the method works. December 21, 2007

**Keywords** Diffusion coefficient · Drift · Mean square estimator · Model selection · Nonparametric estimation · Penalized contrast · Stochastic volatility

**Mathematics Subject Classification (2000)** 62G08 · 62M05 · 62P05

## 1 Introduction

In this paper, we consider a bivariate process  $(X_t, V_t)_{t \geq 0}$  with dynamics described by the following equations:

$$\begin{cases} dX_t = \sqrt{V_t}dB_t, & X_0 = 0, \\ dV_t = b(V_t)dt + \sigma(V_t)dW_t & V_0 = \eta, \quad V_t > 0, \text{ for all } t \geq 0, \end{cases} \quad (1)$$

where  $(B_t, W_t)_{t \geq 0}$  is a standard bidimensional Brownian motion and  $\eta$  is independent of  $(B_t, W_t)_{t \geq 0}$ . Our aim is to propose and study nonparametric estimators of  $b(\cdot)$  and  $\sigma^2(\cdot)$  on the basis of discrete time observations of the process  $X$  only.

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Model (1) was introduced by Hull and White (1987) under the name of Stochastic Volatility model. It is often adopted in finance to model stock prices, stock indexes or short term interest rates: see for instance Hull and White (1987), Anderson and Lund (1997), the review of Stochastic Volatility models in Ghysels *et al.* (1996) or the recent book by Shephard (2005) and the references therein. See also an econometric analysis of the subject in Barndorff-Nielsen and Shephard (2002).

The approach to study model (1) is often parametric: the unknown functions are specified up to a few unknown parameters, see the popular examples of Heston (1993) or Cox, Ingersoll and Ross (1985). General statistical parametric approaches of the problem are studied in Genon-Catalot *et al.* (1999), Hoffmann (2002), Gloter (2007), Aït-Sahalia and Kimmel (2007). A nonparametric estimation of the stationary density of  $V_t$  is studied in Comte and Genon-Catalot (2006). A recent proposal for nonparametric estimation of the drift and diffusion coefficients of  $V$  can be found in Renó (2006), who studies the empirical performance of a Nadaraya-Watson kernel strategy on two parametric simulated examples. Our approach is new and different, and it is based on a nonparametric mean square strategy. We consider the same probabilistic and sampling settings as Gloter (2007) and follow the ideas developed in Comte *et al.* (2006, 2007), where direct or integrated discrete observations of the process  $(V_t)$  are considered. Here, our assumptions ensure that  $(V_t)$  is stationary and we consider discrete time observations  $(X_{\ell\delta})_{1 \leq \ell \leq n+1}$  of the process  $(X_t)$  in the so-called high frequency context:  $\delta$  is small,  $n$  is large and  $n\delta = T$ , the time interval where observations are taken, is large.

We assume that  $n = kN$  and define as it is usual, for  $i = 0, 1, \dots, N-1$ , the realized quadratic variation associated with  $(X_{\ell\delta})_{ik+1 \leq \ell < (i+1)k}$ :

$$\hat{V}_i = \frac{1}{k\delta} \sum_{j=0}^{k-1} \left( X_{(ik+j+1)\delta} - X_{(ik+j)\delta} \right)^2.$$

Setting  $\Delta = k\delta$ ,  $\hat{V}_i$  provides an approximation of the integrated volatility:

$$\bar{V}_i = \frac{1}{\Delta} \int_{i\Delta}^{(i+1)\Delta} V_s ds, \quad (2)$$

which in turn may be, for well chosen  $k, \delta$ , a satisfactory approximation of  $V_{i\Delta}$ . We have in mind to obtain regression-type equations, for  $\ell = 1, 2$ :

$$Y_{i+1}^{(\ell)} = f^{(\ell)}(\hat{V}_i) + \text{noise} + \text{remainder},$$

where

$$f^{(1)} = b, Y_i^{(1)} = \frac{\hat{V}_{i+1} - \hat{V}_i}{\Delta} \text{ and } f^{(2)} = \sigma^2, Y_i^{(2)} = \frac{3(\hat{V}_{i+1} - \hat{V}_i)^2}{2\Delta}. \quad (3)$$

Choosing a collection of finite dimensional spaces, we use the regression-type equations to construct estimators on these spaces. Then, we propose a data driven procedure to select a relevant estimation space in the collection. As it is usual with these methods, the risk of an estimator  $\tilde{f}$  of  $f = b$  or  $\sigma^2$  is measured via  $\mathbb{E}(\|f - \tilde{f}\|_N^2)$  where  $\|f - \tilde{f}\|_N^2 = (1/N) \sum_{i=0}^{N-1} (f - \tilde{f})^2(\hat{V}_i)$ . We obtain risk bounds which can be interpreted as  $n, N$  tend to infinity,  $\delta, \Delta$  tend to 0 and  $T = n\delta = N\Delta$  tends to infinity. These bounds are compared with Hoffmann's (1999) minimax rates in the case of direct observations of

$V$ . For what concerns  $b$ , our method leads to the best rate that can be expected. For what concerns  $\sigma^2$ , no benchmark is available in this asymptotic framework. Indeed, Gloter (2000) and Hoffmann (2002) only treat the case of observations within a fixed length time interval, in a parametric setting. As it is always the case, the rates are different for the two functions.

The paper is organized as follows. Section 2 describes the assumptions on the model and the collection of estimation spaces. In Section 3, the estimators are defined and their risks are studied. Section 4 completes the procedure by the data driven selection of the estimation space. Examples of models and simulation results are presented in Section 5. Lastly, proofs are gathered in Section 6.

## 2 The assumptions

### 2.1 Model assumptions.

Let  $(X_t, V_t)_{t \geq 0}$  be given by (1) and assume that only discrete time observations of  $X$ ,  $(X_{\ell\delta})_{1 \leq \ell \leq n+1}$  are available. We want to estimate the drift function  $b$  and the square of the diffusion coefficient  $\sigma^2$  when  $V$  is stationary and exponentially  $\beta$ -mixing. We assume that the state space of  $(V_t)$  is a known open interval  $(r_0, r_1)$  of  $\mathbb{R}^+$  and consider the following set of assumptions.

- [A1 ]  $0 \leq r_0 < r_1 \leq +\infty$ ,  $\overset{\circ}{I} = (r_0, r_1)$ , with  $\sigma(v) > 0$ , for all  $v \in \overset{\circ}{I}$ . Let  $I = [r_0, r_1] \cap \mathbb{R}$ . The function  $b$  belongs to  $C^1(I)$ ,  $b'$  is bounded on  $I$ ,  $\sigma^2 \in C^2(I)$ ,  $(\sigma^2)'\sigma$  is Lipschitz on  $I$ ,  $(\sigma^2)''$  is bounded on  $I$  and  $\sigma^2(v) \leq \sigma_1^2$  for all  $v$  in  $I$ .
- [A2 ] For all  $v_0, v \in \overset{\circ}{I}$ , the scale density  $s(v) = \exp \left[ -2 \int_{v_0}^v b(u)/\sigma^2(u) du \right]$  satisfies  $\int_{r_0} s(x) dx = +\infty = \int^{r_1} s(x) dx$ , and the speed density  $m(v) = 1/(\sigma^2(v)s(v))$  satisfies  $\int_{r_0}^{r_1} m(v) dv = M < +\infty$ .
- [A3 ]  $\eta \sim \pi$  and  $\forall i, \mathbb{E}(\eta^{2i}) < \infty$ , where  $\pi(v) dv = (m(v)/M) \mathbf{1}_{(r_0, r_1)}(v) dv$ .
- [A4 ] The process  $(V_t)$  is exponentially  $\beta$ -mixing, *i.e.*, there exist constants  $K > 0, \theta > 0$ , such that, for all  $t \geq 0$ ,  $\beta_V(t) \leq K e^{-\theta t}$ .

Under [A1]-[A3],  $(V_t)$  is strictly stationary with marginal distribution  $\pi$ , ergodic and  $\beta$ -mixing, *i.e.*  $\lim_{t \rightarrow +\infty} \beta_V(t) = 0$ . Here,  $\beta_V(t)$  denotes the  $\beta$ -mixing coefficient of  $(V_t)$  and is given by

$$\beta_V(t) = \int_{r_0}^{r_1} \pi(v) dv \|P_t(v, dv') - \pi(v') dv'\|_{TV}.$$

The norm  $\|\cdot\|_{TV}$  is the total variation norm and  $P_t$  denotes the transition probability of  $(V_t)$  (see Genon-Catalot *et al.* (2000)). To prove our main result, we need the stronger mixing condition [A4], which is satisfied in most standard examples. Under [A1]-[A4], for fixed  $\Delta$ ,  $(\bar{V}_i)_{i \geq 0}$  is a strictly stationary process. And we have:

**Proposition 2.1** *Under [A1]-[A4], for fixed  $k$  and  $\delta$ ,  $(\hat{V}_i)_{i \geq 0}$  is strictly stationary and  $\beta_{\hat{V}}(i) \leq c\beta_V(i\Delta)$  for all  $i \geq 1$ .*

## 2.2 Spaces of approximation

The functions  $b$  and  $\sigma^2$  are estimated only on a compact subset  $A$  of the state space  $\overset{\circ}{I}$ . For simplicity and without loss of generality, we assume from now on that

$$A = [0, 1], \text{ and we set } b_A = b1_A, \quad \sigma_A = \sigma 1_A. \quad (4)$$

To estimate  $f = b, \sigma^2$ , we consider a family  $S_m, m \in \mathcal{M}_n$  of finite dimensional subspaces of  $\mathbb{L}_2([0, 1])$  and compute a collection of estimators  $\hat{f}_m$  where for all  $m$ ,  $\hat{f}_m$  belongs to  $S_m$ . Afterwards, a data driven procedure chooses among the collection of estimators the final estimator  $\hat{f}_{\hat{m}}$ .

We consider here simple projection spaces, namely trigonometric spaces,  $S_m, m \in \mathcal{M}_n$ . The space  $S_m$  is linearly spanned in  $\mathbb{L}_2([0, 1])$  by  $\varphi_1, \dots, \varphi_{2m+1}$  with  $\varphi_1(x) = 1_{[0,1]}(x)$ ,  $\varphi_j(x) = \sqrt{2} \cos(2\pi jx) 1_{[0,1]}(x)$  for even  $j$ 's and  $\varphi_j(x) = \sqrt{2} \sin(2\pi jx) 1_{[0,1]}(x)$  for odd  $j$ 's larger than 1. We have  $D_m = 2m + 1 = \dim(S_m) \leq \mathcal{D}_n$  and  $\mathcal{M}_n = \{1, 3, \dots, \mathcal{D}_n\}$ . The largest space in the collection has maximal dimension  $\mathcal{D}_n$ , which is subject to constraints appearing later.

Actually, the theory requires smooth bases and regular wavelet bases would also be adequate.

In connection with the collection of spaces  $S_m$ , we need an additional assumption on the marginal density of the stationary process  $(\hat{V}_i)_{i \geq 0}$ :

[A5 ] The process  $(\hat{V}_i)_{i \geq 0}$  admits a stationary density  $\pi^*$  and there exist two positive constants  $\pi_0^*$  and  $\pi_1^*$  (independent of  $n, \delta$ ) such that  $\forall m \in \mathcal{M}_n, \forall t \in S_m$ ,

$$\pi_0^* \|t\|^2 \leq \mathbb{E}(t^2(\hat{V}_0)) \leq \pi_1^* \|t\|^2. \quad (5)$$

The existence of the density  $\pi^*$  is easy to obtain. The checking of (5) is more technical. See the discussion on [A5] in Section 6.2. Below, we use the notations:

$$\|t\|_{\pi^*}^2 = \int t^2(x) \pi^*(x) dx, \quad \|t\|^2 = \int_0^1 t^2(x) dx \quad \text{and} \quad \|t\|_\infty = \sup_{x \in [0,1]} |t(x)|. \quad (6)$$

## 3 Mean squares estimators of the drift and volatility

### 3.1 Regression equations

Reminding of (3), we first prove the developments, for  $\ell = 1, 2$ :

$$Y_{i+1}^{(\ell)} = f^{(\ell)}(\hat{V}_i) + Z_{i+1}^{(\ell)} + R^{(\ell)}(i+1), \quad (7)$$

where the  $Z_i^{(\ell)}$ 's are noise terms (with martingale properties) and the  $R^{(\ell)}(i)$ 's are negligible residual terms given in Section 6. For the noise terms, we have, for  $\ell = 1$  ( $f^{(1)} = b$ ):

$$Z_i^{(1)} = \frac{1}{\Delta^2} \int_{i\Delta}^{(i+2)\Delta} \psi_{i\Delta}(u) \sigma(V_u) dW_u + (u_{i+1,k} - u_{i,k}) / \Delta,$$

with

$$\psi_{i\Delta}(u) = (u - i\Delta) \mathbf{1}_{[i\Delta, (i+1)\Delta]}(u) + [(i+2)\Delta - u] \mathbf{1}_{[(i+1)\Delta, (i+2)\Delta]}(u). \quad (8)$$

and

$$u_{i,k} = \frac{1}{\Delta} \sum_{j=0}^{k-1} \left[ \left( \int_{(ik+j)\Delta}^{(ik+j+1)\Delta} \sqrt{V_s} dB_s \right)^2 - \int_{(ik+j)\Delta}^{(ik+j+1)\Delta} V_s ds \right].$$

Note that  $\hat{V}_i = \bar{V}_i + u_{i,k}$ .

