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On the sphericity of scaling limits of random planar quadrangulations

Grégory Miermont*

21st December 2007

Abstract

We give a new proof of a theorem by Le Gall & Paulin, showing that scaling limits of random planar quadrangulations are homeomorphic to the 2-sphere. The main geometric tool is a reinforcement of the notion of Gromov-Hausdorff convergence, called 1-regular convergence, that preserves topological properties of metric surfaces.

1 Introduction

A planar map is a combinatorial embedding of a connected graph into the 2-dimensional sphere. Random planar map have drawn much attention in the recent probability literature due to mathematical physics motivations [2] and a powerful encoding of planar maps in terms of labeled trees due to Schaeffer [15, 5]. In turn, scaling limits of labeled trees are well-understood thanks to the works of Aldous, Le Gall and others [1, 8, 9]. Using this line of reasoning, many results have been obtained on the geometric aspects of large random quadrangulations (where faces all have degree 4), and other families of maps. Le Gall [10] showed in particular that scaling limits of random quadrangulations are homeomorphic to the Brownian map introduced by Marckert & Mokkadem [13], and Le Gall & Paulin [11] showed that the topology of the latter is that of the 2-dimensional sphere, hence giving a mathematical content to the claim made by physicists that summing over large random quadrangulations amount to integrating with respect to a (still ill-defined) measure over surfaces.

The aim of this note is to give an alternative proof of Le Gall & Paulin's result. We still strongly rely on the results established by Le Gall [10], but use very different methods from those of [11], where the reasoning uses geodesic laminations and a theorem due to Moore on the topology of quotients of the sphere. We feel that our approach is somewhat more economic, as it only needs certain estimates from [10] and not the technical statements

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of [11, Lemmas 3.1, 3.2] that are necessary to apply Moore’s theorem. On the other hand, this is at the cost of checking that quadrangulations are close to being path metric spaces, which is quite intuitive but needs justification (see definitions below). Our main geometric tool is a reinforcement of Hausdorff convergence, called 1-regular convergence and introduced by Whyburn, and which has the property of conserving the topology of surfaces. We will see that random planar quadrangulations converge 1-regularly, therefore entailing that their limits are of the same topological nature. In the case, considered in this paper, of surfaces with the topology of the sphere, the 1-regularity property is equivalent to [11, Corollary 1], stating that there are no small loops separating large random quadrangulations into two large parts. We prove this by a direct argument rather than obtaining it as a consequence of the theorem.

The basic notations are the following. We let \mathbf{Q}_n be the set of rooted¹ quadrangulations of the sphere with n faces, which is a finite set of cardinality $2 \cdot 3^n (2n)! / (n!(n+2)!)$, see [5]. We let \mathbf{q}_n be a random variable picked uniformly in \mathbf{Q}_n , and endow the set $V(\mathbf{q}_n)$ of its vertices with the usual graph distance d_n^{gr} , i.e. $d_n^{\text{gr}}(x, y)$ is the length of a minimal (geodesic) chain of edges going from x to y .

We briefly give the crucial definitions on the Gromov-Hausdorff topology, referring the interested reader to [4] for more details. The isometry class $[X, d]$ of the metric space (X, d) is the collection of all metric spaces isometric to (X, d) . We let \mathbb{M} be the set of isometry-equivalence classes of *compact* metric spaces. The latter is endowed with the Gromov-Hausdorff distance d_{GH} , where $d_{\text{GH}}(X, X')$ is defined as the least $r > 0$ such that there exist a metric space (Z, δ) and subsets $X, X' \subset Z$ such that $[X, \delta] = X$, $[X', \delta] = X'$, and such that the Hausdorff distance between X and X' in (Z, δ) is less than or equal to r . This turns \mathbb{M} into a complete separable metric space, see [6] (this article focuses on compact \mathbb{R} -trees, which form a closed subspace of \mathbb{M} , but the proofs apply without change to \mathbb{M}).

