



**HAL**  
open science

# The cycling normal form in dual braid monoids

Jean Fromentin

► **To cite this version:**

| Jean Fromentin. The cycling normal form in dual braid monoids. 2007. hal-00200732v1

**HAL Id: hal-00200732**

**<https://hal.science/hal-00200732v1>**

Preprint submitted on 21 Dec 2007 (v1), last revised 9 Dec 2008 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# THE CYCLING NORMAL FORM IN DUAL BRAID MONOIDS

JEAN FROMENTIN

**ABSTRACT.** We introduce the cycling normal form, a new normal form for the dual braid monoid. It is based on a natural operation called the  $\phi$ -splitting, which expresses every dual  $n$ -braid in terms of a finite sequence of braids of dual  $(n-1)$ -braids. We deduce a complete description of the Dehornoy ordering of the dual braid monoids: via the  $\phi$ -splitting, the ordering of dual  $n$ -braids is a lexicographical extension of the ordering of dual  $(n-1)$ -braids. We deduce a new proof for the existence of the braid ordering, and determine the order-type of the braid ordering of dual  $n$ -braids.

Dual braid monoids have been introduced by Birman, Ko, and Lee in [3]. The dual braid monoid  $B_n^{+*}$  is a certain submonoid of the  $n$ -strand braid group  $B_n$ , generated by a family that contains one generator  $\mathbf{a}_{i,j}$  for each pair of integers  $i, j$  between 1 and  $n$ . It is known that the monoid  $B_n^{+*}$  admits a Garside structure, where simple elements correspond to non-crossing partitions of  $n$ —see [2]. In particular, there exists a standard normal form associated with this Garside structure, namely the so-called greedy normal form.

In this paper, we introduce a new normal form called the *cycling normal form*. It stems from investigating the embedding of  $B_{n-1}^{+*}$  in  $B_n^{+*}$ , and follows from the easy observation that each element of  $B_n^{+*}$  admits a maximal right divisor that belongs to  $B_{n-1}^{+*}$ . The key result expresses every braid of  $B_n^{+*}$  in terms of a unique finite sequence of braids of  $B_{n-1}^{+*}$ . In the following statement,  $\phi_n$  denotes the conjugacy automorphism of  $B_n$  associated with the Garside element  $\delta_n$  of  $B_n^{+*}$ , which is  $\sigma_1\sigma_2\dots\sigma_{n-1}$  in terms of the Artin generators  $\sigma_i$ .

**Theorem 1.** *Each braid  $\beta$  in  $B_n^{+*}$  admits a unique decomposition*

$$\beta = \phi_n^{p-1}(\beta_p) \cdot \phi_n^{p-2}(\beta_{p-1}) \cdot \dots \cdot \phi_n(\beta_2) \cdot \beta_1,$$

*such that  $\beta_r$  is, for each  $r$ , the maximal right divisor of  $\phi_n^{p-r}(\beta_p) \cdot \dots \cdot \phi_n(\beta_{r+1}) \cdot \beta_r$  that lies in  $B_{n-1}^{+*}$ .*

The above finite sequence  $(\beta_p, \dots, \beta_1)$  will be called the  $\phi_n$ -splitting of  $\beta$ . A unique normal form is then obtained by a natural recursive definition.

It seems that the  $\phi_n$ -splitting is interesting in itself, and we think it can be applied in various situations. In the current paper, we shall use it to specifically study the ordering of the monoid  $B_n^{+*}$ . It is known since [9] and [15] that the braid group  $B_n$  is left-orderable, by an ordering whose restriction to the positive braid monoid  $B_n^+$  is a well-ordering. Initially introduced by complicated methods of self-distributive algebra, the Dehornoy ordering of braids has then received a lot

of alternative constructions that reflect the many different possible approaches—see [11, 12]. However, this ordering remains a complicated object, and many questions remain open.

Here, we shall study the restriction of the braid ordering to the dual monoid  $B_n^{+*}$ . Our main result is that the ordering of  $B_n^{+*}$  admits a simple construction from the ordering of  $B_{n-1}^{+*}$  via the  $\phi_n$ -splitting of Theorem 1. We prove:

**Theorem 2.** *Assume that  $\beta, \beta'$  belong to  $B_n^{+*}$ . Then  $\beta$  is smaller than  $\beta'$  in the Dehornoy ordering if and only if the  $\phi_n$ -splitting of  $\beta$  is smaller than  $\phi_n$ -splitting of  $\beta'$  with respect to the ShortLex-extension of the ordering on  $B_{n-1}^{+*}$ .*

We recall that, if  $(A, <)$  is an ordered set, a finite sequence  $s$  in  $A$  is said to be ShortLex-smaller than another sequence  $s'$  if either the length of  $s$  is strictly smaller than that of  $s'$ , or the two sequences have the same length and  $s$  is lexicographically smaller than  $s'$ .

Theorem 2 has several consequences. Firstly, it implies that the restriction of the braid ordering to  $B_n^{+*}$  is a well-ordering of ordinal type  $\omega^{\omega^{n-2}}$ . This refines a former result by Laver asserting that the restriction of the ordering to  $B_n^{+*}$  is a well-ordering, but leaving the determination of the ordinal type open. Secondly, the proof of Theorem 2 actually reproves the existence of the braid ordering. More precisely, it gives a new proof for the result that every nontrivial braid can be represented by a so-called  $\sigma$ -positive or  $\sigma$ -negative word (“Property C”). Thirdly, the proof of Theorem 2 turns out to be an algorithm for finding  $\sigma$ -positive word representatives. At the moment, the complexity analysis of this algorithm is not yet completed, but there are good reasons to conjecture that its time complexity is quadratic, and the space complexity is at most quadratic.

Our proof builds on the new approach developed in [8] for the Artin monoid  $B_n^+$ . This method can be easily adapted to the case of  $B_n^{+*}$ . Then, the core of the construction consists in a direct computation that provides the expected  $\sigma$ -positive expression. In contrast to the approach developed by Burckel in the case of  $B_n^+$ , our construction uses no transfinite induction—see [4, 5, 6], and therefore it remains elementary—although quite involved in the general case. This is probably the main advantage of using the Birman–Ko–Lee generators rather than the Artin generators.

The paper is organized as follows. In Section 1, we briefly review the basic properties of the dual braid monoids, prove Theorem 1, and deduce a new normal form result for  $B_n^{+*}$ . In Section 2, we introduce and study a new ordering  $<^*$  on  $B_n^{+*}$  based on the  $\phi_n$ -splitting. In Section 3, we establish that the ordering  $<^*$  of Section 2 actually coincides with the Dehornoy ordering, taking for granted a certain statement called the Key Lemma. Finally, in Section 4, we introduce the technical notion of a stair, which is a certain type of braid word, prove that  $\phi$ -normal words necessarily contain stairs, and deduce the Key Lemma.

## 1. DUAL BRAID MONOIDS

Dual braid monoids have been introduced by J. Birman, K.Y. Ko, and S. Lee in [3]. For each  $n$ , there exists a submonoid  $B_n^{+*}$  of  $B_n$  that properly includes the monoid  $B_n^+$  for  $n \geq 3$ , but still admits a Garside structure, *i.e.*, a good divisibility theory with lcm’s and gcd’s.

In this section, we briefly recall the construction of the dual braid monoid, and its standard Garside structure, with a particular emphasis on the associated cycling

automorphism that will play a significant role in subsequent constructions. We define the notion of  $\phi_n$ -splitting of a braid of  $B_n^{+*}$ , and prove Theorem 1.

**1.1. Presentation of the dual braid monoid.** We follow the usual notation. So  $B_n$  denotes the  $n$ -strand Artin braid group, which admits the presentation—see [1]

$$(1.1) \quad \left\langle \sigma_1, \dots, \sigma_{n-1}; \begin{array}{l} \sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for } |i-j| \geq 2 \\ \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad \text{for } |i-j| = 1 \end{array} \right\rangle.$$

The positive braid monoid  $B_n^+$  is defined to be the monoid with the presentation (1.1). In the standard geometric interpretation, the generator  $\sigma_i$  corresponds to the half-twist where the  $(i+1)$ st strand crosses over the  $i$ th strand. The monoid  $B_n^+$  admits a Garside structure—see [14] and [10, 13] for a general definition. We denote by  $\Delta_n$  the Garside element of  $B_n^+$  and  $\Phi_n$  its flip automorphism.

The dual braid monoid  $B_n^{+*}$  can be introduced as the submonoid of  $B_n$  generated by a new family of generators that consists of certain conjugates of the Artin generators  $\sigma_i$ .

**Definition 1.1.** For  $1 \leq i < j$  we put,

$$(1.2) \quad \mathbf{a}_{i,j} = \sigma_i \dots \sigma_{j-2} \sigma_{j-1} \sigma_{j-2}^{-1} \dots \sigma_i^{-1}.$$

For  $n \geq 2$ , we define the *dual braid monoid*  $B_n^{+*}$  to be the submonoid of  $B_n$  generated by all the elements  $\mathbf{a}_{i,j}$  with  $1 \leq i < j \leq n$ .



FIGURE 1. From the left to the right: diagram of the braids  $\mathbf{a}_{2,3}(= \sigma_3)$ ,  $\mathbf{a}_{1,3}(= \sigma_1 \sigma_2 \sigma_1^{-1})$  and  $\mathbf{a}_{1,4}(= \sigma_1 \sigma_2 \sigma_3 \sigma_2^{-1} \sigma_1^{-1})$ .

In the sequel, we write  $[i, j]$  for the interval  $\{i, \dots, j\}$  of  $\mathbb{N}$ , and we say that  $[i', j']$  is *nested* in  $[i, j]$  if we have  $i < i' < j' < j$ .

**Proposition 1.2** (Birman, Ko, Lee [3]). *The braid monoid  $B_n^{+*}$  is presented by the generators  $\mathbf{a}_{i,j}$  subject to the relations*

$$(1.3) \quad \mathbf{a}_{i,j} \mathbf{a}_{i',j'} = \mathbf{a}_{i',j'} \mathbf{a}_{i,j} \quad \text{for } [i, j] \text{ and } [i', j'] \text{ disjoint or nested,}$$

$$(1.4) \quad \mathbf{a}_{i,j} \mathbf{a}_{j,k} = \mathbf{a}_{j,k} \mathbf{a}_{i,k} = \mathbf{a}_{i,k} \mathbf{a}_{i,j} \quad \text{for } 1 \leq i < j < k \leq n.$$

**Remark 1.3.** Our generators  $\mathbf{a}_{i,j}$  are not those exactly considered in [3]. Precisely, for  $1 \leq s < t \leq n$ , let  $\tilde{\mathbf{a}}_{t,s}$  be  $\sigma_{t-1} \dots \sigma_{s+1} \sigma_s \sigma_{s+1}^{-1} \dots \sigma_{t-1}^{-1}$ . The generators considered in [3] are the  $\tilde{\mathbf{a}}_{t,s}$ , and the dual monoid is defined to be generated by the  $\tilde{\mathbf{a}}_{t,s}$ 's. Now, for all  $s, t$ , we have  $\Delta_n \tilde{\mathbf{a}}_{t,s} \Delta_n^{-1} = \mathbf{a}_{n-t+1, n-s+1}$ , and, therefore, the flip automorphism of  $B_n$  induces an isomorphism of our current  $B_n^{+*}$  to the submonoid of  $B_n$  considered in [3].

The invariance properties of the family of the braids  $\mathbf{a}_{i,j}$  can be visualized by drawing braid diagrams on a cylinder—see Figure 2. Then, it is natural to associate with  $\mathbf{a}_{i,j}$  the chord connecting the vertices  $i$  and  $j$  in a circle with  $n$  marked vertices.

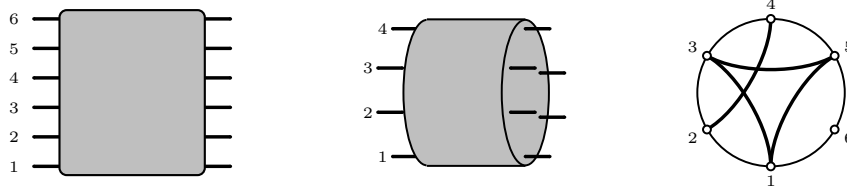


FIGURE 2. Rolling up the usual diagram helps up to see the symmetries of the  $\mathbf{a}_{i,j}$ 's. On the resulting cylinder,  $\mathbf{a}_{i,j}$  naturally corresponds to the chord connecting the vertices  $i$  and  $j$ .

Note that, in the chord representation, the relations of type (1.3) correspond to the fact that, for each chord triangle, the product of two adjacent edges taken clockwise does not depend on the initial vertex: for instance, on Figure 2, the triangle  $(1, 3, 5)$  gives  $\mathbf{a}_{1,3}\mathbf{a}_{3,5} = \mathbf{a}_{3,5}\mathbf{a}_{1,5} = \mathbf{a}_{1,5}\mathbf{a}_{1,3}$ . Relations of type (1.4) say that the generators associated with non-intersecting chords commute: for instance, on Figure 2, we see that  $\mathbf{a}_{2,4}$  and  $\mathbf{a}_{1,5}$  commute.

One of the very nice features with the monoid  $B_n^{+*}$  is that it admits a Garside structure different from that of  $B_n^+$ . This structure is used in [3] to construct a greedy normal form and to solve the word problem efficiently. Here, we shall not really use the whole Garside structure, but only the properties of the Garside element  $\delta_n$  and of the associated conjugation automorphism.

**Definition 1.4.** For  $n \geq 2$ , we define the braid  $\delta_n$  by

$$(1.5) \quad \delta_n = \mathbf{a}_{1,2} \mathbf{a}_{2,3} \dots \mathbf{a}_{n-1,n} = \sigma_1 \sigma_2 \dots \sigma_{n-1}.$$

**Proposition 1.5** (Birman, Ko, Lee [3]). *The dual braid monoid  $B_n^{+*}$  admits a Garside structure associated with the Garside element  $\delta_n$ . The associated inner automorphism  $\phi_n$ —defined by  $\phi_n(\beta) = \delta_n \beta \delta_n^{-1}$ —satisfies*

$$(1.6) \quad \phi_n(\mathbf{a}_{i,j}) = \begin{cases} \mathbf{a}_{i+1,j+1} & \text{for } j < n, \\ \mathbf{a}_{1,n} & \text{for } j = n. \end{cases}$$

In the chord representation the automorphism  $\phi_n$  acts as a clockwise rotation by  $2\pi/n$  in the circle with  $n$  marked vertices—see Figure 3.

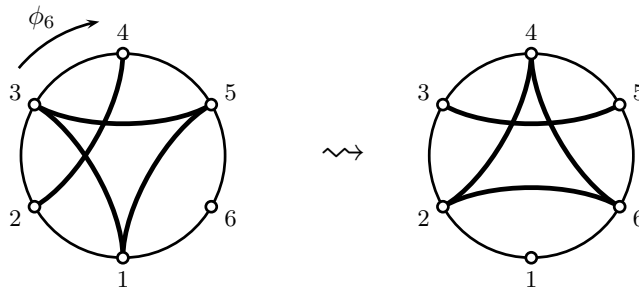


FIGURE 3. Representation of the inner automorphism  $\phi_n$  as a rotation by  $2\pi/n$  clockwise of the marked circle.

The Garside element  $\delta_n$  of  $B_n^{+*}$  is related to the Garside element  $\Delta_n$  of  $B_n^+$  by the relations  $\Delta_n^2 = \delta_n^n$  and  $\Delta_n = \delta_n \Delta_{n-1}$ . The flip automorphism  $\Phi_n$  of  $B_n^+$ , which is the inner automorphism associated with  $\Delta_n$ , is an involution, whereas  $\phi_n$  has order  $n$ , as is visible on the representation by a rotation.

**Lemma 1.6.** *For every  $\beta$  in  $B_n^{+*}$ , there exists  $k \geq 0$  such that  $\delta_n^k \beta$  belongs to  $B_n^+$ .*

*Proof.* Let  $\beta$  a braid of  $B_n^{+*}$ , then  $\beta$  belongs to  $B_n$ . The Garside structure of  $B_n^+$  guarantees the existence of  $k \geq 1$  such that  $\Delta_n^{2k'} \cdot \beta$  belongs to  $B_n^+$ . As  $\Delta_n^2 = \delta_n^n$  holds, the braid  $\delta_n^k \beta$  belongs to  $B_n^+$  for  $k = n k'$ .  $\square$

In the sequel, we shall often use the following braids  $\delta_{i,j}$  and some computational formulas that involve them.

**Definition 1.7.** For  $i \leq j$ , we define  $\delta_{i,j}$  to be  $\mathbf{a}_{i,i+1} \dots \mathbf{a}_{j-1,j} = \sigma_i \dots \sigma_{j-1}$ .

**Proposition 1.8.** *The following equalities are satisfied:*

$$\begin{aligned} (1.7) \quad \mathbf{a}_{i,j} &= \delta_{i,j} \delta_{i,j-1}^{-1}, & \text{for } i \leq j \\ (1.8) \quad \delta_{i,j} &= \delta_{i,k} \delta_{k,j} & \text{for } i \leq k \leq j, \\ (1.9) \quad \delta_{i,j} \delta_{i',j'} &= \delta_{i',j'} \delta_{i,j} & \text{for } i \leq j < i' \leq j', \\ (1.10) \quad \delta_{m,n}^{-1} \mathbf{a}_{i,j} \delta_{m,n} &= \phi_n^{-1}(\phi_m(\mathbf{a}_{i,j})) & \text{for } i < j \leq m \leq n. \end{aligned}$$

*Proof.* Relation (1.7) holds by definition of  $\mathbf{a}_{i,j}$ . Relation (1.8) holds by definition of  $\delta_{i,j}$ . For Relation (1.9), we observe that the Artin generator of greatest index in  $\delta_{i,j}$  is  $\sigma_{j-1}$ , while the Artin generator of lower index in  $\delta_{i',j'}$  is  $\sigma_{i'}$ . Then relation  $j < i'$  implies  $j-1 \leq i' - 2$ , and we can apply the Artin commutativity relation to obtain the expected result. It remains to prove the Relation (1.10). From (1.8), we obtain  $\delta_{1,n} = \delta_{1,m} \delta_{m,n}$ , that implies  $\delta_{m,n} = \delta_{1,m}^{-1} \delta_{1,n}$ , i.e.,  $\delta_m^{-1} \delta_n$ . Then  $\delta_{m,n}^{-1} \mathbf{a}_{i,j} \delta_{m,n}$  becomes  $\delta_n^{-1} \delta_m \mathbf{a}_{i,j} \delta_m^{-1} \delta_n$ . As, by hypothesis,  $\mathbf{a}_{i,j}$  lies in  $B_m^{+*}$ , the underlined factor equals  $\phi_m(\mathbf{a}_{i,j})$ . Finally, from  $B_m^{+*} \subseteq B_n^{+*}$  and  $\phi_m(\mathbf{a}_{i,j}) \in B_m^{+*}$ , we obtain  $\delta_n^{-1} \phi_m(\mathbf{a}_{i,j}) \delta_n = \phi_n^{-1}(\phi_m(\mathbf{a}_{i,j}))$ .  $\square$

**1.2. The  $\phi_n$ -splitting.** It is shown in [8] how to associate with every braid  $\beta$  of  $B_n^+$  a unique sequence of braids in  $B_{n-1}^+$ , called the  $B_{n-1}^+$ -splitting of  $\beta$ , that specifies  $\beta$  completely. In this section, we introduce a similar construction for the dual braid monoid  $B_n^{+*}$ . The automorphisms  $\phi_n$  and  $\Phi_n$  do not coincide: in particular, we saw that  $\Phi_n$  has order 2, while  $\phi_n$  has order  $n$ . Hence, some adaptation of the construction is needed. However, the properties of divisibility in the monoids  $B_n^{+*}$  and  $B_n^+$  are similar, so the needed changes are minor.

The initial observation of [8] is that each braid in the monoid  $B_n^+$  admits a unique, well-defined, maximal right divisor that lies in the submonoid  $B_{n-1}^+$ . The same phenomenon occurs with the dual monoid  $B_n^{+*}$ .

**Lemma 1.9.** *For  $n \geq 3$ , every braid  $\beta$  of  $B_n^{+*}$  admits a maximal right divisor  $\beta_1$  lying in  $B_{n-1}^{+*}$  and the maximal right divisor of  $\beta \beta_1^{-1}$  lying in  $B_{n-1}^{+*}$  is trivial.*

*Proof.* The submonoid  $B_{n-1}^{+*}$  of  $B_n^{+*}$  is closed under right divisor and left lcm. Hence we can apply Lemma 1.12 of [8].  $\square$

**Definition 1.10.** The braid  $\beta_1$  given by Lemma 1.9 is called the  $B_{n-1}^{+*}$ -tail of  $\beta$ .

**Example 1.11.** Let us compute the  $B_2^{+*}$ -tail of  $\delta_3^2$ . By applying twice Relation (1.4) to the word  $\mathbf{a}_{1,2}\mathbf{a}_{2,3}\mathbf{a}_{1,2}\mathbf{a}_{2,3}$  representing  $\delta_3^2$ , we obtain successively  $\mathbf{a}_{1,2}\mathbf{a}_{2,3}\mathbf{a}_{1,3}\mathbf{a}_{1,2}$  and  $\mathbf{a}_{1,2}\mathbf{a}_{1,3}\mathbf{a}_{1,2}^2$ . We claim that  $\mathbf{a}_{1,2}\mathbf{a}_{1,3}$  is not right divisible by  $\mathbf{a}_{1,2}$ , hence the  $B_2^{+*}$ -tail of  $\delta_3^2$  is  $\mathbf{a}_{1,2}^2$ .

By definition of the  $B_{n-1}^{+*}$ -tail, the quotient braid  $\beta'_1 = \beta\beta_1^{-1}$  is right divisible by no  $\mathbf{a}_{i,j}$  with  $j \leq n-1$ , hence, if it is not trivial, it must be right divisible by at least one generator  $\mathbf{a}_{i,n}$ .

**Lemma 1.12.** *Every generator  $\mathbf{a}_{i,j}$  of  $B_n^{+*}$  belongs to  $B_{n-1}^{+*} \cup \phi_n(B_{n-1}^{+*}) \cup \phi_n^2(B_{n-1}^{+*})$ .*

*Proof.* The braids  $\mathbf{a}_{i,j}$  with  $j \leq n-1$  belong to  $B_{n-1}^{+*}$ . Then, the braids  $\mathbf{a}_{i,n}$  with  $i \geq 2$  belong to  $\phi_n(B_{n-1}^{+*})$ , as we can write  $\mathbf{a}_{i,n} = \phi_n(\mathbf{a}_{i,n-1})$ . There remains the braid  $\mathbf{a}_{1,n}$ : it is equal to  $\phi_n(\mathbf{a}_{n-1,n})$ , hence to  $\phi_n^2(\mathbf{a}_{n-2,n-1})$ , and, therefore, it belongs to  $\phi_n^2(B_{n-1}^{+*})$ .  $\square$

In [8], one obtains a distinguished decomposition for every braid in  $B_n^+$  by considering the  $B_{n-1}^+$ -tail and the  $\Phi_n(B_{n+1}^+)$ -tail alternatively. Here, we shall use the  $B_{n-1}^{+*}$ -tail, the  $\phi_n(B_{n-1}^{+*})$ -tail, ..., and the  $\phi_n^{n-1}(B_{n-1}^{+*})$ -tail cyclically to obtain a distinguished decomposition for every braid of  $B_n^{+*}$ .