On the other hand, for  $\ell = 2$  ( $f^{(2)} = \sigma^2$ ), we have  $Z_i^{(2)} = Z_i^{(2,1)} + Z_i^{(2,2)} + Z_i^{(2,3)}$  with

$$Z_i^{(2,1)} = \frac{3}{2\Delta^3} \left[ \left( \int_{i\Delta}^{(i+2)\Delta} \psi_{i\Delta}(s) \sigma(V_s) dW_s \right)^2 - \int_{i\Delta}^{(i+2)\Delta} \psi_{i\Delta}^2(s) \sigma^2(V_s) ds \right],$$

$$\begin{aligned} Z_i^{(2,2)} &= \frac{3}{\Delta} b(V_{i\Delta}) \int_{i\Delta}^{(i+2)\Delta} \psi_{i\Delta}(s) \sigma(V_s) dW_s \\ &\quad + \frac{3}{\Delta^3} \int_{i\Delta}^{(i+2)\Delta} \left( \int_s^{(i+2)\Delta} \psi_{i\Delta}^2(u) du \right) [(\sigma^2)' \sigma](V_s) dW_s, \end{aligned}$$

where  $\psi_{i\Delta}$  is given in (8), and

$$Z_i^{(2,3)} = \frac{3}{\Delta} (\bar{V}_{i+1} - \bar{V}_i) (u_{i+1,k} - u_{i,k}).$$

### 3.2 Mean squares contrast

Equation (7) gives a natural regression equation to estimate  $f^{(\ell)}$ . In light of this, we consider the following contrast, for a function  $t \in S_m$  where  $S_m$  is a space of the collection and for  $\ell = 1, 2$ :

$$\gamma_N^{(\ell)}(t) = \frac{1}{N} \sum_{i=0}^{N-1} [Y_{i+1}^{(\ell)} - t(\hat{V}_i)]^2. \quad (9)$$

Then the estimators are defined as

$$\hat{f}_m^{(\ell)} = \arg \min_{t \in S_m} \gamma_N^{(\ell)}(t). \quad (10)$$

The minimization of  $\gamma_N^{(\ell)}$  over  $S_m$  usually leads to several solutions. In contrast, the random  $\mathbb{R}^N$ -vector  $(\hat{f}_m^{(\ell)}(\hat{V}_0), \dots, \hat{f}_m^{(\ell)}(\hat{V}_{N-1}))'$  is always uniquely defined. Indeed, let us denote by  $\Pi_m$  the orthogonal projection (with respect to the inner product of  $\mathbb{R}^N$ ) onto the subspace of  $\mathbb{R}^N$ ,  $\{(t(\hat{V}_0), \dots, t(\hat{V}_{N-1}))', t \in S_m\}$ , then  $(\hat{f}_m^{(\ell)}(\hat{V}_0), \dots, \hat{f}_m^{(\ell)}(\hat{V}_{N-1}))' = \Pi_m Y^{(\ell)}$  where  $Y^{(\ell)} = (Y_1^{(\ell)}, \dots, Y_N^{(\ell)})'$ . This is the reason why we consider a properly defined risk for  $\hat{f}_m^{(\ell)}$  based on the design points, i.e.

$$\mathbb{E} \left[ \frac{1}{N} \sum_{i=0}^{N-1} (\hat{f}_m^{(\ell)}(\hat{V}_i) - f(\hat{V}_i))^2 \right].$$

Thus, the error is measured via the risk  $\mathbb{E}(\|\hat{f}_m^{(\ell)} - f^{(\ell)}\|_N^2)$  where

$$\|t\|_N^2 = \frac{1}{N} \sum_{i=0}^{N-1} t^2(\hat{V}_i).$$

Let us mention that for a deterministic function  $\mathbb{E}(\|t\|_N^2) = \|t\|_{\pi^*}^2 = \int t^2(x) \pi^*(x) dx$ . Moreover, under Assumption [A5], the norms  $\|\cdot\|$  and  $\|\cdot\|_{\pi^*}$  are equivalent for functions in  $S_m$  (see notations (6)).

The following decomposition of the contrast holds:

$$\gamma_N^{(\ell)}(t) - \gamma_N^{(\ell)}(f^{(\ell)}) = \|t - f^{(\ell)}\|_N^2 - \frac{2}{N} \sum_{i=0}^{N-1} (Y_{i+1}^{(\ell)} - f^{(\ell)}(\hat{V}_i))(f^{(\ell)} - t)(\hat{V}_i)$$

In view of (7), we define the centered empirical processes, for  $\ell = 1, 2$ :

$$\nu_N^{(\ell)}(t) = \frac{1}{N} \sum_{i=0}^{N-1} t(\hat{V}_i^{(\ell)}) Z_{i+1}^{(\ell)},$$

and the residual process:

$$R_N^{(\ell)}(t) = \frac{1}{N} \sum_{i=0}^{N-1} t(\hat{V}_i) R^{(\ell)}(i+1).$$

Then we obtain that

$$\gamma_N^{(\ell)}(t) - \gamma_N^{(\ell)}(f^{(\ell)}) = \|t - f^{(\ell)}\|_N^2 - 2\nu_N^{(\ell)}(t - f^{(\ell)}) - 2R_N^{(\ell)}(t - f^{(\ell)}).$$

Let  $f_m^{(\ell)}$  be the orthogonal projection of  $f^{(\ell)}$  on  $S_m$ . Write simply that  $\gamma_N^{(\ell)}(\hat{f}_m^{(\ell)}) \leq \gamma_N^{(\ell)}(f_m^{(\ell)})$  by definition of the estimator, and therefore that  $\gamma_N^{(\ell)}(\hat{f}_m^{(\ell)}) - \gamma_N^{(\ell)}(f^{(\ell)}) \leq \gamma_N^{(\ell)}(f_m^{(\ell)}) - \gamma_N^{(\ell)}(f^{(\ell)})$ . This yields

$$\|\hat{f}_m^{(\ell)} - f^{(\ell)}\|_N^2 \leq \|f_m^{(\ell)} - f^{(\ell)}\|_N^2 + 2\nu_N^{(\ell)}(\hat{f}_m^{(\ell)} - f_m^{(\ell)}) + 2R_N^{(\ell)}(\hat{f}_m^{(\ell)} - f_m^{(\ell)}).$$

The functions  $\hat{f}_m^{(\ell)}$  and  $f_m^{(\ell)}$  being  $A$ -supported, we can cancel the terms  $\|f \mathbf{1}_{A^c}\|_N^2$  that appears in both sides of the inequality. Therefore, we get

$$\|\hat{f}_m^{(\ell)} - f_A^{(\ell)}\|_N^2 \leq \|f_m^{(\ell)} - f_A^{(\ell)}\|_N^2 + 2\nu_N^{(\ell)}(\hat{f}_m^{(\ell)} - f_m^{(\ell)}) + 2R_N^{(\ell)}(\hat{f}_m^{(\ell)} - f_m^{(\ell)}). \quad (11)$$

Taking expectations and finding upper bounds for

$$\mathbb{E}\left(\sup_{t \in S_m, \|t\|=1} [\nu_N^{(\ell)}(t)]^2\right) \quad \text{and} \quad \mathbb{E}\left(\sup_{t \in S_m, \|t\|=1} [R_N^{(\ell)}(t)]^2\right)$$

will give the rates for the risks of the estimators.

### 3.3 Risk for the collection of drift estimators

For the estimation of  $b$ , we obtain the following result.

**Proposition 3.1** *Assume that  $N\Delta \geq 1$  and  $1/k \leq \Delta$ . Assume that [A1]-[A5] hold and consider a model  $S_m$  in the collection of models with  $\mathcal{D}_n \leq O(\sqrt{N\Delta}/\ln(N))$  where  $\mathcal{D}_n$  is the maximal dimension (see Section 2.2). Then the estimator  $\hat{f}_m^{(1)} = \hat{b}_m$  of  $f^{(1)} = b$  is such that*

$$\mathbb{E}(\|\hat{b}_m - b_A\|_n^2) \leq 7\|b_m - b_A\|_{\pi^*}^2 + K \frac{\mathbb{E}(\sigma^2(V_0))D_m}{N\Delta} + K' \Delta, \quad (12)$$

where  $b_A = b \mathbf{1}_{[0,1]}$  and  $K, K'$  and  $K''$  are some positive constants.

Note that the condition on  $\mathcal{D}_n$  implies that  $\sqrt{N\Delta}/\ln(N)$  must be large enough.

It follows from (12) that it is natural to select the dimension  $D_m$  that leads to the best compromise between the squared bias term  $\|b_m - b_A\|_{\pi^*}^2$  (which decreases when  $D_m$  increases) and the variance term of order  $D_m/(N\Delta)$ .

Now, let us consider the classical high frequency data setting: let  $\Delta = \Delta_n$ ,  $k = k_n$  and  $N = N_n$  be, in addition, such that  $\Delta_n \rightarrow 0$ ,  $N = N_n \rightarrow +\infty$ ,  $N_n\Delta_n/\ln^2(N_n) \rightarrow +\infty$  when  $n \rightarrow +\infty$  and that  $1/(k_n\Delta_n) \leq 1$ . Assume for instance that  $b_A$  belongs to a ball of some Besov space,  $b_A \in \mathcal{B}_{\alpha,2,\infty}([0,1])$ ,  $\alpha \geq 1$ , and that  $\|b_m - b_A\|_{\pi^*}^2 \leq \pi_1^* \|b_m - b_A\|^2$ , then  $\|b_A - b_m\|_{\pi^*}^2 \leq C(\alpha, L, \pi_1^*) D_m^{-2\alpha}$ , for  $\|b_A\|_{\alpha,2,\infty} \leq L$  (see Lemma 12 in Barron *et al.* (1999)). Therefore, if we choose  $D_m = (N_n\Delta_n)^{1/(2\alpha+1)}$ , we obtain

$$\mathbb{E}(\|\hat{b}_m - b_A\|_n^2) \leq C(\alpha, L)(N_n\Delta_n)^{-2\alpha/(2\alpha+1)} + K' \Delta_n. \quad (13)$$

The first term  $(N_n\Delta_n)^{-2\alpha/(2\alpha+1)} = T_n^{-2\alpha/(2\alpha+1)}$  is the optimal nonparametric rate proved by Hoffmann (1999) for direct observation of  $V$ .

Now, let us find conditions under which the last term is negligible. For instance, under the standard condition  $\Delta_n = O(1/(N_n\Delta_n))$ , the term  $\Delta_n$  is negligible with respect to  $(N_n\Delta_n)^{-2\alpha/(2\alpha+1)}$ .

Now, consider the choices  $k_n = 1/\Delta_n$  and  $\delta_n = n^{-c}$ . Let us see if there are possible choices of  $c$  for which all our constraints are fulfilled. To have  $n\delta_n \rightarrow +\infty$  requires  $0 < c < 1$ . As  $\Delta_n = k_n\delta_n = \delta_n/\Delta_n$ , we have  $\Delta_n = \sqrt{\delta_n} = n^{-c/2}$  and  $N_n = n/k_n = n^{1-c/2}$ . Thus,  $\Delta_n \rightarrow 0$  and  $N_n, N_n\Delta_n \rightarrow +\infty$ . Finally, the last constraint to fulfill is that  $N_n\Delta_n^2 = n^{1-3c/2} = O(1)$ . Thus for  $2/3 \leq c < 1$ , the dominating term in (13) is  $(N_n\Delta_n)^{-2\alpha/(2\alpha+1)}$ , *i.e.* the minimax optimal rate. We have obtained a possible “bandwidth” of steps  $\delta_n$ .

