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Abstract

We prove a central limit theorem for linear triangular arrays under weak dependence conditions. Our result is then applied to the study of dependent random variables sampled by a $\mathbb{Z}$-valued transient random walk. This extends the results obtained by Guillotin-Plantard & Schneider (2003). An application to parametric estimation by random sampling is also provided.

Keywords:

Random walks; weak dependence; central limit theorem; dynamical systems; random sampling; parametric estimation.

AMS Subject Classification:

Primary 60F05, 60G50, 62D05; Secondary 37C30, 37E05

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1 Introduction

Let $\{\xi_i\}_{i\in\mathbb{Z}}$ be a sequence of centered, non essentially constant and square integrable real valued random variables. Let $\{a_{n,i}\ ,\ -k_n \le i \le k_n\}$ be a triangular array of real numbers such that for all $n \in \mathbb{N}$, $\sum_{i=-k_n}^{k_n} a_{n,i}^2 > 0$. We are interested in the behaviour of linear triangular arrays of the form

$$X_{n,i} = a_{n,i}\,\xi_i\ ,\ n = 0, 1, \ldots\ ,\ i = -k_n, \ldots, k_n\ , \tag{1.1}$$

where $(k_n)_{n\ge1}$ is a nondecreasing sequence of positive integers satisfying $k_n \xrightarrow[n\to+\infty]{} +\infty$. We work under a weak dependence condition introduced in Dedecker *et al.* (2007). We first prove a central limit theorem for linear triangular arrays of type (1.1) (Theorem 3.1 of Section 3). Applying this result, we then prove a central limit theorem for the partial sums of weakly dependent sequences sampled by a transient $\mathbb{Z}$-valued random walk (Theorem 4.1 of Section 4). This result extends the results obtained by Guillotin-Plantard & Schneider (2003). Peligrad & Utev (1997) derive a central limit theorem for triangular arrays of type (1.1) under mixing conditions. Unfortunately, mixing is a rather restrictive condition, and many simple Markov chains are not mixing. For instance, Andrews (1984) proved that if $(\varepsilon_i)_{i\ge1}$ is independent and identically distributed with marginal $\mathcal{B}(1/2)$, then the stationary solution $(\xi_i)_{i\ge0}$ of the equation

$$\xi_n = \frac{1}{2}(\xi_{n-1} + \varepsilon_n)\ ,\ \xi_0 \text{ independent of } (\varepsilon_i)_{i\ge1} \tag{1.2}$$

is not α-mixing in the sense of Rosenblatt (1956). We have indeed $\alpha(\sigma(\xi_0), \sigma(\xi_n)) = 1/4$ for any n. For any $y \in \mathbb{R}$, let $[y]$ denote the integer part of y. The chain satisfying (1.2) is the Markov chain associated to the dynamical system generated by the map $T(x) = 2x - [2x]$ on the space $[0, 1]$, equipped with the Lebesgue measure, and it is well known that such dynamical systems are not α-mixing in the sense that $\alpha(\sigma(T), \sigma(T^n))$ does not tend to zero as n tends to infinity. Withers (1981) proves triangular central limit theorems under a so-called l-mixing condition, which generalizes the classical notions of mixing (such as strong mixing, absolute regularity, uniform mixing introduced respectively by Rosenblatt (1956), Rozanov & Volkonskii (1959) and Ibragimov (1962)). The idea of l-mixing requires the asymptotic decoupling of the 'past' and the 'future'. The dependence setting used in the present paper (introduced in Dedecker *et al.*, 2007) follows the same idea. In Section 5 we give lots of pertinent examples satisfying our dependence conditions. Coulon-Prieur & Doukhan (2000) proves a triangular central limit theorem under a weaker dependence condition. However, they assume that the random variables ξ_i are uniformly bounded. Their proof is a variation on Lindeberg-Rio's method developed by Rio (1996,1997). Also using a variation on Lindeberg-Rio's method, Bardet *et al.* prove a triangular central limit theorem, requiring moments of order $2 + \delta$, $\delta > 0$. In Section 2, we introduce the dependence setting under which we work in the sequel. Models for which we can compute bounds for our dependence coefficients are presented in Section 5. At least, we give an application to parametric estimation by random sampling in Section 6.

2 Definitions

In this section, we recall the definition of the dependence coefficients which we will use in the sequel. They have first been introduced in Dedecker *et al.* (2007).

On the Euclidean space $\mathbb{R}^m$, we define the metric

$$d_1(x,y) = \sum_{i=1}^{m} |x_i - y_i|.$$

Let $\Lambda = \bigcup_{m\in\mathbb{N}^*} \Lambda_m$ where Λ_m is the set of Lipschitz functions $f : \mathbb{R}^m \to \mathbb{R}$ with respect to the metric d_1. If $f \in \Lambda_m$, we denote by $\mathrm{Lip}(f) := \sup_{x\neq y} \frac{|f(x)-f(y)|}{d_1(x,y)}$ the Lipschitz modulus of f. The set of functions $f \in \Lambda$ such that $\mathrm{Lip}(f) \leq 1$ is denoted by $\tilde{\Lambda}$.

Definition 2.1 *Let ξ be a $\mathbb{R}^m$-valued random variable defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, assumed to be square integrable. For any σ-algebra $\mathcal{M}$ of $\mathcal{A}$, we define the θ_2-dependence coefficient*

$$\theta_2(\mathcal{M}, \xi) = \sup\{\|\mathbb{E}(f(\xi)|\mathcal{M}) - \mathbb{E}(f(\xi))\|_2\,,\ f \in \tilde{\Lambda}\}\,. \tag{2.3}$$

We now define the coefficient $\theta_{k,2}$ for a sequence of σ-algebras and a sequence of $\mathbb{R}$-valued random variables.

Definition 2.2 *Let $(\xi_i)_{i\in\mathbb{Z}}$ be a sequence of square integrable random variables valued in $\mathbb{R}$. Let $(\mathcal{M}_i)_{i\in\mathbb{Z}}$ be a sequence of σ-algebras of $\mathcal{A}$. For any $k \in \mathbb{N}^* \cup \{\infty\}$ and $n \in \mathbb{N}$, we define*

$$\theta_{k,2}(n) = \max_{1\leq l\leq k} \frac{1}{l} \sup\{\theta_2(\mathcal{M}_p, (\xi_{j_1}, \ldots, \xi_{j_l})), p+n \leq j_1 < \ldots < j_l\}$$

and

$$\theta_2(n) = \theta_{\infty,2}(n) = \sup_{k\in\mathbb{N}^*} \theta_{k,2}(n)\,.$$

Definition 2.3 *Let $(\xi_i)_{i\in\mathbb{Z}}$ be a sequence of square integrable random variables valued in $\mathbb{R}$. Let $(\mathcal{M}_i)_{i\in\mathbb{Z}}$ be a sequence of σ-algebras of $\mathcal{A}$. The sequence $(\xi_i)_{i\in\mathbb{Z}}$ is said to be θ_2-weakly dependent with respect to $(\mathcal{M}_i)_{i\in\mathbb{Z}}$ if $\theta_2(n) \xrightarrow[n\to+\infty]{} 0$.*

Remark:
Replacing the $\|\cdot\|_2$ norm in (2.3) by the $\|\cdot\|_1$ norm, we get the θ_1 dependence coefficient first introduced by Doukhan & Louhichi (1999). This weaker coefficient is the one used in Coulon-Prieur & Doukhan (2000).

3 Central limit theorem for triangular arrays of dependent random variables

Let $\{X_{n,i},\ n \in \mathbb{N},\ -k_n \leq i \leq k_n\}$ be a triangular array of type (1.1). We are interested in the asymptotic behaviour of the following sum

$$\Sigma_n = \sum_{i=-k_n}^{k_n} X_{n,i} = \sum_{i=-k_n}^{k_n} a_{n,i}\, \xi_i\,.$$

Let $(\mathcal{M}_i)_{i\in\mathbb{Z}}$ be the sequence of σ-algebras of $\mathcal{A}$ defined by

$$\mathcal{M}_i = \sigma\left(\xi_j\,,\ j \leq i\right)\,,\ i \in \mathbb{Z}\,.$$

In the sequel, the dependence coefficients are defined with respect to the sequence of σ-algebras $(\mathcal{M}_i)_{i\in\mathbb{Z}}$. We denote by σ_n^2 the variance of Σ_n.