We easily obtain the result stated as Theorem 1 in the introduction—an easy result actually.

**Proposition 1.13.** *Assume that  $\beta$  is a nontrivial braid in  $B_n^{+*}$ . Then there exists a unique sequence  $(\beta_p, \dots, \beta_1)$  in  $B_{n-1}^{+*}$  satisfying  $\beta_p \neq 1$  and*

$$(1.11) \quad \beta = \phi_n^{p-1}(\beta_p) \cdot \dots \cdot \phi_n(\beta_2) \cdot \beta_1,$$

$$(1.12) \quad \text{for each } r, \text{ the braid } \beta_r \text{ is the } B_{n-1}^{+*}\text{-tail of } \phi_n^{p-r}(\beta_p) \cdot \dots \cdot \beta_r.$$

*Proof.* The argument is similar to Proposition 1.16 of [8] and we recall it. Starting from  $\beta^{(0)} = \beta$ , we define two sequences of braids, denoted  $\beta^{(r)}$  and  $\beta_r$ , by

$$(1.13) \quad \beta^{(r)} = \phi_n^{-1}(\beta^{(r-1)}\beta_r^{-1}) \text{ and } \beta_r = \text{tail}_{n-1}(\beta^{(r-1)}),$$

where  $\text{tail}_{n-1}$  denotes the  $B_{n-1}^{+*}$ -tail. Firstly, we prove the relations:

$$(1.14) \quad \beta = \phi_n^r(\beta^{(r)}) \cdot \phi_n^{r-1}(\beta_r) \cdot \dots \cdot \beta_1,$$

$$(1.15) \quad \text{tail}_{n-1}(\phi_n(\beta^{(r)})) = 1$$

for every  $r \geq 1$ . Assume  $r = 1$ . Lemma 1.9 implies that the  $B_{n-1}^{+*}$ -tail of  $\beta\beta_1^{-1}$  is trivial. Then, as  $\phi_n(\beta^{(1)})$  equals  $\beta\beta_1^{-1}$ , the  $B_{n-1}^{+*}$ -tail of  $\phi_n(\beta^{(1)})$  is trivial, and the relation  $\beta = \phi_n(\beta^{(1)}) \cdot \beta_1$  holds. Assume  $r \geq 2$ . By construction of  $\beta^{(r)}$ , we have  $\phi_n(\beta^{(r)}) = \beta^{(r-1)}\beta_r^{-1}$ , hence  $\beta^{(r-1)} = \phi_n(\beta^{(r)}) \cdot \beta_r$ . Then we have the relation

$$(1.16) \quad \phi_n^{r-1}(\beta^{(r-1)}) = \phi_n^r(\beta^{(r)}) \cdot \phi_n^{r-1}(\beta_r)$$

On the other hand, by induction hypothesis, we have

$$(1.17) \quad \beta = \phi_n^{r-1}(\beta^{(r-1)}) \cdot \phi_n^{r-2}(\beta_{r-1}) \cdot \dots \cdot \beta_1.$$

Substituting (1.16) in (1.17), we obtain (1.14). As  $\beta_r$  is the  $B_{n-1}^{+*}$ -tail of  $\beta^{(r)}$ , Lemma 1.9 gives (1.15).

By construction, the sequence of right divisors of  $\beta$ ,

$$\beta_1, \phi_n(\beta_2)\beta_1, \phi_n^2(\beta_3)\phi_n(\beta_2)\beta_1, \dots$$

is non-decreasing, and, therefore, it is eventually constant. Hence, by right cancellativity, there exists  $p$  such that for  $r \geq p$ , we have  $\phi_n^{r-1}(\beta_r) \cdots \beta_1 = \phi_n^{p-1}(\beta_p) \cdots \beta_1$ . Then (1.14) implies  $\beta = \phi_n^p(\beta^{(p)}) \phi_n^{p-1}(\beta_p) \cdots \beta_1$ , with  $\beta_p \neq 1$  whenever  $p$  is chosen to be minimal.

By definition of  $p$ , the  $B_{n-1}^{+*}$ -tails of  $\beta^{(p)}$ ,  $\phi_n(\beta^{(p)})$  and  $\phi_n^2(\beta^{(p)})$  are trivial, and Lemma 1.12 implies that  $\beta^{(p)}$  is right divisible by no  $\mathbf{a}_{i,j}$  with  $1 \leq i < j \leq n$ . Hence  $\beta^{(p)}$  is trivial, and we have  $\beta = \phi_n^{p-1}(\beta_p) \cdots \beta_1$ .

The uniqueness of the  $B_{n-1}^{+*}$ -tail implies the uniqueness of the  $\phi_n$ -splitting.  $\square$

**Definition 1.14.** The unique sequence  $(\beta_p, \dots, \beta_1)$  of braids introduced in Proposition 1.13 is called the  $\phi_n$ -splitting of  $\beta$  and its length is called the  $B_{n-1}^{+*}$ -breadth of  $\beta$ .

As the notion of  $B_{n-1}^{+*}$  is fundamental in this paper, we describe several examples.

**Example 1.15.** We start with the  $\phi_n$ -splitting of the generators of  $B_n^{+*}$ , i.e.,  $\mathbf{a}_{i,j}$  with  $1 \leq i < j \leq n$ . For  $j \leq n-1$ , the generator  $\mathbf{a}_{i,j}$  belongs to  $B_{n-1}^{+*}$ , then its  $\phi_n$ -splitting is  $(\mathbf{a}_{i,j})$ . As  $\mathbf{a}_{i,n}$  does not lie in  $B_{n-1}^{+*}$ , the rightmost entry in its  $\phi_n$ -splitting is trivial. As we have  $\phi_n^{-1}(\mathbf{a}_{i,n}) = \mathbf{a}_{i-1,n-1}$  for  $i \geq 2$ , the  $\phi_n$ -splitting of  $\mathbf{a}_{i,n}$  with  $i \geq 2$  is  $(\mathbf{a}_{i-1,n-1}, 1)$ . The braids  $\mathbf{a}_{1,n}$  and  $\phi_n^{-1}(\mathbf{a}_{1,n}) = \mathbf{a}_{n-1,n}$  do not lie in  $B_{n-1}^{+*}$ , but  $\phi_n^{-2}(\mathbf{a}_{1,n}) = \mathbf{a}_{n-2,n-1}$  does. So the  $\phi_n$ -splitting of  $\mathbf{a}_{1,n}$  is  $(\mathbf{a}_{n-2,n-1}, 1, 1)$ . Then the  $\phi_n$ -splitting is

$$(1.18) \quad \begin{cases} (\mathbf{a}_{i,j}) & \text{for } i < j \leq n-1 \\ (\mathbf{a}_{i-1,n-1}, 1) & \text{for } 2 \leq i \text{ and } j = n \\ (\mathbf{a}_{n-2,n-1}, 1, 1) & \text{for } i = 1 \text{ and } j = n \end{cases}$$

**Remark 1.16.** If  $\beta_1$  is the  $B_{n-1}^{+*}$ -tail of a braid  $\beta$ , then, by definition, the  $B_{n-1}^{+*}$ -tail of  $\beta \beta_1^{-1}$  is trivial. Hence, Condition (1.12) is equivalent to

$$(1.19) \quad \text{for each } r \geq 1, \text{ the } B_{n-1}^{+*}\text{-tail of } \phi_n^{p-r} \beta_p \cdots \phi_n \beta_{r+1} \text{ is trivial.}$$

**Example 1.17.** We reconsider Example 1.11 and compute the  $\phi_3$ -splitting of  $\delta_3^2$ . We know that the  $B_2^{+*}$ -tail of  $\delta_3^2$  is  $\mathbf{a}_{1,2}^2$ . We claim that the  $B_2^{+*}$ -tail of  $\phi_3^{-1}(\mathbf{a}_{1,2} \mathbf{a}_{1,3})$  which is  $\mathbf{a}_{1,3} \mathbf{a}_{2,3}$ , is trivial. The  $B_2^{+*}$ -tail of  $\phi_3^{-1}(\mathbf{a}_{1,3} \mathbf{a}_{2,3})$ , which is  $\mathbf{a}_{2,3} \mathbf{a}_{1,2}$ , is  $\mathbf{a}_{1,2}$ . And, finally, the  $B_2^{+*}$ -tail of  $\phi_3^{-1}(\mathbf{a}_{2,3})$ , which is  $\mathbf{a}_{1,2}$ , is obviously  $\mathbf{a}_{1,2}$ . Therefore, the  $\phi_3$ -splitting of  $\delta_3^2$  is  $(\mathbf{a}_{1,2}, \mathbf{a}_{1,2}, 1, \mathbf{a}_{1,2}^2)$ , and its 3-breadth is 4.

**1.3. The  $\phi$ -normal form.** We shall now deduce from the  $\phi_n$ -splitting a unique normal form for the elements of  $B_n^{+*}$ , i.e., identify for each braid  $\beta$  in  $B_n^{+*}$  a distinguished braid word that represents  $\beta$

The principle is as follows. First, each braid of  $B_2^{+*}$  is represented by a unique word  $\mathbf{a}_{1,2}^p$ . Then, the  $\phi_n$ -splitting provides a distinguished decomposition for every braid of  $B_n^{+*}$  in terms of braids of  $B_{n-1}^{+*}$ . Hence, we can obtain a distinguished word representing  $\beta$  starting from words representing the entries of the  $\phi_n$ -splitting of  $\beta$ . In this way, we shall obtain a normal form on  $B_n^{+*}$  by induction on  $n$ .

**Definition 1.18.** A word on the alphabet of  $\mathbf{a}_{i,j}$  with  $i < j \leq n$  is called a *dual  $n$ -braid word*. The collection of all dual  $n$ -braid words is denoted by  ${}^*B_n^+$ . For  $w$  in  ${}^*B_n^+$ , the braid represented by  $w$  is denoted by  $\overline{w}$ . Two dual braid words  $w, w'$  representing the same braid are said to be *equivalent*, denoted  $w \equiv w'$ .



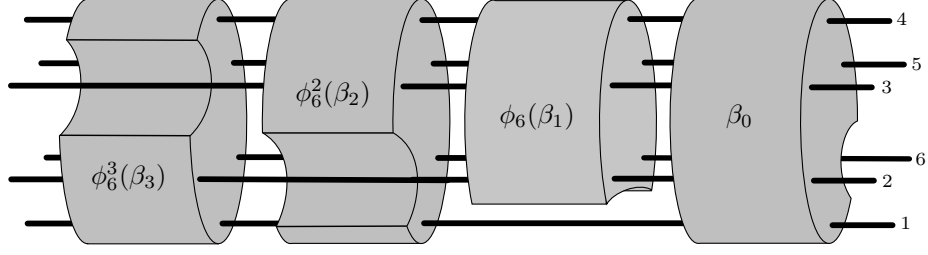


FIGURE 4. The  $\phi_n$ -splitting of a braid of  $B_n^{+*}$ —here  $B_6^{+*}$ . Starting from the right, we take the maximal right divisor that keeps the last strand—here the sixth—unbraided, then rotate counterclockwise by  $2\pi/n$ —here  $2\pi/6$ —and take the maximal right divisor that keeps the first strand unbraided, etc.

**Remark 1.19.** By Proposition 1.5, the application  $\phi_n$  maps a letter to another letter. Then  $\phi_n$  define an application on dual braid words, which is also denoted  $\phi_n$ .

**Definition 1.20.** (i) For  $\beta$  in  $B_2^{+*}$ , the  $\phi$ -normal form of  $\beta$  is defined to be the unique word  $\mathbf{a}_{1,2}^c$  that represents  $\beta$ .

(ii) For  $n \geq 3$  and  $\beta$  in  $B_n^{+*}$ , the  $\phi$ -normal form of  $\beta$  is defined to be the word  $\phi_n^{p-1}(w_p) \dots w_1$ , where  $(\beta_p, \dots, \beta_1)$  is the  $\phi_n$ -splitting of  $\beta$  and, for each  $r$ , the word  $w_r$  is the  $\phi$ -normal form of  $\beta_r$ .

**Example 1.21.** Let us compute the  $\phi$ -normal form of  $\delta_4^2$ . It is easy to check the equality  $\delta_4^2 = \mathbf{a}_{1,2} \mathbf{a}_{1,4} \delta_3^2$ , and that the  $\phi_4$ -splitting of  $\delta_4^2$  is  $(\mathbf{a}_{2,3}, \mathbf{a}_{2,3}, 1, \delta_3^2)$ . The  $\phi_3$ -splitting of  $\mathbf{a}_{2,3}$  is  $(\mathbf{a}_{1,2}, 1)$ , then the  $\phi$ -normal form of  $\mathbf{a}_{2,3}$  is  $\phi_3(\mathbf{a}_{1,2})$  that is  $\mathbf{a}_{2,3}$ . We saw in Example 1.17 that the  $\phi_3$ -splitting of  $\delta_3^2$  is  $(\mathbf{a}_{1,2}, \mathbf{a}_{1,2}, 1, \mathbf{a}_{1,2}^2)$ . Then the  $\phi$ -normal form of  $\delta_3^2$  is  $\phi_3^3(\mathbf{a}_{1,2}) \cdot \phi_3^2(\mathbf{a}_{1,2}) \cdot \phi_3(1) \cdot \mathbf{a}_{2,3}$  that is  $\mathbf{a}_{1,2} \cdot \mathbf{a}_{1,3} \cdot \varepsilon \cdot \mathbf{a}_{1,2} \mathbf{a}_{1,2}$ , i.e.,  $\mathbf{a}_{1,2} \mathbf{a}_{1,3} \mathbf{a}_{1,2} \mathbf{a}_{1,2}$ . Finally the  $\phi$ -normal form of  $\delta_4^2$  is  $\phi_4^3(\mathbf{a}_{2,3}) \cdot \phi_4^2(\mathbf{a}_{2,3}) \cdot \phi_4(1) \cdot \mathbf{a}_{1,2} \mathbf{a}_{1,3} \mathbf{a}_{1,2} \mathbf{a}_{1,2}$  that is  $\mathbf{a}_{1,2} \cdot \mathbf{a}_{1,4} \cdot \varepsilon \cdot \mathbf{a}_{1,2} \mathbf{a}_{1,3} \mathbf{a}_{1,2} \mathbf{a}_{1,2}$ , i.e.,  $\mathbf{a}_{1,2} \mathbf{a}_{1,4} \mathbf{a}_{1,2} \mathbf{a}_{1,3} \mathbf{a}_{1,2} \mathbf{a}_{1,2}$ .

The construction is exactly similar to the alternating normal form in [8]. As for computational issues, all procedures mentioned here for  $B_{n-1}^{+*}$  have the same complexity as their  $B_n^+$ -counterparts. In particular, the  $\phi$ -normal form of an  $n$ -strand braid specified by a dual braid word of length  $\ell$  can be computed in time  $\mathcal{O}(\ell^2)$ .

**1.4. Constraints on the splittings.** We now establish some constraints that are necessarily satisfied by the entries of a  $\phi_n$ -splitting. Indeed, not every sequence of braids in  $B_{n-1}^{+*}$  need to be the  $\phi_n$ -splitting of a braid in  $B_n^{+*}$ , and we shall often use the technical properties that are established below.

As a braid is represented by a unique  $\phi$ -normal word, we can unambiguously use the syntactical properties of its normal form.

**Definition 1.22.** We say that a braid  $\beta$  contains (resp. ends with)  $\mathbf{a}_{i,j}$  if the  $\phi$ -normal form of  $\beta$  does.

With these notions, we can state some constraints that are necessarily satisfied by the entries of a  $\phi_n$ -splitting.

**Lemma 1.23.** *Assume that  $(\beta_p, \dots, \beta_1)$  is a  $\phi_n$ -splitting.*

(i) *For  $r \geq 2$ , the last letter of  $\beta_r$  is  $\mathbf{a}_{i,n-1}$  for some  $i$  except if  $\beta_r$  is trivial.*

(ii) *For  $r \geq 3$ , the braid  $\beta_r$  is non-trivial.*

(iii) *If  $\beta_r = \beta'_r \mathbf{a}_{n-2,n-1}$  holds with  $\beta'_r \neq 1$  for  $r \geq 2$ , then the last letter of  $\beta'_r$  is  $\mathbf{a}_{i,n-1}$  for some  $i$ .*

*Proof.* (i) Let  $\mathbf{a}_{i,j}$  be the last letter of  $\beta_r$  for  $r \geq 2$ . Condition (1.19) implies that the  $B_{n-1}^{+*}$ -tail of  $\phi_n^{p-r+1}(\beta_p) \cdot \dots \cdot \phi_n(\beta_r)$  is trivial, so, *a fortiori*,  $\phi_n(\mathbf{a}_{i,j})$  does not lie in  $B_{n-1}^{+*}$ , *i.e.*,  $j = n-1$ .

(ii) Assume  $p \geq 4$ . By definition of splitting, the braid  $\beta_p$  is not trivial, and (i) implies that its last letter is  $\mathbf{a}_{k,n-1}$  for some  $k$ . Assume that  $\beta_r$  is trivial with  $r = p-1 \geq 3$ . Condition (1.19) implies that the  $B_{n-1}^{+*}$ -tail of  $\phi_n^2(\beta_p) \phi_n(\beta_{p-1})$ , *i.e.*,  $\phi_n^2(\beta_p)$ , is trivial. Hence  $\phi_n^2(\mathbf{a}_{k,n-1})$  does not lie in  $B_{n-1}^{+*}$ , that implies  $k = n-2$ . Now, we state that  $\phi_n^3(\mathbf{a}_{n-2,n-1}) = \mathbf{a}_{1,2}$  quasi-commutes with  $\phi_n(\beta_{p-2})$  and eventually obtain a contradiction. As we have

$$\mathbf{a}_{1,2} \mathbf{a}_{i,j} = \begin{cases} \mathbf{a}_{i,j} \mathbf{a}_{1,2} & \text{for } 2 < i, \\ \mathbf{a}_{1,j} \mathbf{a}_{1,2} & \text{for } 2 = i, \end{cases}$$

there exists  $\beta'$  in  $B_n^{+*}$  satisfying  $\mathbf{a}_{1,2} \phi_n(\beta_{p-1}) = \beta' \mathbf{a}_{1,2}$ . Then  $\mathbf{a}_{1,2}$  is a right divisor of  $\phi_n^3(\beta_p) \cdot \phi_n^2(\beta_{p-1}) \cdot \phi_n(\beta_{p-2})$  with  $p-2 \geq 2$ , and we obtain a contradiction with Condition (1.19). Therefore the braid  $\beta_{p-1}$  is not trivial. We iterate this process with  $r = p-2, \dots, 3$  to prove (ii). The case  $p = 3$  is obvious since, by definition of  $\phi_n$ -splitting,  $\beta_p$  is nontrivial.

(iii) Assume  $\beta_r = \beta'_r \mathbf{a}_{n-2,n-1}$  with  $\beta'_r$  nontrivial. Let  $\mathbf{a}_{i,j}$  be the last letter of  $\beta'_r$ . As we have

$$(1.20) \quad \mathbf{a}_{i,j} \mathbf{a}_{n-2,n-1} = \begin{cases} \mathbf{a}_{n-2,n-1} \mathbf{a}_{i,j} & \text{for } j < n-2, \\ \mathbf{a}_{i,n-1} \mathbf{a}_{i,j} & \text{for } j = n-2, \end{cases}$$

we must have  $j = n-1$ , for otherwise  $\mathbf{a}_{i,j}$  would be a right divisor of  $\beta_r$ , *i.e.*, the  $B_{n-1}^{+*}$ -tail of  $\phi_n(\beta_r)$  would not be trivial, which contradicts Condition (1.19).  $\square$

In the sequel, we shall often consider subsequences consisting of several adjacent entries in a splitting. It will be convenient to have a specific notation for such subsequences.

**Notation 1.24.** Assume that  $\beta$  is a braid of  $B_n^{+*}$ , with  $n \geq 3$ , whose  $n$ -breadth is  $p$ . The braid noted  $\beta_{[q,r]}$  with  $p \geq q \geq r \geq 1$  is defined to be

$$(1.21) \quad \beta_{[q,r]} = \phi_n^{q-1}(\beta_q) \cdot \dots \cdot \phi_n^{r-1}(\beta_r),$$

where  $(\beta_p, \dots, \beta_1)$  is the  $\phi_n$ -splitting of  $\beta$ .

So, in particular,  $\beta_{[1,1]}$  denotes the rightmost entry in the splitting of  $\beta$ , and  $\beta_{[p,1]}$  is all of  $\beta$  if  $\beta$  has breadth  $p$ .

**Lemma 1.25.** *Assume that  $\beta$  is a braid of  $B_n^{+*}$  with  $n \geq 3$ , whose  $n$ -breadth is  $p$ . Then the following holds:*

(i) *For  $p \geq q \geq 1$ , the  $\phi_n$ -splitting of  $\beta_{[q,1]}$  is  $(\beta_q, \dots, \beta_1)$*

(ii) *For  $p \geq q \geq 2$ , the  $\phi_n$ -splitting of  $\beta_{[q,2]}$  is  $(\beta_q, \dots, \beta_2, 1)$ .*

*Proof.* (i) As a right divisor of  $\beta'$  lying in  $B_n^{+*}$  is an right divisor of  $\beta'' \beta'$  for  $\beta''$  in  $B_n^{+*}$ , Condition (1.19) implies

$$(1.22) \quad \text{for each } r \geq 1, \text{ the } B_{n-1}^{+*}\text{-tail of } \phi_n^{q-r}(\beta_q) \cdot \dots \cdot \phi_n(\beta_{r+1}) \text{ is trivial.}$$

(ii) is a consequence of (i) and that Condition (1.19) does not involve the rightmost entry in the  $\phi_n$ -splitting of  $\beta$ , namely  $\beta_1$ .  $\square$

## 2. A LINEAR ORDERING ON $B_n^{+*}$

Our aim in the sequel will be to investigate the restriction of the Dehornoy ordering of braids to the dual monoid  $B_n^{+*}$ . Our approach will be indirect. What we shall do is to introduce an auxiliary ordering on  $B_n^{+*}$  recursively defined using the  $\phi_n$ -splitting, and, at the end, prove that this ordering coincides with the standard ordering.