### 3.4 Risk for the collection of volatility estimators

For the collection of volatility estimators, we have the result

**Proposition 3.2** *Assume that [A1]-[A5] hold and consider a model  $S_m$  in the collection of models with maximal dimension  $\mathcal{D}_n \leq O(\sqrt{N\Delta}/\ln(N))$ . Assume also that  $1/k \leq \Delta$  and  $N\Delta \geq 1$ ,  $\Delta \leq 1$ . Then the estimator  $\hat{f}_m^{(2)} = \hat{\sigma}_m^2$  of  $f^{(2)} = \sigma^2$  is such that*

$$\mathbb{E}(\|\hat{\sigma}_m^2 - \sigma_A^2\|_N^2) \leq 7\|\sigma_m^2 - \sigma_A^2\|_{\pi^*}^2 + K \frac{\mathbb{E}(\sigma^4(V_0))D_m}{N} + K' \text{Res}(D_m, k, \Delta), \quad (14)$$

where the residual term is given by

$$\text{Res}(D_m, k, \Delta) = D_m^2 \Delta^2 + D_m^5 \Delta^3 + \frac{D_m^3}{k^2} + \frac{1}{k^2 \Delta^2}, \quad (15)$$

where  $\sigma_A^2 = \sigma^2 \mathbf{1}_{[0,1]}$ , and  $K, K'$  are some positive constants.

The discussion on rates is much more tedious. Consider the asymptotic setting described for  $b$ . Assume that  $\sigma_A^2$  belongs to a ball of some Besov space,  $\sigma_A^2 \in \mathcal{B}_{\alpha,2,\infty}([0,1])$ , and that  $\|\sigma_m^2 - \sigma_A^2\|_{\pi^*}^2 \leq \pi_1^* \|\sigma_m^2 - \sigma_A^2\|^2$ , then  $\|\sigma_A^2 - \sigma_m^2\|_{\pi^*}^2 \leq C(\alpha, L, \pi_1^*) D_m^{-2\alpha}$ , for  $\|\sigma_A^2\|_{\alpha,2,\infty} \leq L$ . Therefore, if we choose  $D_m = N_n^{1/(2\alpha+1)}$ , and  $k_n \leq 1/\Delta_n$ , we obtain

$$\mathbb{E}(\|\hat{\sigma}_m^2 - \sigma_A^2\|_N^2) \leq C(\alpha, L, \pi_1^*) N_n^{-2\alpha/(2\alpha+1)} + K' \text{Res}(N_n^{1/(2\alpha+1)}, k_n, \Delta_n). \quad (16)$$



The first term  $N_n^{-2\alpha/(2\alpha+1)}$  is the optimal nonparametric rate proved by Hoffmann (1999) when  $N_n$  discrete time observations of  $V$  are available.

For the second term, let us set  $k_n = n^a$ ,  $\Delta_n = n^{-b}$ ,  $\delta_n = n^{-c}$ , and recall that  $n\delta_n = N_n\Delta_n$  and  $n/N_n = k_n$ , so that  $N_n = n^{1-a}$  and  $a+b=c$ . We look for  $a, b$  such that

$$Res(N_n^{1/(2\alpha+1)}, k_n, \Delta_n) \leq N_n^{-2\alpha/(2\alpha+1)}.$$

For this, we take  $1/(k_n^2\Delta_n^2) = N_n^{-2\alpha/(2\alpha+1)}$  which implies  $2(a-b)/(1-a) = 2\alpha/(2\alpha+1)$ . We get

$$a = \frac{(2\alpha+1)c+\alpha}{5\alpha+2}, \quad b = \frac{(3\alpha+1)c-\alpha}{5\alpha+2}.$$

Then we impose  $N_n^{2/(2\alpha+1)}\Delta_n^2 \leq N_n^{-2\alpha/(2\alpha+1)}$  which is equivalent to

$$2b \geq [(2\alpha+2)/(2\alpha+1)](1-a) \Rightarrow c \geq (3\alpha+2)[2(2\alpha+1)].$$

Next  $N_n^{5/(2\alpha+1)}\Delta_n^3 \leq N_n^{-2\alpha/(2\alpha+1)}$  leads to

$$3b \geq [(2\alpha+5)/(2\alpha+1)](1-a) \Rightarrow c \geq (7\alpha+5)/(11\alpha+8).$$

Lastly  $N_n^{3/(2\alpha+1)}/k_n^2 \leq N_n^{-2\alpha/(2\alpha+1)}$  holds for  $-2a \leq -[(3+2\alpha)/(2\alpha+1)](1-a)$ , *i.e.*  $c \geq 2(\alpha+3)/(6\alpha+5)$ .

The optimal dimension has also to fulfill  $N_n^{1/(2\alpha+1)} \leq \mathcal{D}_n \leq \sqrt{N_n\Delta_n}$  *i.e.*  $-(2\alpha-1)/[2(2\alpha+1)](1-a) \leq -b/2$  which implies  $c \leq (5\alpha-2)/(5\alpha)$ . Finally, we must have

$$c \in \left[ \frac{3\alpha+2}{2(2\alpha+1)}, \frac{5\alpha-2}{5\alpha} \right] \xrightarrow{\alpha \rightarrow +\infty} \left[ \frac{3}{4}, 1 \right].$$

This interval is nonempty as soon as  $\alpha > 2$ .

In terms of the initial number  $n$  of observations, the rate is now  $(n^{1-a})^{-2\alpha/(2\alpha+1)}$  where  $1-a$  is at most  $1/2$ , when  $\alpha \rightarrow +\infty$ . This is consistent with Gloter's (2000) result: in the parametric case, he obtains  $n^{-1/2}$  instead of  $n^{-1}$  for the quadratic risk.

#### 4 Data driven estimator of the coefficients

The second step is to ensure an automatic selection of  $D_m$ , which does not use any knowledge on  $f^{(\ell)}$ , and in particular which does not require to know the regularity  $\alpha$ . This selection is standardly done by setting

$$\hat{m}^{(\ell)} = \arg \min_{m \in \mathcal{M}_n} \left[ \gamma_n^{(\ell)}(\hat{f}_m^{(\ell)}) + \text{pen}^{(\ell)}(m) \right], \quad (17)$$

with  $\text{pen}^{(\ell)}(m)$  a penalty to be properly chosen. We denote by  $\tilde{f}^{(\ell)} = \hat{f}_{\hat{m}^{(\ell)}}^{(\ell)}$  the resulting estimator and we need to determine  $\text{pen}$  such that, ideally,

$$\mathbb{E}(\|\tilde{f}^{(\ell)} - f_A^{(\ell)}\|_n^2) \leq C \inf_{m \in \mathcal{M}_n} \left( \|f_A^{(\ell)} - f_m^{(\ell)}\|^2 + \frac{\mathbb{E}(\sigma^{2\ell}(V_0))D_m}{N\Delta^{2-\ell}} \right) + \text{negligible terms},$$

with  $C$  a constant which should not be too large.

#### 4.1 Result for the data driven estimator of $b$

We almost reach this aim for the estimation of  $b$ .

**Theorem 4.1** *Assume that [A1]-[A5] hold,  $1/k \leq \Delta$ ,  $\Delta \leq 1$  and  $N\Delta \geq 1$ . Consider the collection of models with maximal dimension  $\mathcal{D}_n \leq O(\sqrt{N\Delta}/\ln(N))$ . Then the estimator  $\tilde{b} = \hat{f}_{\hat{m}^{(1)}}^{(1)}$  of  $b$  where  $\hat{m}^{(1)}$  is defined by (17) with*

$$\text{pen}^{(1)}(m) \geq \kappa \sigma_1^2 \frac{D_m}{N\Delta}, \quad (18)$$

where  $\kappa$  is a universal constant, is such that

$$\begin{aligned} \mathbb{E}(\|\tilde{b} - b_A\|_n^2) \leq C \inf_{m \in \mathcal{M}_n} \left( \|b_m - b_A\|_{\pi^*}^2 + \text{pen}^{(1)}(m) \right) \\ + K \left( \Delta + \frac{1}{N\Delta} + \frac{1}{\ln^2(N)k\Delta} \right). \end{aligned} \quad (19)$$

For comments on the practical calibration of the penalty, see Section 5.2.

It follows from (19) that the adaptive estimator automatically realizes the bias-variance compromise, provided that the last terms can be neglected as discussed above. Here, the bandwidth for the choices of  $\delta_n$  is slightly narrowed because of a stronger constraint. More precisely, we choose  $1/(k_n \Delta_n) = \Delta_n$  (instead of 1 previously), that is  $k_n = \Delta_n^{-2}$ , so that  $\Delta_n = k_n \delta_n = \Delta_n^{-2} \delta_n^{-1}$ . Therefore  $\Delta_n = \delta_n^{1/3}$  and if  $\delta_n = n^{-c}$ , then  $\Delta_n = n^{-c/3}$ . Also,  $N_n = n/k_n = n^{1-2c/3}$ ,  $N_n \Delta_n = n \delta_n = n^{1-c}$ ,  $N_n \Delta_n^2 = n^{1-4c/3}$ . Hence if  $3/4 < c < 1$ , we have altogether:  $N_n$ ,  $N_n \Delta_n / \ln^2(N_n)$  tend to infinity with  $n$ ,  $\Delta_n$ ,  $N_n \Delta_n^2$  tend to zero.

In that case, whenever  $b_A$  belongs to some Besov ball (see (13)), and if  $\|b_m - b_A\|_{\pi^*}^2 \leq \pi_1^* \|b_m - b_A\|^2$ , then  $\tilde{b}$  achieves the optimal corresponding nonparametric rate. Note that, in the parametric framework, Gloter (2007) obtains an efficient estimation of  $b$  in the same asymptotic context.

#### 4.2 Result for the data driven estimator of the volatility

We can prove the following Theorem.

**Theorem 4.2** *Assume that [A1]-[A5] hold,  $1/k \leq \Delta$ ,  $\Delta \leq 1$  and  $N\Delta \geq 1$ . Consider the collection of models with maximal dimension  $\mathcal{D}_n \leq \sqrt{N\Delta}/\ln(N)$ . Then the estimator  $\tilde{\sigma}^2 = \hat{f}_{\hat{m}^{(2)}}^{(2)}$  of  $\sigma^2$  where  $\hat{m}^{(2)}$  is defined by (17) with*

$$\text{pen}^{(2)}(m) \geq \kappa \sigma_1^4 \frac{D_m}{N}, \quad (20)$$

where  $\kappa$  is a universal constant, is such that

$$\mathbb{E}(\|\tilde{\sigma}^2 - \sigma_A^2\|_N^2) \leq C \inf_{m \in \mathcal{M}_n} \left( \|\sigma_m^2 - \sigma_A^2\|_{\pi^*}^2 + \text{pen}^{(2)}(m) \right) + C' \widetilde{Res}(N, k, \Delta), \quad (21)$$

where

$$\widetilde{Res}(N, k, \Delta) = N\Delta^3 + N^{5/2} \Delta^{11/2} + \frac{(N\Delta)^{3/2}}{k^2} + \frac{1}{k^2 \Delta^2}. \quad (22)$$

Now, if  $\sigma_A^2$  belongs to a ball of some Besov space,  $\sigma_A^2 \in \mathcal{B}_{\alpha,2,\infty}([0,1])$ , then automatically,

$$\inf_{m \in \mathcal{M}_n} \left( \|\sigma_m^2 - \sigma_A^2\|_{\pi^*}^2 + \text{pen}^{(2)}(m) \right) = O(N_n^{-2\alpha/(2\alpha+1)})$$

without requiring the knowledge of  $\alpha$ . Therefore,

$$\mathbb{E}(\|\check{\sigma}_m^2 - \sigma_A^2\|_N^2) \leq C(\alpha, L)N_n^{-2\alpha/(2\alpha+1)} + C' \widetilde{\text{Res}}(N_n, k_n, \Delta_n).$$

It remains to study the residual term. Notice that we do not know the optimal min-max rate for estimating  $\sigma^2$ , under our set of assumptions on the models and on the asymptotic framework. However, Gloter (2000) and Hoffmann (2002), with observations within a fixed length time interval, obtain the parametric rate  $n^{-1/2}$  (in variance). Taking this as a benchmark, we try to make the residual less than  $O(n^{-1/2})$ . Let us set  $k_n = n^a$ ,  $\Delta_n = n^{-b}$ , hence  $N_n = n/k_n = n^{1-a}$  and  $N_n \Delta_n = n^{1-(a+b)}$ . This yields that  $1-a-3b, (5-5a-11b)/2, (3-7a-3b)/2, 2(b-a)$  must all be less than or equal to  $-1/2$ , in association with  $a+b < 1$  and  $N_n^{1/(2\alpha+1)} \leq \sqrt{N_n \Delta_n}$ . This set of constraint is not empty (e.g.  $a = 9/16, b = 5/16$  fits).