Theorem 1 ([11]) *A limit in distribution of $[V(\mathbf{q}_n), n^{-1/4} d_n^{\text{gr}}]$ for the Gromov-Hausdorff topology, where $n \rightarrow \infty$ along some subsequence, is homeomorphic to the 2-sphere.*

Remarks. • One of the main open questions in the topic of scaling limits of random quadrangulations is to uniquely characterize the limit, i.e. to get rid of the somewhat annoying “along some subsequence” in the previous statement.

• To be perfectly accurate, Le Gall & Paulin showed the same result for uniform $2k$ -angulations (maps with degree- $2k$ faces) with n faces. Our methods also apply in this setting (and possibly to more general families of maps), but we will restrict ourselves to the case of quadrangulations for the sake of brevity.

• In the work in preparation [14], we provide a generalization of this result to higher genera, in the framework of Boltzmann-Gibbs distributions on quadrangulations rather than uniform laws.

¹Which means that one oriented edge of the quadrangulation is distinguished as the root

As we are quite strongly relying on Le Gall's results in [10], we will mainly focus on the new aspects of our approach. As a consequence, this paper contains two statements whose proofs will not be detailed (Proposition 2 and Lemma 2), because they are implicit in [10] and follow directly from the arguments therein, and also because their accurate proof would need a space-consuming introduction to continuum tree and snake formalisms. Taking them for granted, the proofs should in a large part be accessible to readers with no particular acquaintance with continuum trees or Schaeffer's bijection.

2 Gromov-Hausdorff convergence and regularity

We say that a metric space (X, d) is a path metric space if every two points $x, y \in X$ can be joined by a path isometric to a real segment (with length $d(x, y)$). We let PM be the set of isometry classes of compact path metric spaces, and the latter is a closed subspace of $(\mathbb{M}, d_{\text{GH}})$, see [4, Theorem 7.5.1]. One of the main tools needed in this article is a notion that reinforces the convergence in the metric space $(\text{PM}, d_{\text{GH}})$, which was introduced by Whyburn in 1935 and was extensively studied in the years 1940's. Our main source is Begle [3].

Definition 1 *Let $(X_n, n \geq 1)$ be a sequence of spaces in PM converging to a limit X . We say that X_n converges 1-regularly to X if for every $\varepsilon > 0$, one can find $\delta, N > 0$ such that for all $n \geq N$, every loop in X_n with diameter $\leq \delta$ is homotopic to 0 in its ε -neighborhood.*

There are a couple of slight differences between this definition and that in [3]. In the latter reference, the setting is that X_n are compact subsets of a common compact space, converging in the Hausdorff sense to a limiting set X . This is not restrictive as Gromov-Hausdorff convergence entails Hausdorff convergence of representative spaces in a common compact space, see for instance [7, Lemma A.1]. It is also assumed in the definition of 1-regular convergence that for every $\varepsilon > 0$, there exists $\delta, N > 0$ such that any two point that lie at distance $\leq \delta$ are in a connected subset of X_n of diameter $\leq \varepsilon$, but this condition is tautologically satisfied for path metric spaces. Last, the definition in [3] is stated in terms of homology, so our definition in terms of homotopy is in fact stronger.

The following theorem is due to Whyburn, see [3, Theorem 6] and comments before.

Theorem 2 *Let $(X_n, n \geq 1)$ be a sequence of elements of PM that are all homeomorphic to \mathbb{S}^2 . Assume that X_n converges to X for the Gromov-Hausdorff distance, where X is not reduced to a point, and that the convergence is 1-regular. Then X is homeomorphic to \mathbb{S}^2 as well.*

3 Quadrangulations

Rooted quadrangulations are maps whose faces all have degree 4, and their set is denoted by $\mathbf{Q} := \bigcup_{n \geq 1} \mathbf{Q}_n$ with the notations of the Introduction. For $\mathbf{q} \in \mathbf{Q}$ we let

$V(\mathbf{q}), E(\mathbf{q}), F(\mathbf{q})$ be the set of vertices, edges and faces of \mathbf{q} , and denote by $d_{\mathbf{q}}^{\text{gr}}$ the graph distance on $V(\mathbf{q})$.