In this section, we briefly recall the definition of the standard ordering on  $B_n$ , introduce our new ordering  $<^*$ , and prove some preparatory results about this ordering.

**2.1. The Dehornoy ordering of braids.** We recall the definition of the braid ordering in terms of  $\sigma$ -positive braid words.

**Definition 2.1.** (i) A braid word  $w$  (in the letters  $\sigma_i$ ) is said to be  $\sigma_i$ -positive if  $w$  contains at least one  $\sigma_i$ , no  $\sigma_i^{-1}$ , and no letter  $\sigma_j^\pm$  with  $j > i$ .

(ii) A braid  $\beta$  is said to be  $\sigma_i$ -positive if, among the braid words representing  $\beta$ , at least one is a  $\sigma_i$ -positive word.

(iii) A braid is said to be  $\sigma_i$ -negative if its inverse is  $\sigma_i$ -positive.

(iv) If  $\beta, \beta'$  are braids, we declare that  $\beta < \beta'$  is true if the braid  $\beta^{-1}\beta'$  is  $\sigma_i$ -positive for some  $i$ .

**Example 2.2.** Let us show that  $\mathbf{a}_{i,j} < \mathbf{a}_{i',j'}$  holds if and only if either  $i < j$ , or  $j = j'$  and  $i > i'$ . Assume  $j < j'$ . Then, by the definition of  $\mathbf{a}_{i,j}$ , the quotient  $\mathbf{a}_{i,j}^{-1} \mathbf{a}_{i',j'}$  is represented by

$$\sigma_i \dots \sigma_{j-1} \sigma_j \sigma_{j-1}^{-1} \dots \sigma_i^{-1} \cdot \sigma_{i'} \dots \sigma_{j'-1} \sigma_{j'} \sigma_{j'-1}^{-1} \dots \sigma_i^{-1},$$

whose generator of greatest index, namely  $\sigma_{j'}$ , appears only positively. Hence we have  $\mathbf{a}_{i,j} < \mathbf{a}_{i',j'}$ . Assume  $j = j'$  and  $i > i'$ . Using (1.7), we decompose  $\mathbf{a}_{i,j}$  as  $\delta_{i,j} \delta_{i,j-1}^{-1}$  and  $\mathbf{a}_{i',j}$  as  $\delta_{i',i-1} \delta_{i-1,i} \delta_{i,j} \delta_{i',j-1}^{-1}$ , i.e.,  $\delta_{i',i-1} \mathbf{a}_{i-1,i} \delta_{i,j} \delta_{i',j-1}^{-1}$ , and we obtain

$$\mathbf{a}_{i,j}^{-1} \mathbf{a}_{i',j} = \delta_{i,j} \delta_{i,j}^{-1} \cdot \delta_{i',i-1} \mathbf{a}_{i-1,i} \delta_{i,j} \delta_{i',j-1}^{-1}$$

By (1.10), the underlined factor becomes  $\phi_j^{-1}(\phi_i(\delta_{i',i-1} \mathbf{a}_{i-1,i}))$ , that is equal to  $\phi_j^{-1}(\delta_{i'+1,i} \mathbf{a}_{1,i})$ , i.e.,  $\delta_{i',i-1} \mathbf{a}_{i-1,j}$ . Hence the quotient  $\mathbf{a}_{i,j}^{-1} \cdot \mathbf{a}_{i',j'}$  is represented by

$$\delta_{i,j-1} \delta_{i',i-1} \mathbf{a}_{i-1,j} \delta_{i',j-1}^{-1} = \sigma_i \dots \sigma_{j-2} \sigma_{i'} \dots \sigma_{i-2} \sigma_{i-1} \dots \sigma_{j-1} \sigma_j \sigma_{j-2}^{-1} \dots \sigma_{i-1}^{-1} \sigma_{j-1}^{-1} \dots \sigma_{i'}^{-1},$$

where the generator of greatest index, i.e.,  $\sigma_j$  appears only positively, and the relation  $\mathbf{a}_{i,j} < \mathbf{a}_{i',j}$  holds (for  $i > i'$ ). Thus we obtained the well-ordered sequence

$$\mathbf{a}_{1,2} < \mathbf{a}_{2,3} < \mathbf{a}_{1,3} < \mathbf{a}_{3,4} < \mathbf{a}_{2,4} < \mathbf{a}_{1,4} < \dots < \mathbf{a}_{n-1,n} < \dots < \mathbf{a}_{1,n}.$$

**Proposition 2.3.** [11] *For each  $n$ , the relation  $<$  is a linear ordering of  $B_n$  that is invariant under left multiplication.*

**Remark 2.4.** It should be noted that we use here the so-called upper version of the ordering, in which one takes into account the generator  $\sigma_i$  with greatest index, and not the lower version, in which one considers the generator with lowest index. This choice is necessary here, because we wish that  $B_{n-1}^{+*}$  is an initial segment of  $B_n^{+*}$ , which would not be true if we were considering the lower version of the ordering.

Following [11], we recall that, in Proposition 2.3, the non-trivial parts are :

**Property A.** Every  $\sigma$ -positive braid is non-trivial.

**Property C.** Every braid is  $\sigma$ -positive or  $\sigma$ -negative.

In the sequel, we shall never use Proposition 2.3, *i.e.*, we shall not pre-suppose that the relation  $<$  of Definition 2.1 is a linear ordering. The only properties of  $<$  we shall use are the following trivial facts—plus Property A, but exclusively for the corollaries.

**Lemma 2.5.** *The relation  $<$  is transitive, and invariant under left-multiplication.*

**2.2. The  $<^*$ -ordering.** As was said in the introduction, we aim at proving results about the restriction of the ordering  $<$  to the dual braid monoid  $B_n^{+*}$ , but we shall do that indirectly, first introducing an auxiliary ordering  $<^*$ , and eventually proving that the latter coincides with the original braid ordering.

The construction of the auxiliary ordering  $<^*$  is recursive, based on the  $\phi_n$ -splitting of Definition 1.14.

**Definition 2.6.** For  $n \geq 2$ , we recursively define  $<_n^*$  on  $B_n^{+*}$  as follows:

- (i) For  $\beta, \beta'$  in  $B_2^{+*}$ , we declare that  $\beta <_2^* \beta'$  is true if we have  $\beta = \mathbf{a}_{1,2}^p$  and  $\beta' = \mathbf{a}_{1,2}^q$  with  $p < q$ ;
- (ii) For  $\beta, \beta'$  in  $B_n^{+*}$  with  $n \geq 3$ , we declare that  $\beta <_n^* \beta'$  is true if the  $\phi_n$ -splitting of  $\beta$  is smaller than this of  $\beta'$  for the **ShortLex**-extension of  $<_{n-1}^*$ .

We recall that a sequence  $s$  is said to be **ShortLex**-smaller than  $s'$  if the length of  $s$  is strictly smaller than the length of  $s'$ , or the lengths are equal and  $s$  is lexicographically smaller than  $s'$  (for some prescribed ordering on the set of entries).

**Example 2.7.** As was seen in Example 1.15, the  $\phi_n$ -splitting of  $\mathbf{a}_{i,j}$  of  $B_n^{+*}$  is

$$(2.1) \quad \begin{cases} (\mathbf{a}_{i,j}) & \text{for } i < j \leq n-1 \\ (\mathbf{a}_{i-1,n-1}, 1) & \text{for } 2 \leq i \text{ and } j = n \\ (\mathbf{a}_{n-2,n-1}, 1, 1) & \text{for } i = 1 \text{ and } j = n \end{cases}$$

Hence  $\mathbf{a}_{1,n}$  is the  $<^*$ -greatest generator of  $B_n^{+*}$ , and we have  $\mathbf{a}_{i,j} <_n^* \mathbf{a}_{k,n}$  for all  $i < j < n$  and  $k < n$ . Then, an easy induction gives

$$1 <_n^* \mathbf{a}_{1,2} <_n^* \mathbf{a}_{2,3} <_n^* \mathbf{a}_{1,3} <_n^* \mathbf{a}_{3,4} <_n^* \mathbf{a}_{2,4} <_n^* \mathbf{a}_{1,4} <_n^* \dots <_n^* \mathbf{a}_{n-1,n} <_n^* \dots <_n^* \mathbf{a}_{1,n}.$$

So, we can see that the relations  $<$  and  $<_n^*$  agree on the generators of  $B_n^{+*}$ .

**Proposition 2.8.** *For  $n \geq 2$ , the relation  $<_n^*$  is a well-ordering of  $B_n^{+*}$ . For each braid  $\beta$ , the immediate  $<_n^*$ -successor of  $\beta$  is  $\beta \mathbf{a}_{1,2}$ , *i.e.*,  $\beta \sigma_1$ .*

*Proof.* The ordered monoid  $(B_2^{+*}, <_2^*)$  is isomorphic to  $\mathbb{N}$  with the usual ordering that is a well-ordering. As the **ShortLex**-extension of a well-ordering is itself a well-ordering, we prove inductively on  $n$  that  $<_n^*$  is a well-ordering. The second result is an immediate consequence that if the  $\phi_n$ -splitting of a braid  $\beta$  is  $(\beta_p, \dots, \beta_1)$  then the  $\phi_n$ -splitting of  $\beta \mathbf{a}_{1,2}$  is  $(\beta_p, \dots, \beta_1 \mathbf{a}_{1,2})$ .  $\square$

The connection between the ordering  $<_{n-1}^*$  and the restriction of  $<_n^*$  to  $B_{n-1}^{+*}$  is easily described: the next result states that  $B_{n-1}^{+*}$  is an initial segment of  $B_n^{+*}$ .

**Proposition 2.9.** *For  $n \geq 3$ , the monoid  $B_{n-1}^{+*}$  is the initial segment of  $(B_n^{+*}, <_n^*)$  determined by  $\mathbf{a}_{n-1,n}$ , i.e., we have  $B_{n-1}^{+*} = \{\beta \in B_n^{+*} \mid \beta <_n^* \mathbf{a}_{n-1,n}\}$ . Moreover the braid  $\mathbf{a}_{n-1,n}$  is the smallest of  $n$ -breadth 2.*

*Proof.* First, by construction, every braid of  $n$ -breadth 1 belongs to  $B_{n-1}^{+*}$ . On the other hand, the  $\phi_n$ -splitting of  $\mathbf{a}_{n-1,n}$  is  $(\mathbf{a}_{n-2,n-1}, 1)$ . Let  $\beta$  be a braid of  $B_{n-1}^{+*}$  that satisfies  $\beta <_n^* \mathbf{a}_{n-1,n}$ . We saw in Example 1.15 that the  $n$ -breadth of  $\mathbf{a}_{n-2,n-1}$  is 2, so, by definition of  $<_n^*$ , the  $n$ -breadth of  $\beta$  is at most 2. We shall show that, in fact, it must be at most 1, i.e.,  $\beta$  lies in  $B_{n-1}^{+*}$ .

Assume first  $n = 3$ . By definition, every  $\phi_3$ -splitting of length 2 has the form  $(\mathbf{a}_{1,2}^p, \mathbf{a}_{1,2}^q)$  with  $p \neq 0$ . Hence the braid  $\mathbf{a}_{2,3}$ , whose  $\phi_3$ -splitting is  $(\mathbf{a}_{1,2}, 1)$ , is the  $<_3^*$ -smallest element of  $B_3^{+*}$  with 3-breadth equal to 2. So, by definition of the **ShortLex**-extension,  $\beta <_3^* \mathbf{a}_{1,2}$  implies that the 3-breadth of  $\beta$  is 1, and, therefore, that  $\beta$  lies in  $B_2^{+*}$ .

Assume now  $n > 3$ . Assume for a contradiction that the  $n$ -breadth of  $\beta$  is 2. Let  $(\beta_2, \beta_1)$  be the  $\phi_n$ -splitting of  $\beta$ . As the  $\phi_n$ -splitting of  $\mathbf{a}_{n-1,n}$  is  $(\mathbf{a}_{n-2,n-1}, 1)$ , and  $\beta_1 <_{n-1}^* 1$  is impossible, the hypothesis  $\beta <_n^* \mathbf{a}_{n-1,n}$  implies  $\beta_2 <_{n-1}^* \mathbf{a}_{n-2,n-1}$ . By induction hypothesis, this implies that  $\beta_2$  lies in  $B_{n-2}^{+*}$ , hence  $\phi_n(\beta_2)$  lies in  $B_{n-1}^{+*}$ . This contradicts Condition (1.19). So the hypothesis that  $\beta$  has  $n$ -breadth 2 is contradictory, and  $\beta$  necessarily lies in  $B_{n-1}^{+*}$ .  $\square$

Building on the compatibility result of Proposition 2.9, we can now drop the subscript in the ordering  $<_n^*$ , and, from now on, write  $<^*$  for arbitrary braids in  $B_\infty^{+*}$ , which is an inductive limit of the  $B_n^{+*}$ .

**2.3. The braids  $\widehat{\delta}_{n,p}$ .** By definition, for  $r < p$ , every braid in  $B_n^{+*}$  that has  $n$ -breadth  $r$  is  $<^*$ -smaller than every braid that has  $n$ -breadth  $p$ . As the ordering  $<^*$  is a well-ordering, there must exist, for each  $p$ , a  $<^*$ -smallest braid with  $n$ -breadth  $p$ . It is easy to identify these braids, which are sorts of separators for  $<^*$ , and which will play an important role in the sequel.

Clearly, the least upper bound of all braids with  $n$ -breadth 1 is  $\mathbf{a}_{1,2}$ , i.e.,  $\sigma_1$ . Proposition 2.9 implies that the least upper bound of all braids with  $n$ -breadth 2 is  $\mathbf{a}_{n-1,n}$ . From  $n$ -breadth 3, a periodic pattern appears.

**Definition 2.10.** For  $q \geq 1$ , we define  $\widehat{\delta}_{n,q} = \phi_n^{q+1}(\mathbf{a}_{n-2,n-1}) \cdot \dots \cdot \phi_n^2(\mathbf{a}_{n-2,n-1})$ .

For instance, we find  $\widehat{\delta}_{6,4} = \phi_6^5(\mathbf{a}_{4,5}) \cdot \phi_6^4(\mathbf{a}_{4,5}) \cdot \phi_6^3(\mathbf{a}_{4,5}) \cdot \phi_6^2(\mathbf{a}_{4,5})$  that is equal to  $\mathbf{a}_{3,4} \mathbf{a}_{2,3} \mathbf{a}_{1,2} \mathbf{a}_{1,6}$  and  $\widehat{\delta}_{5,3} = \mathbf{a}_{2,3} \mathbf{a}_{1,2} \mathbf{a}_{1,5}$ .

**Proposition 2.11.** *For all  $n \geq 3$  and  $q \geq 1$ ,*

- (i) *the  $\phi_n$ -splitting of  $\widehat{\delta}_{n,q}$  is the length  $q+2$  sequence  $(\mathbf{a}_{n-2,n-1}, \dots, \mathbf{a}_{n-2,n-1}, 1, 1)$ .*
- (ii) *the relation  $\widehat{\delta}_{n,q} = \delta_n^q \delta_{n-1}^{-q}$  holds;*
- (iii) *the braid  $\widehat{\delta}_{n,q}$  is the  $<^*$ -smallest braid in  $B_n^{+*}$  that has  $n$ -breadth  $q+2$ —hence it is the least upper bound of all braids of  $n$ -breadth  $\leq q+1$ .*

*Proof.* (i) By definition of  $\widehat{\delta}_{n,q}$  it is sufficient to show that the length  $q+2$  sequence  $(\mathbf{a}_{n-2,n-1}, \dots, \mathbf{a}_{n-2,n-1}, 1, 1)$  is a  $\phi_n$ -splitting. First, we observe that there does not exist relation  $\mathbf{a}_{n-1,n} \mathbf{a}_{n-2,n-1} = \dots$  in the presentation of the monoid  $B_n^{+*}$ . Then the word  $\mathbf{a}_{n-1,n} \mathbf{a}_{n-2,n-1} = \phi_n(\mathbf{a}_{n-2,n-1}) \mathbf{a}_{n-2,n-1}$  is alone in its equivalence class

under the relation of  $B_n^{+*}$ . An easy induction, using the symmetries of the presentation of  $B_n^{+*}$  under  $\phi_n$ , establishes that the word  $w = \phi_n^{r-1}(\mathbf{a}_{n-2,n-1}) \cdots \mathbf{a}_{n-2,n-1}$  is also alone in its equivalence class for all  $r$ . In particular the braid  $\bar{w}$  is right divisible by only one generator, namely  $\mathbf{a}_{n-2,n-1}$ . For  $p \geq 3$  we put  $\beta_p = \dots = \beta_3 = \mathbf{a}_{n-2,n-1}$  and  $\beta_2 = \beta_1 = 1$ . We have to prove that  $(\beta_p, \dots, \beta_1)$  is a  $\phi_n$ -splitting. As every entry lies in  $B_{n-1}^{+*}$ , it remains to prove Condition (1.19). By the above remark the only generator that divides  $\phi_n^{p-r}(\beta_p) \cdots \phi_n(\beta_{r+1})$  is  $\phi_n(\mathbf{a}_{n-2,n-1})$ , *i.e.*,  $\mathbf{a}_{n-1,n}$  for  $r \geq 2$  and  $\phi_n^2(\mathbf{a}_{n-2,n-1})$ , *i.e.*,  $\mathbf{a}_{1,n}$  for  $r = 1$ . Therefore, for  $p \geq 3$ , the length  $p$  sequence  $(\mathbf{a}_{n-2,n-1}, \dots, \mathbf{a}_{n-2,n-1}, 1, 1)$  is a  $\phi_n$ -splitting.

(ii) We use an induction on  $q$ . Assume  $q = 1$ . By (1.7), we have  $\mathbf{a}_{1,n} = \delta_n \delta_{n-1}^{-1}$ . Using the relation  $\phi_n(\beta) \delta_n = \delta_n \beta$ , we deduce

$$\widehat{\delta}_{n,1} = \phi_n^2(\mathbf{a}_{n-2,n-1}) = \mathbf{a}_{1,n} = \delta_n \delta_{n-1}^{-1}.$$

Assume now  $q \geq 2$ . Using the induction hypothesis and pushing  $\delta_n^{q-1}$  to the left, we find:

$$\begin{aligned} \widehat{\delta}_{n,q} &= \phi_n^{q+1}(\mathbf{a}_{n-2,n-1}) \widehat{\delta}_{n,q-1} = \phi_n^{q+1}(\mathbf{a}_{n-2,n-1}) \delta_n^{q-1} \delta_{n-1}^{-q+1} \\ &= \delta_n^{q-1} \phi_n^2(\mathbf{a}_{n-2,n-1}) \delta_{n-1}^{-q+1} \\ &= \delta_n^{q-1} \mathbf{a}_{1,n} \delta_{n-1}^{-q+1} \\ &= \delta_n^{q-1} \delta_n \delta_{n-1}^{-1} \delta_{n-1}^{-q+1} = \delta_n^q \delta_{n-1}^{-q}, \end{aligned}$$

which is the expected formula.

(iii) Let  $(\beta_{q+2}, \dots, \beta_1)$  be the  $\phi_n$ -splitting of  $\beta$  lying in  $B_n^{+*}$  satisfying  $\beta \leq^* \widehat{\delta}_{n,q}$ . By definition of  $\leq^*$ , we have the relation  $\beta_{q+2} \leq^* \mathbf{a}_{n-2,n-1}$ . As the last letter of  $\beta_{q+2}$  is  $\mathbf{a}_{i,n-1}$ , by Lemma 1.23 (i) and (ii), the  $(n-1)$ -breadth of  $\beta_{q+2}$  is at least 2. Then Proposition 2.9 implies that  $\beta_{q+2}$  is equal to  $\mathbf{a}_{n-2,n-1}$ . By definition of  $\leq^*$  again, the relation  $\beta_{[q+1,1]} \leq^* \widehat{\delta}_{n,q-1}$  holds. Hence, inductively, we obtain  $\beta_r = \mathbf{a}_{n-2,n-1}$  for  $r \geq 3$  and  $\beta_{[2,1]} \leq^* 1$ , which implies  $\beta_{[2,1]} = 1$ . This is the expected result.  $\square$

Owing to Proposition 2.11, it is coherent to extend Definition 2.10 by  $\widehat{\delta}_{n,0} = \mathbf{a}_{n-1,n}$ . In this way, the result of Proposition 2.11(iii) extends to the case  $r = 0$ .

**Lemma 2.12.** *Assume  $n \geq 3$ . Then  $p < q$  implies  $\widehat{\delta}_{n,p} < \widehat{\delta}_{n,q}$ .*

*Proof.* Let  $p$  and  $q$  be integers satisfying  $0 < p < q$ . By point (ii), we have

$$(2.2) \quad \widehat{\delta}_{n,p}^{-1} \cdot \widehat{\delta}_{n,q} = \delta_{n-1}^p \delta_n^{-p} \cdot \delta_n^q \delta_{n-1}^{-q} = \delta_{n-1}^p \underline{\delta_n^{q-p} \delta_{n-1}^{-q}}.$$

where the underlined factor is  $\sigma_{n-1}$ -positive and the other does not involve the  $n$ th strand, *i.e.*,  $\widehat{\delta}_{n,p}^{-1} \cdot \widehat{\delta}_{n,q}$  is  $\sigma_{n-1}$ -positive. It remains to establish the result for  $p = 0$ . With the previous case, by transitivity of  $<$ , we can suppose  $p = 0$  and  $q = 1$ . Using Proposition 2.11 (ii) and inserting  $\delta_n \delta_{n-1}^{-1}$  on the left, we obtain

$$(2.3) \quad \widehat{\delta}_{n,0}^{-1} \widehat{\delta}_{n,1} = \mathbf{a}_{n-1,n}^{-1} \delta_n \delta_{n-1}^{-1} = \delta_n \underline{\delta_n^{-1} \mathbf{a}_{n-1,n}^{-1} \delta_n} \delta_{n-1}^{-1}.$$

By definition of  $\phi_n$ , the underlined factor in (2.3) is  $\phi_n^{-1}(\mathbf{a}_{n-1,n}^{-1})$  that is equal to  $\mathbf{a}_{n-2,n-1}^{-1}$ , and it follows that  $\widehat{\delta}_{n,0}^{-1} \widehat{\delta}_{n,1}$  is  $\sigma_{n-1}$ -positive.  $\square$

**Remark 2.13.** By Proposition 2.11 (ii), for  $p \geq 1$ , the quotient  $\widehat{\delta}_{n,p}^{-1} \delta_n^p$  is equal to  $\delta_{n-1}^p$ , which is  $\sigma$ -positive. This implies  $\widehat{\delta}_{n,p} < \delta_n^p$ . As  $\delta_{n-1}^p$  lies in  $B_{n-1}^{+*}$ , for

$p \geq 1$ , the  $\phi_n$ -splitting of  $\delta_n^p$  is the length  $p+2$  sequence

$$(\mathbf{a}_{n-2, n-1}, \dots, \mathbf{a}_{n-2, n-1}, 1, \delta_{n-1}^p).$$

So we see in these examples that the orderings  $<^*$  and  $<$  agree on the separators  $\widehat{\delta}_{n,p}$  and on the powers of the Garside element.