## 5 Examples and numerical simulation results

In this section, we consider examples of diffusions and implement the estimation algorithm on simulated data for the stochastic volatility model  $X$  given by (1).

### 5.1 Simulated paths

We consider the processes  $V_t^{(i)}$  for  $i = 1, \dots, 4$  specified by the couples of functions  $b_i, \sigma_i^2$ ,  $i = 1, \dots, 4$ :

1.  $b_1(x) = x \left( -\theta \ln(x) + \frac{1}{2} c^2 \right)$ ,  $\sigma_1^2(x) = c^2 x^2$  which corresponds to  $\exp(U_t)$  for  $U_t$  an Ornstein-Uhlenbeck process,  $dU_t = -\theta U_t dt + c dW_t$ . Whatever the chosen step,  $U_t$  is exactly simulated as an autoregressive process of order 1. We took  $\theta = 1$  and  $c = 0.75$ .
2.  $b_2(x) = b_0(x-2)$ ,  $\sigma_2^2(x) = \sigma_0^2(x-2)$ , where  $b_0(x) = -(1-x^2) \left[ c^2 x + \frac{\theta}{2} \ln \left( \frac{1+x}{1-x} \right) \right]$  and  $\sigma_0^2(x) = c(1-x^2)$  are the diffusion coefficients of the process  $\text{th}(U_t)$  ( $\text{th}(x) = (e^x - e^{-x})/(e^x + e^{-x})$ , with the same parameters as for case 1). The process  $V_t^{(2)}$  corresponds to  $\text{th}(U_t) + 2$  which is a positive bounded process.
3.  $b_3(x) = x(b_0(\ln(x)) + \frac{1}{2} \sigma_0^2(\ln(x)))$  and  $\sigma_3^2(x) = x^2 \sigma_0^2(\ln(x))$  which corresponds to the process  $V_t^{(3)} = \exp(\text{th}(U_t))$ .
4.  $b_4(x) = dc^2/4 - \theta x$ ,  $\sigma_4^2(x) = c^2 x$  which corresponds to the Cox-Ingersoll-Ross process. A discrete time sample is obtained in an exact way by taking the Euclidean norm of a  $d$ -dimensional Ornstein-Uhlenbeck process with parameters  $-\theta/2$  and  $c/2$ . We took  $d = 9$ ,  $\theta = 0.75$  and  $c = 1/3$ .

We obtain samples of discrete observations of the processes  $(V_{\ell\delta'}^{(j)})_{1 \leq \ell \leq N'}$  for  $j = 1, \dots, 4$  with  $\delta' = \delta/10$ ,  $N'\delta' = T$ , from which we generate  $(X_{\ell\delta}^{(j)})_{1 \leq \ell \leq n}$ , by using that

$$X_{\ell\delta} - X_{(\ell-1)\delta} = \sqrt{\int_{(\ell-1)\delta}^{\ell\delta} V_s ds} \varepsilon_\ell,$$

		$k = 150$	$k = 200$	$k = 250$	$k = 300$	$k = 500$
$b$	mean (std)	$1,70.10^{-3}$ ( $5,38.10^{-4}$ )	$1,87.10^{-3}$ ( $5,06.10^{-4}$ )	$1,95.10^{-3}$ ( $4,93.10^{-4}$ )	$2,1.10^{-3}$ ( $4,92.10^{-4}$ )	$2,91.10^{-3}$ ( $4,68.10^{-4}$ )
$\sigma^2$	mean (std)	$14,8.10^{-5}$ ( $3,26.10^{-5}$ )	$6,23.10^{-5}$ ( $2,26.10^{-5}$ )	$8,77.10^{-5}$ ( $3,74.10^{-5}$ )	$15,3.10^{-5}$ ( $4,0.10^{-5}$ )	$28,6.10^{-5}$ ( $3,39.10^{-5}$ )

**Table 1** Mean squared errors (with standard deviations in parenthesis) for the estimation of  $b$  and  $\sigma^2$ , 100 paths of the CIR process, different values of  $k$  for the quadratic variation, when using the trigonometric basis.

	Process	$V_t^{(1)}$ [T]	$V_t^{(2)}$ [T]	$V_t^{(3)}$ [T]	$V_t^{(4)}$ [T]	$V_t^{(4)}$ [GP]
$b$	mean (std)	$4,08.10^{-2}$ ( $6,89.10^{-3}$ )	$7,51.10^{-2}$ ( $8,56.10^{-3}$ )	$7,05.10^{-2}$ ( $8,12.10^{-3}$ )	$1,95.10^{-3}$ ( $4,93.10^{-4}$ )	$1,04.10^{-3}$ ( $2,89.10^{-4}$ )
$\sigma^2$	mean (std)	$1,42.10^{-1}$ ( $3,47.10^{-2}$ )	$1,89.10^{-2}$ ( $1,54.10^{-3}$ )	$8,32.10^{-2}$ ( $1,61.10^{-2}$ )	$8,77.10^{-5}$ ( $3,74.10^{-5}$ )	$4,61.10^{-5}$ ( $3,19.10^{-5}$ )

**Table 2** Mean squared errors (with standard deviations in parenthesis) for the estimation of  $b$  and  $\sigma^2$ , 100 paths of the processes  $V_t^{(i)}$ ,  $i = 1, \dots, 4$  when using the trigonometric basis (except the last column, piecewise polynomial basis),  $k = 250$ .

with  $(\varepsilon_\ell)$  i.i.d.  $\mathcal{N}(0,1)$  independent of  $(V_s, s \geq 0)$ . Approximations of the integrated processes are computed by discrete integration (with a trapeze method).

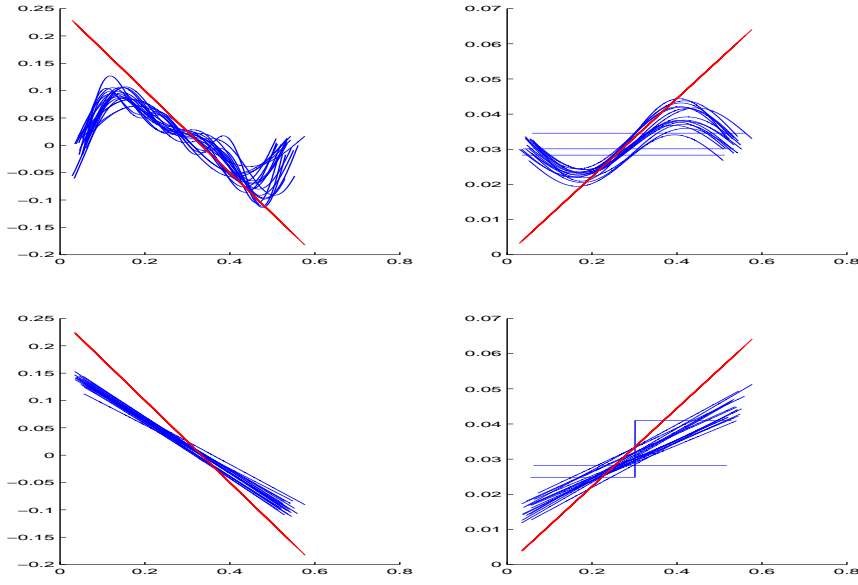
The generated  $V_{j\delta'}^{(i)}$ ,  $i = 1, \dots, 4$  samples have length  $N' = 5.10^6$ , for a step  $\delta' = 1000/5.10^6 = 2.10^{-4}$ , and the integrated process is computed using 10 data, therefore, we obtain  $n = 5.10^5$  and  $\delta = 2.10^{-3}$ , for  $T = n\delta = 1000$ . Different values of  $k$  are used, but the best value,  $k = 250$ , corresponds to  $\Delta = k\delta = 0.5$  and  $N = 2000$  data for the same  $T$ .

## 5.2 Estimation algorithms and numerical results

We use the algorithm of Comte and Rozenholc (2004). The precise calibration of penalties is difficult and done for the trigonometric basis but also for a general piecewise polynomial basis, described in detail in Comte *et al* (2006). Additive correcting terms are involved in the penalty. Such terms avoid under-penalization and are in accordance with the fact that the theorems provide lower bounds for the penalty. The correcting terms are asymptotically negligible and do not affect the rate of convergence. For the trigonometric polynomial collection (denoted by [T]), the drift penalty ( $i = 1$ ) and the diffusion penalty ( $i = 2$ ) are given by

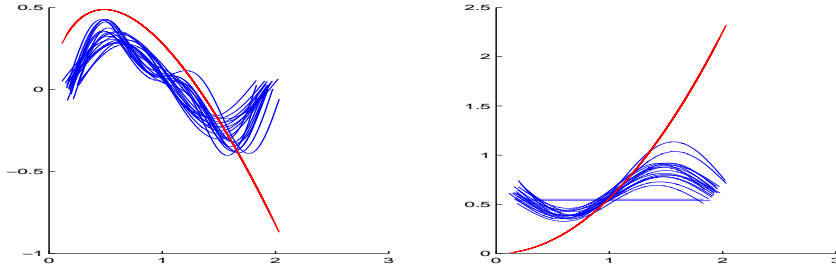
$$2 \frac{\hat{s}_i^2}{n} \left( D_m + \ln^{2.5}(D_m + 1) \right), \text{ with } D_m \text{ at most } [N\Delta / \ln^{1.5}(N)].$$

For the penalty when considering general piecewise polynomial bases (denoted by [GP]), we refer the reader to Comte *et al.* (2006). The constants  $\kappa_1$  and  $\kappa_2$  in both drift and diffusion penalties have been set equal to 2. The term  $\hat{s}_1^2$  replaces  $\sigma_1^2/\Delta$  for the estimation of  $b$  and  $\hat{s}_2^2$  replaces  $\sigma_1^4$  for the estimation of  $\sigma^2$ . Let us first explain



**Fig. 1** Estimation of  $b$  (left) and  $\sigma^2$  (right) for 20 paths of the CIR process with the trigonometric basis (top) and the piecewise polynomial basis (bottom),  $k = 250$ .

how  $\hat{s}_2^2$  is obtained. We run once the estimation algorithm of  $\sigma^2$  with the basis [T] and with a preliminary penalty where  $\hat{s}_2^2$  is taken equal to  $2 \max_m (\gamma_n^{(2)}(\hat{\sigma}_m^2))$ . This gives a preliminary estimator  $\hat{\sigma}_0^2$ . Afterwards, we take  $\hat{s}_2$  equal to twice the 99.5%-quantile of  $\hat{\sigma}_0^2$ . The use of the quantile is here to avoid extreme values. We get  $\hat{\sigma}^2$ . We use this estimate and set  $\hat{s}_1^2 = \max_{0 \leq k \leq N-1} (\hat{\sigma}^2(\hat{V}_k))/\Delta$  for the penalty of  $b$ . The results given



**Fig. 2** Estimation of  $b$  (left) and  $\sigma^2$  (right) for 20 paths of the process  $V_t^{(1)}$  (exponential Ornstein Uhlenbeck) with the trigonometric basis,  $k = 250$ .

by our algorithm are described in Figure 1 and 2. We plot in Figure 1 the true function (thick curve) and 20 estimated functions (thin curves) in the case  $b$  and  $\sigma^2$  when using

first the basis [T] and then the basis [GP], in the case of the CIR process. We can see that the trigonometric basis finds the right slope in the central part of the interval, whereas the basis [GP] in general selects only one bin and a straight curve, but with a slightly too small slope. The same type of result holds in Figure 2 for the exponential Ornstein Uhlenbeck process. For comparison with direct or integrated observations of  $V$ , we refer to Comte *et al.* (2006,2007). It is not surprising that in the case of a stochastic volatility model, empirical results are less satisfactory and require a large number of observations.

We also give in Tables 1 and 2 results of Monte-Carlo type experiments. In Table 1, we show the results of the estimation procedure with the basis [T] and the CIR process when choosing different values of  $k$  for building the quadratic variation. Clearly, there is an optimal value. If  $k$  is too large, there are not enough observations left for the estimation algorithm. If  $k$  is too small, bias phenomena appear, related to the violation of the theoretical assumptions (mainly  $1/k \leq \Delta$ ). We repeated the experiment for the other processes and obtained analogous results. In general, for this sample size, the choice  $k = 250$  seems to be relevant. In Table 2, we can see from the last two columns that the basis [GP] seems to be better than [T], at least for the CIR process. The errors are computed as the mean over 100 simulated paths of the empirical errors (e.g.  $(1/N \sum_{i=0}^{N-1} [b(\hat{V}_i) - \tilde{b}(\hat{V}_i)]^2 \text{ for } b)$ ).