3.1 A metric surface representation

One of the issues that must be addressed in order to apply Theorem 2 is that the metric space $[V(\mathbf{q}), d_{\mathbf{q}}^{\text{gr}}]$ is not a surface, rather, it is a finite metric space. We take care of this by constructing a particular graphical representative of \mathbf{q} which is a path metric space whose restriction to the vertices of the graph is isometric to $(V(\mathbf{q}), d_{\mathbf{q}}^{\text{gr}})$.

Let $(X_f, d_f), f \in F(\mathbf{q})$ be copies of the emptied unit cube “with bottom removed”

$$X_f = [0, 1]^3 \setminus (0, 1)^2 \times [0, 1),$$

endowed with the intrinsic metric d_f inherited from the Euclidean metric (i.e. the distance between two points of X_f is the minimal Euclidean length of a path in X_f). Obviously each (X_f, d_f) is a path metric space homeomorphic to a closed disk of \mathbb{R}^2 . For each face $f \in F(\mathbf{q})$, we label the four incident half-edges turning counterclockwise as (e_1, e_2, e_3, e_4) , where the labeling is arbitrary among the 4 possible labelings preserving the cyclic order. Then define

$$\begin{aligned} c_{e_1}(t) &= (t, 0, 0)_f & , & & 0 \leq t \leq 1 \\ c_{e_2}(t) &= (1, t, 0)_f & , & & 0 \leq t \leq 1 \\ c_{e_3}(t) &= (1 - t, 1, 0)_f & , & & 0 \leq t \leq 1 \\ c_{e_4}(t) &= (0, 1 - t, 0)_f & , & & 0 \leq t \leq 1. \end{aligned}$$

In these notations, we keep the subscript f to differentiate points of different spaces X_f . In this way, for every $e \in E(\mathbf{q})$, we have defined a path c_e of length 1 which goes along one of the four edges of the boundary $\partial X_f = ([0, 1]^2 \setminus (0, 1)^2) \times \{0\}$, where f is the face incident to e .

We then define an equivalence relation \sim on the disjoint union $\coprod_{f \in F(\mathbf{q})} X_f$, as the coarsest equivalence relation such that for every $e \in E(\mathbf{q})$, and every $t \in [0, 1]$, we have $c_e(t) \sim c_{\bar{e}}(1 - t)$. By identifying points of the same class, we glue the boundaries of the spaces X_f together in a way that is consistent with the map structure. More precisely, the topological quotient $\mathcal{S}_{\mathbf{q}} := \coprod_{f \in F(\mathbf{q})} X_f / \sim$ is a 2-dimensional cell complex whose 1-skeleton $\mathcal{E}_{\mathbf{q}}$ is a graph representation of \mathbf{q} , and where the faces are the interiors of the spaces X_f . In particular, $\mathcal{S}_{\mathbf{q}}$ is homeomorphic to \mathbb{S}^2 . We let $\mathcal{V}_{\mathbf{q}}$ be the 0-skeleton of this complex, i.e. the vertices of the graph. We call the 1-cells and 0-cells of $\mathcal{E}_{\mathbf{q}}$ and $\mathcal{V}_{\mathbf{q}}$ the edges and vertices of $\mathcal{S}_{\mathbf{q}}$.

We next endow the disjoint union $\coprod_{f \in F(\mathbf{q})} X_f$ with the largest pseudo-metric $D_{\mathbf{q}}$ that is compatible with $d_f, f \in F(\mathbf{q})$ and with \sim , in the sense that $D_{\mathbf{q}}(x, y) \leq d_f(x, y)$ for $x, y \in X_f$, and $D_{\mathbf{q}}(x, y) = 0$ for $x \sim y$. Therefore, the function $D_{\mathbf{q}} : \coprod_{f \in F(\mathbf{q})} X_f \times \coprod_{f \in F(\mathbf{q})} X_f \rightarrow \mathbb{R}_+$ is compatible with the equivalence relation, and its quotient mapping $d_{\mathbf{q}}$ defines a pseudo-metric on the quotient space $\mathcal{S}_{\mathbf{q}}$.