### 3. THE MAIN RESULT

We shall now state our main result, which connects the orderings  $<$  and  $<^*$ , and list its consequences. The main technical step of the proof is stated here as Key Lemma. Its proof is postponed to the subsequent sections.

**3.1. The connection between orderings, and its applications.** At this point, we have two *a priori* unrelated linear orderings of the dual braid monoid  $B_{n-1}^{+*}$ , namely the Dehornoy ordering, and the ordering  $<^*$  constructed in Section 2 by a recursive definition. The main result is the following:

**Theorem 3.1.** *For all braids  $\beta, \beta'$  in  $B_n^{+*}$ , the relation  $\beta <^* \beta'$  implies  $\beta < \beta'$ .*

Before starting the proof of this result, we list a few direct consequences.

First we obtain a new proof of Property **C**.

**Corollary 3.2** (Property **C**). *Every non-trivial braid is  $\sigma$ -positive or  $\sigma$ -negative.*

*Proof.* Assume that  $\beta$  is a non-trivial braid of  $B_n$ . First, as  $B_n$  is the fraction group of  $B_n^{+*}$ , there exist two braids  $\beta'$  and  $\beta''$  in  $B_n^{+*}$  such that  $\beta$  equals  $\beta'^{-1}\beta''$ . As  $\beta$  is assumed to be non-trivial, we have  $\beta' \neq \beta''$ . As  $<^*$  is a strict linear ordering, one of  $\beta' <^* \beta''$  or  $\beta'' <^* \beta'$  holds. In the first case, Theorem 3.1 implies that  $\beta'^{-1}\beta''$ , *i.e.*,  $\beta$ , is  $\sigma$ -positive. In the second case, Theorem 3.1 implies that  $\beta''^{-1}\beta'$  is  $\sigma$ -positive, hence  $\beta$  is  $\sigma$ -negative.  $\square$

**Corollary 3.3.** *The relation  $<^*$  coincide with the restriction of  $<$  to  $B_n^{+*}$ .*

*Proof.* Let  $\beta, \beta'$  belong to  $B_n^{+*}$ . By Theorem 3.1,  $\beta <^* \beta'$  implies  $\beta < \beta'$ . Conversely, assume  $\beta \not<^* \beta'$ . As  $<^*$  is a linear ordering, we have either  $\beta' <^* \beta$ , hence  $\beta' < \beta$ , or  $\beta = \beta'$ . In both cases, Property **A** implies that  $\beta < \beta'$  fails. So the equivalence is established.  $\square$

By construction, the ordering  $<$  is invariant under left-multiplication. So it follows from Corollary 3.3 that the ordering  $<^*$  is invariant as well: note that the latter result is *not* obvious at all from the direct definition of that relation.

Another direct consequence is the structural result mentionned as Theorem 2 in the introduction:

**Corollary 3.4.** *The Dehornoy ordering of the dual braid monoid  $B_n^{+*}$  admits the following recursive characterization: for  $\beta, \beta'$  in  $B_n^{+*}$ , the relation  $\beta < \beta'$  holds if and only if, denoting by  $(\beta_p, \dots, \beta_1)$  and  $(\beta'_p, \dots, \beta'_1)$  the  $\phi_n$ -splittings of  $\beta$  and  $\beta'$ , the sequence  $(\beta_p, \dots, \beta_1)$  is smaller than the sequence  $(\beta'_p, \dots, \beta'_1)$  with respect to the ShortLex-extension of the ordering of  $B_{n-1}^{+*}$ .*

*Proof.* Indeed, the above characterization is just the recursive definition of the ordering  $<^*$ .  $\square$

Finally, we obtain a new proof of Laver's result, together with a determination of the order type.

**Corollary 3.5.** *The Dehornoy ordering of  $B_n^{+*}$  is a well-ordering, and its order type is the ordinal  $\omega^{\omega^{n-2}}$ .*

*Proof.* It is standard that, if  $(X, <)$  is a well-ordering of ordinal type  $\lambda$ , then the **ShortLex**-extension of  $<$  on the set of all finite sequences in  $X$  is a well-ordering of ordinal type  $\lambda^\omega$ . The ordinal type of  $<^*$  on  $B_2^{+*}$  is  $\omega$ , the ordering type of the standard ordering of natural numbers. So, an immediate induction shows that, for each  $n \geq 2$ , the ordinal type of  $<^*$  on  $B_n^{+*}$  is at most  $\omega^{\omega^{n-2}}$ .

A priori, we only have an upper bound, because it is not true that every sequence of braids in  $B_{n-1}^{+*}$  is the splitting of a braid in  $B_n^{+*}$ . However, by construction, the monoid  $B_n^{+*}$  includes the positive braid monoid  $B_n^+$ , and it was shown in [4]—or, alternatively in [7]—that the ordinal type of the restriction of the Dehornoy ordering to  $B_n^+$  is  $\omega^{\omega^{n-2}}$ . Hence the ordinal type of its restriction to  $B_n^{+*}$  is at least that ordinal, and, finally, we have equality.  $\square$

**3.2. Splitting the problem.** The proof of Theorem 3.1 consists in computing a  $\sigma$ -positive word representing the quotient  $\beta^{-1}\beta'$  when the braids  $\beta$  and  $\beta'$  of  $B_n^{+*}$  satisfy  $\beta <^* \beta'$ . The computation is direct and effective—so, in particular, it results in an algorithm for finding  $\sigma$ -positive representatives.

However, describing the computation and proving its correctness is a rather delicate process, and we shall split it into several steps.

So, our problem is to prove

$$(3.1) \quad \beta <^* \beta' \quad \text{implies} \quad \beta < \beta'$$

The first step consists in replacing the initial problem that involves two arbitrary braids  $\beta, \beta'$  with two problems, each of which only involves one braid. To this end, we shall use the separators  $\widehat{\delta}_{n,r}$  of Definition 2.10, and show how to reduce the problem to comparing an arbitrary braid with the special braids  $\widehat{\delta}_{n,r}$ .

In particular, we shall address the questions of proving that

$$(3.2) \quad \beta <^* \widehat{\delta}_{n,q} \quad \text{implies} \quad \beta < \widehat{\delta}_{n,q}$$

$$(3.3) \quad \widehat{\delta}_{n,p} \leq^* \beta \quad \text{implies} \quad \widehat{\delta}_{n,p} \leq \beta$$

We have three things to do: proving (3.2), proving (3.3), and showing how to deduce the general result (3.1).

We shall see now that the first of these three tasks is easy.

**Proposition 3.6.** *For  $\beta$  in  $B_n^{+*}$  with  $n$ -breadth at most  $q+1$ , the braid  $\beta^{-1}\widehat{\delta}_{n,q}$  is  $\sigma_{n-1}$ -positive.*

*Proof.* Let  $(\beta_p, \dots, \beta_1)$  be the  $\phi_n$ -splitting of  $\beta$  with  $p \leq q+1$ . If  $q = 0$  holds the result is immediate since  $\widehat{\delta}_{n,0} = \mathbf{a}_{n-1,n}$  is  $\sigma_{n-1}$ -positive and  $\beta$  lies in  $B_{n-1}^{+*}$ . Assume  $q > 0$ . Then we have

$$\beta^{-1}\widehat{\delta}_{n,q} = \beta^{-1} \cdot \delta_n^q \cdot \delta_{n-1}^{-q} = \beta_1^{-1} \cdot \phi_n(\beta_2^{-1}) \cdot \dots \cdot \phi_n^{p-1}(\beta_p^{-1}) \cdot \delta_n^q \cdot \delta_{n-1}^{-q}.$$

Using the relation  $\phi_n^k(\beta') \cdot \delta_n^k = \delta_n^k \cdot \beta'$  that follows from the definition of  $\phi_n$  for every nonnegative  $k$ , we push the  $p-1$  factors  $\delta_n$  to the left and dispatch them



between the  $\beta_i^{-1}$ :

$$\begin{aligned} \beta^{-1} \widehat{\delta}_{n,q} &= \beta_1^{-1} \cdot \phi_n(\beta_2^{-1}) \cdot \dots \cdot \phi_n^{p-2}(\beta_{p-1}^{-1}) \cdot \phi_n^{p-1}(\beta_p^{-1}) \cdot \delta_n^{p-1} \cdot \delta_n^{q-p+1} \cdot \delta_{n-1}^{-q} \\ &= \beta_1^{-1} \cdot \phi_n(\beta_2^{-1}) \cdot \dots \cdot \phi_n^{p-2}(\beta_{p-1}^{-1}) \cdot \delta_n^{p-2} \cdot \delta_n \cdot \beta_p^{-1} \cdot \delta_n^{q-p+1} \cdot \delta_{n-1}^{-q} \\ &\dots \\ &= \beta_1^{-1} \cdot \delta_n \cdot \beta_2^{-1} \cdot \dots \cdot \beta_{p-1}^{-1} \cdot \delta_n \cdot \beta_p^{-1} \cdot \delta_n^{q-p+1} \cdot \delta_{n-1}^{-q}. \end{aligned}$$

Each braid  $\beta_r$  belongs to  $B_{n-1}^{+*}$ , and so its inverse  $\beta_r^{-1}$  does not involve the  $n$ th strand. For the same reason the braid  $\delta_{n-1}^{-q+2}$  does not involve the  $n$ th strand. On the other hand, the braid  $\delta_n$  is  $\sigma_{n-1}$ -positive, and occurs at least once in the above decomposition of  $\beta^{-1} \widehat{\delta}_{n,q}$ . Hence the latter is  $\sigma_{n-1}$ -positive.  $\square$

**Corollary 3.7.** *Implication (3.2) is true.*

*Proof.* Assume  $\beta <^* \widehat{\delta}_{n,q}$ . By Proposition 2.11(iii), the  $n$ -breadth of  $\beta$  is at most  $q+1$ . By Proposition 3.6, the braid  $\beta^{-1} \widehat{\delta}_{n,q}$  is  $\sigma$ -positive, hence  $\beta < \widehat{\delta}_{n,q}$  is true.  $\square$

As the following example shows, we cannot hope to prove (3.3) using the argument of Proposition 3.6.

**Example 3.8.** Let  $\beta$  be the braid  $\mathbf{a}_{2,4} \mathbf{a}_{1,3} \mathbf{a}_{2,4} \mathbf{a}_{3,4} \mathbf{a}_{1,2}$ . The  $\phi_4$ -splitting of  $\beta$  is  $(\mathbf{a}_{1,3}, \mathbf{a}_{1,3}, \mathbf{a}_{1,3} \mathbf{a}_{2,3}, \mathbf{a}_{1,2})$ , so its 4-breadth is 4. We try to prove that  $\widehat{\delta}_{4,2}^{-1} \beta$  is  $\sigma$ -positive or trivial. We start from

$$\widehat{\delta}_{4,2}^{-1} \beta = \delta_3^2 \cdot \delta_4^{-2} \cdot \beta = \delta_3^2 \cdot \delta_4^{-2} \cdot \phi_4^3(\mathbf{a}_{1,3}) \cdot \phi_4^2(\mathbf{a}_{1,3}) \cdot \phi_4(\mathbf{a}_{1,3} \mathbf{a}_{2,3}) \cdot \mathbf{a}_{1,2}.$$

Following the scheme of Proposition 3.6 we push the 2 factors  $\delta_4^{-1}$  to the right and dispatch them between the  $\phi_4(\beta_r)$ :

$$\begin{aligned} \widehat{\delta}_{4,2}^{-1} \beta &= \delta_3^2 \cdot \delta_4^{-2} \cdot \phi_4^3(\mathbf{a}_{1,3}) \cdot \phi_4^2(\mathbf{a}_{1,3}) \cdot \phi_4(\mathbf{a}_{1,3} \mathbf{a}_{2,3}) \cdot \mathbf{a}_{1,2} \\ &= \delta_3^2 \cdot \phi_4(\mathbf{a}_{1,3}) \cdot \delta_4^{-1} \cdot \delta_4^{-1} \cdot \phi_4^2(\mathbf{a}_{1,3}) \cdot \phi_4(\mathbf{a}_{1,3} \mathbf{a}_{2,3}) \cdot \mathbf{a}_{1,2} \\ (3.4) \quad &= \delta_3^2 \cdot \underbrace{\phi_4(\mathbf{a}_{1,3}) \cdot \delta_4^{-1}} \cdot \underbrace{\phi_4(\mathbf{a}_{1,3}) \cdot \delta_4^{-1}} \cdot \phi_4(\mathbf{a}_{1,3} \mathbf{a}_{2,3}) \cdot \mathbf{a}_{1,2}. \end{aligned}$$

Now we could conclude if the underlined factors were  $\sigma_3$ -positive. But this is just false: we have  $\phi_4(\mathbf{a}_{1,3}) \cdot \delta_4^{-1} = \mathbf{a}_{2,4} \cdot \delta_4^{-1} = \mathbf{a}_{3,4}^{-1} \mathbf{a}_{1,2}^{-1}$ , so the underlined factors are  $\sigma_3$ -negative.

Nevertheless, we observe now that, in good cases, the previous argument works.

Assume  $\beta$  is a braid of  $B_n^{+*}$ . Let  $(\beta_p, \dots, \beta_1)$  be the  $\phi_n$ -splitting of  $\beta$ . Then we consider the following conditions

$$(*) \quad \text{for } r \geq 3, \text{ the last letter of } \beta_r \text{ is } \mathbf{a}_{n-2, n-1}.$$

**Proposition 3.9.** *Assume that  $\beta$  satisfies the conditions of (\*). Then Implication (3.3) is true for  $\beta$ .*

*Proof.* Let  $(\beta_p, \dots, \beta_1)$  be the  $\phi_n$ -splitting of  $\beta$ . By hypothesis,  $\beta$  satisfies the conditions of (\*). Then, for  $r \geq 3$ , we denote by  $\beta'_r$  the braid  $\beta_r \mathbf{a}_{n-2, n-1}^{-1}$ . As in Example 3.8, we obtain

$$\widehat{\delta}_{n, p-2}^{-1} \beta = \delta_{n-1}^{p-2} \cdot \phi_n(\beta'_p \mathbf{a}_{n-2, n-1}) \cdot \delta_n^{-1} \cdot \dots \cdot \phi_n(\beta'_3 \mathbf{a}_{n-2, n-1}) \cdot \delta_n^{-1} \cdot \phi_n(\beta_2) \cdot \beta_1.$$

As  $\phi_n(\mathbf{a}_{n-2,n-1}) \cdot \delta_n^{-1}$  equals  $\mathbf{a}_{n-1,n} \delta_n^{-1}$  that is  $\delta_{n-1}^{-1}$ , we obtain

$$(3.5) \quad \widehat{\delta}_{n,p-2}^{-1} \beta = \delta_{n-1}^{p-2} \cdot \phi_n(\beta'_p) \cdot \delta_{n-1}^{-1} \cdot \dots \cdot \phi_n(\beta'_3) \cdot \delta_{n-1}^{-1} \cdot \phi_n(\beta_2) \cdot \beta_1,$$

If  $\beta_{[p,3]}$  equals  $\widehat{\delta}_{n,p-2}$ , by definition of  $\widehat{\delta}_{n,p-2}$ , the braid  $\beta'_r$  is trivial for  $r \geq 3$ . Then the quotient  $\widehat{\delta}_{n,p-2}^{-1} \beta$  equals  $\phi_n(\beta_2) \cdot \beta_1$  that is  $\sigma$ -positive or trivial. Otherwise, if  $\beta_{[p,3]} \neq \widehat{\delta}_{n,p-2}$  holds, by Lemma 1.23 (iii), one of the braids  $\beta'_p, \dots, \beta'_3$  ends by  $\mathbf{a}_{i,n-1}$  for some  $i$ , i.e., is  $\sigma_{n-2}$ -positive. Then, by (3.5), the quotient  $\widehat{\delta}_{n,p-2} \beta$  is  $\sigma_{n-1}$ -positive.  $\square$

**Corollary 3.10.** *For  $n = 3$ , Implication (3.3) is true for  $\beta$ .*

*Proof.* The braid  $\mathbf{a}_{1,2}$  is the only generator of the form  $\mathbf{a}_{i,n-1}$  for  $n = 3$ . So Condition (\*) is satisfied for every braid  $\beta$  in  $B_3^{**}$ .  $\square$

**3.3. Dangerous elements.** As we shown in Example 3.8, we cannot prove Implication (3.3) for  $n \geq 4$  by just pushing the factors  $\delta_n^{-1}$  to the right in the  $\phi_n$ -splitting of  $\beta$  and obtain the desired expression. We shall now identify the fragments in  $\beta_r$  that can result in the quotient  $\phi_n(\beta) \delta_n^{-1}$  being  $\sigma_{n-1}$ -negative.

**Example 3.11.** In Example 3.8 we prove that the quotient is  $\phi_4(\mathbf{a}_{1,3}) \cdot \delta_4^{-1}$  is  $\sigma_3$ -negative. In the other hand the quotient  $\phi_4(\mathbf{a}_{2,3}) \cdot \delta_4^{-1}$  is equal to  $\delta_3^{-1}$  that is not  $\sigma_3$ -negative. We have to isolate what are the differences between  $\mathbf{a}_{1,3}$  and  $\mathbf{a}_{2,3}$ . First, we observe that the braid  $\mathbf{a}_{1,3}$  can be decomposed as  $\mathbf{a}_{1,2} \mathbf{a}_{2,3} \mathbf{a}_{1,2}^{-1}$ , where  $\mathbf{a}_{2,3}$  and a negative part, i.e.,  $\mathbf{a}_{1,2}^{-1}$ , occur. The idea consists to split  $\mathbf{a}_{1,3}$  in two factors and obtain

$$(3.6) \quad \phi_4(\mathbf{a}_{1,3}) \cdot \delta_4^{-1} = \phi_4(\mathbf{a}_{1,2} \mathbf{a}_{2,3}) \cdot \delta_4^{-1} \cdot \phi_4^2(\mathbf{a}_{1,2}^{-1}) = \phi_4(\mathbf{a}_{1,2}) \cdot \delta_3^{-1} \cdot \phi_4^2(\mathbf{a}_{1,2}^{-1}),$$

We substitute the first underlined factor of (3.4) by  $\phi_4(\mathbf{a}_{1,2}) \cdot \delta_3^{-1} \cdot \phi_4^2(\mathbf{a}_{1,2}^{-1})$  to obtain

$$(3.7) \quad \begin{aligned} \widehat{\delta}_{4,2}^{-1} \beta &= \delta_3^2 \cdot \phi_4(\mathbf{a}_{1,3}) \cdot \delta_4^{-1} \cdot \phi_4(\mathbf{a}_{1,3}) \cdot \delta_4^{-1} \cdot \phi_4(\mathbf{a}_{1,3} \mathbf{a}_{2,3}) \cdot \mathbf{a}_{1,2} \\ &= \delta_3^2 \cdot \phi_4(\mathbf{a}_{1,2}) \cdot \delta_3^{-1} \cdot \phi_4^2(\mathbf{a}_{1,2}^{-1}) \cdot \phi_4(\mathbf{a}_{1,3}) \cdot \delta_4^{-1} \cdot \phi_4(\mathbf{a}_{1,3} \mathbf{a}_{2,3}) \cdot \mathbf{a}_{1,2} \\ &= \delta_3^2 \cdot \phi_4(\mathbf{a}_{1,2}) \cdot \delta_3^{-1} \cdot \phi_4(\underline{\phi_4(\mathbf{a}_{1,2}^{-1})} \cdot \mathbf{a}_{1,3}) \cdot \delta_4^{-1} \cdot \phi_4(\mathbf{a}_{1,3} \mathbf{a}_{2,3}) \cdot \mathbf{a}_{1,2} \end{aligned}$$

Now we would apply anymore this method, the only obstruction is that the underlined factor is not exactly the third terms of the  $\phi_4$ -splitting of  $\beta$ , but they are not very different as show the following computation:

$$(3.8) \quad \phi_4(\mathbf{a}_{1,2}^{-1}) \cdot \mathbf{a}_{1,3} = \mathbf{a}_{1,3} \mathbf{a}_{1,2}^{-1} = \mathbf{a}_{1,2} \mathbf{a}_{2,3} \mathbf{a}_{1,2}^{-2}.$$

So, the negative part of  $\phi_4(\mathbf{a}_{1,2}^{-1}) \cdot \mathbf{a}_{1,3}$  is  $\mathbf{a}_{1,2}^{-2}$ . We substitute the underlined factor of (3.7) by (3.8) to obtain

$$\begin{aligned} \widehat{\delta}_{4,2}^{-1} \beta &= \delta_3^2 \cdot \phi_4(\mathbf{a}_{1,2}) \cdot \delta_3^{-1} \cdot \phi_4(\mathbf{a}_{1,2} \mathbf{a}_{2,3} \mathbf{a}_{1,2}^{-2}) \cdot \delta_4^{-1} \cdot \phi_4(\mathbf{a}_{1,3} \mathbf{a}_{2,3}) \cdot \mathbf{a}_{1,2} \\ &= \delta_3^2 \cdot \phi_4(\mathbf{a}_{1,2}) \cdot \delta_3^{-1} \cdot \phi_4(\mathbf{a}_{1,2} \mathbf{a}_{2,3}) \cdot \delta_4^{-1} \cdot \phi_4^2(\mathbf{a}_{1,2}^{-2}) \cdot \phi_4(\mathbf{a}_{1,3} \mathbf{a}_{2,3}) \cdot \mathbf{a}_{1,2} \\ &= \delta_3^2 \cdot \phi_4(\mathbf{a}_{1,2}) \cdot \delta_3^{-1} \cdot \phi_4(\mathbf{a}_{1,2}) \cdot \delta_3^{-1} \cdot \phi_4(\phi_4(\mathbf{a}_{1,2}^{-2}) \cdot \mathbf{a}_{1,3} \mathbf{a}_{2,3}) \cdot \mathbf{a}_{1,2} \end{aligned}$$

An immediate generalization of (3.8) gives  $\phi_4(\mathbf{a}_{1,2}^{-2}) \cdot \mathbf{a}_{1,3} = \mathbf{a}_{1,3} \cdot \mathbf{a}_{1,2}^{-2}$ . Then we obtain:

$$\begin{aligned} \widehat{\delta}_{4,2}^{-1} \beta &= \delta_3^2 \cdot \phi_4(\mathbf{a}_{1,2}) \cdot \delta_3^{-1} \cdot \phi_4(\mathbf{a}_{1,2}) \cdot \delta_3^{-1} \cdot \phi_4(\mathbf{a}_{1,3} \mathbf{a}_{1,2}^{-2}) \cdot \mathbf{a}_{1,2} \\ &= \delta_3^2 \cdot \mathbf{a}_{2,3} \cdot \delta_3^{-1} \cdot \mathbf{a}_{2,3} \cdot \delta_3^{-1} \cdot \underline{\mathbf{a}_{2,4}} \cdot \mathbf{a}_{2,3}^{-2} \cdot \mathbf{a}_{1,2}, \end{aligned}$$

that is  $\sigma_3$ -positive as the underlined factor is  $\sigma_3$ -positive and the other braid can be express with  $\sigma_1^\pm$  and  $\sigma_2^\pm$ .