## 6 Discussion on the assumptions and proofs

### 6.1 Proof of Proposition 2.1

We start with some preliminaries. Let  $I_t = \int_0^t V_s ds$ . The joint process  $(V_t, I_t)_{t \geq 0}$  is a two dimensional diffusion satisfying:

$$\begin{cases} dV_t = b(V_t)dt + \sigma(V_t)dW_t, & V_0 = \eta, \\ dI_t = V_t dt, & I_0 = 0 \end{cases}$$

Under regularity assumptions on  $b$  and  $\sigma$ , this process admits a transition density, say  $q_t(v_0, i_0; v, i)$  for the conditional density of  $(V_t, I_t)$  given  $V_0 = v_0, I_0 = i_0$ . This density is w.r.t. the Lebesgue measure on  $(0, +\infty)^2$  (see Rogers and Williams (2000)). We assume that these assumptions hold.

Now, let us set

$$J_{\ell\delta} = \int_{(\ell-1)\delta}^{\ell\delta} V_s ds, \quad \ell \geq 1. \quad (23)$$

The discrete time process  $(V_{\ell\delta}, J_{\ell\delta})_{\ell \geq 1}$  is strictly stationary and Markov. Its one step transition operator is given by the density:

$$(v, j) \rightarrow q_\delta(v_0, 0; v, j) := q_\delta(v_0; v, j).$$

Its stationary density is given by  $\int \pi(v_0) dv_0 q_\delta(v_0; v, j) := \pi_\delta(v, j)$ .

Let us set, for  $\ell \geq 1$ ,

$$Z_\ell = X_{\ell\delta} - X_{(\ell-1)\delta} \quad (24)$$

and define  $\varepsilon_\ell$  by the relation:  $Z_\ell = J_{\ell\delta}^{1/2} \varepsilon_\ell$ . Conditionally on  $(V_t)_{t \geq 0}$ , the random variables (r.v.)  $Z_\ell, \ell \geq 1$  are independent and  $Z_\ell$  has distribution  $\mathcal{N}(0, J_{\ell\delta})$ . Consequently,

the r.v.  $(\varepsilon_\ell, \ell \geq 1)$  are i.i.d. with distribution  $\mathcal{N}(0, 1)$  and the sequence  $(\varepsilon_\ell, \ell \geq 1)$  is independent of  $(V_t)_{t \geq 0}$ . Hence  $(Z_\ell)_{\ell \geq 1}$  and  $(\hat{V}_i)_{i \geq 0}$  are strictly stationary processes. From the preliminaries and the above remarks, we deduce that the process  $(V_{\ell\delta}, J_{\ell\delta}, \varepsilon_\ell)_{\ell \geq 1}$  is stationary Markov. Its  $\ell$ -step transition operator is given by:

$$Q_\ell^\delta(v_0; dv, dj, du) = q_\delta^{(\ell)}(v_0; v, j)n(u)dvdjdu$$

where  $q_\delta^{(\ell)}(v_0; v, j)$  is the  $\ell$ -step transition density of  $(V_{\ell\delta}, J_{\ell\delta})$  and  $n(u)$  is the standard gaussian density. The stationary density of  $(V_{\ell\delta}, J_{\ell\delta}, \varepsilon_\ell)_{\ell \geq 1}$  is  $\pi_\delta(v, j)n(u)$ . Hence

$$\begin{aligned} \|Q_\delta^{(\ell)}(v_0; dv, dj, du) - \pi_\delta(v, j)n(u)dvdjdu\|_{TV} &= \int |q_\delta^{(\ell)}(v_0; v, j) - \pi_\delta(v, j)|n(u)dvdjdu \\ &= \int |q_\delta^{(\ell)}(v_0; v, j) - \pi_\delta(v, j)|dvdj. \end{aligned}$$

We may now use the representation of the  $\beta$ -mixing coefficient of strictly stationary Markov processes (see e.g. Genon-Catalot *et al.* (2000)) to compute

$$\begin{aligned} \beta_{V_{\delta}, J_{\delta}, \varepsilon}(\ell) &= \int \pi_\delta(v_0, j_0)n(u_0)du_0dv_0dj_0 \|Q_\delta^{(\ell)}(v_0; dv, dj, du) - \pi_\delta(v, j)n(u)dvdjdu\|_{TV} \\ &= \beta_{V_{\delta}, J_{\delta}}(\ell). \end{aligned}$$

Now, we have  $\beta_Z(\ell) \leq \beta_{V_{\delta}, J_{\delta}, \varepsilon}(\ell) = \beta_{V_{\delta}, J_{\delta}}(\ell) \leq \beta_V((\ell - 1)\delta)$ . Finally,

$$\beta_{\hat{V}}(i) \leq \beta_Z(ik) \leq \beta_V((ik - 1)\delta) \leq c\beta_V(i\Delta). \quad \square$$

## 6.2 Discussion on the assumptions

Actually, Assumption [A3] is too strong. We only need the existence of moments up to a certain order. Let us now discuss [A5]. Using the representation

$$\hat{V}_0 = \frac{1}{k\delta} \sum_{\ell=1}^k J_{\ell\delta} \varepsilon_\ell^2,$$

we see that  $\hat{V}_0$  has a conditional density given  $(V_t, t \geq 0)$ . Integrating this density w.r.t. the distribution of  $(J_{\ell\delta}, \ell = 1, \dots, k)$ , we get that  $\hat{V}_0$  has a density  $\pi^*$ . However the formula for  $\pi^*$  is untractable.

On the other hand, we can obtain (5) by another approach. We have

$$t^2(\bar{V}_0) = t^2(V_0) + (\bar{V}_0 - V_0)(t^2)'(V_0) + \frac{1}{2}(\bar{V}_0 - V_0)^2 \int_0^1 (t^2)''(V_0 + u(\bar{V}_0 - V_0))du.$$

Now we use that, for any  $t \in S_m$ , there exists some constant  $C$  such that

$$\|(t^2)'\|_\infty \leq CD_m^2 \|t\|^2 \text{ and } \|(t^2)''\|_\infty \leq CD_m^3 \|t\|^2.$$

Noting that  $|\mathbb{E}(\bar{V}_0 - V_0 | \mathcal{F}_0)| = O(\Delta)$ , we get  $|\mathbb{E}[(\bar{V}_0 - V_0)(t^2)'(V_0)]| \leq CD_m^2 \Delta \|t\|^2 = O(D_m^2 \Delta)$ . On the other hand,

$$\begin{aligned} \left| \mathbb{E} \left[ (\bar{V}_0 - V_0)^2 \int_0^1 (t^2)''(V_0 + u(\bar{V}_0 - V_0))du \right] \right| &\leq \|(t^2)''\|_\infty \mathbb{E}[(\bar{V}_0 - V_0)^2] \\ &\leq CD_m^3 \Delta \|t\|^2. \end{aligned}$$

It follows that  $|\mathbb{E}(t^2(\bar{V}_0) - t^2(V_0))| \leq C\Delta D_m^3 \|t\|^2$ . Next,

$$\begin{aligned} t^2(\hat{V}_0) &= t^2(\bar{V}_0) + (\hat{V}_0 - \bar{V}_0)(t^2)'(V_0) + (\hat{V}_0 - \bar{V}_0)[(t^2)'(\bar{V}_0) - (t^2)'(V_0)] \\ &\quad + \frac{1}{2}(\hat{V}_0 - \bar{V}_0)^2 \int_0^1 (t^2)''(\bar{V}_0 + u(\hat{V}_0 - \bar{V}_0))du. \end{aligned}$$

By Gloter's (2007) Proposition 3.1, we have  $|\mathbb{E}[(\hat{V}_0 - \bar{V}_0)|V_0]| \leq c\delta(1 + V_0)^c$  and  $\mathbb{E}[\|\hat{V}_0 - \bar{V}_0\|^2] \leq c/k$ . Hence

$$|\mathbb{E}(t^2(\hat{V}_0) - t^2(\bar{V}_0))| \leq C\|t\|^2(\Delta D_m^2 + \frac{\sqrt{\Delta}D_m^3}{\sqrt{k}} + \frac{D_m^3}{k}).$$

Since  $1/k \leq \Delta$

$$|\mathbb{E}(t^2(\hat{V}_0) - t^2(V_0))| \leq C\|t\|^2 \Delta D_m^3.$$

As there exist two positive constants  $\pi_0, \pi_1$  such that  $\forall v \in A, \pi_0 \leq \pi(v) \leq \pi_1$ , we obtain

$$(\pi_0 - C\Delta D_n^3)\|t\|^2 \leq \|t\|_{\pi^*}^2 \leq (\pi_1 + C\Delta D_n^3)\|t\|^2.$$

Under the constraint that  $\Delta D_n^3 = o(1)$ , we get (5) for  $n$  large enough. This constraint is compatible with the other ones, see the discussion after Theorem 4.1.

### 6.3 Definition of the residuals and their properties

We have

$$R^{(1)}(i+1) = b(\bar{V}_i) - b(\hat{V}_i) + R_*^{(1)}((i+1)\Delta)$$

where  $R_*^{(1)}$  is the residual term for  $b$  studied in Comte *et al.* (2006, Proposition 3.1) and defined by

$$R_*^{(1)}((i+1)\Delta) = b(V_{(i+1)\Delta}) - b(\bar{V}_i) + \frac{1}{\Delta^2} \int_{(i+1)\Delta}^{(i+3)\Delta} \psi_{(i+1)\Delta}(s)(b(V_s) - b(V_{(i+1)\Delta}))ds.$$

On the other hand,

$$R^{(2)}(i+1) = \frac{3}{2} \frac{(u_{i+1,k} - u_{i,k})^2}{\Delta} + [\sigma^2(V_{(i+1)\Delta}) - \sigma^2(\hat{V}_i)] + R_*^{(2)}((i+1)\Delta)$$

where  $R_*^{(2)}$  is the residual term for  $\sigma^2$  studied in Comte *et al.* (2006, Propositions 4.1, 4.2 and 4.3) defined by  $R_*^{(2)} = \sum_{m=1}^3 R_*^{(2,m)}$  with

$$\begin{aligned} R_*^{(2,1)}(i\Delta) &= \frac{3}{2\Delta^3} \left( \int_{i\Delta}^{(i+2)\Delta} \psi_{i\Delta}(s)b(V_s)ds \right)^2 \\ R_*^{(2,2)}(i\Delta) &= \frac{3}{\Delta^3} \left( \int_{i\Delta}^{(i+2)\Delta} \psi_{i\Delta}(u)(b(V_u) - b(V_{i\Delta}))du \right) \left( \int_{i\Delta}^{(i+2)\Delta} \psi_{i\Delta}(u)\sigma(V_u)dW_u \right) \\ R_*^{(2,3)}(i\Delta) &= \frac{3}{2\Delta^3} \int_{i\Delta}^{(i+2)\Delta} \left( \int_s^{(i+2)\Delta} \psi_{i\Delta}^2(u)du \right) \tau_{b,\sigma}(V_s)ds, \end{aligned}$$

where  $\tau_{b,\sigma} = (\sigma^2/2)(\sigma^2)'' + b(\sigma^2)'$ . This decomposition is obtained by applying Ito's formula and Fubini's theorem.

We may now summarize the following useful results, proved in Comte *et al.* (2006, Propositions 3.1, 4.1, 4.2 and 4.3):



**Lemma 6.1** Under Assumptions [A1]-[A2]-[A3],

1. For  $\ell = 1, 2$ , for  $m = 1, 2$ , for all  $i$ ,  $\mathbb{E}\{[R_*^{(\ell)}(i\Delta)]^{2m}\} \leq c\Delta^{2m\ell}$  where  $c$  is a constant.
2. Let  $Z_*^{(1)}(i) = (1/\Delta^2) \int_{i\Delta}^{(i+2)\Delta} \psi_{i\Delta}(s) \sigma(V_s) dW_s$ . For all  $i$ ,  $\mathbb{E}([Z_*^{(1)}(i)]^2) \leq (2/3\Delta) \mathbb{E}(\sigma^2(V_0))$ .
3. For all  $i$ ,  $\mathbb{E}([Z_i^{(2,1)}]^2) \leq c_1 \mathbb{E}(\sigma^4(V_0))$  and  $\mathbb{E}([Z_i^{(2,2)}]^2) \leq c_2 \sigma_1^2 \Delta$ .

We also need the following result:

**Lemma 6.2** Under assumptions [A1]-[A3], for any integer  $i$ ,  $\mathbb{E}[(\bar{V}_i - \hat{V}_i)^2] = \mathbb{E}(u_{i,k}^2) \leq 2\mathbb{E}(V_0^2)/k$  and  $\mathbb{E}[(\bar{V}_i - \hat{V}_i)^4] = \mathbb{E}(u_{i,k}^4) \leq 56\mathbb{E}(V_0^4)/k^2$ .