Proposition 1 *The space $(\mathcal{S}_{\mathbf{q}}, d_{\mathbf{q}})$ is a path metric space homeomorphic to \mathbb{S}^2 . Moreover, the restriction of $\mathcal{S}_{\mathbf{q}}$ to the set $\mathcal{V}_{\mathbf{q}}$ is isometric to $(V(\mathbf{q}), d_{\mathbf{q}}^{\text{gr}})$, and any geodesic path in $\mathcal{S}_{\mathbf{q}}$ between two elements of $\mathcal{V}_{\mathbf{q}}$ is a concatenation of edges of $\mathcal{E}_{\mathbf{q}}$. Last,*

$$d_{\text{GH}}([V(\mathbf{q}), d_{\mathbf{q}}^{\text{gr}}], [\mathcal{S}_{\mathbf{q}}, d_{\mathbf{q}}]) \leq 3.$$

Proof. What we first have to check is that $d_{\mathbf{q}}$ is a true metric on $\mathcal{S}_{\mathbf{q}}$, i.e. it separates points. To see this, we use the fact [4, Theorem 3.1.27] that $D_{\mathbf{q}}$ admits the constructive expression:

$$D_{\mathbf{q}}(a, b) = \inf \left\{ \sum_{i=0}^n d(x_i, y_i) : n \geq 0, x_0 = a, y_n = b, y_i \sim x_{i+1} \right\},$$

where we have set $d(x, y) = d_f(x, y)$ if $x, y \in X_f$ for some f , and $d(x, y) = \infty$ otherwise. It follows that for $a \in X_f \setminus \partial X_f$, and for $b \neq a$, $D_{\mathbf{q}}(a, b) > \min(d(a, b), d_f(a, \partial X_f)) > 0$, so a and b are separated.

It remains to treat the case $a \in \partial X_f$ for some f . The crucial observation is that a shortest path in X_f between two points of ∂X_f is entirely contained in ∂X_f . Therefore, the distance $D_{\mathbf{q}}(a, b)$ is always larger than the length of a path with values in the edges $\Pi \partial X_f / \sim$ of $\mathcal{S}_{\mathbf{q}}$, where all edges have total length 1. In particular, points in distinct classes are at positive distance. One deduces that $d_{\mathbf{q}}$ is a true distance on $\mathcal{S}_{\mathbf{q}}$, and by the compactness of the latter, $(\mathcal{S}_{\mathbf{q}}, d_{\mathbf{q}})$ is homeomorphic to \mathbb{S}^2 [4, Exercise 3.1.14].

From this same observation, we obtain that a shortest path between vertices of $\mathcal{S}_{\mathbf{q}}$ is a shortest path of edges, i.e. is the geodesic distance for the (combinatorial) graph distance. Thus, $(\mathcal{V}_{\mathbf{q}}, d_{\mathbf{q}})$ is indeed isometric to $(V(\mathbf{q}), d_{\mathbf{q}}^{\text{gr}})$. The last statement follows immediately from this and the fact that $\text{diam}(X_f, d_f) \leq 3$, entailing that $\mathcal{V}_{\mathbf{q}}$ is 3-dense in $(\mathcal{S}_{\mathbf{q}}, d_{\mathbf{q}})$, i.e. its 3-neighborhood in $(\mathcal{S}_{\mathbf{q}}, d_{\mathbf{q}})$ equals $\mathcal{S}_{\mathbf{q}}$. \square