We now introduce the general form of the braids that play the role of the negative fragments  $\mathbf{a}_{1,2}^{-1}$  and  $\mathbf{a}_{1,2}^{-2}$  of Example 3.11, *i.e.*, that possibly endanger the expected  $\sigma$ -positivity.

**Definition 3.12.** Assume  $i \leq n-1$ . A braid  $\beta$  of  $B_{n-1}^{+*}$  is said to be  $\mathbf{a}_{i,n}$ -dangerous if it admits a decomposition of the form

$$(3.9) \quad \beta_0^+ \delta_{i,n-1}^{-1} \beta_1^+ \cdots \beta_{d-1}^+ \delta_{i,n-1}^{-1} \beta_d^+$$

where  $\beta_0^+, \dots, \beta_d^+$  belongs to  $B_{n-1}^+$ .

For instance  $\mathbf{a}_{1,2}^{-1}$  and  $\mathbf{a}_{1,2}^{-2}$  are  $\mathbf{a}_{1,3}$ -dangerous. By (1.7) each generator  $\mathbf{a}_{i,n}$  can be decomposed as  $\delta_{i,n-1} \cdot \mathbf{a}_{n-1,n} \cdot \delta_{i,n-1}^{-1}$ . Then the negative part of  $\mathbf{a}_{i,n}$  is an  $\mathbf{a}_{i,n}$ -dangerous braid.

A braid can be  $\mathbf{a}_{i,n}$ -dangerous for many  $i$ .

**Lemma 3.13.** An  $\mathbf{a}_{i,n}$ -dangerous braid is also  $\mathbf{a}_{j,n}$ -dangerous for  $j \leq i$ .

*Proof.* Let  $\beta$  be an  $\mathbf{a}_{i,n}$ -dangerous braid. By definition, we have

$$\beta_0^+ \delta_{i,n-1}^{-1} \beta_1^+ \cdots \beta_{d-1}^+ \delta_{i,n-1}^{-1} \beta_d^+$$

with  $\beta_0^+, \dots, \beta_d^+$  belongs to  $B_{n-1}^+$ . Using the relation  $\delta_{j,n-1} = \delta_{j,i} \delta_{i,n-1}$ , we obtain

$$\begin{aligned} \beta &= \beta_0^+ \delta_{j,n-1}^{-1} \delta_{j,i} \beta_1^+ \cdots \delta_{j,i} \beta_{d-1}^+ \delta_{j,n-1}^{-1} \delta_{j,i} \beta_d^+ \\ &= \gamma_0^+ \delta_{j,n-1}^{-1} \gamma_1^+ \cdots \gamma_{d-1}^+ \delta_{j,n-1}^{-1} \gamma_d^+ \end{aligned}$$

with  $\gamma_0^+ = \beta_0^+$  and  $\gamma_k^+ = \delta_{i,j} \beta_k^+$  for  $k = 1, \dots, d$ . As  $\delta_{i,j}$  together with  $\beta_k^+$  lies in  $B_{n-1}^+$  the braids  $\gamma_k^+$  belong to  $B_{n-1}^+$ , that prove  $\beta$  is  $\mathbf{a}_{j,n}$ -dangerous.  $\square$

**3.4. The notion of a  $\sigma_{i,n-1}$ -positive braid.** We wish now state the Key Lemma, that will eventually allow us to prove both Implication (3.3) and Theorem (3.1), we have to introduce one more family of braids, namely a refinement of the notion of  $\sigma_{n-1}$ -positive braid in which we identify a dangerous part.

First, it is convenient to consider the following weakening of  $\sigma_i$ -positivity.

**Definition 3.14.** A braid  $\beta$  is said to be  $\sigma_i$ -nonnegative if it is  $\sigma_i$ -positive, or it belongs to  $B_i$ .

For instance the trivial braid,  $\sigma_1^{-1}$  and  $\sigma_2 \sigma_1^{-1}$  are  $\sigma_2$ -nonnegative. Note that an  $\mathbf{a}_{i,n}$ -dangerous braid is also  $\sigma_{n-1}$ -nonnegative.

**Definition 3.15.** Assume  $i \leq n-1$ .

(i) A braid  $\beta$  lying in  $B_n^{+*}$  is called  $\sigma_{i,n-1}$ -positive with  $i \leq n-2$  if it admits a decomposition of the form

$$(3.10) \quad \beta^* \cdot \mathbf{a}_{n-1,n} \cdot \beta_* = \beta^* \cdot \sigma_{n-1} \cdot \beta_*$$

where  $\beta^*$  is  $\sigma_{n-1}$ -nonnegative and  $\beta_*$  is  $\mathbf{a}_{i,n}$ -dangerous.

(ii) A braid  $\beta$  lying in  $B_n^{+*}$  is called  $\sigma_{n-1,n-1}$ -positive if it admits a decomposition of the form

$$(3.11) \quad \beta^* \cdot \mathbf{a}_{n-1,n} \cdot \beta_* \cdot \mathbf{a}_{n-1,n} = \beta^* \cdot \sigma_{n-1} \cdot \beta_* \cdot \sigma_{n-1}$$

where  $\beta^*$  is  $\sigma_{n-1}$ -nonnegative and  $\beta_*$  is  $\sigma_{j,n-1}$ -positive for some  $j$ .

The following properties directly follow from Definitions 3.15 and 3.14.

- Lemma 3.16.** (i) Every  $\sigma_{i,n-1}$ -positive braid is  $\sigma_{n-1}$ -positive.  
(ii) If  $\beta$  is  $\sigma_{n-1}$ -nonnegative and  $\beta'$  is  $\sigma_{i,n-1}$ -positive, then  $\beta\beta'$  is  $\sigma_{i,n-1}$ -positive.  
(iii) An  $\sigma_{i,n-1}$ -positive braid is also  $\sigma_{j,n-1}$ -positive for  $j \leq i$ .

*Proof.* (i) As  $\beta^*$  and  $\beta_*$  are  $\sigma_{n-1}$ -nonnegative, the braids (3.10) and (3.11) are  $\sigma_{n-1}$ -positive. (ii) The product of two  $\sigma_{n-1}$ -positive braid is  $\sigma_{n-1}$ -positive, so is  $\beta\beta^*$ . (iii) Immediate by Lemma 3.13.  $\square$

**Example 3.17.** Every braid of  $B_{n-1}^{+*}$  that is right divisible by  $\mathbf{a}_{i,n}$  with  $i < n-1$  is  $\sigma_{i,n-1}$ -positive. Indeed, by definition, such an element admits an expression of the form  $\beta \mathbf{a}_{i,n}$  where  $\beta$  is an element of  $B_{n-1}^{+*}$ , and the generator  $\mathbf{a}_{i,n}$  can be decomposed as  $\delta_{i,n-1} \cdot \mathbf{a}_{n-1,n} \cdot \delta_{i,n-1}^{-1}$ . Then we put  $\beta^* = \beta \delta_{i,n-1}$  and  $\beta_* = \delta_{i,n-1}$ , and we conclude that  $\beta$  is  $\sigma_{i,n-1}$ -positive.

In general a braid that is right divisible by  $\mathbf{a}_{n-1,n}$  is not  $\sigma_{n-1,n}$ -positive. Indeed  $\mathbf{a}_{n-1,n}$  is not  $\sigma_{n-1,n}$ -positive.

**Lemma 3.18.** Every  $\beta$  braid of  $B_n^{+*}$  different from  $\mathbf{a}_{n-1,n}$ , whose last letter is  $\mathbf{a}_{k,n}$ , is  $\sigma_{k,n-1}$ -positive.

*Proof.* If the last letter of  $\beta$  is  $\mathbf{a}_{k,n-1}$ , then  $\beta$  is right divisible by  $\mathbf{a}_{k,n-1}$ . Thus the case  $k < n-1$  has been observed in Example 3.17. Assume  $k = n-1$ . By hypothesis the  $\phi$ -normal form of  $\beta$  ending by  $\mathbf{a}_{n-1,n}$ , so the  $B_{n-1}^{+*}$ -tail of  $\beta$  is trivial. Let  $w$  be the dual braid word such that the  $\phi$ -normal word of  $\beta$  is  $w\mathbf{a}_{n-1,n}$ . The hypothesis  $\beta \neq \mathbf{a}_{n-1,n}$  guarantees that  $w$  is not empty. We shall prove that  $w$  ends by  $\mathbf{a}_{i,n-1}$  for some  $i$ . We decompose  $w$  as  $w' \mathbf{a}_{i,j}$ . The  $B_{n-1}^{+*}$ -tail of  $\beta$  is trivial, hence the relations

$$(3.12) \quad \mathbf{a}_{i,j} \mathbf{a}_{n-1,n} = \begin{cases} \mathbf{a}_{n-1,n} \mathbf{a}_{i,j} & \text{for } j < n-1, \\ \mathbf{a}_{i,n} \mathbf{a}_{i,j} & \text{for } j = n-1, \end{cases}$$

imply  $j = n$ . Therefore the  $\phi$ -normal form of  $\beta$  is  $w' \mathbf{a}_{i,n} \mathbf{a}_{n-1,n}$ . Let  $\beta'$  to be the braid represented by  $w'$ . By (1.7), we can decompose  $\mathbf{a}_{i,n}$  as  $\delta_{i,n-1} \cdot \mathbf{a}_{n-1,n} \cdot \delta_{i,n-1}^{-1}$ . Then  $\beta$  admits a decomposition of the form (3.11) with  $\beta^* = \beta' \cdot \delta_{i,n-1}$  and  $\beta_* = \delta_{i,n-1}^{-1}$ .  $\square$

**3.5. The Key Lemma.** We recall that our aim is to prove Implication (3.3) in the general case, and to show how to deduce Implication (3.1). Actually, we shall establish a unique statement, which is a refinement of Implication (3.3):

**Proposition 3.19 (Key Lemma).** Assume that  $(\beta_p, \dots, \beta_2, 1)$  is the  $\phi_n$ -splitting of a braid  $\beta$  in  $B_n^{+*}$  with  $p \geq 2$ . Let  $\mathbf{a}_{i-1,n-1}$  be the last letter of  $\beta_p$  and  $\mathbf{a}_{j,n}$  is the one of  $\beta$ . Then, for every  $\sigma_{i-1,n-2}$ -positive braid  $\beta'$ , the braid

$$(3.13) \quad \delta_n^{-p+2} \cdot \phi_n^{p-1}(\beta') \cdot \phi_n^{p-2}(\beta_{p-1}) \cdot \dots \cdot \phi_n(\beta_2)$$

is  $\sigma_{j,n-1}$ -positive.

We postpone the proof of Proposition 3.19 to Section 4, and show now how it implies Implication (3.3). We begin with an easy preparatory lemma.

**Lemma 3.20.** *Assume  $n \geq 3$ . Let  $\beta$  be a braid of  $B_n^{+*}$ , and let  $(\beta_q, \dots, \beta_1)$  be its splitting. Assume  $q \geq 3$ . Let  $r$  to be the greatest index satisfying  $\beta_r \neq \mathbf{a}_{n-2, n-1}$  if such an index exists, and 0 otherwise. Then we have*

$$(3.14) \quad \delta_n^{-q+2} \beta = \begin{cases} \delta_{n-1}^{-q+r} \delta_n^{-r+2} \beta_{[r,1]} & \text{for } r \geq 3, \\ \delta_{n-1}^{-q+2} \phi_n(\beta_2) \beta_1 & \text{for } r \leq 2. \end{cases}$$

*Proof.* Pushing  $q-r$  factors of  $\delta_n^{-q+2}$  to the right in the decomposition of  $\delta_n^{-q+2} \beta$ , we obtain

$$\begin{aligned} \delta_n^{-q+2} \beta &= \delta_n^{-q+2} \phi_n^{q-1}(\beta_q) \cdot \dots \cdot \phi_n^r(\beta_{r+1}) \cdot \phi_n^{r-1}(\beta_r) \cdot \dots \cdot \beta_1 \\ &= \phi_n(\beta_q) \delta_n^{-1} \delta_n^{-q+3} \cdot \dots \cdot \phi_n^r(\beta_{r+1}) \cdot \phi_n^{r-1}(\beta_r) \cdot \dots \cdot \beta_1 \\ &\dots \\ &= \phi_n(\beta_q) \delta_n^{-1} \cdot \dots \cdot \phi_n(\beta_{r+1}) \delta_n^{-1} \delta_n^{-r+2} \cdot \phi_n^{r-1}(\beta_r) \cdot \dots \cdot \beta_1 \\ &= \phi_n(\mathbf{a}_{n-2, n-1}) \delta_n^{-1} \cdot \dots \cdot \phi_n(\mathbf{a}_{n-2, n-1}) \delta_n^{-1} \delta_n^{-r+2} \cdot \phi_n^{r-1}(\beta_r) \cdot \dots \cdot \beta_1. \end{aligned}$$

Using the relation  $\phi_n(\mathbf{a}_{n-2, n-1}) \delta_n^{-1} = \delta_{n-1}^{-1}$ , the expected result follows.  $\square$

Now, we prove the following result than is an adaptation of the Key Lemma in a special case.

**Lemma 3.21.** *Assume that  $(\beta_p, \dots, \beta_2, 1)$  is the  $\phi_n$ -splitting of a braid in  $B_n^{+*}$  with  $p \leq 2$  and that the last letter of  $\beta_p$  is  $\mathbf{a}_{n-2, n-1}$ . Let  $\mathbf{a}_{j, n}$  be the last letter of  $\beta$ . Then the braid*

$$(3.15) \quad \delta_n^{-p+2} \cdot \phi_n^{p-1}(\mathbf{a}_{n-2, n-1}) \cdot \phi_n^{p-2}(\beta_{p-1}) \cdot \dots \cdot \phi_n(\beta_2),$$

*is  $\sigma_{j, n-1}$ -positive for  $\beta_{[p-1, 2]} \neq \phi_n^{p-2}(\mathbf{a}_{n-2, n-1}) \cdot \dots \cdot \phi_n^2(\mathbf{a}_{n-2, n-1})$ , and equals  $\delta_{n-1}^{-p+2}$  otherwise.*

*Proof.* Let  $r$  be the greatest integer satisfying  $\beta_r \neq \mathbf{a}_{n-2, n-1}$  and  $r \leq p-1$ . If  $r \leq 1$  holds, the braid  $\beta_{[p-1, 2]}$  equals  $\phi_n^{p-2}(\mathbf{a}_{n-2, n-1}) \cdot \dots \cdot \phi_n^2(\mathbf{a}_{n-2, n-1})$ , and, by Lemma 3.20, the braid (3.15) equals  $\delta_{n-1}^{-q+2}$ , which is the expected result in this case. Assume  $r \geq 2$ . By Lemma 3.20, the braid (3.15) equals  $\delta_{n-1}^{-p+r} \delta_n^{-r+2} \beta_{[r, 2]}$ . By definition of  $r$ , Lemma 3.18 guarantees that  $\beta_r$  is  $\sigma_{i-1, n-1}$ -positive for some  $i$ . Then Proposition 3.19 implies that  $\delta_n^{-r+2} \beta_{[r, 2]}$  is  $\sigma_{j, n-1}$ -positive, and so is the braid (4.15), since  $\delta_{n-1}^{-1}$  is  $\sigma_{n-1}$ -nonnegative.  $\square$

Then the Key Lemma (Proposition 3.19) implies the second of our expected implications.

**Corollary 3.22.** *For  $n \geq 3$ , Implication (3.3) is true.*

*Proof.* Let  $\beta$  be a braid satisfying  $\widehat{\delta}_{n, p} \leq^* \beta$  and let  $(\beta_q, \dots, \beta_1)$  be its  $\phi_n$ -splitting. By definition of  $\leq^*$ , we have  $p+2 \leq q$ . Assume first  $p \geq 1$ . As we have

$$(3.16) \quad \widehat{\delta}_{n, p}^{-1} \beta = \widehat{\delta}_{n, p}^{-1} \widehat{\delta}_{n, q-2} \cdot \widehat{\delta}_{n, q-2}^{-1} \beta,$$

and as the not underlined factor is  $\sigma$ -positive by Proposition 2.11 (iv), we can suppose  $q = p+2$ . Using Proposition 2.11 (ii), we establish

$$(3.17) \quad \widehat{\delta}_{n, q-2}^{-1} \beta = \delta_{n-1}^{q-2} \delta_n^{-q+2} \beta,$$

which, by Lemma 3.21, is  $\sigma$ -positive.

Assume now  $p = 1$ . As in the previous case, we can suppose  $q = 2$ . Let  $(\beta_2, \beta_1)$  be the  $\phi_n$ -splitting of  $\beta$ , then we obtain

$$\widehat{\delta}_{n,0}^{-1} \beta = \mathbf{a}_{n-1,n} \phi_n(\beta_2) \cdot \beta_1 = \phi_n(\mathbf{a}_{n-2,n-1}^{-1}) \phi_n(\beta_2) \beta_1 = \phi_n(\mathbf{a}_{n-2,n-1}^{-1} \beta_2) \cdot \beta_1.$$

We use induction on  $n$ . For  $n = 3$ , there exists  $e \geq 1$  such that  $\beta_2 = \mathbf{a}_{1,2}^e$ , then  $\mathbf{a}_{1,2}^{-1} \beta_2$  equals  $\mathbf{a}_{1,2}^{e-1}$  that is  $\sigma_1$ -positive. For  $n > 3$ , by Lemma 1.23, the braid  $\beta_2$  have a  $(n-1)$ -breath at least 2 and  $\mathbf{a}_{n-2,n-1}$  is  $\widehat{\delta}_{n,0}$ . Then, by induction hypothesis,  $\mathbf{a}_{n-2,n-1}^{-1} \beta_2$  is  $\sigma$ -positive, and so is  $\widehat{\delta}_{n-1,0}^{-1} \beta$ .  $\square$

**3.6. Proof of Theorem 3.1.** At this point, we saw that, provided we take the Key Lemma for granted, both Implications (3.2) and (3.3) are true, and it is not hard to deduce that Implication (3.1), which is our goal, is true in the case when the breadth of  $\beta$  is smaller than the breadth of  $\beta'$ , *i.e.*, the “Short”-case in the ShortLex-ordering.

So, there remains to treat the “Lex”-case, *i.e.*, the case when  $\beta$  and  $\beta'$  have the same breadth, and this is what we shall do now. Actually, as was already mentioned when stating the Key Lemma, in order to maintain an induction hypothesis, we shall prove a stronger implication: instead of merely proving that the quotient braid  $\beta^{-1}\beta'$  is  $\sigma$ -positive, we shall prove the more precise conclusion that  $\beta^{-1}\beta'$  is  $\sigma_{i,n-1}$ -positive for some  $i$  related to the last letter in  $\beta'$ . That is why we shall have to reconsider both the Short- and the Lex-cases.

The main result is the following one—note that, as the case  $\beta = 1$  is trivially satisfied, it contains Theorem 3.1.

**Proposition 3.23.** *For  $\beta, \beta'$  in  $B_n^{+*}$  with  $\beta \neq 1$ , the relation  $\beta <^* \beta'$  implies  $\beta < \beta'$ . Moreover, if the  $B_{n-1}^{+*}$ -tails of  $\beta, \beta'$  are trivial and  $\beta^{-1}\beta' \neq \mathbf{a}_{n-1,n}$  the braid  $\beta^{-1}\beta'$  is  $\sigma_{i,n-1}$ -positive, where  $\mathbf{a}_{i,n}$  is the last letter of  $\beta'$ .*

*Proof of Proposition 3.23 from the Key Lemma.* We use an induction on  $n$ . For  $n = 2$ , everything is obvious, as both  $<$  and  $<^*$  identify with the standard ordering of the natural numbers.

From now on, we assume  $n \geq 3$ . Let  $\beta$  and  $\beta'$  be elements of  $B_n^{+*}$  satisfying  $\beta <^* \beta'$  and  $\beta \neq 1$ , hence  $\beta' \neq 1$  as well. Let  $(\beta_p, \dots, \beta_1)$  and  $(\beta'_q, \dots, \beta'_1)$  be the  $\phi_n$ -splittings of  $\beta$  and  $\beta'$ .

Firstly, we observe that, when the  $n$ -breadth of  $\beta$  and  $\beta'$  is at least 2, it is sufficient to prove that  $\beta^{-1}\beta'_{[q,2]}$  is  $\sigma_{i,n-1}$ -positive, where  $\mathbf{a}_{i,n}$  is the last letter of  $\beta'_{[r,2]}$ . Indeed, if this is done, then the  $\sigma_{i,n-1}$ -positivity of  $\beta^{-1}\beta'_{[q,2]}$ , plus the  $\sigma$ -positivity of  $\beta'_1$ , imply the  $\sigma$ -positivity of  $\beta^{-1}\beta'$ . Moreover, if the  $B_{n-1}^{+*}$ -tail of  $\beta'$  is trivial, then  $\beta'$  equals  $\beta'_{[q,2]}$ , and the  $\sigma_{i,n-1}$ -positivity of  $\beta^{-1}\beta'_{[q,2]}$  is that of  $\beta^{-1}\beta'$ .

Assume first  $p = q$ . By definition of  $<^*$ , there exists an integer  $r$  satisfying

$$(3.18) \quad \beta_t = \beta'_t \quad \text{for } t > r, \text{ and } \beta_r <^* \beta'_r.$$

We consider three cases : the generic case  $r \geq 3$ , plus the special cases  $r = 2$  and  $r = 1$ .