**Proof of Lemma 6.2.** This follows from Proposition 3.1 p.504 in Gloter (2007).  $\square$

#### 6.4 Proof of Propositions 3.1 and 3.2

For sake of brevity, we give both proofs at the same time. The main difference lies in the orders of the expectations and in the appearance of a specific term in the study of the estimator of  $\sigma^2$ . Let us thus define  $R_{**}^{(\ell)}$  for  $\ell = 1, 2$  as  $R_{**}^{(1)} = R^{(1)}$  and

$$R_{**}^{(2)}(i+1) = R^{(2)}(i+1) - [\sigma^2(V_{(i+1)\Delta}) - \sigma^2(\hat{V}_i)].$$

Moreover let  $T_N^{(1)}(t) = 0$  and

$$T_N^{(2)}(t) = \frac{1}{N} \sum_{i=0}^{N-1} (\sigma^2(V_{(i+1)\Delta}) - \sigma^2(\hat{V}_i)) t(\hat{V}_i).$$

Let us consider the set

$$\Omega_N = \left\{ \omega / \left| \frac{\|t\|_N^2}{\|t\|_{\pi^*}^2} - 1 \right| \leq \frac{1}{2}, \quad \forall t \in \cup_{m, m' \in \mathcal{M}_n} (S_m + S_{m'}) / \{0\} \right\}. \quad (25)$$

On  $\Omega_N$ ,  $\|t\|_{\pi^*} \leq \sqrt{2}\|t\|_N$ . From (11), we deduce

$$\begin{aligned} \|\hat{f}_m^{(\ell)} - f_A^{(\ell)}\|_N^2 &\leq \|f_m^{(\ell)} - f_A^{(\ell)}\|_N^2 + \frac{1}{8} \|\hat{f}_m^{(\ell)} - f_m^{(\ell)}\|_{\pi^*}^2 + 16 \sup_{t \in S_m, \|t\|_{\pi^*}=1} [\nu_N^{(\ell)}]^2(t) \\ &\quad + 16 \sup_{t \in S_m, \|t\|_{\pi^*}=1} [T_N^{(\ell)}(t)]^2 \\ &\quad + \frac{1}{8} \|\hat{f}_m^{(\ell)} - f_m^{(\ell)}\|_N^2 + \frac{8}{N} \sum_{i=0}^{N-1} [R_{**}^{(\ell)}(i+1)]^2 \\ &\leq \|f_m^{(\ell)} - f_A^{(\ell)}\|_N^2 + \frac{3}{8} \|\hat{f}_m^{(\ell)} - f_m^{(\ell)}\|_N^2 + 16 \sup_{t \in S_m, \|t\|_{\pi^*}=1} [\nu_N^{(\ell)}]^2(t) \\ &\quad + \frac{16}{\pi_0^*} \sup_{t \in S_m, \|t\|=1} [T_N^{(\ell)}(t)]^2 + \frac{8}{N} \sum_{i=0}^{N-1} [R_{**}^{(\ell)}(i+1)]^2. \end{aligned}$$

In the last line above, we use the lower bound  $\pi_0^*$  introduced in [A5].

Setting  $B_m(0, 1) = \{t \in S_m, \|t\| = 1\}$  and  $B_m^{\pi^*}(0, 1) = \{t \in S_m, \|t\|_{\pi^*} = 1\}$ , the following holds on the set  $\Omega_N$ :

$$\frac{1}{4} \|\hat{f}_m^{(\ell)} - f_A^{(\ell)}\|_N^2 \leq \frac{7}{4} \|f_m^{(\ell)} - f_A^{(\ell)}\|_N^2 + 16 \sup_{t \in B_m^{\pi^*}(0, 1)} [\nu_N^{(\ell)}]^2(t) + \frac{16}{\pi_0^*} \sup_{t \in B_m(0, 1)} [T_N^{(\ell)}(t)]^2 + \frac{8}{N} \sum_{i=0}^{N-1} [R_{**}^{(\ell)}(i+1)]^2.$$

We have the following result:

**Lemma 6.3** *Under assumptions [A1]-[A3] and [A5], if  $1/k \leq \Delta$ , we have, for  $\ell = 1, 2$*

$$\mathbb{E} \left( \sup_{t \in B_{m^*}^{\pi^*}(0,1)} [\nu_N^{(\ell)}]^2(t) \right) \leq K \frac{C_\ell D_m}{N \Delta^{2-\ell}},$$

with  $C_\ell = \mathbb{E}(\sigma^{2\ell}(V_0))$ .

The Lipschitz condition on  $b$  and Lemma 6.2 imply that

$$\mathbb{E}[(b(\bar{V}_i) - b(\hat{\bar{V}}_i))^2] \leq c_b \mathbb{E}[(\bar{V}_i - \hat{\bar{V}}_i)^2] \leq 2c_b \mathbb{E}(V_0^2)/k.$$

Consequently, there exists a constant  $c$  such that

$$\mathbb{E} \left( \frac{8}{N} \sum_{i=0}^{N-1} [R_{**}^{(1)}(i+1)]^2 \right) \leq c(\Delta + k^{-1}).$$

Thus

$$\mathbb{E}(\|\hat{b}_m - b_A\|_N^2 \mathbf{1}_{\Omega_N}) \leq 7\|b_m - b\|_{\pi^*}^2 + \frac{32}{\pi_0^*} \mathbb{E} \left( \sup_{t \in S_m, \|t\|=1} [\nu_N^{(1)}(t)]^2 \right) + c''(\Delta + k^{-1}).$$

By gathering all bounds, we find

$$\mathbb{E}(\|\hat{b}_m - b\|_N^2 \mathbf{1}_{\Omega_N}) \leq 7\|b_m - b\|_{\pi^*}^2 + K \frac{\mathbb{E}(\sigma^2(V_0)) D_m}{N \Delta} (1 + \frac{1}{k \Delta}) + K'(\Delta + k^{-1}).$$

On the other hand, Lemma 6.1 and Lemma 6.2 imply that

$$\begin{aligned} \mathbb{E} \left( \frac{1}{N} \sum_{i=0}^{N-1} [R_{**}^{(2)}(i+1)]^2 \right) &\leq 2 \mathbb{E} \left[ \frac{1}{N} \sum_{i=0}^{N-1} \left( [R_*^{(2)}(i+1)]^2 + \frac{9}{4} \frac{(u_{i+1,k} - u_{i,k})^4}{\Delta^2} \right) \right] \\ &\leq 2c\Delta^2 + \frac{36}{\Delta^2} \mathbb{E}(u_{1,k}^4) \leq C(\Delta^2 + \frac{1}{k^2 \Delta^2}). \end{aligned}$$

Next we need to bound  $\mathbb{E} \left( \sup_{t \in S_m, \|t\|=1} [T_N^{(2)}(t)]^2 \right)$ . This is obtained in the following Lemma:

**Lemma 6.4** *Under the Assumptions of Proposition 3.2 and if  $1/k \leq \Delta$ , there exists a constant  $C$  such that*

$$\mathbb{E} \left( \sup_{t \in S_m, \|t\|=1} [T_N^{(2)}(t)]^2 \right) \leq C(D_m^2 \Delta^2 + D_m^5 \Delta^3 + D_m^3/k^2 + D_m/(Nk)).$$

We can use Lemma 6.1 in Comte *et al.* (2005) to obtain that, if  $\mathcal{D}_n \leq C\sqrt{N\Delta}/\ln(N)$ , then

$$\mathbb{P}(\Omega_N^c) \leq \frac{c}{N^4}.$$

This enables to check that  $\mathbb{E}(\|\hat{f}_m^{(\ell)} - f^{(\ell)}\|_N^2 \mathbf{1}_{\Omega_N^c}) \leq c/N$  using the same lines as the analogous proof given p.532 in Comte *et al.* (2007). For this reason, details are omitted.  $\square$

## 6.5 Proof of Lemma 6.3.

Case  $\ell = 1$ . Next, let us define  $\mathcal{F}_t = \sigma((W_s, B_s), 0 \leq s \leq t, \eta)$ . We can use martingale properties to see that,  $\forall t \in S_m$ ,

$$\mathbb{E}(t(\hat{V}_i)Z_{i+1}^{(1)}) = \mathbb{E}(\mathbb{E}(t(\hat{V}_i)Z_{i+1}^{(1)}|\mathcal{F}_{(i+1)\Delta})) = \mathbb{E}(t(\hat{V}_i)\mathbb{E}(Z_{i+1}^{(1)}|\mathcal{F}_{(i+1)\Delta})) = 0$$

because the last conditional expectation is zero. Moreover, the same tool shows that the covariance term  $\mathbb{E}(t(\hat{V}_i)t(\hat{V}_\ell)Z_{i+1}^{(1)}Z_{\ell+1}^{(1)})$  for  $\ell \geq i+2$  is also null by inserting a conditional expectation given  $\mathcal{F}_{(\ell+1)\Delta}$ . Consequently, it is now easy to see that

$$\begin{aligned} \mathbb{E}\left(\sup_{t \in S_m, \|t\|=1} [\nu_N^{(1)}(t)]^2\right) &\leq \sum_{j=1}^{D_m} \mathbb{E}[\nu_N^2(\varphi_j)] \leq \sum_{j=1}^{D_m} \text{Var}\left[\frac{1}{N} \sum_{i=0}^{N-1} \varphi_j(\hat{V}_i)Z_{i+1}^{(1)}\right] \\ &\leq \frac{2}{N} \sum_{j=1}^{D_m} \text{Var}\left(\varphi_j(\hat{V}_1)Z_2^{(1)}\right) \\ &\leq \frac{2}{N} \sum_{j=1}^{D_m} \mathbb{E}(\varphi_j^2(\hat{V}_1)Z_2^{(1)2}) \leq \frac{2D_m \mathbb{E}[(Z_2^{(1)})^2]}{N}. \end{aligned}$$

Now, Lemma 6.2 implies that  $\mathbb{E}[(u_{i+2,k} - u_{i+1,k})^2/\Delta^2] = \mathbb{E}[(u_{i+2,k}^2 + u_{i+1,k}^2)/\Delta^2] \leq c/(k\Delta^2)$ . Then, applying also Lemma 6.1 (ii), it follows that, with

$$\mathbb{E}\left(\sup_{t \in S_m, \|t\|=1} [\nu_N^{(1)}(t)]^2\right) \leq K \frac{D_m}{N\Delta} \left(1 + \frac{1}{k\Delta}\right).$$

Case  $\ell = 2$ . Next, for the martingale terms, we write

$$\begin{aligned} \mathbb{E}\left(\sup_{t \in B_m^{\pi_m^*}(0,1)} [\nu_N^{(2)}(t)]^2\right) &\leq \frac{1}{\pi_0^*} \mathbb{E}\left(\sup_{t \in B_m(0,1)} [\nu_N^{(2)}(t)]^2\right) \leq \frac{1}{\pi_0^*} \sum_{j=1}^{D_m} \mathbb{E}([\nu_n^{(2)}(\varphi_j)]^2) \\ &= \frac{1}{\pi_0^*} \sum_{j=1}^{D_m} \mathbb{E}\left(\frac{1}{N} \sum_{i=0}^{N-1} \varphi_j(\hat{V}_i)Z_{i+1}^{(2)}\right)^2 \\ &\leq \frac{2}{\pi_0^*} \sum_{j=1}^{D_m} \mathbb{E}\left[\left(\frac{1}{N} \sum_{i=0}^{N-1} \varphi_j(\hat{V}_i)(Z_{i+1}^{(2,1)} + Z_{i+1}^{(2,2)})\right)^2\right. \\ &\quad \left.+ \left(\frac{9}{N\Delta} \sum_{i=0}^{N-1} \varphi_j(\hat{V}_i)(\bar{V}_{i+2} - \bar{V}_i)(u_{i+2,k} - u_{i+1,k})\right)^2\right] \end{aligned}$$

Both terms are bounded separately. For the first one, we use that, for  $r = 1, 2$

$$\text{cov}(\varphi_j(\hat{V}_i)Z_{i+1}^{(2,r)}, \varphi_j(\hat{V}_\ell)Z_{\ell+1}^{(2,r)}) = 0$$

if  $\ell \geq i + 2$ , by inserting a conditional expectation with respect to  $\mathcal{F}_{(\ell+1)\Delta}$ . Now, for  $r = 1, 2$ ,

$$\begin{aligned} & \sum_{j=1}^{D_m} \mathbb{E} \left[ \left( \frac{1}{N} \sum_{i=0}^{N-1} \varphi_j(\hat{V}_i) Z_{i+1}^{(2,r)} \right)^2 \right] \leq \frac{1}{N^2} \sum_{j=1}^{D_m} \mathbb{E} \left( \sum_{0 \leq i, \ell \leq N-1} \varphi_j(\hat{V}_i) Z_{i+1}^{(2,r)} \varphi_j(\hat{V}_\ell) Z_{\ell+1}^{(2,r)} \right) \\ &= \frac{1}{N^2} \sum_{j=1}^{D_m} \mathbb{E} \left\{ \sum_{i=0}^{N-1} \left[ \varphi_j^2(\hat{V}_i) [Z_{i+1}^{(2,r)}]^2 + \varphi_j(\hat{V}_i) Z_{i+1}^{(2,r)} \varphi_j(\hat{V}_{i+1}) Z_{i+2}^{(2,r)} \right] \right\} \\ &\leq \frac{2}{N} \left\| \sum_{j=1}^{D_m} \varphi_j^2 \right\|_\infty \mathbb{E}[(Z_2^{(2,r)})^2] \leq 2 \frac{D_m}{N} [\tilde{c}_1 \mathbb{E}(\sigma^4(V_0)) + \tilde{c}_2 \Delta] \end{aligned}$$

by using Lemma 6.1.