Theorem 3.1
Assume that the following conditions are satisfied :

(A_1) (i) $\liminf_{n\to+\infty} \frac{\sigma_n^2}{\sum_{i=-k_n}^{k_n} a_{n,i}^2} > 0,$

(ii) $\lim_{n\to+\infty} \sigma_n^{-1} \ \max_{-k_n\le i\le k_n} |a_{n,i}| = 0.$

(A_2) $\{\xi_i^2\}_{i\in\mathbb{Z}}$ *is an uniformly integrable family.*

(A_3) $\theta_2^{\xi}(\cdot)$ *is bounded above by a non-negative function* $g(\cdot)$ *such that*

$$x \mapsto x^{3/2}\ g(x) \text{ is non-increasing,}$$

$$\exists\ 0<\varepsilon<1\ ,\ \sum_{i=0}^{\infty} 2^{\frac{3i}{2}} g(2^{i\varepsilon}) < \infty.$$

Then, as n *tends to infinity,* $\frac{\Sigma_n}{\sigma_n}$ *converges in distribution to* $\mathcal{N}(0,1)$*.*

Remark:

- Theorem 2.2 (c) in Peligrad & Utev (1997) yields a central limit theorem for strongly mixing linear triangular arrays of type (1.1). They assume that $\{|\xi_i|^{2+\delta}\}$ is uniformly integrable for a certain $\delta > 0$. Such an assumption is also required for Theorem 2.1 in Withers (1981) for l-mixing arrays. In Coulon-Prieur & Doukhan (2000), the random variables ξ_i are assumed to be uniformly bounded.
- The proof of Theorem 2.2 (c) in Peligrad & Utev (1997) relies on a variation on Theorem 4.1 in Utev (1990) (see Theorem B in Peligrad & Utev, 1997). The proof of Theorem 3.1, which is postponed to the Appendix, also makes use of a variation on Theorem 4.1 in Utev (1990) (see also Utev, 1991).
- If $\theta_2^{\xi}(n) = \mathcal{O}\left(n^{-a}\right)$ for some positive a, condition (A_3) holds for $a > 3/2$.

4 Central limit theorem for the sum of dependent random variables sampled by a transient random walk

4.1 The main result

Let $(E,\mathcal{E},\mu)$ be a probability space, and $T : E \mapsto E$ a bijective bimeasurable transformation preserving the probability μ. We define the stationary sequence $(\xi_i)_{i\in\mathbb{Z}} = (T^i)_{i\in\mathbb{Z}}$ from (E,μ) to E. Let $(X_i)_{i\ge1}$ be a sequence of independent and identically distributed random variables defined on a probability space $(\Omega,\mathcal{A},\mathbb{P})$ with values in $\mathbb{Z}$ and

$$S_n = \sum_{i=1}^{n} X_i,\ n\ge1, \qquad S_0 \equiv 0\ .$$

For $f \in \mathbb{L}^1(\mu)$ and $\omega \in \Omega$, we are interested in the sampled ergodic sum

$$\sum_{k=0}^{n-1} f\circ\xi_{S_k(\omega)}\ .$$

By applying Birkhoff's ergodic Theorem to the skew-product:

$$U \;:\; \begin{array}{l} \Omega \times E \to \Omega \times E \\ (\omega, x) \mapsto (\sigma\omega, T^{\omega_1}x) \end{array}$$

where σ is the shift on the path space $\Omega = \mathbb{Z}^{\mathbb{N}}$, we obtain that for every function $f \in \mathbb{L}^1(\mu)$, the sampled ergodic sum converges $\mathbb{P} \otimes \mu$-almost surely. A natural question is to know if the random walk is universally representative for $\mathbb{L}^p$, $p > 1$ in the following sense: there exists a subset Ω_0 of Ω of probability one such that for every $\omega \in \Omega_0$, for every dynamical system $(E, \mathcal{E}, \mu, T)$, for every $f \in \mathbb{L}^p$, $p > 1$, the sampled ergodic average converges μ-almost surely. The answer can be found in Lacey *et al.* (1994) if the X_i's are square integrable: The random walk is universally representative for $\mathbb{L}^p$, $p > 1$ if and only if the expectation of X_1 is not equal to 0 which corresponds to the case where the random walk is transient. In that case, it seems natural to study the fluctuations of the sampled ergodic averages around the limit. From Lacey's theorem (1991), for any $H \in (0, 1)$, there exists some function $f \in \mathbb{L}^2(\mathbb{P} \otimes \mu)$ such that the finite-dimensional distributions of the process

$$\frac{1}{n^H} \sum_{k=0}^{[nt]-1} f \circ U^k(\omega, x)$$

converge to the finite dimensional distributions of a self-similar process. Unfortunately, this convergence on the product space does not imply the convergence in distribution for a given path of the random walk. A first answer to this question is given in Guillotin-Plantard & Schneider (2003) where the technique of martingale differences is used. Let us recall that this method consists (under convenient conditions) of decomposing the function f as the sum of a function g generating a sequence of martingale differences and a cocycle $h - h \circ T$. In the standard case, the central limit theorem for the ergodic sum is deduced from central limit theorems for the sums of martingale differences, the term corresponding to the cocycle being negligeable in probability. In Guillotin-Plantard & Schneider (2003), only functions f generating a sequence of martingale differences are considered. In this section, where we prove a central limit theorem for θ_2-weakly dependent random variables sampled by a transient random walk, this reasoning does not hold anymore. We apply Theorem 3.1 of Section 3.

In the sequel, the random walk $(S_n)_{n\geq 0}$ is assumed to be transient. In particular, for every $x \in \mathbb{Z}$, the Green function

$$G(0, x) = \sum_{k=0}^{+\infty} \mathbb{P}(S_k = x)$$

is finite. For example, it is the case if the random variable X_1 is assumed with finite absolute mean and nonzero mean. It is also possible to choose the random variables $(X_i)_{i\geq 1}$ centered and for every $x \in \mathbb{R}$,

$$\mathbb{P}(n^{-1/\alpha} S_n \leq x) \xrightarrow[n\to+\infty]{} F_\alpha(x)\,,$$

where F_α is the distribution function of a stable law with index $\alpha \in (0, 1)$. Stone (1966) has proved a local limit theorem for this kind of random walks from which the transience can be deduced. The expectation with respect to the measure μ (resp. with respect to $\mathbb{P}$, $\mathbb{P} \otimes \mu$) will be denoted in the sequel by $\mathbb{E}_\mu$ (resp. by $\mathbb{E}_{\mathbb{P}}$, $\mathbb{E}$).
For every function $f \in \mathbb{L}^2(\mu)$ such that $\mathbb{E}_\mu(f) = 0$, we define

$$\sigma^2(f) = 2 \sum_{x\in\mathbb{Z}} G(0, x)\mathbb{E}_\mu(f f \circ T^x) - \mathbb{E}_\mu(f^2)\,.$$

Let us now state our main result whose proof is deferred to Subsection 4.3.

Theorem 4.1
Let f be a function in $\mathbb{L}^2(\mu)$ such that $\mathbb{E}_\mu(f) = 0$. Assume that $(f \circ T^x)_{x\in\mathbb{Z}}$ satisfies assumption (A_3) of Theorem 3.1. Assume that $\sigma^2(f)$ is finite and positive.
Then, for $\mathbb{P}$-almost every $\omega \in \Omega$,

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n} f \circ T^{S_k(\omega)} \xrightarrow[n\to+\infty]{} \mathcal{N}(0, \sigma^2(f)) \quad \textit{in distribution.}$$

Remark:
1. In the particular case where $(f \circ T^x)_{x\in\mathbb{Z}}$ is a sequence of martingale differences, we recognize Theorem 3.2 of Guillotin-Plantard & Schneider (2003). Indeed, assumptions are satisfied using orthogonality of the $f \circ T^x$'s and then, $\sigma^2(f) = (2G(0,0) - 1)\mathbb{E}_\mu(f^2)$.
2. The stationarity assumption on the sequence $(\xi_i)_{i\in\mathbb{Z}}$ can be relaxed by a stationarity assumption of order 2, that is:

$$\forall\, i \in \mathbb{Z}\,,\ \mathrm{Var}\,\xi_i = \mathrm{Var}\,\xi_1 \ \text{ and}$$

$$\forall\, i < j\,,\ \mathrm{Cov}(\xi_i, \xi_j) = \mathrm{Cov}(\xi_1, \xi_{1+j-i})\,.$$