– Case  $r \geq 3$ . Let  $\beta''$  be the quotient  $\beta_r^{-1}\beta'_r$ . Then we express the quotient  $\beta^{-1}\beta'$  as the product of three factors:

$$(3.19) \quad \begin{aligned} \beta^{-1}\beta'_{[r,2]} &= \beta_{[r-1,1]}^{-1} \cdot \phi_n^{r-1}(\beta'') \cdot \beta'_{[r-1,2]} \\ &= (\beta_{[r-1,1]}^{-1} \cdot \widehat{\delta}_{n,r-2}) \cdot \delta_{n-1}^{r-2} \cdot (\delta_n^{-r+2} \cdot \phi_n^{r-1}(\beta'') \cdot \beta'_{[r-1,2]}). \end{aligned}$$

By Lemma 1.25, the  $n$ -breadth of  $\beta_{[r-1,1]}$  is  $r-1$ , hence, by Lemma 3.6, the left factor in (3.19), namely  $\beta_{[r-1,1]}^{-1} \widehat{\delta}_{n,r-2}$ , is  $\sigma_{n-1}$ -positive. As the middle factor is  $\sigma$ -positive, it remains the right factor of (3.19). As, by Lemma 1.23 (ii),  $\beta_r$  and  $\beta'_r$  are not trivial, Lemma 1.23 implies that the last letter of  $\beta_r$  does not lie in  $B_{n-1}^{+*}$ . Then the last letter of  $\beta'_r$  is  $\mathbf{a}_{j-1,n-1}$  for some  $j \leq n-1$ . Assume  $\beta'' \neq \mathbf{a}_{n-2,n-1}$ . The induction hypothesis implies that  $\beta''$  is  $\sigma_{j-1,n-2}$ -positive. Then the Key Lemma implies that last right of (3.19) is  $\sigma_{i,n-1}$ -positive. Therefore Relation (3.19) together with Lemma 3.16 (ii) implies that the braid  $\beta^{-1} \beta'_{[r,2]}$  is  $\sigma_{i,n-1}$ -positive.

Assume now  $\beta'' = \mathbf{a}_{n-2,n-1}$ . For  $\beta'_{[r-1,2]} \neq \phi_n^{r-2}(\mathbf{a}_{n-2,n-1}) \cdot \dots \cdot \phi_n^2(\mathbf{a}_{n-2,n-1})$ , Lemma 3.21 implies that the right factor in (3.19) is  $\sigma_{i,n-1}$ -positive. Assume  $\phi_n^{r-2}(\mathbf{a}_{n-2,n-1}) \cdot \dots \cdot \phi_n^2(\mathbf{a}_{n-2,n-1})$ . Then  $i$  equals 1, and (3.19) becomes

$$(3.20) \quad \beta_1^{-1} \cdot \delta_n \beta_2^{-1} \cdot \dots \cdot \delta_n \beta_{r-1}^{-1}$$

By Lemma 1.6, there exists  $k \geq 0$  satisfying  $\delta_{n-1}^k \beta_{r-1}^{-1} = \beta'_{r-1}$  with  $\beta'_{r-1} \in B_{n-1}^+$ . Then we obtain

$$\begin{aligned} \beta^{-1} \beta'_{[r,2]} &= \beta_1^{-1} \cdot \delta_n \beta_2^{-1} \cdot \dots \cdot \delta_n \delta_{n-1}^{-k} \beta'_{r-1} \\ &= \beta_1^{-1} \cdot \delta_n \beta_2^{-1} \cdot \dots \cdot \delta_n \delta_{n-1}^{-1} \delta_{n-1}^{-k+1} \beta'_{r-1} \\ &= \beta_1^{-1} \cdot \delta_n \beta_2^{-1} \cdot \dots \cdot \underline{\mathbf{a}_{1,n} \delta_{n-1}^{-k+1} \beta'_{r-1}}, \end{aligned}$$

where the underlined factor is  $\sigma_{1,n-1}$ -positive. Thus Lemma 3.16 (ii) implies that the braid  $\beta^{-1} \beta'_{[r,2]}$  is  $\sigma_{i,n-1}$ -positive.

– Case  $r = 2$ . Let  $\mathbf{a}_{j-1,n-1}$  be the last letter of  $\beta'_2$ . Then we have  $\beta^{-1} \beta'_{[r,2]} = \beta_1^{-1} \cdot \phi_n(\beta_2^{-1} \beta'_2)$ . For  $\beta_2 \neq 1$ , the induction hypothesis implies that  $\beta_2^{-1} \beta'_2$  is  $\sigma_{j-1,n-2}$ -positive. For  $\beta_2 = 1$  with  $\beta'_2 \neq \mathbf{a}_{n-2,n-1}$ , Lemma 3.18 implies that  $\beta'_2$  is  $\sigma_{j-1,n-2}$ -positive. In these both cases, Proposition 3.19 implies that the quotient  $\beta^{-1} \beta'_{[r,2]}$  is  $\sigma_{i,n-1}$ -positive. For  $\beta_2 = 1$  with  $\beta'_2 = \mathbf{a}_{n-2,n-1}$ , the quotient braid  $\beta_2^{-1} \beta'_2$  is  $\sigma_{n-2}$ -positive. Then  $\beta^{-1} \beta'_{[r,2]}$  is  $\sigma_{n-1}$ -positive. We prove nothing more since the quotient  $\beta^{-1} \beta'$  is  $\mathbf{a}_{n-1,n}$ .

– Case  $r = 1$ . By induction hypothesis, the quotient  $\beta_1^{-1} \beta'_1$  is  $\sigma_k$ -positive with  $k \leq n-2$ , then  $\beta^{-1} \beta' = \phi_n(\beta_1^{-1} \beta'_1)$  is  $\sigma$ -positive, actually  $\sigma_{k+1}$ -positive. We prove nothing more since the last letter of  $\beta'$  lies in  $B_{n-1}^{+*}$ .

Assume now  $p < q$ .

– Case  $p = 1$ . Then  $\beta$  lies in  $B_{n-1}^{+*}$  but  $\beta'$  does not, that implies  $\beta^{-1} \beta'$  is  $\sigma_{n-1}$ -positive, which is the expected result as the last letter of  $\beta$  lies also in  $B_{n-1}^{+*}$ .

– Case  $p \geq 2$  with  $\beta'_{[q,2]} \neq \widehat{\delta}_{n,q-2}$ . Then we have the decomposition

$$\beta^{-1} \beta'_{[q,2]} = (\beta^{-1} \widehat{\delta}_{n,q-2}) \cdot (\widehat{\delta}_{n,q-2}^{-1} \beta'_{[q,2]}).$$

Corollary 3.22 implies that  $\widehat{\delta}_{n,q-2}^{-1} \beta'_{[q,2]}$  is  $\sigma_{i,n-1}$ -positive, where  $\mathbf{a}_{i-1,n-2}$  is the last letter of  $\beta'_2$ . In the other hand, by Proposition 3.6,  $\beta^{-1} \widehat{\delta}_{n,q-2}$  is  $\sigma_{n-1}$ -positive. Therefore, by Lemma 3.16 (ii), the quotient  $\beta^{-1} \beta'_{[q,2]}$  is  $\sigma_{i,n-1}$ -positive.

– Case  $p \geq 2$  with  $\beta'_{[q,2]} = \widehat{\delta}_{n,q-2}$ . Then the last letter of  $\beta'_{[q,2]}$  is  $\mathbf{a}_{1,n}$ . Using the

definition of  $\phi_n$  and pushing the power of  $\delta_n$  to the left, we find

$$\begin{aligned}
\beta^{-1}\beta'_{[q,2]} &= \beta_1^{-1} \cdot \phi_n(\beta_2^{-1}) \cdot \dots \cdot \phi_n^{p-1}(\beta_p^{-1}) \cdot \delta_n^{q-1} \delta_{n-1}^{-q+2} \\
&= \beta_1^{-1} \cdot \phi_n(\beta_2^{-1}) \cdot \dots \cdot \phi_n^{p-1}(\beta_p^{-1}) \cdot \delta_n^{p-1} \cdot \delta_n^{q-p-1} \delta_{n-1}^{-q+2} \\
(3.21) \quad &= \beta_1^{-1} \cdot \delta_n \beta_2^{-1} \cdot \dots \cdot \delta_n \beta_p^{-1} \cdot \underline{\delta_n^{q-p-1} \delta_{n-1}^{-q+2}}
\end{aligned}$$

The not underlined factor is  $\sigma_{n-1}$ -positive as  $\delta_n$  is  $\sigma_{n-1}$ -positive and every  $\beta_i$  lies in  $B_{n-1}^{+*}$ . It remains to prove that the underlined factor is  $\sigma_{1,n-1}$ -positive. For  $q \geq p+2$ , we have  $\delta_n^{q-p-1} \delta_{n-1}^{-q+2} = \delta_n^{q-p-2} \mathbf{a}_{1,n} \delta_{n-1}^{-q+3}$  that is  $\sigma_{1,n-1}$ -positive, and so is  $\beta^{-1}\beta'_{[q,2]}$ , by Lemma 3.16 (ii). Otherwise, *i.e.*, for  $q = p+1$ , Lemma 1.6 implies that there exists  $k \geq 0$  satisfying  $\delta_{n-1}^k \beta_p^{-1} = \beta_p^+ \in B_{n-1}^+$ , and (3.21) becomes

$$\begin{aligned}
\beta^{-1}\beta'_{[q,2]} &= \beta_1^{-1} \cdot \delta_n \beta_2^{-1} \cdot \dots \cdot \delta_n \beta_{p-1}^{-1} \cdot \delta_n \delta_{n-1}^{-k} \beta_p^+ \cdot \delta_{n-1}^{-q+2}, \\
(3.22) \quad &= \beta_1^{-1} \cdot \delta_n \beta_2^{-1} \cdot \dots \cdot \delta_n \beta_{p-1}^{-1} \cdot \underline{\mathbf{a}_{1,n} \delta_{n-1}^{-k+1} \beta_p^+ \cdot \delta_{n-1}^{-q+2}}.
\end{aligned}$$

The underlined factor is  $\sigma_{1,n-1}$ -positive, and so is  $\beta^{-1}\beta'_{[q,2]}$  as the not underlined factor is  $\sigma_{n-1}$ -positive.  $\square$

Thus, at this point, the main argument is completed, and we are left with proving the Key Lemma, *i.e.*, Proposition 3.19.

#### 4. PROOF OF THE KEY LEMMA

The Key Lemma, which claims that certain specific braids are  $\sigma_{i,n-1}$ -positive, is true because  $\phi_n$ -splittings have to satisfy some constraints. We already saw a few such constraints in Lemma 1.23. We shall now establish more. The main result is, under specific circumstances, the entries in a  $\phi_n$ -splitting must contain specific letters that will be called barriers, and such barriers organize into what will be called stairs.

**4.1. Barriers.** If  $\beta$  is a braid of  $B_n^{+*}$  of  $n$ -breadth at least 3, and  $(\beta_p, \dots, \beta_1)$  is its  $\phi_n$ -splitting, then, by Lemma 1.23, the last letter of  $\beta_r$  for  $r \geq 3$  is some letter  $\mathbf{a}_{k-1,n-1}$ : in this case, we shall see that the braid  $\beta_{r-1}$  cannot be an arbitrary braid of  $B_{n-1}^{+*}$ , but it has to satisfy constraints involving the integer  $k$ , namely containing a letter called an  $\mathbf{a}_{k,n}$ -barrier—and this will be the point in the sequel.

**Definition 4.1.** The letter  $\mathbf{a}_{i,j}$  is called  $\mathbf{a}_{k,n}$ -barrier if we have

$$(4.1) \quad 1 \leq i < k < j \leq n-1.$$

There exists no  $\mathbf{a}_{k,n}$ -barrier with  $n \leq 3$ ; the only  $\mathbf{a}_{k,n}$ -barrier for  $n = 4$  is  $\mathbf{a}_{1,3}$ , which is an  $\mathbf{a}_{2,4}$ -barrier.

By definition, if the letter  $s$  is an  $\mathbf{a}_{k,n}$ -barrier, then there exists in the presentation of  $B_n^{+*}$  given in Proposition 1.2 no relation of the form  $\mathbf{a}_{k,n} \cdot s = s' \cdot \mathbf{a}_{k,n}$  allowing one to quasi-commute  $s$  and  $\mathbf{a}_{k,n}$ : so, in some sense,  $s$  acts as a barrier to prevent the migration of  $\mathbf{a}_{k,n}$  to the right. This fact will be crucial in proving that, if  $\beta$  contains a  $\mathbf{a}_{k,n}$ -barrier and has last letter  $\mathbf{a}_{i,n}$ , then  $\delta_n^{-1} \cdot \mathbf{a}_{k,n} \cdot \beta$  is  $\sigma_{i,n-1}$ -positive

**Lemma 4.2.** Assume that  $\beta$  is a braid of  $B_{n-1}^{+*}$  such that the  $B_{n-1}^{+*}$ -tail of  $\phi_n(\mathbf{a}_{k,n}\beta)$  is trivial for  $k \leq n-2$ . Then  $\beta$  contains an  $\mathbf{a}_{k,n}$ -barrier.



*Proof.* We assume that  $\beta$  contains no  $\mathbf{a}_{k,n}$ -barrier, and shall eventually obtain a contradiction. Let  $w = \mathbf{a}_{k,n}w'$  where  $w'$  is the  $\phi$ -normal form of  $\beta$ , and let  $S$  be the set of all letters  $\mathbf{a}_{i,j}$  with  $k < j \leq n-1$ . As we have  $\phi_n(B_{n-2}^{+*}) \subset B_{n-1}^{+*}$ , the last letter of  $w'$  is some  $\mathbf{a}_{i,n-1}$ , which is an element of  $S$  because, by hypothesis, it is not an  $\mathbf{a}_{k,n}$ -barrier. Then the longest suffix  $v$  of  $w$  that lie in  $S^*$  is nonempty. Let  $u$  the associated prefix. As the first letter of  $w$  is  $\mathbf{a}_{k,n}$ , which is not in  $S$ , the word  $u$  is nonempty. Let  $\mathbf{a}_{g,h}$  be the last letter of  $u$ . Two cases are possible: either  $u$  has length 1, and  $\mathbf{a}_{g,h}$  is  $\mathbf{a}_{k,n}$ , or it has length 2 at least, and then  $\mathbf{a}_{g,h}$  is a letter from  $w'$ , hence it satisfies  $h \leq k$  as, by construction, it does not lie in  $S$ . In both cases, we have

$$(4.2) \quad \phi_n(\mathbf{a}_{g,h}) \in B_{n-1}^{+*}.$$

On the other hand, no letter  $\mathbf{a}_{i,j}$  in  $v$  is an  $\mathbf{a}_{k,n}$ -barrier, hence it satisfies  $k \leq i < j \leq n-1$ . By the relations

$$\mathbf{a}_{g,h}\mathbf{a}_{i,j} \equiv \begin{cases} \mathbf{a}_{i,j}\mathbf{a}_{g,h}, & \text{for } k < i \text{ or } h < k & \text{by (1.3),} \\ \mathbf{a}_{g,j}\mathbf{a}_{g,h}, & \text{for } i = h = k & \text{by (1.4),} \\ \mathbf{a}_{j,n}\mathbf{a}_{g,h}, & \text{for } i = g = k & \text{by (1.4).} \end{cases}$$

the letter  $\mathbf{a}_{g,h}$  quasi-commutes with  $v$ , *i.e.*, there exists a word  $w'$  satisfying  $\mathbf{a}_{g,h}v \equiv v'\mathbf{a}_{g,h}$ . Therefore,  $\mathbf{a}_{k,n}\beta$  is right divisible by  $\mathbf{a}_{g,h}$ . It follows that  $\phi_n(\mathbf{a}_{g,h})$  is an right divisor of  $\phi_n(\mathbf{a}_{k,n}\beta)$ , then (4.2) contradicts the hypothesis.  $\square$

We saw in Example 3.11 that  $\mathbf{a}_{2,3}^{-1}\mathbf{a}_{1,3}$  equals  $\mathbf{a}_{1,3}\mathbf{a}_{1,2}^{-1}$ , a  $\sigma_2$ -positive expression. With our current terminology, this shows how the barrier  $\mathbf{a}_{1,3}$  can protect us—namely guarantee  $\sigma$ -positivity—against the dangerous factor  $\mathbf{a}_{2,3}^{-1}$ . This protection phenomenon takes the following general form.

**Lemma 4.3.** *Assume that  $\mathbf{a}_{i,j}$  is an  $\mathbf{a}_{k,n}$ -barrier. Then we have*

$$(4.3) \quad \delta_{k,n-1}^{-1} \mathbf{a}_{i,j} = \mathbf{a}_{i,n-1} \delta_{i,k-1} \delta_{i,j-1}^{-1} \delta_{j,n-1}^{-1}.$$

*Proof.* We decompose  $\mathbf{a}_{i,j}$  as  $\delta_{i,k} \delta_{k,j} \delta_{i,j-1}^{-1}$  by (1.7) and (1.8), and we obtain

$$\begin{aligned} \delta_{k,n-1}^{-1} \mathbf{a}_{i,j} &= \delta_{k,n-1}^{-1} \delta_{i,k} \delta_{k,j} \delta_{i,j-1}^{-1} \delta_{j,n-1} \delta_{j,n-1}^{-1} \\ &= \delta_{k,n-1}^{-1} \delta_{i,k} \delta_{k,j} \delta_{j,n-1} \delta_{i,j-1}^{-1} \delta_{j,n-1}^{-1} && \text{by (1.9)} \\ &= \delta_{k,n-1}^{-1} \delta_{i,k} \delta_{k,n-1} \delta_{i,j-1}^{-1} \delta_{j,n-1}^{-1} && \text{by (1.8)} \\ &= \delta_{k,n-1}^{-1} \delta_{i,k} \delta_{i,k-1}^{-1} \delta_{i,k-1} \delta_{k,n-1} \delta_{i,j-1}^{-1} \delta_{j,n-1}^{-1} \\ &= \delta_{k,n-1}^{-1} \mathbf{a}_{i,k} \delta_{i,k-1} \delta_{k,n-1} \delta_{i,j-1}^{-1} \delta_{j,n-1}^{-1} && \text{by (1.7)} \\ &= \delta_{k,n-1}^{-1} \mathbf{a}_{i,k} \delta_{k,n-1} \delta_{i,k-1} \delta_{i,j-1}^{-1} \delta_{j,n-1}^{-1} && \text{by (1.9)} \\ &= \phi_{n-1}^{-1}(\phi_k(\mathbf{a}_{i,k})) \delta_{i,k-1} \delta_{i,j-1}^{-1} \delta_{j,n-1}^{-1} && \text{by (1.10)} \\ &= \mathbf{a}_{i,n-1} \delta_{i,k-1} \delta_{i,j-1}^{-1} \delta_{j,n-1}^{-1}, && \text{by (1.6)} \end{aligned}$$

which is the expected result.  $\square$

Our aim would be to prove that braids of the form  $\delta_n^{-1} \cdot \phi_n^2(\mathbf{a}_{k-1,n-1}) \cdot \phi_n(\beta)$  are  $\sigma$ -positive whenever  $\beta$  belongs to  $B_{n-1}^{+*}$  and contains an  $\mathbf{a}_{k,n}$ -barrier. We decompose

the letter  $\mathbf{a}_{k-1,n-1}$  as  $\delta_{k-1,n-2} \mathbf{a}_{n-2,n-1} \delta_{k-1,n-2}^{-1}$  and we push  $\delta_n^{-1}$  to the right of  $\mathbf{a}_{n-1,n}$  in order to obtain

$$\begin{aligned} \delta_n^{-1} \cdot \phi_n^2(\mathbf{a}_{k,n}) \cdot \phi_n(\beta) &= \phi_n(\delta_{k-1,n-2} \mathbf{a}_{n-2,n-1}) \delta_n^{-1} \phi_n(\delta_{k,n-1}^{-1} \beta), \\ (4.4) \qquad \qquad \qquad &= \phi_n(\delta_{k-1,n-2}) \delta_{n-1}^{-1} \phi_n(\delta_{k,n-1}^{-1} \beta). \end{aligned}$$

Let us suppose that the  $\phi$ -normal of  $\beta$  is  $\mathbf{a}_{i,j}w$ , where  $\mathbf{a}_{i,j}$  is an  $\mathbf{a}_{k,n}$ -barrier. Then we try to apply Lemma 4.3 in order to show that the quotient  $\delta_{k,n-1}^{-1} \mathbf{a}_{i,j}$  is  $\sigma_{n-2}$ -positive. Here comes a problem.

**Example 4.4.** By Definition 4.1, the letter  $\mathbf{a}_{1,3}$  is an  $\mathbf{a}_{2,4}$  barrier and Lemma 4.3 implies

$$\delta_{2,4}^{-1} \mathbf{a}_{1,3} = \mathbf{a}_{1,4} \delta_{1,1} \delta_{1,2}^{-1} \delta_{3,4}^{-1} = \mathbf{a}_{1,4} \mathbf{a}_{1,2}^{-1} \mathbf{a}_{3,4}^{-1}.$$

By relation (1.3), the braid  $\mathbf{a}_{1,4} \mathbf{a}_{1,2}^{-1} \mathbf{a}_{3,4}^{-1}$  is equal to  $\mathbf{a}_{1,4} \mathbf{a}_{3,4}^{-1} \mathbf{a}_{1,2}^{-1}$ . As the relation  $\mathbf{a}_{3,4} \mathbf{a}_{1,4} = \mathbf{a}_{1,3} \mathbf{a}_{3,4}$  holds, see (1.4), the braid  $\mathbf{a}_{1,4} \mathbf{a}_{3,4}^{-1}$  is equal to  $\mathbf{a}_{3,4}^{-1} \mathbf{a}_{1,3}$ , that is  $\sigma_3$ -negative, and so is  $\delta_{2,4}^{-1} \mathbf{a}_{1,3}$ : in that case, the barrier  $\mathbf{a}_{1,3}$  did *not* protect us against the dangerous element  $\delta_{2,4}^{-1}$ .

We are thus led to partition barriers into two subclasses:

**Definition 4.5.** A  $\mathbf{a}_{k,n}$ -barrier  $\mathbf{a}_{i,j}$  is said to be *strong* if we have  $j = n-1$ , otherwise, for  $j < n-1$ , it said to be *weak*.

**Lemma 4.6.** Assume that  $\mathbf{a}_{i,j}$  is a strong  $\mathbf{a}_{k,n}$ -barrier. Then the braid  $\delta_{k,n-1}^{-1} \mathbf{a}_{i,j}$  is  $\sigma_{n-2}$ -positive.

*Proof.* By Lemma 4.3, we obtain

$$\delta_{k,n-1}^{-1} \mathbf{a}_{i,j} = \mathbf{a}_{i,n-1} \delta_{i,k-1} \delta_{i,n-2}^{-1},$$

where the only negative factor, namely  $\delta_{i,n-2}^{-1}$ , does not involve the  $(n-1)$ th strand while  $\mathbf{a}_{i,n-1}$  does.  $\square$

In the sequel, we shall use the symbol  $\delta_{i,j}$  for the braid word  $\sigma_i \dots \sigma_{j-1}$  and for the braid represented by its word.

Before addressing the problem of weak barriers, we conclude this section with an additional technical result. Above we considered the question of proving that  $\delta_{k,n-1}^{-1} \mathbf{a}_{i,j}$  is possibly  $\sigma$ -positive. Actually, in the general case, we have to consider a more general situation of the type  $\delta_{k,n-1}^{-1} w \mathbf{a}_{i,j}$ , where  $w$  is a word containing no  $\mathbf{a}_{k,n}$ -barrier. This is not difficult, because a braid containing no  $\mathbf{a}_{k,n}$ -barrier has good properties under conjugation by  $\delta_{k,n-1}^{-1}$ .