3.2 Tree encoding of quadrangulations

We briefly introduce the second main ingredient, the Schaeffer bijection. Let \mathbf{T}_n be the set of pairs (\mathbf{t}, \mathbf{l}) where \mathbf{t} is a rooted planar tree with n edges, and \mathbf{l} is a function from the set of vertices of \mathbf{t} to $\mathbb{N} = \{1, 2, \dots\}$, such that $|\mathbf{l}(x) - \mathbf{l}(y)| \leq 1$ if x and y are neighbors. Then the set \mathbf{Q}_n is in one-to-one correspondence with \mathbf{T}_n . More precisely, this correspondence is such that given a graph representation of $\mathbf{q} \in \mathbf{Q}_n$ on a surface, the corresponding $(\mathbf{t}, \mathbf{l}) \in \mathbf{T}_n$ can be realized as a graph whose vertices are $V(\mathbf{t}) = V(\mathbf{q}) \setminus \{x_*\}$, where x_* is the origin vertex of the root edge, and \mathbf{l} is the restriction to $V(\mathbf{t})$ of the function $\mathbf{l}(x) = d_{\mathbf{q}}^{\text{gr}}(x, x_*)$, $x \in V(\mathbf{q})$. Moreover, the edges of \mathbf{t} and \mathbf{q} only intersect at vertices. The root vertex of \mathbf{t} is the tip of the root edge of \mathbf{q} , so it lies at $d_{\mathbf{q}}^{\text{gr}}$ -distance 1 from x_* .

Let $x(0)$ be the root vertex of \mathbf{t} , and given $\{x(0), \dots, x(i)\}$, and let $x(i+1)$ be the first child² of $x(i)$ not in $\{x(0), \dots, x(i)\}$ if there is any, or the parent of $x(i)$ if there is not. This procedure stops at $i = 2n$, where we are back to the root and have explored all vertices

²For the natural order inherited from the planar structure of \mathbf{t}

of the tree. We let $C_i = d_{\mathbf{t}}^{\text{gr}}(x(i), x(0))$, and $L_i = \mathbf{l}(x(i))$. Both C and L are extended by linear interpolation between integer times into continuous functions, still called C, L , with duration $2n$. The contour process C of \mathbf{t} is the usual Harris walk encoding of the rooted tree \mathbf{t} , and the pair (C, L) determines (\mathbf{t}, \mathbf{l}) completely. In the sequel, we will use the fact that $x(i)$ can be identified with a vertex of \mathbf{q} .

A simple consequence (see [10, Lemma 3.1]) of the construction is that for $i < j$,

$$d_{\mathbf{q}}^{\text{gr}}(x(i), x(j)) \leq L_i + L_j - 2 \min_{i \leq k \leq j} L_k + 2. \quad (1)$$

3.3 Estimates on the lengths of geodesics

Our last ingredient is a slight rewriting of the estimates of Le Gall [10] on geodesic paths in quadrangulations in terms of encoding processes. Precisely, let C^n, L^n be the contour and label process of a uniform random element \mathbf{t}_n of \mathbb{T}_n , and let \mathbf{q}_n be the quadrangulation that is the image of this element by Schaeffer's bijection. In particular, \mathbf{q}_n is a random uniform element of \mathbf{Q}_n . Also, recall that a graphical representation \mathcal{T}_n of \mathbf{t}_n can be drawn on the representation $\mathcal{S}_{\mathbf{q}_n}$ of Sect. 3.1, in such a way that the vertices of \mathcal{T}_n are $\mathcal{V}_{\mathbf{q}_n} \setminus \{x_*\}$, where x_* is the root vertex, and \mathcal{T}_n intersects edges $\mathcal{E}_{\mathbf{q}_n}$ only at vertices. For simplicity we let $V_n = V(\mathbf{q}_n)$, $d_n^{\text{gr}} = d_{\mathbf{q}_n}^{\text{gr}}$, $\mathcal{S}_n = \mathcal{S}_{\mathbf{q}_n}$, $d_n = d_{\mathbf{q}_n}$.