4.2 Computation of the variance

The random walk $(S_n)_{n\geq 0}$ is defined as in the previous section. The local time of the random walk is then defined for every $x \in \mathbb{Z}$ by

$$N_n(x) = \sum_{i=0}^{n} \mathbf{1}_{\{S_i = x\}}\,.$$

The self-intersection local time is defined for every $x \in \mathbb{Z}$ by

$$\alpha(n, x) = \sum_{i,j=0}^{n} \mathbf{1}_{\{S_i - S_j = x\}}$$

and can be rewritten using the definition of the local time as

$$\alpha(n, x) = \sum_{y\in\mathbb{Z}} N_n(y + x)\ N_n(y)\,.$$

Let f be a function in $\mathbb{L}^2(\mu)$ such that $\mathbb{E}_\mu(f) = 0$. For every $\omega \in \Omega$,

$$\sum_{k=0}^{n} f \circ T^{S_k(\omega)} = \sum_{x\in\mathbb{Z}} N_n(x)(\omega) f \circ T^x\,.$$

In order to apply results of Theorem 3.1, we need to study, for any fixed $\omega \in \Omega$, the asymptotic behaviour of the variance of this sum, namely

$$\sigma_n^2(f) = \mathbb{E}_\mu\left(\left|\sum_{k=0}^{n} f \circ T^{S_k(\omega)}\right|^2\right)\,.$$

The variable ω will be omitted in the next calculations. We have

$$\begin{aligned}
\sigma_n^2(f) &= \mathbb{E}_\mu \left| \sum_{x\in\mathbb{Z}} N_n(x) f\circ T^x \right|^2 \\
&= \sum_{x,y\in\mathbb{Z}} N_n(x)N_n(y)\mathbb{E}_\mu(f\circ T^{x-y}\ f) \\
&= \sum_{y,z\in\mathbb{Z}} N_n(y+z)N_n(y)\mathbb{E}_\mu(f\circ T^{z}\ f) \\
&= \sum_{z\in\mathbb{Z}} \alpha(n,z)\mathbb{E}_\mu(f\circ T^{z}\ f)\,.
\end{aligned}$$

We are now able to prove the following proposition:

Proposition 4.1 *If $\sum_{x\in\mathbb{Z}} G(0,x)\mathbb{E}_\mu(f\ f\circ T^x) < +\infty$, then*

$$\frac{\sigma_n^2(f)}{n} \xrightarrow[n\to+\infty]{\mathbb{P}\text{-}a.s.} \sigma^2(f)\,.$$

Proof of Proposition 4.1:
Let us assume first that the function f is positive. For every $0\le m<n$, we denote by $W_{m,n}$ the random variable

$$-\sum_{i,j=m}^{n}\sum_{x\in\mathbb{Z}} 1_{\{S_i-S_j=x\}}\mathbb{E}_\mu(f\ f\circ T^x)\,.$$

Then, since f is positive, for every k,m,n such that $0\le k<m<n$,

$$W_{k,n}\le W_{k,m}+W_{m,n}\,,$$

that is $(W_{m,n})_{m,n\ge 0}$ is a subadditive sequence. Then,

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}(W_{0,n}) &= -\sum_{i,j=0}^{n}\mathbb{E}_{\mathbb{P}}\Big(\sum_{x\in\mathbb{Z}} 1_{\{S_i-S_j=x\}}\mathbb{E}_\mu(f\ f\circ T^x)\Big) \\
&= -\sum_{i,j=0}^{n}\sum_{x\in\mathbb{Z}}\mathbb{P}(S_i-S_j=x)\mathbb{E}_\mu(f\ f\circ T^x), \quad \text{by Fubini Theorem} \\
&= -\Big((n+1)\mathbb{E}_\mu(f^2)+2\sum_{i=1}^{n}\sum_{j=0}^{i-1}\sum_{x\in\mathbb{Z}}\mathbb{P}(S_{i-j}=x)\mathbb{E}_\mu(f\ f\circ T^x)\Big) \\
&= -\Big((n+1)\mathbb{E}_\mu(f^2)+2\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{x\in\mathbb{Z}}\mathbb{P}(S_{j}=x)\mathbb{E}_\mu(f\ f\circ T^x)\Big)\,.
\end{aligned}$$

Now, using that

$$\lim_{i\to+\infty}\sum_{j=1}^{i}\sum_{x\in\mathbb{Z}}\mathbb{P}(S_j=x)\mathbb{E}_\mu(f\ f\circ T^x)=\sum_{x\in\mathbb{Z}} G(0,x)\mathbb{E}_\mu(f\ f\circ T^x)-\mathbb{E}_\mu(f^2),$$

we conclude that

$$\lim_{n\to+\infty}\frac{\mathbb{E}_{\mathbb{P}}(W_{0,n})}{n}=\mathbb{E}_\mu(f^2)-2\sum_{x\in\mathbb{Z}} G(0,x)\mathbb{E}_\mu(f\ f\circ T^x)<\infty\,.$$

So the sequence $(W_{m,n})_{m,n\geq 0}$ satisfies all the conditions of Theorem 5 in Kingman (1968). Hence

$$\frac{W_{0,n}}{n} \xrightarrow[n\to+\infty]{\mathbb{P}\text{-a.s.}} \mathbb{E}_\mu(f^2) - 2\sum_{x\in\mathbb{Z}} G(0,x)\mathbb{E}_\mu(f\ f\circ T^x)\,.$$

By remarking that $W_{0,n} = -\sigma_n^2(f)$, Proposition 4.1 follows for positive functions f. If the function f is not positive, we can decompose it as

$$f = f1_{\{f\geq 0\}} - (-f)1_{\{f<0\}}$$

and, for all $x\in\mathbb{Z}$, $f\circ T^x$ as

$$f\circ T^x = f\circ T^x 1_{\{f\circ T^x\geq 0\}} - (-f\circ T^x)1_{\{f\circ T^x<0\}}\,.$$

Then, a simple calculation yields

$$\begin{aligned}\mathbb{E}_\mu(f\ f\circ T^x) &= \mathbb{E}_\mu(f\ f\circ T^x\ 1_{\{f\geq 0; f\circ T^x\geq 0\}}) + \mathbb{E}_\mu((-f)(-f\circ T^x)1_{\{f<0; f\circ T^x<0\}}) \\ &- \mathbb{E}_\mu(f\ (-f\circ T^x)\ 1_{\{f\geq 0; f\circ T^x<0\}}) - \mathbb{E}_\mu((-f)\ (f\circ T^x)\ 1_{\{f<0; f\circ T^x\geq 0\}})\,.\end{aligned}$$

By applying the previous reasoning at each of the four positive terms of the right-hand side, Proposition 4.1 follows. △

Remark:
Let us consider the unsymmetric random walk on nearest neighbours with $p>q$. Then, for $x\geq 0$,

$$G(0,x) = (p-q)^{-1}\,,$$

and for $x\leq -1$,

$$G(0,x) = (p-q)^{-1}\left(\frac{p}{q}\right)^x\,.$$

A simple calculation gives

$$\sigma^2(h-h\circ T) = 2\sum_{x\in\mathbb{Z}}\left[2G(0,x)-G(0,x+1)-G(0,x-1)\right]\mathbb{E}_\mu(h\ h\circ T^x) - 2\mathbb{E}_\mu(h^2) + 2\mathbb{E}_\mu(h\ h\circ T)$$

$$= -2\frac{p-1}{p}\mathbb{E}_\mu(h^2) + 2\mathbb{E}_\mu(h\ h\circ T) - 2\frac{(p-q)}{pq}\sum_{x\geq 1}\left(\frac{q}{p}\right)^x\mathbb{E}_\mu(h\ h\circ T^x)\,.$$

4.3 Proof of Theorem 4.1

Let us define $M_n = \max_{0\leq k\leq n}|S_k|$. First note that

$$\sum_{k=0}^n f\circ T^{S_k} = \sum_{|x|\leq M_n} N_n(x) f\circ T^x\,.$$

We want to apply Theorem 3.1 to the triangular array

$$\left\{X_{n,i} = \frac{N_n(i)}{\sqrt{n}} f\circ T^i,\ n\in\mathbb{N},\ -M_n\leq i\leq M_n\right\}\,. \tag{4.4}$$

As f belongs to $\mathbb{L}^2(\mu)$, the family $\left\{\left(f \circ T^i\right)^2\right\}_{i\in\mathbb{Z}}$ is uniformly integrable as it is stationary. It remains to prove that assumption (A_1) of Theorem 3.1 is satisfied for the triangular array defined by (4.4).