For the second part, let us define the filtration generated by  $B$  and the whole path of  $V$ , i.e.

$$\mathcal{G}_t^V = \sigma(V_s, s \in \mathbb{R}^+, B_s, s \leq t) = \sigma(W_s, s \in \mathbb{R}^+, B_s, s \leq t, \eta).$$

Now we observe that

$$\begin{aligned} \mathbb{E}(t(\hat{V}_i)(\bar{V}_{i+2} - \bar{V}_{i+1})u_{i+1,k}) &= \mathbb{E} \left[ \mathbb{E}(t(\hat{V}_i)(\bar{V}_{i+2} - \bar{V}_{i+1})u_{i+1,k}) | \mathcal{G}_{(i+1)\Delta}^V \right] \\ &= \mathbb{E} \left[ t(\hat{V}_i)(\bar{V}_{i+2} - \bar{V}_{i+1}) \mathbb{E}(u_{i+1,k}) | \mathcal{G}_{(i+1)\Delta}^V \right] \\ &= 0 \end{aligned}$$

as  $\mathbb{E}(u_{i+1,k}) | \mathcal{G}_{(i+1)\Delta}^V = 0$ . Moreover for any  $\ell > i$ ,

$$\mathbb{E}(t(\hat{V}_i)(\bar{V}_{i+2} - \bar{V}_{i+1})u_{i+1,k} t(\hat{V}_\ell)(\bar{V}_{\ell+2} - \bar{V}_{\ell+1})u_{\ell+1,k}) = 0$$

by inserting a conditional expectation with respect to  $\mathcal{G}_{(\ell+1)\Delta}^V$ . The last remark is that one can easily see that

$$\mathbb{E}[(\bar{V}_{i+1} - \bar{V}_i)^4] \leq \frac{1}{\Delta^4} \mathbb{E} \left[ \left( \int_{(i+1)\Delta}^{(i+2)\Delta} (V_s - V_{s-\Delta}) ds \right)^4 \right] \leq C \Delta^2.$$

Now we have

$$\begin{aligned} \sum_{j=1}^{D_m} \mathbb{E} \left( \frac{1}{N\Delta} \sum_{i=0}^{N-1} \varphi_j(\hat{V}_i)(\bar{V}_{i+2} - \bar{V}_i)u_{i+1,k} \right)^2 &= \frac{1}{N^2 \Delta^2} \sum_{j=1}^{D_m} \sum_{i=0}^{N-1} \mathbb{E} \left( \varphi_j^2(\hat{V}_i)(\bar{V}_{i+2} - \bar{V}_i)^2 u_{i+1,k}^2 \right) \\ &\leq \frac{D_m}{N \Delta^2} \mathbb{E}^{1/2}[(\bar{V}_2 - \bar{V}_1)^4] \mathbb{E}^{1/2}[u_{2,k}^4] \\ &\leq C \frac{D_m}{N} \frac{1}{k\Delta}. \end{aligned}$$

The second part of this term can be treated in the same way, and it follows that if  $1/k \leq \Delta$ , then this term is less than  $C' D_m/N$ .  $\square$

## 6.6 Proof of Lemma 6.4.

Let us recall that we know from Comte *et al.* (2006) that

$$T_N^*(t) = \frac{1}{N} \sum_{i=0}^{N-1} (\sigma^2(V_{(i+1)\Delta}) - \sigma^2(\bar{V}_i)) t(\bar{V}_i)$$

is such that

$$\mathbb{E}(\sup_{t \in B_m(0,1)} [T_N^*(t)]^2) \leq C(D_m^2 \Delta^2 + D_m^5 \Delta^3).$$

Here, we write that  $T_N^{(2)}(t) = T_N^{(2,1)}(t) + T_N^{(2,2)}(t) + T_N^{(2,3)}(t) + T_N^*(t)$  with

$$T_N^{(2,1)}(t) = \frac{1}{N} \sum_{i=0}^{N-1} [t(\hat{V}_i) - t(\bar{V}_i)] [\sigma^2(\hat{V}_i) - \sigma^2(\bar{V}_i)], \quad T_N^{(2,2)}(t) = \frac{1}{N} \sum_{i=0}^{N-1} t(\bar{V}_i) [\sigma^2(\hat{V}_i) - \sigma^2(\bar{V}_i)],$$

$$T_N^{(2,3)}(t) = \frac{1}{N} \sum_{i=0}^{N-1} [t(\hat{V}_i) - t(\bar{V}_i)] [\sigma^2(\bar{V}_i) - \sigma^2(V_{(i+1)\Delta})].$$

We shall use the following decompositions obtained by the Taylor formula:

$$\sigma^2(\hat{V}_i) - \sigma^2(\bar{V}_i) = (\hat{V}_i - \bar{V}_i)(\sigma^2)'(\bar{V}_i) + R_i, \quad t(\hat{V}_i) - t(\bar{V}_i) = (\hat{V}_i - \bar{V}_i)t'(\bar{V}_i) + S_i(t)$$

with  $\mathbb{E}(R_i^2) \leq C/k^2$  and  $\mathbb{E}(R_i^4) \leq C/k^4$  if  $(\sigma^2)''$  is bounded, and  $\mathbb{E}(\sup_{t \in B_m(0,1)} S_i(t)^2) \leq CD_m^5/k^2$ ,  $\mathbb{E}^{1/2}(\sup_{t \in B_m(0,1)} S_i(t)^4) \leq CD_m^5/k^2$  because  $\|t'\|_\infty^2 \leq CD_m^5\|t\|^2$ . Now, the three terms can be studied as follows. First

$$\begin{aligned} T_N^{(2,1)}(t) &= \frac{1}{N} \sum_{i=0}^{N-1} (\hat{V}_i - \bar{V}_i)^2 (t')(\bar{V}_i) (\sigma^2)'(\bar{V}_i) + \frac{1}{N} \sum_{i=0}^{N-1} (\hat{V}_i - \bar{V}_i) t'(\bar{V}_i) R_i \\ &\quad + \frac{1}{N} \sum_{i=0}^{N-1} (\hat{V}_i - \bar{V}_i) (\sigma^2)'(\bar{V}_i) S_i(t) + \frac{1}{N} \sum_{i=0}^{N-1} R_i S_i(t) \\ &:= T_N^{(2,1,1)}(t) + T_N^{(2,1,2)}(t) + T_N^{(2,1,3)}(t) + T_N^{(2,1,4)}(t), \end{aligned}$$

and we bound each term successively. Clearly by Schwarz inequality applied to each term, we find,

$$\mathbb{E}(\sup_{t \in B_m(0,1)} [T_N^{(2,1,1)}(t)]^2) \leq C \mathbb{E}^{1/2}(\bar{V}_1^4) \frac{D_m^3}{k^2}$$

using that  $\|t'\|_\infty^2 \leq CD_m^3\|t\|^2$ ,

$$\mathbb{E}(\sup_{t \in B_m(0,1)} [T_N^{(2,1,2)}(t)]^2) \leq C \frac{D_m^3}{k^3}, \quad \mathbb{E}(\sup_{t \in B_m(0,1)} [T_N^{(2,1,3)}(t)]^2) \leq C \mathbb{E}^{1/2}(\bar{V}_1^4) \frac{D_m^5}{k^3},$$

and

$$\mathbb{E}(\sup_{t \in B_m(0,1)} [T_N^{(2,1,4)}(t)]^2) \leq C \frac{D_m^5}{k^4}.$$

Therefore, if  $1/k \leq \Delta$ ,  $\mathbb{E}(\sup_{t \in B_m(0,1)} [T_N^{(2,1)}(t)]^2) \leq C(D_m^3/k^2 + D_m^5/k^3)$ .

Next, we write that

$$\begin{aligned} T_N^{(2,2)}(t) &= \frac{1}{N} \sum_{i=0}^{N-1} t(\bar{V}_i)(\sigma^2)'(\bar{V}_i)(\hat{V}_i) - \bar{V}_i + \frac{1}{N} \sum_{i=0}^{N-1} t(\bar{V}_i)R_i \\ &= T_N^{(2,2,1)}(t) + T_N^{(2,2,2)}(t). \end{aligned}$$

We obtain easily that

$$\mathbb{E}(\sup_{t \in B_m(0,1)} [T_N^{(2,2,2)}(t)]^2) \leq \mathbb{E}(\sup_{t \in B_m(0,1)} \|t\|_\infty^2 \frac{1}{N} \sum_{i=1}^N R_i^2) \leq \Phi_0^2 D_m \mathbb{E}(R_1^2) \leq C D_m / k^2,$$

a term which is negligible with respect to the previous ones.

Then  $(\hat{V}_i - \bar{V}_i)\psi(\bar{V}_i)$  is a martingale increment with respect to the filtration  $(\mathcal{G}_t^V)$ , for any measurable function  $\psi$ . In particular,

$$\begin{aligned} \mathbb{E}[(\hat{V}_i - \bar{V}_i)\psi(\bar{V}_i)] &= \mathbb{E}[\mathbb{E}[(\hat{V}_i - \bar{V}_i)\psi(\bar{V}_i)|\mathcal{G}_{i\Delta}^V]] \\ &= \mathbb{E}[\psi(\bar{V}_i)\mathbb{E}[(\hat{V}_i - \bar{V}_i)|\mathcal{G}_{i\Delta}^V]] = 0 \end{aligned}$$

since  $\mathbb{E}(\hat{V}_i|\mathcal{G}_{i\Delta}^V) = \bar{V}_i$ . In the same way, for  $i < \ell$ ,

$$\mathbb{E}\left((\hat{V}_i - \bar{V}_i)\psi(\bar{V}_i)(\hat{V}_\ell - \bar{V}_\ell)\psi(\bar{V}_\ell)\right) = 0$$

by inserting a conditional expectation with respect to  $\mathcal{G}_{\ell\Delta}^V$ . Therefore

$$\begin{aligned} \mathbb{E}(\sup_{t \in B_m(0,1)} [T_N^{(2,2,1)}(t)]^2) &\leq \sum_{j=1}^{D_m} \mathbb{E}\left(\frac{1}{N} \sum_{i=0}^{N-1} \varphi_j(\bar{V}_i)(\sigma^2)'(\bar{V}_i)(\hat{V}_i - \bar{V}_i)\right)^2 \\ &= \sum_{j=1}^{D_m} \frac{1}{N} \mathbb{E}\left(\varphi_j(\bar{V}_1)(\sigma^2)'(\bar{V}_1)(\hat{V}_1 - \bar{V}_1)\right)^2 \\ &\leq \frac{1}{N} \mathbb{E}\left(\left(\sum_{j=1}^{D_m} \varphi_j^2(\bar{V}_1)\right)[(\sigma^2)'(\bar{V}_1)]^2(\hat{V}_1 - \bar{V}_1)^2\right) \\ &\leq \frac{D_m}{N} \mathbb{E}^{1/2}[(\sigma^2)'(\bar{V}_1)^4] \mathbb{E}^{1/2}[u_{1,k}^4] \leq C \mathbb{E}^{1/2}(\bar{V}_1^4) \frac{D_m}{Nk}. \end{aligned}$$

For the last term, we write  $T_N^{(2,3)}(t) = T_N^{(2,3,1)}(t) + T_N^{(2,3,2)}(t)$  where

$$T_N^{(2,3,1)}(t) = (1/N) \sum_{i=0}^{N-1} (\hat{V}_i - \bar{V}_i)t'(\bar{V}_i)(\sigma^2(\bar{V}_i) - \sigma^2(V_{(i+1)\Delta})),$$

$$T_N^{(2,3,2)}(t) = (1/N) \sum_{i=0}^{N-1} S_i(t)(\sigma^2(\bar{V}_i) - \sigma^2(V_{(i+1)\Delta})).$$