**Lemma 4.7.** Assume  $\beta$  is a braid of  $B_{n-1}^{+*}$  that can be represented by a word containing no  $\mathbf{a}_{k,n}$ -barrier. Then there exists a word  $w$  representing  $\delta_{k,n-1}^{-1} \beta \delta_{k,n-1}$ , such that every  $\mathbf{a}_{k,n}$ -barrier occurring in  $w$  is a strong barrier.

*Proof.* Let  $w'$  be a braid word representing  $\beta$  that contains no  $\mathbf{a}_{k,n}$ -barrier. If  $w'$  is empty, the result is obvious, since  $\beta$  is trivial. Assume that  $w'$  consists of one letter  $\mathbf{a}_{i,j}$ . By hypothesis, the letter  $\mathbf{a}_{i,j}$  is not an  $\mathbf{a}_{k,n}$ -barrier. Hence the integers  $i$  et  $j$  satisfy either  $1 \leq i < j \leq k$ , or  $k \leq i < j \leq n-1$ . Assume  $1 \leq i < j \leq k$  holds. From (1.10), we deduce

$$\delta_{k,n-1}^{-1} \mathbf{a}_{i,j} \delta_{k,n-1} = \phi_{n-1}^{-1}(\phi_k(\mathbf{a}_{i,j})),$$

which is  $\mathbf{a}_{i,n-1}$  for  $j = k$ , and  $\mathbf{a}_{i,j}$  for  $j < k$ . In this case, only  $\mathbf{a}_{i,n}$  can be an  $\mathbf{a}_{k,n}$ -barrier, and it is a strong one. Assume  $k \leq i < j \leq n-1$  holds. By definition of  $\phi_{n-1}$  and  $\phi_{n-k-2}$ , we obtain

$$\begin{aligned} \delta_{k,n-1}^{-1} \mathbf{a}_{i,j} \delta_{k,n-1} &= \phi_{n-1}^{k-1} (\delta_{n-k-2}^{-1} \mathbf{a}_{i-k-1,j-k-1} \delta_{n-k-2}) \\ &= \phi_{n-1}^{k-1} (\phi_{n-k-2}^{-1} (\mathbf{a}_{i-k-1,j-k-1})), \end{aligned}$$

which is  $\mathbf{a}_{j-1,n-1}$  for  $i = k$ , and  $\mathbf{a}_{i-1,j-1}$  for  $i > k$ . The letters  $\mathbf{a}_{j-1,n-1}$  and  $\mathbf{a}_{i-1,j-1}$  are not  $\mathbf{a}_{k,n}$ -barriers, since  $j-1 \geq k$  and  $i-1 \geq k$  hold respectively in this case.

Assume that  $w$  is a word of length  $\ell$  with  $\ell \geq 1$ . We decompose  $w'$  as  $\alpha_1 \dots \alpha_\ell$ , where the letters  $\alpha_i$  are not  $\mathbf{a}_{k+1,n}$ -barrier. Then we obtain

$$(4.5) \quad \delta_{k,n-1}^{-1} w' \delta_{k,n-1} = \delta_{k,n-1}^{-1} \alpha_1 \delta_{k,n-1} \dots \delta_{k,n-1}^{-1} \alpha_\ell \delta_{k,n-1},$$

using the above result.  $\square$

**4.2. Stairs.** We have seen above that every braid  $\beta$  of  $B_{n-1}^{+*}$  such that the  $\phi_n$ -splitting of  $\phi_n(\mathbf{a}_{k,n}\beta)$  is trivial, contains an  $\mathbf{a}_{k,n}$ -barrier. As was explained, we intend to use such a barrier to guarantee that, when we multiply by a  $\delta_{k,n-1}^{-1}$  on the left of  $\beta$ , then the negative factors cannot destroy the overall  $\sigma_{n-2}$ -positivity. Actually, we saw that only strong barriers achieve this goal: as shown in Example 4.4, a weak barrier is not sufficient.

Fortunately, we shall now see that, under the same hypotheses about the braid  $\beta$ , the latter does not only contain one barrier, but also a sequence of overlapping barriers that, taken altogether, will achieve the expected protection against dangerous elements. Such sequences of overlapping barriers are what we shall call stairs.

**Notation 4.8.** We note  $\underline{B}_n$  the set of the word on the alphabet  $\{\sigma_1^\pm, \dots, \sigma_{n-1}^\pm\}$ , and  $\underline{B}_n^+$  the set of the word on the alphabet  $\{\sigma_1, \dots, \sigma_{n-1}\}$ .

**Definition 4.9.** We say that a word  $w$  of  $\underline{B}_{n-1}$  is an  $\mathbf{a}_{k,n}$ -stair of height  $t$  lent on  $\mathbf{a}_{i-1,n-1}$ , if there exists a decomposition

$$(4.6) \quad w = w_0 s_1 w_1 \dots w_{t-1} s_t w_t,$$

and a sequence  $k = k_1 < k_2 < \dots < k_{t+1} = n-1$  such that

- (i) for each  $r \leq t$ , the letter  $s_r$  is an  $\mathbf{a}_{k_r,n}$ -barrier of the form  $\mathbf{a}_{\dots, k_r+1}$ ,
- (ii) for each  $r < t$ , the word  $w_r$  lies in  ${}^* \underline{B}_{n-1}^+$  and contains no  $\mathbf{a}_{k_r,n}$ -barrier,
- (iii) the word  $w$  represents a  $\sigma_{i-1,n-2}$ -positive braid.

**Example 4.10.** Let  $\beta$  be the braid  $\mathbf{a}_{1,3}\mathbf{a}_{4,5}\mathbf{a}_{2,4}\mathbf{a}_{2,4}\mathbf{a}_{3,5}\mathbf{a}_{4,5}$ . We claim that the  $\phi_5$ -splitting of  $\beta$  is  $(\mathbf{a}_{1,4}, \mathbf{a}_{3,4}\mathbf{a}_{1,3}\mathbf{a}_{1,3}\mathbf{a}_{2,4}\mathbf{a}_{3,4}, 1) = (\beta_3, \beta_2, \beta_1)$  and that the  $\phi$ -normal word of  $\beta_2$  is  $u = \mathbf{a}_{3,4}\mathbf{a}_{1,3}\mathbf{a}_{1,3}\mathbf{a}_{2,4}\mathbf{a}_{3,4}$ . We shall see that the  $\phi$ -normal word of  $\beta_2$  is an  $\mathbf{a}_{2,5}$ -stair lent on  $\mathbf{a}_{3,4}$ . We recall that the  $\phi$ -normal form of  $\beta_2$  is  $u = \mathbf{a}_{3,4}\mathbf{a}_{1,3}\mathbf{a}_{1,3}\mathbf{a}_{2,4}\mathbf{a}_{3,4}$ . From the left, the first  $\mathbf{a}_{2,5}$ -barrier occurring in  $u$  is  $\mathbf{a}_{1,3}$ . Then we put  $w_0 = \mathbf{a}_{3,4}$  and  $s_1 = \mathbf{a}_{1,3}$ . The remaining word is  $v = \mathbf{a}_{1,3}\mathbf{a}_{2,4}\mathbf{a}_{3,4}$ . From the left, the first  $\mathbf{a}_{1,5}$ -barrier occurring in  $v$  is  $\mathbf{a}_{2,4}$ . Then we put  $w_1 = \mathbf{a}_{1,3}$  and  $s_2 = \mathbf{a}_{2,4}$ . The latter barrier involves the 4th strand, so the decomposition ends with  $w_2 = \mathbf{a}_{3,4}$ . As  $\beta_2$  is different from  $\mathbf{a}_{3,4}$ , Lemma 3.18 implies that the braid  $\beta_2$  is  $\sigma_{3,3}$ -positive. Hence the  $\phi$ -normal form of  $\beta_2$  is an  $\mathbf{a}_{2,5}$ -stair of height 2 lent on  $\mathbf{a}_{3,4}$ .

It will be helpful to represent the generators  $\mathbf{a}_{i,j}$  as a vertical line from the  $i$ th line to the  $j$ th line on an  $n$ -line stove.

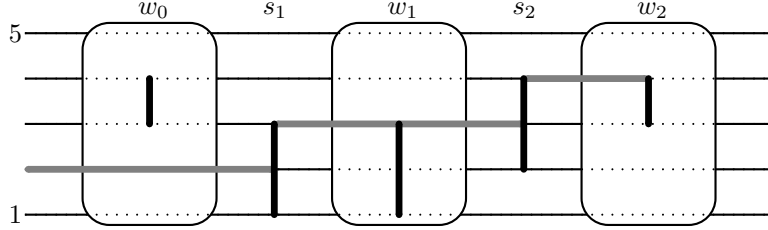


FIGURE 5. An  $\mathbf{a}_{2,5}$ -stair. The gray line represents the structure of the stair.

In Example 4.10, we have seen that the  $\phi$ -normal form of  $\beta_2$  is an  $\mathbf{a}_{2,5}$ -stair let on  $\mathbf{a}_{3,4}$ , where  $\phi_5^{-1}(\mathbf{a}_{2,5}) = \mathbf{a}_{1,4}$  is the last letter of  $\beta_3$  and  $\mathbf{a}_{3,4}$  is the last letter of  $\beta_2$ . This is actually a general result, that we shall prove in Proposition 4.12 below. In order to do that, we begin with a preparatory lemma that shows that, after certain letters in a  $\phi$ -normal form, there always exist a barrier. Applying this lemma repeatedly will allow us to obtain a stair.

**Lemma 4.11.** *Assume that  $w$  lies in  ${}^*B_{n-1}^+$  and that  $\mathbf{a}_{i,j}$  is a letter with  $j \leq n-2$  such that the  $B_{n-1}^*$ -tail of  $\phi_n(\mathbf{a}_{i,j}\bar{w})$  is trivial. Then  $w$  contains an  $\mathbf{a}_{j,n}$ -barrier.*

*Proof.* As, by hypothesis, the  $B_{n-1}^*$ -tail of  $\phi_n(\mathbf{a}_{i,j}\bar{w})$  is trivial, the last letter of  $w$  exists and is some  $\mathbf{a}_{m,n-1}$ . Let  $S$  be the set of the letters  $\mathbf{a}_{k,\ell}$  with  $j < \ell$ . Let  $v$  the greatest suffix of  $\mathbf{a}_{i,j}w$  lying in  $S^*$ , and,  $u$  the associated prefix. As  $\mathbf{a}_{i,j}$  is not in  $S$  and the last letter of  $w$  is, the words  $u$  and  $v$  are not empty. Let  $\mathbf{a}_{g,h}$  be the last letter of  $u$ , which, by construction, satisfies  $h \leq j$ . Assume that every letter  $\mathbf{a}_{k,\ell}$  of  $v$  satisfies  $k \geq j$ . Using the relations

$$\mathbf{a}_{g,h}\mathbf{a}_{k,\ell} = \begin{cases} \mathbf{a}_{k,\ell}\mathbf{a}_{g,h} & \text{for } k > j \text{ or } h < j, \\ \mathbf{a}_{g,\ell}\mathbf{a}_{g,h} & \text{for } k = h = j, \end{cases}$$

we establish  $\mathbf{a}_{g,h}v \equiv v'\mathbf{a}_{g,h}$ , where  $v'$  is a word of  ${}^*B_n^+$ . It follows that  $\phi_n(\mathbf{a}_{g,h})$  is an right divisor of  $\phi_n(\mathbf{a}_{i,j}\bar{w})$ , which contradicts the hypothesis on  $w$ , since the relation  $h \leq j < n-1$  implies  $\phi_n(\mathbf{a}_{g,h}) \in B_{n-1}^*$ .  $\square$

We can now show that, provided the braid  $\beta$  satisfies a mild condition, its  $\phi$ -normal form is a stair.

**Proposition 4.12.** *Assume that  $\beta$  belongs to  $B_{n-1}^*$  and  $k \leq n-2$  is an integer such that the  $B_{n-1}^*$ -tail of  $\phi_n(\mathbf{a}_{k,n}\beta)$  is trivial. Let  $\mathbf{a}_{i,n-1}$  be the last letter of  $\beta$ . Then, the  $\phi$ -normal form of  $\beta$  is an  $\mathbf{a}_{k,n}$ -stair lent on  $\mathbf{a}_{i,n-1}$ .*

*Proof.* Lemma 4.2 guarantees that the  $\phi$ -normal form of  $\beta$  can be decompose as  $w_0s_1w'_1$  where  $w_0$  contains no  $\mathbf{a}_{k,n}$ -barrier and  $s_1$  is an  $\mathbf{a}_{k,n}$ -barrier, namely  $\mathbf{a}_{g,h}$ . As the  $B_{n-1}^*$ -tail of  $\phi_n(\mathbf{a}_{k,n}\beta)$  is trivial, the  $B_{n-1}^*$ -tail of  $\phi_n(\mathbf{a}_{g,h}\bar{w}'_1)$  is trivial.. Assume that  $s_1$  is not a strong barrier, *i.e.*,  $h < n-1$  holds, Lemma 4.11 implies that  $w'_1$  can be decompose as  $w_1s_2w'_2$ , where  $w_2$  contains no  $\mathbf{a}_{h,n}$ -barrier, and the letter  $s_2$  is an  $\mathbf{a}_{h,n}$ -barrier. We continue this process until we find a decomposition  $w_0s_1w_1 \dots s_pw'_p$  with  $s_p$  a strong barrier, and we put  $w'_p = w_p$ . If  $w_p$  is empty then

$s_p$  is the last letter of  $\beta$ , then the last letter of  $w'$  is this of  $\beta$ . In the both cases the last letter of  $s_p w_p$  is  $\mathbf{a}_{i-1, n-1}$ . As  $\mathbf{a}_{n-2, n-1}$  is not a barrier, the braid  $\beta$  is not  $\mathbf{a}_{n-2, n-1}$ . Then Lemma 3.18 implies that  $\beta$  is  $\sigma_i n-1$ -positive.  $\square$

In this way, we obtained the expected constraint about the entries in a  $\phi_n$ -splitting: they contain stairs.

**Corollary 4.13.** *Assume that  $\beta$  belongs to  $B_n^{+*}$ . Let  $(\beta_p, \dots, \beta_1)$  be the  $\phi_n$ -splitting of  $\beta$ . Assume  $p \geq r \geq 3$  and that the last letter of  $\beta_r$  is  $\mathbf{a}_{k-1, n-1}$  with  $k \leq n-2$ . Let  $\mathbf{a}_{i-1, n-1}$  be the last letter of  $\beta_{r-1}$ . Then the  $\phi$ -normal form of  $\beta_{r-1}$  is an  $\mathbf{a}_{k, n}$ -stair lent on  $\mathbf{a}_{i-1, n-1}$ .*

*Proof.* As a right divisor of  $\beta$  is an right divisor of  $\beta' \beta$ , Condition (1.19) implies that the  $B_{n-1}^{+*}$ -tail of  $\phi_n^2(\mathbf{a}_{k-1, n-1}) \phi_n(\beta_{r-1})$  is trivial, i.e., the  $B_n^{+*}$ -tail of  $\phi_n(\mathbf{a}_{k, n} \beta_{r-1})$  is trivial. Then Proposition 4.12 implies that the  $\phi$ -normal form of  $\beta_r$  is an  $\mathbf{a}_{k, n}$ -stair lent on  $\mathbf{a}_{i-1, n-1}$ .  $\square$

**4.3. Stairs protect against dangerous elements.** We are now ready to prove that stairs have the expected property, namely that a  $\mathbf{a}_{k, n}$ -stair protects against an  $\mathbf{a}_{k, n}$ -dangerous element, this meaning that, if  $\beta_*$  is  $\mathbf{a}_{k, n}$ -dangerous and  $\beta$  is represented by an  $\mathbf{a}_{k, n}$ -stair, then the quotient  $\beta_* \beta$  is  $\sigma_{n-2}$ -positive—as stated in Proposition 4.16 below.

The proof will use an induction on the height of the stair. We begin with the case of height 1.

**Lemma 4.14.** *Assume that  $\beta_*$  is  $\mathbf{a}_{k, n}$ -dangerous and  $\beta$  can be represented by an  $\mathbf{a}_{k, n}$ -stair of height 1 lent on  $\mathbf{a}_{j-1, n-1}$ . Then  $\beta_* \beta$  is represented by an  $\mathbf{a}_{k, n}$ -stair of height 1 lent on  $\mathbf{a}_{j-1, n-1}$ .*

*Proof.* Let  $w$  be an  $\mathbf{a}_{k, n}$ -stair of height 1 lent on  $\mathbf{a}_{j-1, n-1}$  that represents  $\beta$ . By definition of dangerous, it is sufficient to prove that, for  $u \in \underline{B}_{n-1}^+$  or  $u = \delta_{k-1, n-1}^{-1}$  there exists an  $\mathbf{a}_{k, n}$ -stair  $w'$  lent on  $\mathbf{a}_{j-1, n-1}$  of height 1 satisfying  $u w \equiv w' v$ . By hypothesis, the word  $w$  can be decomposed as  $w_0 s_1 w_1$  where  $w_0$  contains no  $\mathbf{a}_{k, n}$ -barrier and  $s_1$  is an  $\mathbf{a}_{k, n}$ -barrier.

– Case  $u \in \underline{B}_{n-1}^+$ . Then  $u$  contains no  $\mathbf{a}_{k, n}$ -barrier, actually no barrier. Therefore the word  $u w$  is an  $\mathbf{a}_{k, n}$ -stair of height 1 lent on  $\mathbf{a}_{j-1, n-1}$ .

– Case  $u = \delta_{k-1, n-1}^{-1}$ . By Lemma 4.7, there exists a word  $w'_0$  lying in  $\underline{B}_{n-1}^+$  such that  $\delta_{k-1, n-1}^{-1} w_0$  is equivalent to  $w'_0 \delta_{k-1, n-1}^{-1}$ . Moreover,  $w'_0$  contains no  $\mathbf{a}_{k, n}$ -barrier, except maybe strong ones. Let us denote by  $\mathbf{a}_{i, n-1}$  the letter  $s_1$ . Then, by Lemma 4.3, we have  $\delta_{k-1, n-1}^{-1} \mathbf{a}_{i, n-1} \equiv \mathbf{a}_{i, n-1} \delta_{i, k-1} \delta_{i, n-2}^{-1}$ , and we obtain

$$(4.7) \quad \delta_{k-1, n-1}^{-1} w \equiv w' = w'_0 \underline{\mathbf{a}_{i, n-1} \delta_{i, k-1} \delta_{i, n-2}^{-1}} w_1,$$

that is an  $\mathbf{a}_{k, n}$ -stair. It remains to prove that  $w'$  represents a  $\sigma_{j-1, n-2}$ -positive braid. If  $w_1$  is empty or an  $\mathbf{a}_{j-1, n-1}$ -dangerous word, then  $i = j-1$  holds and it follows that the underlined subword in (4.7) represents an  $\sigma_{j-1, n-2}$ -positive braid. Otherwise  $w_1$  represents an  $\sigma_{j-1, n-2}$ -positive braid. In both case, the remaining property is satisfied as the not underlined factor is positive.  $\square$

**Example 4.15.** We consider  $B_5^{+*}$ . Let  $\beta_*$  be  $\delta_{2,4}^{-2}$  and  $\beta$  be  $\mathbf{a}_{1,4}$ . The letter  $\mathbf{a}_{1,4}$  is an  $\mathbf{a}_{2,5}$ -barrier. As, by Lemma 3.18, the braid  $\beta$  is  $\sigma_{1,3}$ -positive, it is represented by an  $\mathbf{a}_{2,5}$ -stair of height 1 lent on  $\mathbf{a}_{1,4}$ . We shall prove that  $\delta_{2,4}^{-2} \cdot \mathbf{a}_{1,4}$  is also an

$\mathbf{a}_{2,5}$ -stair of height 1 lent on  $\mathbf{a}_{1,4}$ . By Lemma 4.3, the quotient  $\delta_{2,4}^{-1} \cdot \mathbf{a}_{1,4}$  is equal to  $\mathbf{a}_{1,4} \delta_{1,3}^{-1}$ . Then, again by Lemma 4.3, the braid  $\delta_{2,4}^{-2} \cdot \mathbf{a}_{1,4}$  becomes  $\mathbf{a}_{1,4} \delta_{1,3}^{-2}$ . The latter can be represented by  $\varepsilon \cdot \mathbf{a}_{1,4} \cdot \delta_{1,3}^{-2}$ , that is an  $\mathbf{a}_{2,5}$ -stair of height 1 lent on  $\mathbf{a}_{1,4}$ .

Here comes the main statement of this section:

**Proposition 4.16.** *Assume that  $\beta_*$  is  $\mathbf{a}_{k,n}$ -dangerous and  $\beta$  can be represented by an  $\mathbf{a}_{k,n}$ -stair lent on  $\mathbf{a}_{j-1,n-1}$ . Then  $\beta_* \beta$  is a  $\sigma_{j-1,n-2}$ -positive braid.*

*Proof.* Let  $w$  be an  $\mathbf{a}_{k,n}$ -stair lent on  $\mathbf{a}_{j-1,n-1}$  representing  $\beta$ . We use an induction on the height  $h$  of  $w$ . By definition,  $\beta_*$  is represented by  $u_1^+ \delta_{k,n-1}^{-1} u_2^+ \dots u_{d-1}^+ \delta_{k,n-1}^{-1} u_d^+$  with  $u_r^+ \in \underline{B}_{n-1}^+$  for  $r = 1, \dots, d$ . Lemma 4.3 implies directly the case  $h = 1$ .