Proof of $(A_1)(i)$**:**
First, by Proposition 3.1. in Guillotin-Plantard & Schneider (2003), $\sum_{i=-M_n}^{M_n} a_{n,i}^2 = \frac{\alpha(n,0)}{n}$ converges $\mathbb{P}$-almost surely to $2G(0,0)-1$ as n goes to infinity. Then, by Proposition 4.1, we know that $\sigma_n^2(f)/n$ converges to $\sigma^2(f)$, which is assumed to be positive. Hence $(A_1)(i)$ is satisfied. △

Proof of $(A_1)(ii)$**:**
Now, by Proposition 3.2. in Guillotin-Plantard & Schneider (2003), we know that for every $\rho > 0$,

$$\max_{-M_n\le i\le M_n} |a_{n,i}| = \frac{1}{\sqrt{n}} \max_{i\in\mathbb{Z}} N_n(i) = o\left(n^{\rho-\frac{1}{2}}\right) \mathbb{P}-\text{almost surely.}$$

So $\sqrt{\frac{n}{\sigma_n^2(f)}}\ \max_{-M_n\le i\le M_n} |a_{n,i}|$ tends to zero $\mathbb{P}-$almost surely and assumption $(A_1)(ii)$ is satisfied. △

Hence Theorem 3.1 applied to $\sum_{i=-M_n}^{M_n} a_{n,i}\ f\circ T^i$, Proposition 4.1 and Slutsky Lemma yield the result. △

5 Examples

In this section, we present examples for which we can compute upper bounds for $\theta_2(n)$ for any $n \ge 1$. We refer to Chapter 3 in Dedecker & al (2007) and references therein for more details.

5.1 Example 1: causal functions of stationary sequences

Let $(E, \mathcal{E}, \mathbb{Q})$ be a probability space. Let $(\varepsilon_i)_{i\in\mathbb{Z}}$ be a stationary sequence of random variables with values in a measurable space $\mathcal{S}$. Assume that there exists a real valued function H defined on a subset of $\mathcal{S}^{\mathbb{N}}$, such that $H(\varepsilon_0, \varepsilon_{-1}, \varepsilon_{-2}, \ldots,)$ is defined almost surely. The stationary sequence $(\xi_n)_{n\in\mathbb{Z}}$ defined by $\xi_n = H(\varepsilon_n, \varepsilon_{n-1}, \varepsilon_{n-2}, \ldots)$ is called a causal function of $(\varepsilon_i)_{i\in\mathbb{Z}}$.

Assume that there exists a stationary sequence $({\varepsilon_i}')_{i\in\mathbb{Z}}$ distributed as $(\varepsilon_i)_{i\in\mathbb{Z}}$ and independent of $(\varepsilon_i)_{i\le 0}$. Define $\xi_n^* = H({\varepsilon_n}', {\varepsilon_{n-1}}', {\varepsilon_{n-2}}', \ldots)$. Clearly, ξ_n^* is independent of $\mathcal{M}_0 = \sigma(\xi_i\,,\ i \le 0)$ and distributed as ξ_n. Let $(\delta_2(i))_{i>0}$ be a non increasing sequence such that

$$\|\mathbb{E}\left(|\xi_i - \xi_i^*| \mid \mathcal{M}_0\right)\|_2 \le \delta_2(i)\,. \tag{5.5}$$

Then the coefficient θ_2 of the sequence $(\xi_n)_{n\ge 0}$ satisfies

$$\theta_2(i) \le \delta_2(i)\,. \tag{5.6}$$

Let us consider the particular case where the sequence of innovations $(\varepsilon_i)_{i\in\mathbb{Z}}$ is absolutely regular in the sense of Volkonskii & Rozanov (1959). Then, according to Theorem 4.4.7 in Berbee (1979), if E is rich enough, there exists $(\varepsilon_i')_{i\in\mathbb{Z}}$ distributed as $(\varepsilon_i)_{i\in\mathbb{Z}}$ and independent of $(\varepsilon_i)_{i\le 0}$ such that

$$\mathbb{Q}(\varepsilon_i \ne \varepsilon_i' \text{ for some } i \ge k \mid \mathcal{F}_0) = \frac{1}{2}\left\|\mathbb{Q}_{\tilde{\varepsilon}_k|\mathcal{F}_0} - \mathbb{Q}_{\tilde{\varepsilon}_k}\right\|_v,$$

where $\tilde{\varepsilon}_k = (\varepsilon_k, \varepsilon_{k+1}, \ldots)$, $\mathcal{F}_0 = \sigma(\varepsilon_i\,,\, i \leq 0)$, and $\|\cdot\|_v$ is the variation norm. In particular if the sequence $(\varepsilon_i)_{i\in\mathbb{Z}}$ is idependent and identically distributed, it suffices to take $\varepsilon'_i = \varepsilon_i$ for $i > 0$ and $\varepsilon'_i - \varepsilon''_i$ for $i \leq 0$, where $(\varepsilon''_i)_{i\in\mathbb{Z}}$ is an independent copy of $(\varepsilon_i)_{i\in\mathbb{Z}}$.

Application to causal linear processes:
In that case, $\xi_n = \sum_{j\geq 0} a_j \varepsilon_{n-j}$, where $(a_j)_{j\geq 0}$ is a sequence of real numbers. We can choose

$$\delta_2(i) \geq \|\varepsilon_0 - \varepsilon'_0\|_2 \sum_{j\geq i} |a_j| + \sum_{j=0}^{i-1} |a_j| \|\varepsilon_{i-j} - \varepsilon'_{i-j}\|_2\,.$$

From Proposition 2.3 in Merlevède & Peligrad (2002), we obtain that

$$\delta_2(i) \leq \|\varepsilon_0 - \varepsilon'_0\|_2 \sum_{j\geq i} |a_j| + \sum_{j=0}^{i-1} |a_j| \Big(2^2 \int_0^{\beta(\sigma(\varepsilon_k, k\leq 0), \sigma(\varepsilon_k, k\geq i-j))} Q^2_{\varepsilon_0}(u) \Big)^{1/2} du,$$

where Q_{ε_0} is the generalized inverse of the tail function $x \mapsto \mathbb{Q}(|\varepsilon_0| > x)$.

5.2 Example 2: iterated random functions

Let $(\xi_n)_{n\geq 0}$ be a real valued stationary Markov chain, such that $\xi_n = F(\xi_{n-1}, \varepsilon_n)$ for some measurable function F and some independent and identically distributed sequence $(\varepsilon_i)_{i>0}$ independent of ξ_0. Let ξ_0^* be a random variable distributed as ξ_0 and independent of $(\xi_0, (\varepsilon_i)_{i>0})$. Define $\xi_n^* = F(\xi_{n-1}^*, \varepsilon_n)$. The sequence $(\xi_n^*)_{n\geq 0}$ is distributed as $(\xi_n)_{n\geq 0}$ and independent of ξ_0. Let $\mathcal{M}_i = \sigma(\xi_j, 0 \leq j \leq i)$. As in Example 1, define the sequence $(\delta_2(i))_{i>0}$ by (5.5). The coefficient θ_2 of the sequence $(\xi_n)_{n\geq 0}$ satisfies the bound (5.6) of Example 1.

Let μ be the distribution of ξ_0 and $(\xi_n^x)_{n\geq 0}$ be the chain starting from $\xi_0^x = x$. With these notations, we can choose $\delta_2(i)$ such that

$$\delta_2(i) \geq \|\xi_i - \xi_i^*\|_2 = \Big(\int \int \| |\xi_i^x - \xi_i^y\|_2^2 \mu(dx)\mu(dy) \Big)^{1/2}.$$

For instance, if there exists a sequence $(d_2(i))_{i\geq 0}$ of positive numbers such that

$$\|\xi_i^x - \xi_i^y\|_2 \leq d_2(i)|x-y|,$$

then we can take $\delta_2(i) = d_2(i)\|\xi_0 - \xi_0^*\|_2$. For example, in the usual case where $\|F(x,\varepsilon_0) - F(y,\varepsilon_0)\|_2 \leq \kappa|x-y|$ for some $\kappa < 1$, we can take $d_2(i) = \kappa^i$.

An important example is $\xi_n = f(\xi_{n-1}) + \varepsilon_n$ for some κ-Lipschitz function f. If ξ_0 has a moment of order 2, then $\delta_2(i) \leq \kappa^i \|\xi_0 - \xi_0^*\|_2$.

5.3 Example 3: dynamical systems on $[0,1]$

Let $I = [0,1]$, T be a map from I to I and define $X_i = T^i$. If μ is invariant by T, the sequence $(X_i)_{i\geq 0}$ of random variables from (I,μ) to I is strictly stationary.

For any finite measure ν on I, we use the notations $\nu(h) = \int_I h(x)\nu(dx)$. For any finite signed measure ν on I, let $\|\nu\| = |\nu|(I)$ be the total variation of ν. Denote by $\|g\|_{1,\lambda}$ the $\mathbb{L}^1$-norm with respect to the Lebesgue measure λ on I.