The main result of [9] says that the convergence in distribution in $\mathcal{C}([0, 1], \mathbb{R})^2$ holds:

$$\left(\left(\frac{1}{\sqrt{2n}} C_{2nt}^n \right)_{0 \leq t \leq 1}, \left(\left(\frac{9}{8n} \right)^{1/4} L_{2nt}^n \right)_{0 \leq t \leq 1} \right) \xrightarrow[n \rightarrow \infty]{(d)} (\bar{\mathfrak{e}}, \bar{\mathcal{Z}}), \quad (2)$$

where $(\bar{\mathfrak{e}}, \bar{\mathcal{Z}})$ is the Brownian snake conditioned to be positive introduced by Le Gall & Weill [12]. Moreover, it is shown in [10] that the laws of $[V_n, n^{-1/4} d_n^{\text{gr}}]$ form a relatively compact family in the set of probability measures on \mathbb{M} endowed with the weak topology. Since V_n is 3-dense in \mathcal{S}_n , the same holds for $[\mathcal{S}_n, n^{-1/4} d_n]$. We argue as in [10], and assume by Skorokhod's theorem that the trees \mathbf{t}_n (hence also the quadrangulations \mathbf{q}_n) are defined on a common probability space on which we have, almost-surely

- $[\mathcal{S}_n, n^{-1/4} d_n] \rightarrow [S, d]$, some random limiting space in PM , along some subsequence $n_k \rightarrow \infty$, and
- the convergence (2) holds a.s. along this subsequence.

From this point on, we will always assume that n is taken along this subsequence. In particular, we have that $\text{diam } S = \lim_n n^{-1/4} \text{diam } \mathcal{S}_n \geq \lim_n \sup n^{-1/4} L^n = \sup \bar{\mathcal{Z}} > 0$ a.s., so S is not reduced to a point and Theorem 2 may be applied if we check that the convergence is 1-regular. We are going to rely on proposition 4.2 of [10], which can be rephrased as follows.

Proposition 2 *The following property is true with probability 1. Let i_n, j_n be integers such that $i_n/2n \rightarrow s, j_n/2n \rightarrow t$ in $[0, 1]$, where s, t satisfy*

$$\bar{\mathfrak{e}}_s = \inf_{s \wedge t \leq u \leq s \vee t} \bar{\mathfrak{e}}_u < \bar{\mathfrak{e}}_t.$$

For $n \geq 1$, let γ_n be a path in \mathbf{q}_n between $x(i_n)$ and $x(j_n)$ with the notation of Sect. 3.2. Then it holds that

$$\liminf_{n \rightarrow \infty} n^{-1/4} \text{length}(\gamma_n) > 0.$$

In [10], this proposition was a first step in the proof of the fact that a limit in distribution of (V_n, d_n^{gr}) can be expressed as a quotient of the continuum tree with contour function \bar{e} : this lemma says that two points of the latter such that one is an ancestor of the other are not identified. Le Gall completed this study by exactly characterizing which are the points that are identified.

4 Proof of Theorem 1

Lemma 1 *Almost-surely, for every $\varepsilon > 0$, there exists a $\delta \in (0, \varepsilon)$ such that for n large enough, any simple loop γ_n made of edges of \mathcal{S}_n , with diameter $\leq n^{1/4}\delta$, splits \mathcal{S}_n in two Jordan domains, one of which has diameter $\leq n^{1/4}\varepsilon$.*

Proof. We argue by contradiction, assuming there exist simple loops γ_n made of edges of \mathcal{S}_n , with diameters $o(n^{1/4})$ as $n \rightarrow \infty$, such that the two Jordan domains bounded by γ_n are of diameters $\geq n^{1/4}\varepsilon$, where $\varepsilon > 0$ is some fixed constant. Let l_n be the minimal label on γ_n , i.e. the distance from the root vertex x_* to γ_n . Then all the labels of vertices that are in a connected component D_n of $\mathcal{S}_n \setminus \gamma_n$ not containing x_* are all larger than l_n , since a geodesic from x_* to any such vertex must pass through γ_n .