Moreover, we know from Comte *et al.* (2006) that  $\mathbb{E}[(\sigma^2(\bar{V}_i) - \sigma^2(V_{(i+1)\Delta}))^2] \leq \mathbb{E}^{1/2}[(\sigma^2(\bar{V}_i) - \sigma^2(V_{(i+1)\Delta}))^4] \leq C\Delta$ . Now, for  $T_N^{(2,3,1)}(t)$ , we proceed as for  $T_N^{(2,2,1)}(t)$  since both have the same martingale property w.r.t.  $\mathcal{G}_s^V$ . We get

$$\begin{aligned} \mathbb{E}\left(\sup_{t \in B_m(0,1)} [T_N^{(2,3,1)}(t)]^2\right) &\leq \sum_{j=1}^{D_m} \mathbb{E}\left(\frac{1}{N} \sum_{i=0}^{N-1} \varphi'_j(\bar{V}_i)(\hat{V}_i - \bar{V}_i)(\sigma^2(\bar{V}_i) - \sigma^2(V_{(i+1)\Delta}))\right)^2 \\ &\leq \frac{1}{N} \sum_{j=1}^{D_m} \mathbb{E}\left((\varphi'_j)^2(\bar{V}_1)(\hat{V}_1 - \bar{V}_1)^2(\sigma^2(\bar{V}_1) - \sigma^2(V_{2\Delta}))^2\right) \\ &\leq \frac{CD_m^3}{N} \mathbb{E}^{1/2}(u_{1,k}^4) \mathbb{E}^{1/2}[(\sigma^2(\bar{V}_1) - \sigma^2(V_{2\Delta}))^4] \\ &\leq C \frac{D_m^3 \Delta}{Nk} \end{aligned}$$

as  $\sum_j (\varphi'_j)^2(x) \leq CD_m^3$ . Using  $D_m^2 \leq N\Delta$  and  $1/k \leq \Delta$  implies  $\mathbb{E}(\sup_{t \in B_m(0,1)} [T_N^{(2,3,1)}(t)]^2) \leq CD_m \Delta^3$ . On the other hand,  $\mathbb{E}(\sup_{t \in B_m(0,1)} [T_N^{(2,3,2)}(t)]^2) \leq CD_m^5 \Delta/k^2 \leq CD_m^5 \Delta^3$ , as  $1/k \leq \Delta$ .

By gathering and comparing all terms and assuming that  $1/k \leq \Delta$ , we obtain the bound given in Lemma 6.4.  $\square$

## 6.7 Proof of Theorem 4.1

The proof of this theorem relies on the following Bernstein-type Inequality:

**Lemma 6.5** *Under the assumptions of Theorem 4.1, for any positive numbers  $\epsilon$  and  $v$ , we have*

$$\mathbb{P}\left[\sum_{i=0}^{N-1} t(\hat{V}_i)Z_{(i+1)\Delta}^{(1)} \geq N\epsilon, \|t\|_N^2 \leq v^2\right] \leq \exp\left(-\frac{N\Delta\epsilon^2}{2\sigma_1^2 v^2}\right).$$

**Proof of Lemma 6.5:** Noting that  $W$  is a Brownian motion with respect to the augmented filtration  $\mathcal{F}_s = \sigma((B_u, W_u), u \leq s, \eta)$ , the proof is obtained as the analogous proof in Comte *et al.* (2007), Lemma 2 p.533.  $\square$

Now we turn to the proof of Theorem 4.1.

As in the proof of Proposition 3.1, we have to split  $\|\tilde{b} - b_A\|_N^2 = \|\tilde{b} - b_A\|_N^2 \mathbf{1}_{\Omega_N} + \|\tilde{b} - b_A\|_N^2 \mathbf{1}_{\Omega_N^c}$ . For the study on  $\Omega_N^c$ , the end of the proof of Proposition 3.1 can be used.

Now, we focus on what happens on  $\Omega_N$ . From the definition of  $\tilde{b}$ , we have,  $\forall m \in \mathcal{M}_n$ ,  $\gamma_N(\hat{b}_{\hat{m}}) + \text{pen}(\hat{m}) \leq \gamma_N(b_m) + \text{pen}(m)$ . We proceed as in the proof of Proposition 3.1 with some additional penalty terms and obtain

$$\begin{aligned} \mathbb{E}(\|\hat{b}_{\hat{m}} - b_A\|_N^2 \mathbf{1}_{\Omega_N}) &\leq 7\|b_m - b_A\|_{\pi^*}^2 + \text{pen}(m) + 32\mathbb{E}\left(\sup_{t \in S_m + S_{\hat{m}}, \|t\|_{\pi^*}=1} [\nu_N^{(1)}(t)]^2 \mathbf{1}_{\Omega_N}\right) \\ &\quad - \mathbb{E}(\text{pen}(\hat{m})) + 32c'\Delta. \end{aligned}$$

The difficulty here is to control the supremum of  $\nu_N^{(1)}(t)$  on a random ball (which depends on the random  $\hat{m}$ ). This is done by setting  $\nu_N^{(1)} = \nu_N^{(1,1)} + \nu_N^{(1,2)}$ , with

$$\nu_N^{(1,1)}(t) = \frac{1}{N} \sum_{i=0}^{N-1} Z_{(i+1)\Delta}^{(1)} t(\hat{V}_i), \quad \nu_N^{(1,2)}(t) = \frac{1}{N} \sum_{i=0}^{N-1} t(\hat{V}_i) \left( \frac{u_{i+2,k} - u_{i+1,k}}{\Delta} \right).$$

We use the martingale property of  $\nu_N^{(1,1)}(t)$  and a rough bound for  $\nu_N^{(1,2)}(t)$  as follows.

For  $\nu_N^{(1,2)}$ , we simply write, as previously

$$\begin{aligned} \mathbb{E} \left( \sup_{t \in S_m + S_{\hat{m}}, \|t\|_{\pi^*} = 1} [\nu_n^{(1,2)}(t)]^2 \right) &\leq \frac{1}{\pi_0^*} \mathbb{E} \left( \sup_{t \in \mathcal{S}_n, \|t\| = 1} [\nu_n^{(1,2)}(t)]^2 \right) \\ &\leq \frac{1}{\pi_0^*} \sum_{j=1}^{\mathcal{D}_n} \mathbb{E}[(\nu_N^{(2)}(\varphi_j))^2] \\ &\leq \frac{4\mathcal{D}_n}{\pi_0^* N} \mathbb{E}[(u_{1,k}/\Delta)^2] \leq \frac{4\mathcal{D}_n}{\pi_0^* N k_n \Delta^2} \leq \frac{4}{\pi_0^*} \frac{1}{k_n \Delta}. \end{aligned}$$

For  $\nu_N^{(1,1)}$ , let us denote by

$$G_m(m') = \sup_{t \in S_m + S_{m'}, \|t\|_{\pi^*} = 1} \nu_N^{(1,1)}(t)$$

the quantity to be studied. Introducing a function  $p(m, m')$ , we first write

$$\begin{aligned} G_m^2(\hat{m}) \mathbf{1}_{\Omega_N} &\leq [(G_m^2(\hat{m}) - p(m, \hat{m})) \mathbf{1}_{\Omega_N}]_+ + p(m, \hat{m}) \\ &\leq \sum_{m' \in \mathcal{M}_n} [(G_m^2(m') - p(m, m')) \mathbf{1}_{\Omega_N}]_+ + p(m, \hat{m}). \end{aligned}$$

Then  $\text{pen}$  is chosen such that  $32p(m, m') \leq \text{pen}(m) + \text{pen}(m')$ . More precisely, the next Proposition determines the choice of  $p(m, m')$  which in turn will fix the penalty.

**Proposition 6.1** *Under the assumptions of Theorem 4.1, there exists a numerical constant  $\kappa_1$  such that, for  $p(m, m') = \kappa_1 \sigma_1^2 (D_m + D_{m'}) / (n\Delta)$ , we have*

$$\mathbb{E}[(G_m^2(m') - p(m, m')) \mathbf{1}_{\Omega_N}]_+ \leq c \sigma_1^2 \frac{e^{-D_{m'}}}{N\Delta}.$$

**Proof of Proposition 6.1.** The result of Proposition 6.1 follows from the inequality of Lemma 6.5 by the  $\mathbb{L}^2$ -chaining technique used in Baraud et al. (2001b) (see Section 7 p.44-47, Lemma 7.1, with  $s^2 = \sigma_1^2/\Delta$ ).  $\square$

It is easy to see that the result of Theorem 4.1 follows from Proposition 6.1 with  $\text{pen}(m) = \kappa \sigma_1^2 D_m / (N\Delta)$ .  $\square$

## 6.8 Proof of Theorem 4.2

The lines of the proof are the same as the ones of Theorem 4.1. Moreover, they follow closely the analogous proof of Theorem 2 p.524 in Comte *et al.* (2007), see also Comte *et al.* (2006). Therefore, we omit it.



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## References

1. Aït-Sahalia, Y. and Kimmel, R.: Maximum likelihood estimation of stochastic volatility models. *J. Finance Econ.* **83**, 413-452 (2007).
2. Andersen, T., Lund, J.: Estimating continuous-time stochastic volatility models of the short-term interest rate. *J. Econom.* **77**, 343-377 (1997).
3. Barndorff-Nielsen, O. E., Shephard, N.: Econometric analysis of realized volatility and its use in estimating stochastic volatility models. *J. R. Stat. Soc. Ser. B Stat. Methodol.* **64**, 253-280 (2002).
4. Barron, A.R., Birgé, L., Massart, P.: Risk bounds for model selection via penalization. *Probab. Theory Related Fields* **113**, 301-413 (1999).
5. Comte, F., Genon-Catalot, V. Penalized projection estimator for volatility density. *Scand. J. Statist.* **33**, 875-893 (2006).
6. Comte, F., Genon-Catalot, V., Rozenholc, Y.: Penalized nonparametric mean square estimation of the coefficients of diffusion processes. *Bernoulli* **13**, 514-543 (2007).
7. Comte, F., Genon-Catalot, V., Rozenholc, Y.: Nonparametric estimation for a discretely observed integrated diffusion model. Preprint MAP5, 2006-11 (2006).
8. Comte, F., Rozenholc, Y.: A new algorithm for fixed design regression and denoising. *Ann. Inst. Statist. Math.* **56**, 449-473 (2004).
9. Cox, J. C., Ingersoll, J. E., Jr., Ross, S. A.: A theory of the term structure of interest rates. *Econometrica* **53**, 385-407 (1985).
10. Genon-Catalot, V., Jeantheau, T., Larédo, C.: Parameter estimation for discretely observed stochastic volatility models. *Bernoulli* **5**, 855-872 (1999).
11. Genon-Catalot, V., Jeantheau, T., Larédo, C.: Stochastic volatility models as hidden Markov models and statistical applications. *Bernoulli*, **6**, 1051-1079 (2000).
12. Ghysels, E., Harvey, A., Renault, E.: Stochastic volatility. In: Maddala, G. (Ed.) *Handbook of Statistics*, vol. 14, North-Holland, Amsterdam, 119-191 (1996).
13. Gloter, A.: Estimation of the volatility diffusion coefficient for a stochastic volatility model. (French) *C. R. Acad. Sci. Paris, Sr. I Math.* **330**, 3, 243-248 (2000).
14. Gloter, A.: Efficient estimation of drift parameters in stochastic volatility models. *Finance Stoch.*, **11**, 495-519 (2007).
15. Heston, S.: A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Rev. Financial Stud.* **6**, 327-343 (1993).
16. Hoffmann, M.: Adaptive estimation in diffusion processes. *Stochastic Process. Appl.* **79**, 135-163 (1999).
17. Hoffmann, M.: Rate of convergence for parametric estimation in a stochastic volatility model. *Stochastic Process. Appl.* **97**, 147-170 (2002).
18. Hull, J., White, A.: The pricing of options on assets with stochastic volatility. *J. of Finance* **42**, 281-300 (1987).
19. Renò, R.: Nonparametric estimation of stochastic volatility models. *Economic Letters* **90**, 390-395 (2006).
20. Rogers, L.C.G., Williams, D.: *Diffusions, Markov processes, and martingales*. Vol. 2, Reprint of the second (1994) edition, Cambridge Univ. Press, Cambridge (2000).
21. Shephard, N.: *Stochastic volatility. Selected readings*. (Advanced texts in Econometrics). Oxford University Press, London (2005).