Covariance inequalities. In many interesting cases, one can prove that, for any BV function h and any k in $\mathbb{L}^1(I, \mu)$,

$$|\operatorname{Cov}(h(X_0), k(X_n))| \leq a_n \|k(X_n)\|_1 (\|h\|_{1,\lambda} + \|dh\|) , \tag{5.7}$$

for some nonincreasing sequence a_n tending to zero as n tends to infinity.

Spectral gap. Define the operator $\mathcal{L}$ from $\mathbb{L}^1(I, \lambda)$ to $\mathbb{L}^1(I, \lambda)$ *via* the equality

$$\int_0^1 \mathcal{L}(h)(x)k(x)d\lambda(x) = \int_0^1 h(x)(k \circ T)(x)d\lambda(x) \quad \text{where } h \in \mathbb{L}^1(I, \lambda) \text{ and } k \in \mathbb{L}^\infty(I, \lambda).$$

The operator $\mathcal{L}$ is called the Perron-Frobenius operator of T. In many interesting cases, the spectral analysis of $\mathcal{L}$ in the Banach space of BV-functions equiped with the norm $\|h\|_v = \|dh\| + \|h\|_{1,\lambda}$ can be done by using the Theorem of Ionescu-Tulcea and Marinescu (see Lasota and Yorke (1974) and Hofbauer and Keller (1982)). Assume that 1 is a simple eigenvalue of $\mathcal{L}$ and that the rest of the spectrum is contained in a closed disk of radius strictly smaller than one. Then there exists a unique T-invariant absolutely continuous probability μ whose density f_μ is BV, and

$$\mathcal{L}^n(h) = \lambda(h)f_\mu + \Psi^n(h) \quad \text{with} \quad \|\Psi^n(h)\|_v \leq K\rho^n \|h\|_v. \tag{5.8}$$

for some $0 \leq \rho < 1$ and $K > 0$. Assume moreover that:

$$I_* = \{f_\mu \neq 0\} \text{ is an interval, and there exists } \gamma > 0 \text{ such that } f_\mu > \gamma^{-1} \text{ on } I_*. \tag{5.9}$$

Without loss of generality assume that $I_* = I$ (otherwise, take the restriction to I_* in what follows). Define now the Markov kernel associated to T by

$$P(h)(x) = \frac{\mathcal{L}(f_\mu h)(x)}{f_\mu(x)}. \tag{5.10}$$

It is easy to check (see for instance Barbour *et al.* (2000)) that $(X_0, X_1, \ldots, X_n)$ has the same distribution as $(Y_n, Y_{n-1}, \ldots, Y_0)$ where $(Y_i)_{i \geq 0}$ is a stationary Markov chain with invariant distribution μ and transition kernel P. Since $\|fg\|_\infty \leq \|fg\|_v \leq 2\|f\|_v\|g\|_v$, we infer that, taking $C = 2K\gamma(\|df_\mu\| + 1)$,

$$P^n(h) = \mu(h) + g_n \quad \text{with} \quad \|g_n\|_\infty \leq C\rho^n \|h\|_v. \tag{5.11}$$

This estimate implies (5.7) with $a_n = C\rho^n$ (see Dedecker & Prieur, 2005).

Expanding maps: Let $([a_i, a_{i+1}[)_{1 \leq i \leq N}$ be a finite partition of $[0, 1[$. We make the same assumptions on T as in Collet *et al* (2002).

1. For each $1 \leq j \leq N$, the restriction T_j of T to $]a_j, a_{j+1}[$ is strictly monotonic and can be extented to a function $\overline{T}_j$ belonging to $C^2([a_j, a_{j+1}])$.

2. Let I_n be the set where $(T^n)'$ is defined. There exists $A > 0$ and $s > 1$ such that $\inf_{x \in I_n} |(T^n)'(x)| > As^n$.

3. The map T is topologically mixing: for any two nonempty open sets U, V, there exists $n_0 \geq 1$ such that $T^{-n}(U) \cap V \neq \emptyset$ for all $n \geq n_0$.

If T satisfies 1.,2. and 3.,then (5.8) holds. Assume furthermore that (5.9) holds (see Morita (1994) for sufficient conditions). Then, arguing as in Example 4 in Section 7 of Dedecker & Prieur (2005), we can prove that for the Markov chain $(Y_i)_{i\geq 0}$ and the σ-algebras $\mathcal{M}_i = \sigma(Y_j\,,\, j \leq i)$, there exists a positive constant C such that $\theta_2(i) \leq C\rho^i$.

Remark:
In examples 2 and 3, the sequences are indexed by $\mathbb{N}$ and not by $\mathbb{Z}$. However, using existence Theorem of Kolmogorov (see Theorem 0.2.7 in Dacunha-Castelle & Duflo, 1983), if $(X_i)_{i\in\mathbb{N}}$ is a stationary process indexed by $\mathbb{N}$, there exists a stationary sequence $(Y_i)_{i\in\mathbb{Z}}$ indexed by $\mathbb{Z}$ such that for any $k \leq l \in \mathbb{Z}$, both marginals $(Y_k, \ldots, Y_l)$ and $(X_0, \ldots, X_{l-k})$ have the same distribution. Moreover, in examples 2 and 3, the sequences are Markovian, hence $\theta_2^Y(n) = \theta_2^X(n)$ for any $n \geq 1$. We then apply Theorem 4.1 to the sequence $(Y_i)_{i\in\mathbb{Z}}$. The limit variance can be rewritten as

$$\sigma^2(f) = 2\sum_{x\in\mathbb{Z}} G(0,x)\,\mathrm{Cov}(f(X_0), f(X_{|x|})) - \mathrm{Var}(f(X_0))\,.$$

6 Application to parametric estimation by random sampling

We investigate in this section the problem of parametric estimation by random sampling for second order stationary processes. We assume that we observe a stationary process $(\xi_i)_{i\in\mathbb{N}}$ at random times S_n, $n \geq 0$, where $(S_n)_{n\geq 0}$ is a non negative increasing random walk satisfying the assumptions of Section 4. In the case where the marginal expectation of the process $(\xi_i)_{i\in\mathbb{N}}$, m, is unknown, Deniau *et al.* estimate it using the sampled empirical mean $\hat{m}_n = \frac{1}{n}\sum_{i=1}^n \xi_{S_i}$. They measure the quality of this estimator by considering the following quadratic criterion function:

$$a(S) = \lim_{n\to+\infty} (n\,\mathrm{Var}\,\hat{m}_n)\,.$$

In the case where $(\mathrm{Cov}(\xi_1, \xi_{n+1}))_{n\in\mathbb{N}}$ is in l^1, we have

$$a(S) = \sum_{k=-\infty}^{+\infty} \mathrm{Cov}(\xi_{S_1}, \xi_{S_{|k|+1}}) < \infty\,.$$

We then get Corollary 6.1 below, which gives the asymptotic behaviour of the estimate $\hat{m}_n$ after centering and normalization.

Corollary 6.1 *Let us keep the assumptions of Section 4 on the random walk $(S_n)_{n\in\mathbb{N}}$ and on the process $(\xi_i)_{i\in\mathbb{N}}$. Assume moreover that $S_0 = 0$ and that $(S_{n+1} - S_n)_{n\in\mathbb{N}}$ takes its values in $\mathbb{N}^*$. Then, for $\mathbb{P}$-almost every $\omega \in \Omega$,*

$$\sqrt{n}\,(\hat{m}_n - m) \xrightarrow[n\to+\infty]{} \mathcal{N}(0, a(S))\,.$$

Proof of Corollary 6.1:
Corollary 6.1 can be deduced from Theorem 4.1 of Section 4 applied to $f = Id - m$. We have indeed $\sigma^2(f) = a(S)$. △

Let $\kappa \in \mathbb{R}_+$, $\kappa \geq 1$. A κ-optimal law for the step $S_{n+1} - S_n$ is a distribution which minimizes $a(S)$ under the constraint $\mathbb{E}(S_{n+1} - S_n) \leq \kappa$. We say that the sampling rate is less or equal to $1/\kappa$. Deniau *et al.* (1988) give sufficient conditions for the existence of a κ-optimal law of the step $S_{n+1} - S_n$. In the case of an A.R.(1) model $\xi_n = \rho\xi_{n-1} + \varepsilon_n$, $|\rho| < 1$, with ε a white noise, we know from Taga (1965) that there exists a unique κ-optimal law L_0 given by :

- $L_0 = \delta_1$ if $\rho < 0$ (the sampled process is the process itself),
- $L_0 = [1 - (\kappa - [\kappa])]\ \delta_{[\kappa]} + (\kappa - [\kappa])\ \delta_{[\kappa]+1}$ if $\rho > 0$ ($[\cdot]$ denotes the integer part).