The intuitive idea of the proof is the following. Starting from the root of the tree \mathcal{T}_n , follow a maximal simple path in \mathcal{T}_n that enters in D_n at some stage. If all such paths remained in D_n after entering, then all the descendants of the entrance vertices would have labels larger than that of the entrance vertex, a property of zero probability under the limiting Brownian snake measure, see [10, Lemma 2.2] and Lemma 2 below. Thus, some of these paths must go out of D_n after entering, but they can do it only by passing through γ_n , which entails that strict ancestors in \mathcal{T}_n will be at distance $o(n^{1/4})$, and this is prohibited by Proposition 2. This is summed up in Figure 1, which gathers some of the notations to come.

We proceed to the rigorous proof. Take a vertex y_n in D_n . As a vertex of \mathcal{T}_n , it can be written in the form $x(j_n)$ for some j_n . Let j'_n be the first integer $j \geq j_n$ such that $x(j)$ is at d_n -distance ≤ 1 from γ_n . Such a j exists because of the way edges of \mathcal{T}_n are drawn (entailing that the ancestral path in \mathcal{T}_n from x_n to the root of \mathcal{T}_n must itself pass at distance ≤ 1 from γ_n , since the root of \mathcal{T}_n is at distance 1 from x_* and x_n lies in D_n) and the label $\mathbf{l}(x(j'_n))$ is at most $\max_{z \in \gamma_n} \mathbf{l}(z) + 1 = l_n + o(n^{1/4})$. Moreover, for $k \in [j_n, j'_n]$, the vertex $x(k)$ is in D_n , and in particular, its label is $\geq l_n$. Applying the bound (1) to the times j_n, j'_n , we get that $d_n(y_n, x(j'_n)) \leq \mathbf{l}(y_n) + l_n - 2l_n + o(n^{1/4})$. Since by hypothesis the diameter of D_n is at least $n^{1/4}\varepsilon$, it is thus possible to choose y_n with label $\mathbf{l}(y_n) \geq l_n + n^{1/4}\varepsilon/2$.

We let x_n be the first ancestor of y_n in \mathcal{T}_n lying at d_n -distance ≤ 1 from γ_n , so that $\mathbf{l}(x_n) = l_n + o(n^{1/4})$. Take $i_n < j_n$ such that i_n is a time encoding x_n , so that

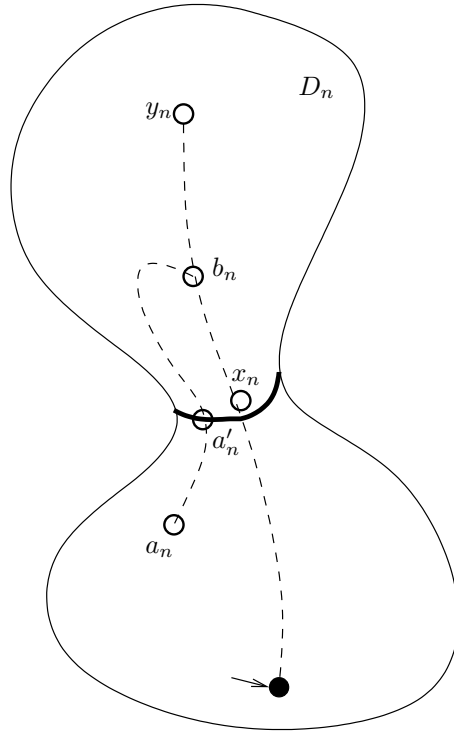


Figure 1: Illustration of the proof. The surface \mathcal{S}_n is depicted as a sphere with a bottleneck circled by γ_n (thick line). The root edge of the quadrangulation is drawn at the bottom, and the tree \mathcal{T}_n originates from its tip. In dashed lines are represented the two branches of \mathcal{T}_n that are useful in the proof: one enters the component D_n , and the other goes out after entering, identifying strict ancestors in the limit

$C_{i_n}^n = \inf_{i_n \leq r \leq j_n} C_r^n$. Up to further extraction, we may and will assume that

$$(9/8n)^{1/4} l_n \rightarrow l, \quad i_n/2n \rightarrow s, \quad j_n/2n \rightarrow t.$$