Assume  $h \geq 2$ . Then  $w$  can be decomposed as  $w_0 s_1 w_1 \dots s_h w_h$ . Let

$$(4.8) \quad w' = w_0 s_1, \quad w'' = w_1 s_2 w_2 \dots s_t w_t, \quad u' = u_1^+ \delta_{k,n-1}^{-1} u_2^+ \dots u_{d-2}^+ \delta_{k,n-1}^{-1} u_{d-1}^+.$$

As  $u_d^+$  lies in  $\underline{B}_{n-1}^+$ , it contains no  $\mathbf{a}_{k,n}$ -barrier, actually no barrier, and so is  $w'_0 = u_d^+ w_0$ . Lemma 4.7 implies the existence of a word  $w''_0$  lying in  $\underline{B}_{n-1}^+$  such that  $\delta_{k,n-1}^{-1} w'_0 \equiv w''_0 \delta_{k,n-1}^{-1}$  holds and  $w''_0$  contains no  $\mathbf{a}_{k,n}$ -barrier, excepted strong ones. Let us denote by  $\mathbf{a}_{g,h}$  the letter  $s_1$ . As the height of  $w$  is at least 2, the letter  $s_1$  is not a strong barrier. Then we find  $\delta_{k,n-1}^{-1} w' \equiv w''_0 \delta_{k,n-1}^{-1} \mathbf{a}_{g,h}$ , and, by Lemma 4.3, we obtain

$$(4.9) \quad \delta_{k,n-1}^{-1} w' \equiv (w''_0 \mathbf{a}_{g,n-1} \delta_{g,k-1} \delta_{g,h-1}^{-1}) \delta_{h,n-1}^{-1} = v \delta_{h,n-1}^{-1}.$$

We note that the word  $v$  is an  $\mathbf{a}_{k,n}$ -stair of height 1 lent on  $\mathbf{a}_{g,n-1}$ . Then, by Lemma 4.14, the braid  $\beta' = \bar{u}' \bar{v}$  is an  $\mathbf{a}_{k,n}$ -stair lent on  $\mathbf{a}_{g,n-1}$ , that is, in particular,  $\sigma_{n-2}$ -nonnegative. In the other hand,  $w''$  is an  $\mathbf{a}_{h,n}$ -stair of height  $h-1$  lent on  $\mathbf{a}_{j-1,n-1}$ , and  $\delta_{h,n-1}^{-1}$  is  $\mathbf{a}_{h,n}$ -dangerous. Then, by induction hypothesis,  $\delta_{h,n-1}^{-1} w''$  represents an  $\sigma_{j-1,n-2}$ -positive braid. Finally, Lemma 3.16 (ii) implies the expected result.  $\square$

**Example 4.17.** We prove in Example (4.10) that the word  $u = \mathbf{a}_{3,4} \mathbf{a}_{1,3} \mathbf{a}_{1,3} \mathbf{a}_{2,4} \mathbf{a}_{3,4}$  is an  $\mathbf{a}_{2,5}$ -stair of height 2 lent on  $\mathbf{a}_{3,4}$ . Let us show that  $\delta_{2,4}^{-3} \cdot \bar{u}$  is  $\sigma_{3,3}$ -positive.

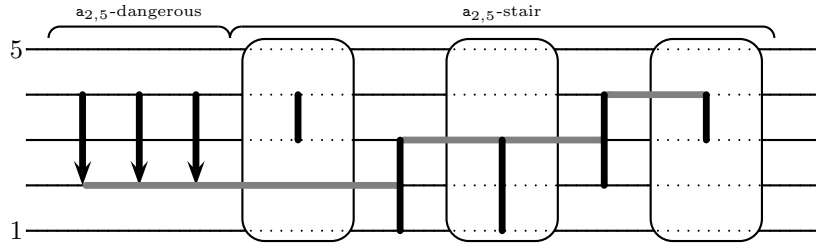


FIGURE 6. Encounter of the  $\mathbf{a}_{2,5}$ -dangerous braid  $\delta_{2,4}^{-3}$  and the  $\mathbf{a}_{2,5}$ -stair  $u$ : the vertical descending arrow from the  $j$ th line to the  $i$ th line of the stave represents the braid  $\delta_{i,j}^{-1}$ .

The letter  $\mathbf{a}_{3,4}$  is not an  $\mathbf{a}_{2,5}$ -barrier. Then we have to cross throughout it. By Lemma 4.7, we obtain  $\delta_{2,4}^{-1} \mathbf{a}_{3,4} = \mathbf{a}_{2,3} \delta_{2,4}^{-1}$ , next  $\delta_{2,4}^{-1} \mathbf{a}_{2,3} = \mathbf{a}_{2,4} \delta_{2,4}^{-1}$ , and finally  $\delta_{2,4}^{-1} \mathbf{a}_{2,4} = \mathbf{a}_{3,4} \delta_{2,4}^{-1}$ . Then  $\delta_{2,4}^{-3} \cdot \bar{u}$  is equivalent to the word  $\mathbf{a}_{3,4} \cdot \delta_{2,4}^{-3} \cdot \underline{\mathbf{a}_{1,3} \mathbf{a}_{1,3} \mathbf{a}_{2,4} \mathbf{a}_{3,4}}$ ,

where the underlined factor is an  $\mathbf{a}_{2,5}$ -stair lent on  $\mathbf{a}_{3,4}$ . The letter  $\mathbf{a}_{1,3}$  is an  $\mathbf{a}_{2,5}$ -barrier. Lemma 4.3 implies that  $\delta_{2,4}^{-1} \mathbf{a}_{1,3}$  is equal to  $\mathbf{a}_{1,4} \delta_{1,2}^{-1} \delta_{3,4}^{-1}$ . Then we obtain

$$(4.10) \quad \delta_{2,4}^{-3} \cdot \bar{u} = \mathbf{a}_{3,4} \cdot \delta_{2,4}^{-2} \underline{\mathbf{a}_{1,4}} \cdot \delta_{1,2}^{-1} \cdot \delta_{3,4}^{-1} \underline{\mathbf{a}_{1,3} \mathbf{a}_{2,4} \mathbf{a}_{3,4}} \quad (\text{see Figure 7})$$

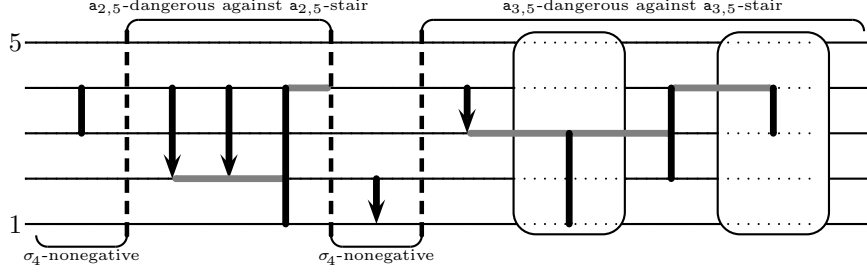


FIGURE 7. Emerging of two stair: when the dangerous part crosses the first barrier, a new stair appears, and, then, it absorbs remaining old dangerous fragment, while a new dangerous fragment appears.

The first underlined factor in (4.10) is an  $\mathbf{a}_{2,5}$ -stair of height 1 and Example 4.15 gives  $\delta_{2,4}^{-2} \mathbf{a}_{1,4} = \mathbf{a}_{1,4} \cdot \delta_{1,3}^{-2}$ . As  $u$  is an  $\mathbf{a}_{2,5}$ -stair of height 2 lent on  $\mathbf{a}_{3,4}$ , the second underlined factor in (4.10) is an  $\mathbf{a}_{3,5}$ -stair of height 1 lent on  $\mathbf{a}_{3,4}$ . By Lemma 4.7, the quotient  $\delta_{3,4}^{-1} \mathbf{a}_{1,3}$  equals  $\mathbf{a}_{1,4} \delta_{3,4}^{-1}$ . As  $\mathbf{a}_{2,4}$  is an  $\mathbf{a}_{3,5}$ -barrier, Lemma 4.3 implies  $\delta_{3,4}^{-1} \mathbf{a}_{2,4} = \mathbf{a}_{2,4} \delta_{2,3}^{-1}$ . Finally we obtain

$$(4.11) \quad \delta_{2,4}^{-3} \cdot \bar{u} = \mathbf{a}_{3,4} \cdot \mathbf{a}_{1,4} \delta_{1,3}^{-2} \cdot \delta_{1,2}^{-1} \cdot \mathbf{a}_{1,4} \mathbf{a}_{2,4} \delta_{2,3}^{-1} \cdot \mathbf{a}_{3,4},$$

which is  $\sigma_{3,3}$ -positive since  $\mathbf{a}_{2,4} \delta_{2,3}^{-1}$  is  $\sigma_{2,3}$ -positive.

**Corollary 4.18.** *Assume that  $\beta$  belongs to  $B_n^{+*}$ . Let  $(\beta_p, \dots, \beta_1)$  be the  $\phi_n$ -splitting of  $\beta$ . Assume  $p \geq r \geq 3$  and that the last letter of  $\beta_r$  is  $\mathbf{a}_{k-1, n-1}$  with  $k \leq n-2$ . Let  $\mathbf{a}_{i-1, n-1}$  be the last letter of  $\beta_{r-1}$ . Then for each  $\mathbf{a}_{k,n}$ -dangerous braid  $\beta_*$ , the braid  $\beta_* \beta_{r-1}$  is  $\sigma_{i-1, n-2}$ -positive.*

*Proof.* Corollary 4.13 guarantees that  $\beta_{r-1}$  is an  $\mathbf{a}_{k,n}$ -stair lent on  $\mathbf{a}_{i-1, n-1}$ . Then Proposition 4.16 implies that  $\beta_* \beta_{r-1}$  is  $\sigma_{i-1, n-2}$ -positive.  $\square$

Thus, at this point, we know that we have stairs, and that stair achieve their protecting role against dangerous elements.

**4.4. Proof of the Key Lemma.** From that point, proving the Key Lemma 3.19 is easy. We recall the statement we wish to establish.

**Proposition (Key Lemma).** *Assume that  $(\beta_p, \dots, \beta_2, 1)$  is the  $\phi_n$ -splitting of a braid  $\beta$  in  $B_n^{+*}$  with  $p \geq 2$ . Let  $\mathbf{a}_{i-1, n-1}$  be the last letter of  $\beta_p$  and  $\mathbf{a}_{j,n}$  this of  $\beta$ . Then for every  $\sigma_{i-1, n-2}$ -positive braid  $\beta'$ , the braid*

$$(4.12) \quad \delta_n^{-p+2} \cdot \phi_n^{p-1}(\beta') \cdot \phi_n^{p-2}(\beta_{p-1}) \cdot \dots \cdot \phi_n(\beta_2)$$

*is  $\sigma_{j, n-1}$ -positive.*

The proof of the Key Lemma (Proposition 3.19) uses an induction on the  $n$ -breadth  $p$  of the input braid  $\beta$ .

We shall treat the cases  $p \leq 3$  separately, because the entries  $\beta_2$  and  $\beta_1$  of the  $\phi_n$ -splitting of  $\beta$  may be trivial, and do not obey the same constraints as the other entries. For instance, if  $\beta$  has  $n$ -breadth 3 with  $\beta_2 = \beta_1 = 1$ , then the braid  $\phi_n(\beta_2) \cdot \beta_1$  does not satisfy the hypotheses of the Key Lemma, and, therefore, such cases must remain outside the main induction.

*Proof of Proposition 3.19 from Corollary 4.18.* We consider four cases. We start with the generic one.

– Case  $p \geq 4$ , or  $p = 3$  with  $j \neq 1$ . Then the braid  $\beta_{[p-1,2]}$  is not trivial. Assume  $i \leq n-2$ . By hypothesis,  $\beta'$  can be decomposed as  $\beta'^* \mathbf{a}_{n-2,n-1} \beta'_*$ , where  $\beta'^*$  is  $\sigma_{n-2}$ -nonnegative and  $\beta'_*$  is  $\mathbf{a}_{i-1,n-1}$ -dangerous. First, we compute

$$\begin{aligned} \delta_n^{-p+2} \phi_n^{p-1}(\beta'^* \mathbf{a}_{n-2,n-1}) &= \phi_n(\beta'^* \mathbf{a}_{n-2,n-1}) \delta_n^{-1} \delta_n^{-p+3} \\ &= \phi_n(\beta'^*) \delta_{n-1}^{-1} \delta_n^{-p+3}, \end{aligned}$$

which is  $\sigma_{n-1}$ -nonnegative. Then we are now left with

$$(4.13) \quad \delta_n^{-p+3} \cdot \underbrace{\phi_n^{p-1}(\beta'_*) \cdot \phi_n^{p-2}(\beta_{p-1})}_{\text{underlined}} \cdot \dots \cdot \phi_n(\beta_2).$$

The underlined factor in (4.13) is equal to  $\phi_n^{p-1}(\phi_n(\beta'_*) \cdot \beta_{p-1})$ . As  $\phi_n(\beta'_*)$  is  $\mathbf{a}_{i,n}$ -dangerous, Corollary 4.18 implies that  $\beta'' = \phi_n(\beta'_*) \cdot \beta_{p-1}$  is  $\sigma_{j-1,n-2}$ -positive, where  $\mathbf{a}_{j-1,n-1}$  is the last letter of  $\beta_{p-1}$ . The expression (4.13) becomes

$$(4.14) \quad \delta_n^{-p+3} \cdot \phi_n^{p-2}(\beta'') \cdot \phi_n^{p-3}(\beta_{p-2}) \cdot \dots \cdot \phi_n(\beta_2),$$

that is exactly of the form (4.12). Therefore, we conclude using the induction hypothesis. Assume  $i = n-1$ . Then, by Definition 3.15, the braid  $\beta'$  can be decomposed as  $\beta'' \cdot \mathbf{a}_{n-2,n-1}$ , where  $\beta''$  is a  $\sigma_{k-1,n-2}$ -positive braid for some  $k$ . Hence the relation (4.12) becomes

$$\begin{aligned} \delta_n^{-p+2} \cdot \phi_n^{p-1}(\beta') \cdot \beta_{[p-1,2]} &= \delta_n^{-p+2} \cdot \phi_n^{p-1}(\beta'' \cdot \mathbf{a}_{n-2,n-1}) \cdot \beta_{[p-1,2]} \\ &= \phi_n(\beta'' \cdot \mathbf{a}_{n-2,n-1}) \cdot \delta_n^{-1} \cdot \delta_n^{-p+3} \cdot \beta_{[p-1,2]} \\ (4.15) \quad &= \phi_n(\beta'') \cdot \delta_{n-1}^{-1} \cdot \underbrace{\delta_n^{-p+3} \cdot \beta_{[p-1,2]}}_{\text{underlined}} \end{aligned}$$

Assume  $\beta_{[p-1,3]} = \widehat{\delta}_{n,p-3}$ . Then relation (4.15) becomes

$$(4.16) \quad \delta_n^{-p+2} \cdot \phi_n^{p-1}(\beta') \cdot \beta_{[p-1,2]} = \underbrace{\phi_n(\beta'') \cdot \delta_{n-1}^{-p+2}}_{\text{underlined}} \cdot \phi_n(\beta_2).$$

As  $\beta''$  is  $\sigma_{k-1,n-2}$ -positive, the braid  $\phi_n(\beta'')$  is  $\sigma_{k,n-1}$ -positive, and also  $\sigma_{1,n-1}$ -positive, by Lemma 3.16 (iii). Then the underlined factor is  $\sigma_{1,n-1}$ -positive. If  $\beta_2$  is trivial, the last letter of  $\beta$  is  $\mathbf{a}_{1,n}$ , i.e.,  $j = 1$  holds, and the braid (4.16) is  $\sigma_{1,n-1}$ -positive. If the last letter of  $\beta_2$  is  $\mathbf{a}_{n-2,n-1}$ , i.e.,  $j = n-1$  holds, then the braid (4.16) is  $\sigma_{n-1,n-1}$ -positive. If the last letter of  $\beta_2$  is not  $\mathbf{a}_{n-2,n-1}$ , then, by Lemma 1.23, the braid  $\beta_2$  is  $\sigma_{j-1,n-1}$ -positive and (4.16) is  $\sigma_{j,n-1}$ -positive since the underlined factor is, in particular,  $\sigma_{n-1}$ -nonnegative. Assume  $\beta_{[p-1,3]} \neq \widehat{\delta}_{n,p-3}$ . Let  $r$  be the greatest index such that  $\beta_r \neq \mathbf{a}_{n-2,n-1}$ . Assume  $r \geq 3$ . Then, by (3.14), the underlined factor in (4.15) becomes

$$(4.17) \quad \delta_{n-1}^{-p+r+1} \cdot \underbrace{\delta_n^{-r+2} \cdot \beta_{[r,2]}}_{\text{underlined}}.$$



The not underlined factor in (4.15) and (4.17) are  $\sigma_{n-1}$ -nonnegative. Lemma 1.23 implies that the braid  $\beta_r$  is  $\mathbf{a}_{k-1, n-1}$ -dangerous, where  $\mathbf{a}_{k-1, n-1}$  is the last letter of  $\beta$ . As the underlined factor in (4.17) is exactly of the form (4.12) with  $r$  replacing  $p$ , we conclude by induction hypothesis.

– Case  $p = 3$  with  $j = 1$ . Then the braids  $\beta_1$  and  $\beta_2$  are trivial and the last letter of  $\beta_3$  is  $\mathbf{a}_{n-2, n-1}$ , *i.e.*,  $i = n-1$  holds. Hence we can decompose  $\beta'$  as  $\beta'' \mathbf{a}_{n-2, n-1}$  where  $\beta''$  is  $\sigma_{k-1, n-2}$ -positive for some  $k$ . Then (4.12) becomes

$$(4.18) \quad \delta_n^{-1} \cdot \phi_n^2(\beta') = \phi_n(\beta') \cdot \delta_n^{-1} = \phi_n(\beta'' \cdot \mathbf{a}_{n-2, n-1}) \cdot \delta_n^{-1} = \phi_n(\beta'') \cdot \delta_n^{-1}.$$

As  $\beta''$  is  $\sigma_{k-1, n-2}$ -positive, the braid  $\phi_n(\beta'')$  is  $\sigma_{k, n-1}$ -positive, and also  $\sigma_{1, n-1}$ -positive, by Lemma 3.16 (iii). Therefore, as  $\phi_n(\beta'')$  is  $\sigma_{n-1}$ -nonnegative, the braid  $\delta_n^{-1} \cdot \phi_n^2(\beta')$  is  $\sigma_{1, n-1}$ -positive, which is the expected result.

– Case  $p = 2$ . Then we have  $\beta = \phi_n(\beta_2)$  and the last letter of  $\beta_2$  is  $\mathbf{a}_{j-1, n-1}$ . The braid (4.12) becomes  $\phi_n(\beta')$ . As  $\beta'$  is  $\sigma_{j-1, n-2}$ -positive, the braid  $\phi_n(\beta')$  is  $\sigma_{j, n-1}$ -positive, which is the expected result.  $\square$

So the Key Lemma is proved, and so are Theorem 3.1 and its corollaries listed in Section 3.1.

## 5. OPEN QUESTIONS.

We conclude with a few open questions directly inspired by the previous results.

**5.1. The  $\phi_n$ -splitting.** With the  $\phi_n$ -splitting, we associated with every braid of  $B_n^{+*}$  a unique sequence in  $B_{n-1}^{+*}$ . In this paper, we have seen that a  $\phi_n$ -splitting satisfies various constraints: the latter are necessary conditions, but, so far, we do not know whether they are sufficient.

**Question 5.1.** *Is there a simple characterization of the sequences in  $B_{n-1}^{+*}$  that are the  $\phi_n$ -splitting of a braid in  $B_n^{+*}$ ?*

Of course, we can recognize whether a sequence  $(\beta_p, \dots, \beta_1)$  is a splitting by computing its product (with the necessary rotations inserted) and determining its splitting, but, here, we ask for a direct criterion, for instance a syntactic one, in the vein of the following easy result, that we state without proof:

**Proposition 5.2.** *A sequence  $(\mathbf{a}_{1,2}^{e_p}, \dots, \mathbf{a}_{1,2}^{e_1})$  is a  $B_2^{+*}$ -splitting if and only if the condition  $e_r \geq 1$  is satisfied for  $r \geq 3$ .*

The monoid  $B_n^{+*}$  includes the monoid  $B_n^+$  for  $n \geq 3$ . Then we can compute the  $\phi_n$ -splitting of a braid lying in  $B_n^+$ .

**Question 5.3.** *What are the specific properties of the  $\phi_n$ -splitting of a braid belonging to  $B_n^+$ ?*

**5.2. Sigma-representative word.** The main open questions involve the complexity of the process described in this paper. Indeed, our proof is effective, and, starting with two braid words in the generators  $\mathbf{a}_{i,j}$ —hence, in particular, starting with two braid words in the generators  $\sigma_i$ —we obtain a  $\sigma$ -positive or  $\sigma$ -negative representative of the quotient.

**Conjecture 5.4.** *For each  $n$ , the algorithm associated with the current construction, running on  $n$ -braid words of length  $\ell$ , terminates in time  $O(\ell^2)$ , and returns a final word of length at most  $O(\ell^2)$ .*

Proving this conjecture amounts to precisely analysing all steps of the computation. In principle, this task is easy, but it requires some care, and we keep it for further research.

## ACKNOWLEDGMENT

The author wishes to thank Patrick Dehornoy for his help in writing the paper.

## REFERENCES

- [1] E. Artin, *Theory of Braids*, Annals of Math. **48** (1947) 101126.
- [2] D. Bessis, *The dual braid monoid*, Ann. Scient. Ec. Norm. Sup. **36** (2003), 647-683.
- [3] J. Birman, K.H. Ko & S.J. Lee, *A new approach to the word problem in the braid groups*, Advances in Math. **139-2** (1998) 322-353.
- [4] S. Burckel, *L'ordre total sur les tresses positives*, Ph.D. Thesis, Université de Caen, (1994).
- [5] S. Burckel, *The wellordering on positive braids*, J. Pure Appl. Algebra, **120** (1994), no 1, 1-17.
- [6] S. Burckel, *Computation of the ordinal of braids*, Order **16** (1999) 291-304.
- [7] L. Carlucci, P. Dehornoy, A. Weiermann, *Unprovability statements involving braids*, arXiv: math.LO/0711.3785.
- [8] P. Dehornoy, *Alternating normal forms for braids and locally Garside monoids*, arXiv: math.GR/0702592.
- [9] P. Dehornoy, *Braid groups and left distributive operations*, Trans. Amer. Math. Soc. **345** (1994), no. 2, 293-304.
- [10] P. Dehornoy, *Groupes de Garside*, Ann. Scient. Ec. Norm. Sup. **35** (2002) 267306.
- [11] P. Dehornoy, I. Dynnikov, D. Rolfsen, B. Wiest, *Why are braids orderable ?*, Soc. Math. de France, Panorama et Synthèse **14** (2002), xiii + 190pp.
- [12] P. Dehornoy, I. Dynnikov, D. Rolfsen, B. Wiest, *Ordering Braids*, Mathematical Surveys and Monographs, Amer. Math. Soc., to appear.
- [13] F. Digne, J. Michel, *Garside and locally Garside categories*, Preprint, arXiv: math.GR/0612652.
- [14] F. A. Garside, *The braid group and other groups*, Quart. J. Math. Oxford **20** (1969) 235-254.
- [15] R. Laver, *Braid group actions on left distributive structures and well-orderings in the braid group*, J. Pure Appl. Algebra **108** (1996), no. 1, 81-98.

LABORATOIRE DE MATHÉMATIQUES NICOLAS ORESME, UMR 6139 CNRS, UNIVERSITÉ DE CAEN  
BP 5186, 14032 CAEN, FRANCE

*E-mail address:* [jean.fromentin@math.unicaen.fr](mailto:jean.fromentin@math.unicaen.fr)