7 Appendix

This section is devoted to the proof of Theorem 3.1 of Section 3.

Proof of Theorem 3.1:
In order to prove Theorem 3.1, we first use a classical truncation argument. For any $M > 0$, we define :

$$\varphi_M \ : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \varphi_M(x) = (x \wedge M) \vee (-M) \end{cases}$$

and

$$\varphi^M \ : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \varphi^M(x) = x - \varphi_M(x)\,. \end{cases}$$

We now prove the following Lindeberg condition :

$$\sigma_n^{-2} \sum_{i=-k_n}^{k_n} \mathbb{E}\left((\varphi^{\varepsilon\sigma_n}(X_{n,i}))^2 \right) \xrightarrow[n\to+\infty]{} 0\,. \tag{7.12}$$

We have, for n large enough,

$$\begin{aligned} \sigma_n^{-2} \textstyle\sum_{i=-k_n}^{k_n} \mathbb{E}\left((\varphi^{\varepsilon\sigma_n}(X_{n,i}))^2 \right) &\leq \sigma_n^{-2} \textstyle\sum_{i=-k_n}^{k_n} a_{n,i}^2 \mathbb{E}\left(\xi_i^2 \mathbf{1}_{|\xi_i||a_{n,i}|>\varepsilon\sigma_n} \right) \\ &\leq \sigma_n^{-2} \textstyle\sum_{i=-k_n}^{k_n} a_{n,i}^2 \mathbb{E}\left(\xi_i^2 \mathbf{1}_{|\xi_i|>\varepsilon\sigma_n/\max_j |a_{n,j}|} \right)\,. \end{aligned}$$

The last right hand term in the above inequalities is bounded by

$$\max_{-k_n\leq i\leq k_n} \left(\mathbb{E}\left(\xi_i^2 \mathbf{1}_{|\xi_i|>\varepsilon\sigma_n/\max_j |a_{n,j}|} \right) \right) \sigma_n^{-2} \sum_{i=-k_n}^{k_n} a_{n,i}^2\,, \tag{7.13}$$

which tends to zero as n goes to infinity, by assumptions (A_1) and (A_2).

By (7.13) we find a sequence of positive numbers $(\varepsilon_n)_{n\geq 1}$ such that $\varepsilon_n \xrightarrow[n\to+\infty]{} 0$, and

$$\max_{-k_n\leq i\leq k_n} \left(\mathbb{E}\left(\xi_i^2 \mathbf{1}_{|\xi_i|>\varepsilon_n\sigma_n/\max_j |a_{n,j}|} \right) \right) \sigma_n^{-2} \sum_{i=-k_n}^{k_n} a_{n,i}^2 \xrightarrow[n\to+\infty]{} 0\,. \tag{7.14}$$

Let us now prove that (7.14) yields

$$\sigma_n^{-2}\, \mathrm{Var} \left(\sum_{i=-k_n}^{k_n} \varphi^{\varepsilon_n\sigma_n}(X_{n,i}) \right) \xrightarrow[n\to+\infty]{} 0\,. \tag{7.15}$$

To prove (7.15), we need the following Lemma :

Lemma 7.1 *Assume that* $(\eta_i)_{i\in\mathbb{Z}}$ *is centered and satisfies conditions* (A_2) *and* (A_3) *of Theorem 3.1, then for any reals* $-k_n \le a \le b \le k_n$*,*

$$\operatorname{Var}\left(\sum_{i=a}^{b} a_{n,i}\eta_i\right) \le C\sum_{i=a}^{b} a_{n,i}^2\,,$$

with $C = \sup_{i\in\mathbb{Z}}\left(\mathbb{E}\eta_i^2\right) + 2\sqrt{\sup_{i\in\mathbb{Z}}\left(\mathbb{E}\eta_i^2\right)}\ \sum_{l=1}^{\infty}\theta_{1,2}(l)$.

Before proving Lemma 7.1, we finish the proof of (7.15).

For any fixed $n\ge 0$, and any $-k_n\le i\le k_n$ such that $a_{n,i}\ne 0$, define :

$$V_{n,i} = \varphi^{\varepsilon_n\sigma_n/|a_{n,i}|}(\xi_i) - \mathbb{E}\left(\varphi^{\varepsilon_n\sigma_n/|a_{n,i}|}(\xi_i)\right)\,.$$

If $a_{n,i}=0$, let $V_{n,i}=0$. As for any fixed $n\ge 0$, $-k_n\le i\le k_n$, the function

$$x\mapsto \varphi^{\varepsilon_n\sigma_n/|a_{n,i}|}(x) - \mathbb{E}\left(\varphi^{\varepsilon_n\sigma_n/|a_{n,i}|}(\xi_i)\right)$$

is 1-Lipschitz, we have for all $l\ge 1$, for all $k\ge 1$,

$$\theta_{k,2}^{V_{\cdot,n}}(l) \le \theta_{k,2}^{\xi}(l)\,,$$

where $V_{\cdot,n} = (V_{n,i})_{-k_n\le i\le k_n}$ and $\xi=(\xi_i)_{i\in\mathbb{Z}}$. Hence, for any fixed n, the sequence $(V_{n,i})_{-k_n\le i\le k_n}$ satisfies assumptions (A_2) and (A_3) of Theorem 3.1. Moreover, as

$$a_{n,i}\varphi^{\varepsilon_n\sigma_n/|a_{n,i}|}(\xi_i) = \varphi^{\varepsilon_n\sigma_n}(a_{n,i}\xi_i),$$

applying Lemma 7.1 yields

$$\sigma_n^{-2}\operatorname{Var}\Big(\sum_{i=-k_n}^{k_n}\varphi^{\varepsilon_n\sigma_n}(a_{n,i}\xi_i)\Big) \le C_n\sigma_n^{-2}\sum_{i=-k_n}^{k_n} a_{n,i}^2\,, \tag{7.16}$$

with $C_n = \sup_{-k_n\le i\le k_n}\left(\mathbb{E}V_{n,i}^2\right) + 2\sqrt{\sup_{-k_n\le i\le k_n}\left(\mathbb{E}V_{n,i}^2\right)}\ \sum_{l=1}^{\infty}\theta_{1,2}^{\xi}(l)$. It remains to prove that the right hand term in (7.16) converges to 0 as n goes to infinity.

We have, for n large enough,

$$\mathbb{E}V_{n,i}^2 \le \mathbb{E}\left(\xi_i^2\mathbf{1}_{|\xi_i|>\varepsilon_n\sigma_n/\max_j|a_{n,j}|}\right)\,. \tag{7.17}$$

Hence we conclude with (7.14).

Proof of Lemma 7.1:

$$\begin{aligned}\operatorname{Var}\Big(\sum_{j=a}^{b} a_{n,j}\eta_j\Big) &= \sum_{j=a}^{b} a_{n,j}^2\operatorname{Var}(\eta_j) + \sum_{i=a}^{b}\sum_{j=a;j\ne i}^{b} a_{n,i}\ a_{n,j}\operatorname{Cov}(\eta_i,\eta_j)\\ &\le \sum_{j=a}^{b} a_{n,j}^2\operatorname{Var}(\eta_j) + \sum_{i=a}^{b} a_{n,i}^2\sum_{j=a;j\ne i}^{b}|\operatorname{Cov}(\eta_i,\eta_j)|\end{aligned}$$

by remarking that $|a_{n,i}|\ |a_{n,j}| \le \frac{1}{2}(a_{n,i}^2 + a_{n,j}^2)$.
Then for any $j > i$, using Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned} |\operatorname{Cov}(\eta_i, \eta_j)| &= |\mathbb{E}\left(\eta_i \, \mathbb{E}\left(\eta_j | \mathcal{M}_i\right)\right)| \\ &\le ||\eta_i||_2 \ \ ||\mathbb{E}\left(\eta_j|\mathcal{M}_i\right)||_2 \\ &\le ||\eta_i||_2 \ \theta_{1,2}(j-i). \end{aligned}$$

As $(\eta_i)_{i\in\mathbb{Z}}$ is centered, and as $(\eta_i^2)_{i\in\mathbb{Z}}$ is uniformly integrable, we deduce that

$$\operatorname{Var}\Big(\sum_{j=a}^{b} a_{n,j}\eta_j\Big) \le C \sum_{j=a}^{b} a_{n,j}^2 \,,$$

with $C = \sup_{i\in\mathbb{Z}}\left(\mathbb{E}\eta_i^2\right) + 2\sqrt{\sup_{i\in\mathbb{Z}}\left(\mathbb{E}\eta_i^2\right)}\ \sum_{l=1}^{\infty}\theta_{1,2}(l)$ which is finite from assumptions (A_2) and (A_3). $\triangle$

Define $Z_{n,i}$ by

$$\frac{\varphi_{\varepsilon_n\sigma_n}(X_{n,i}) - \mathbb{E}\left(\varphi_{\varepsilon_n\sigma_n}(X_{n,i})\right)}{\sqrt{\operatorname{Var}\left(\sum_{i=-k_n}^{k_n}\varphi_{\varepsilon_n\sigma_n}(X_{n,i})\right)}} \equiv \frac{\varphi_{\varepsilon_n\sigma_n}(X_{n,i}) - \mathbb{E}\left(\varphi_{\varepsilon_n\sigma_n}(X_{n,i})\right)}{\sigma_n'}\,.$$

By (7.15) we conclude that, to prove Theorem 3.1, it is enough to prove it for the truncated sequence $(Z_{n,i})_{n\ge 0,-k_n\le i\le k_n}$, that is to show that

$$\sum_{i=-k_n}^{k_n} Z_{n,i} \xrightarrow[n\to+\infty]{\mathcal{D}} \mathcal{N}(0,1)\,. \tag{7.18}$$

The proof is now a variation on the proof of Theorem 4.1 in Utev (1990). Let

$$d_t(X,Y) = \left|\mathbb{E}e^{itX} - \mathbb{E}e^{itY}\right|\,.$$

To prove Theorem 3.1, it is enough to prove that for all t,

$$d_t\left(\sum_{i=-k_n}^{k_n} Z_{n,i}, \eta\right) \xrightarrow[n\to+\infty]{} 0\,,$$

with η the standard normal distribution. We first need some simple properties of the distance d_t. Let X, X_1, X_2, Y_1, Y_2 be random variables with zero means and finite second moments. We assume that the random variables Y_1, Y_2 are independent and that the distribution of X_j coincides with that of Y_j, $j = 1, 2$. We define $A_t(X) = d_t\left(X, \eta\sqrt{\mathbb{E}X^2}\right)$. We have then the following inequalities :

Lemma 7.2 (Lemma 4.3 in Utev, 1990)

$$A_t(X) \le \frac{2}{3}|t|^3\mathbb{E}|X|^3,$$

$$A_t(Y_1+Y_2) \le A_t(Y_1) + A_t(Y_2),$$

$$d_t(X_1+X_2,X_1) \le \frac{t^2}{2}\left(\mathbb{E}X_2^2 + (\mathbb{E}X_1^2\mathbb{E}X_2^2)^{1/2}\right),$$

$$d_t(\eta a,\eta b) \le \frac{t^2}{2}|a^2-b^2|.$$

We next need the following lemma :

Lemma 7.3 *Let $0<\varepsilon<1$. There exists some positive constant $C(\varepsilon)$ such that for all $a\in\mathbb{Z}$, for all $v\in\mathbb{N}^*$, $A_t\left(\sum_{i=a+1}^{a+v} Z_{n,i}\right)$ is bounded by*

$$C(\varepsilon)\left(|t|^3h^{2/\varepsilon}\sum_{i=a+1}^{a+v}\mathbb{E}\left(|Z_{n,i}|^3\right)+t^2\left(h^{\frac{\varepsilon-1}{2}}+\sum_{j\ :2^j\ge h^{1/\varepsilon}}2^{\frac{3j}{2}}g(2^{j\varepsilon})\right)\sigma'^{-2}_n\sum_{i=a+1}^{a+v}a_{n,i}^2\right),$$

where h is an arbitrary positive natural number and with g introduced in Assumption (A_3) of Theorem 3.1.

Before proving Lemma 7.3, we achieve the proof of Theorem 3.1. By Lemma 7.3, we have

$$d_t\left(\sum_{i=-k_n}^{k_n} Z_{n,i},\eta\right) = A_t\left(\sum_{i=-k_n}^{k_n} Z_{n,i}\right) \le C(t,\varepsilon)\left(h^{2/\varepsilon}\sum_{i=-k_n}^{k_n}\mathbb{E}(|Z_{n,i}|^3)+\delta(h)\right),$$

with $\delta(h)=\left(h^{\frac{\varepsilon-1}{2}}+\sum_{j\ :2^j\ge h^{\frac{1}{\varepsilon}}}2^{\frac{3j}{2}}g(2^{j\varepsilon})\right)\sigma'^{-2}_n\sum_{i=-k_n}^{k_n}a_{n,i}^2$.

From (7.15) we deduce that $\sigma_n/\sigma'_n \xrightarrow[n\to+\infty]{} 1$.

Hence using assumptions $(A_1)(i)$ and (A_3) we get $\delta(h)\xrightarrow[h\to+\infty]{} 0$.

On the other hand we have

$$\sum_{i=-k_n}^{k_n}\mathbb{E}(|Z_{n,i}|^3) \le 2\,\frac{\varepsilon_n\sigma_n}{\sigma'_n}\sum_{i=-k_n}^{k_n}\mathrm{Var}(Z_{n,i}). \tag{7.19}$$

For any fixed $n\ge 0$, and any $-k_n\le i\le k_n$ such that $a_{n,i}\ne 0$, define :

$$W_{n,i}=\varphi_{\varepsilon_n\sigma_n/|a_{n,i}|}(\xi_i)-\mathbb{E}\left(\varphi_{\varepsilon_n\sigma_n/|a_{n,i}|}(\xi_i)\right).$$

If $a_{n,i}=0$, let $W_{n,i}=0$. As for any fixed $n\ge 0$, $-k_n\le i\le k_n$, the function

$$x\mapsto \varphi_{\varepsilon_n\sigma_n/|a_{n,i}|}(x)-\mathbb{E}\left(\varphi_{\varepsilon_n\sigma_n/|a_{n,i}|}(\xi_i)\right)$$

is 1-Lipschitz, we have for all $l\ge 1$, for all $k\ge 1$,

$$\theta_{k,2}^{W_{\cdot,n}}(l)\le\theta_{k,2}^{\xi}(l),$$

where $W_{\cdot,n}=(W_{n,i})_{-k_n\le i\le k_n}$ and $\xi=(\xi_i)_{i\in\mathbb{Z}}$. Hence, arguing as for the proof of Lemma 7.1, we prove that for any fixed n, the sequence $(W_{n,i})_{-k_n\le i\le k_n}$ satisfies assumptions (A_2) and (A_3) of Theorem 3.1. Therefore, for any reals $-k_n\le a\le b\le k_n$,

$$\mathrm{Var}\left(\sum_{i=a}^{b} Z_{n,i}\right)\le C\sigma'^{-2}_n\sum_{i=a}^{b}a_{n,i}^2 \tag{7.20}$$

with $C = \sup_{i\in\mathbb{Z}} \left(2\mathbb{E}\xi_i^2\right) + 2\sqrt{\sup_{i\in\mathbb{Z}} \left(2\mathbb{E}\xi_i^2\right)}\ \sum_{l=1}^{\infty} \theta_{1,2}(l)$ which is finite from assumptions (A_2) and (A_3). Applying (7.20) with $a = b$, we get that the right hand term of (7.19) is bounded by $2\ \varepsilon_n \frac{\sigma_n}{\sigma'_n} \sum_{i=-k_n}^{k_n} \frac{C a_{n,i}^2}{\sigma'^2_n}$, which tends to zero as n tends to infinity, using assumption $(A_1)(i)$ and the fact that $\varepsilon_n \xrightarrow[n\to+\infty]{} 0$.

Consequently

$$\inf_{h\geq 1} \left(h^{2/\varepsilon} \sum_{i=-k_n}^{k_n} \mathbb{E}(|Z_{n,i}|^3) + \delta(h) \right) \xrightarrow[n\to+\infty]{} 0.$$

It achieves the proof of Theorem 3.1. $\triangle$

Proof of Lemma 7.3: Let $h \in \mathbb{N}^*$. Let $0 < \varepsilon < 1$. In the following, C, $C(\varepsilon)$ denote constants which may vary from line to line. Let κ_ε be a positive constant greater than 1 which will be precised further. Let $v < \kappa_\varepsilon\ h^{\frac{1}{\varepsilon}}$. We have

$$A_t\left(\sum_{i=a+1}^{a+v} Z_{n,i}\right) \leq \frac{2}{3}|t|^3\mathbb{E}\left|\sum_{i=a+1}^{a+v} Z_{n,i}\right|^3 \leq \frac{2}{3}\ \kappa_\varepsilon^2\ |t|^3\ h^{2/\varepsilon} \sum_{i=a+1}^{a+v} \mathbb{E}(|Z_{n,i}|^3) \tag{7.21}$$

since $|x|^3$ is a convex function.

Let now $v \geq \kappa_\varepsilon\ h^{\frac{1}{\varepsilon}}$. Without loss of generality, assume that $a = 0$. Let $\delta_\varepsilon = (1-\varepsilon^2+2\varepsilon)/2$. Define then

$$m = [v^\varepsilon],\ B = \left\{u \in \mathbb{N}\ :\ 2^{-1}(v - [v^{\delta_\varepsilon}]) \leq um \leq 2^{-1}v\right\},$$

$$A = \left\{u \in \mathbb{N}\ : 0 \leq u \leq v,\ \sum_{i=um+1}^{(u+1)m} a_{n,i}^2 \leq (m/v)^\varepsilon \sum_{i=1}^{v} a_{n,i}^2\right\}.$$

Following Utev (1991) we prove that, for $0 < \varepsilon < 1$, $A \cap B$ is not wide for v greater than κ_ε. We have indeed

$$|A\cap B| = |B| - |\overline{A}\cap B| \geq |B| - |\overline{A}| \geq \frac{v^{(1-\varepsilon^2)/2}}{2}\left(1 - 4v^{-(1-\varepsilon)^2/2}\right) - \frac{3}{2}\ ,$$

where $\overline{A}$ denotes the complementary of the set A. We can find κ_ε large enough so that $|A\cap B|$ be positive.

Let $u \in A\cap B$. We start from the following simple identity

$$\begin{aligned} Q &\equiv \sum_{i=1}^{v} Z_{n,i} \\ &= \sum_{i=1}^{um} Z_{n,i} + \sum_{i=um+1}^{(u+1)m} Z_{n,i} + \sum_{i=(u+1)m+1}^{v} Z_{n,i} \\ &\equiv Q_1 + Q_2 + Q_3. \end{aligned} \tag{7.22}$$

By Lemma 7.2,

$$d_t(Q, Q_1+Q_3) = d_t(Q, Q-Q_2) \leq \frac{t^2}{2}\left(\mathbb{E}Q_2^2 + (\mathbb{E}Q_2^2\mathbb{E}Q^2)^{1/2}\right). \tag{7.23}$$

Using (7.23) and (7.20), we get

$$d_t(Q, Q_1 + Q_3) \leq Ct^2 v^{\frac{(\varepsilon-1)\varepsilon}{2}} {\sigma'}_n^{-2} \sum_{i=1}^{v} a_{n,i}^2. \tag{7.24}$$

Now, given the random variables Q_1 and Q_3, we define two independent random variables g_1 and g_3 such that the distribution of g_i coincides with that of Q_i, $i = 1, 3$. We have

$$\begin{aligned} d_t(Q_1 + Q_3, g_1 + g_3) &= \left|\mathbb{E}(e^{itQ_1} - 1)(e^{itQ_3} - 1) - \mathbb{E}(e^{itQ_1} - 1)\mathbb{E}(e^{itQ_3} - 1)\right| \\ &\leq \left\|\mathbb{E}(e^{itQ_1} - 1)\right\|_2 \left\|\mathbb{E}\left(e^{itQ_3} - 1 - \mathbb{E}(e^{itQ_3} - 1) \mid \mathcal{M}_{um}\right)\right\|_2 \\ &\leq 2|t| \, \|\textstyle\sum_{i=1}^{um} Z_{n,i}\|_2 \, v \, |t| \, {\sigma'}_n^{-1} \left(\sum_{i=(u+1)m+1}^{v} |a_{n,i}|\right) \theta_2^{\xi}(m+1) \\ &\leq C \, t^2 \, v^{3/2} \, {\sigma'}_n^{-2} \left(\textstyle\sum_{i=1}^{v} a_{n,i}^2\right) g(v^{\varepsilon}), \end{aligned}$$

by (7.20), Definition 2.2 and Assumption (A_3) of Theorem 3.1. Hence

$$d_t(Q_1 + Q_3, g_1 + g_3) \leq Ct^2 f(v) {\sigma'}_n^{-2} \sum_{i=1}^{v} a_{n,i}^2, \tag{7.25}$$

where $f(v) = v^{3/2} \, g(v^{\varepsilon})$ is non-increasing by assumption (A_3) of Theorem 3.1.

We also have by Lemma 7.2

$$A_t(g_1 + g_3) \leq A_t(g_1) + A_t(g_3). \tag{7.26}$$

Finally, still by Lemma 7.2, and using Definition 2.2, we have

$$\begin{aligned} d_t\left(\eta\sqrt{\mathbb{E}(Q^2)}, \eta\sqrt{\mathbb{E}\left((g_1 + g_3)^2\right)}\right) &\leq \frac{t^2}{2}\left|\mathbb{E}(Q^2) - \mathbb{E}\left((g_1 + g_3)^2\right)\right| \\ &\leq \frac{t^2}{2}\left|\mathbb{E}(Q_2^2) + 2\mathbb{E}(Q_1Q_2) + 2\mathbb{E}(Q_2Q_3) + 2\mathbb{E}(Q_1Q_3)\right| \\ &\leq Ct^2\left(v^{\frac{(\varepsilon-1)\varepsilon}{2}} + f(v)\right) {\sigma'}_n^{-2} \sum_{i=1}^{v} a_{n,i}^2. \end{aligned} \tag{7.27}$$

Combining (7.24)-(7.27), we get the following recurrent inequality :

$$\begin{aligned} A_t\left(\textstyle\sum_{i=1}^{v} Z_{n,i}\right) &\leq A_t\left(\textstyle\sum_{i=1}^{um} Z_{n,i}\right) + A_t\left(\textstyle\sum_{i=(u+1)m+1}^{v} Z_{n,i}\right) \\ &+ Ct^2\left(v^{\frac{(\varepsilon-1)\varepsilon}{2}} + f(v)\right) {\sigma'}_n^{-2} \textstyle\sum_{i=1}^{v} a_{n,i}^2 \end{aligned}$$

for $v \geq \kappa_\varepsilon \, h^{\frac{1}{\varepsilon}} \geq \kappa_\varepsilon$.

We then need the following Lemma, which is a variation on Lemma 1.2. in Utev (1991).

Lemma 7.4 *For every $\varepsilon \in \,]0, 1[$, denote $\delta_\varepsilon = (1 - \varepsilon^2 + 2\varepsilon)/2$. Let a non-decreasing sequence of non-negative numbers $a(n)$ be specified, such that there exist non-increasing sequences of non-negative numbers $\varepsilon(k)$, $\gamma(k)$ and a sequence of naturals $T(k)$, satisfying conditions*

$$T(k) \leq 2^{-1}(k + [k^{\delta_\varepsilon}]),$$

$$a(k) \le \max_{k_0 \le s \le k} \left(a(T(s)) + \gamma(s)\right)$$

for any $k \ge k_0$ with an arbitrary $k_0 \in \mathbb{N}^*$. Then

$$a(n) \le a(n_0) + 2 \sum_{k_0 \le 2^j \le n} \gamma(2^j) \ ,$$

for any $n \ge k_0$, where one can take $n_0 = 2^c$ with $c > \frac{2-\delta_\varepsilon}{1-\delta_\varepsilon}$.

Proof of Lemma 7.4: The proof follows essentially the same lines as the proof of Lemma 1.2. in Utev (1991) and therefore is omitted here. △

We now apply Lemma 7.4 above with

- ⋆ $k_0 = \kappa_\varepsilon \ h^{\frac{1}{\varepsilon}}$,
- ⋆ for $k \ge k_0$, $T(k) = \max\{u_k m_k, k - u_k m_k - m_k\}$ where u_k and m_k are defined from k as u and m from v (see the proof of $A \cap B$ not wide),
- ⋆ $c < \frac{\ln(\kappa_\varepsilon)}{\ln(2)}$ (we may need to enlarge κ_ε),
- ⋆ for $s \ge k_0$, $\gamma(s) = C \ t^2 \ \left(s^{\frac{\varepsilon(\varepsilon-1)}{2}} + f(s)\right)$,
- ⋆ for $s \ge k_0$, $a(s) = \sup_{l \in \mathbb{Z}} \max_{k_0 \le i \le s} \dfrac{A_t\left(\sum_{j=l+1}^{l+i} Z_{n,j}\right)}{\sigma'^{-2}_n \sum_{j=l+1}^{l+i} a_{n,j}^2}$.

Applying Lemma 7.4 yields the statement of Lemma 7.3. △

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