Then $s \leq t$ and $\bar{\epsilon}_s \leq \bar{\epsilon}_u$ for $u \in [s, t]$. More precisely, we have $\bar{Z}_s = l$ and $\bar{Z}_t \geq l + (9/8)^{1/4} \varepsilon/2$, which implies $s < t$, and $\bar{\epsilon}_s < \bar{\epsilon}_t$. In terms of the continuum tree encoded by $\bar{\epsilon}$, this amounts to the fact that s, t encode two vertices such that the first is an ancestor of the second, and that are not the same because the snake \bar{Z} takes distinct values at these points. We will need the following technical statement:

Lemma 2 *Assume that $s > 0$. With probability 1, there exist $\eta > 0$ and integers i'_n, k_n, r_n with $i_n \leq i'_n < k_n < r_n < j_n$ such that $i'_n/2n \rightarrow s' \in [s, t)$, that satisfy for n large enough:*

$$C_{i'_n}^m = \inf_{i'_n \leq r \leq j_n} C_r^m, \quad C_{r_n}^m = \inf_{k_n \leq r \leq j_n} C_r^m,$$

and

$$(2n)^{-1/2} C_{i'_n}^m \rightarrow \bar{\epsilon}_{s'} = \bar{\epsilon}_s,$$

so that i'_n, k_n, r_n encode vertices x'_n, a_n, b_n of \mathcal{T}_n such that $x_n \prec x'_n \prec b_n \prec y_n$ and $b_n \prec a_n$, where \prec denotes “is an ancestor of”. Moreover, $d_n(x_n, x'_n) = o(n^{1/4})$ and the labels satisfy

$$\mathbf{l}(b_n) = L_{r_n}^n > L_{i_n}^n + \eta n^{1/4} = \mathbf{l}(x_n) + \eta n^{1/4},$$

and

$$\mathbf{l}(a_n) = L_{k_n}^n \leq L_{i_n}^n - \eta n^{1/4} = \mathbf{l}(x_n) - \eta n^{1/4}.$$

The statement says roughly the following: there exist subtrees of \mathcal{T}_n branching on a vertex b_n of the ancestral line from x_n to y_n that attain labels that are significantly smaller (in the scale $n^{1/4}$) than $\mathbf{l}(y_n)$, but such that $\mathbf{l}(b_n)$ is significantly larger than $\mathbf{l}(y_n)$. As mentioned in the Introduction, a rigorous proof needs some prerequisites on continuum trees, and is really a re-writing of the proof of [10, Proposition 4.2, pp.649–650], using the fact [10, Lemma 2.2] that the positive Brownian snake has no increase points except 0. We only explain in detail the role of i'_n in the statement. Introducing $i'_n \neq i_n$ may be necessary if it happens that two macroscopic subtrees branch just above x_n . This happens if $\bar{\epsilon}$ attains a local minimum equal to $\bar{\epsilon}_s$ at a time $s' \in (s, t)$, which is the most pathological situation that can occur since local minima of a Brownian motion are pairwise distinct and realized only once. In this case, we take i'_n so that $i'_n/2n \rightarrow s'$ and C_{2n}^n achieves a local minimum at i'_n . Then, the vertex encoded by i'_n is encoded by another time $i''_n < i'_n$ such that $i''_n/2n \rightarrow s$, which together with (1) implies the property $d_n(x_n, x'_n) = o(n^{1/4})$. If $\bar{\epsilon}_s < \bar{\epsilon}_u$ for every $u \in (s, t]$, we simply take $i'_n = i_n$.

Now back to the proof of Lemma 1. Because of the property of the label of a_n , it does not lie in D_n , however, its ancestor b_n does because it is on the ancestral path from x_n to y_n . Hence some ancestor of a_n must belong to γ_n , and let a'_n be the youngest of these (the highest in the tree), and take $k'_n \in (k_n, r_n)$ encoding a'_n . Since x_n is at distance at most 1 from γ_n , we obtain that $d_n(a'_n, x_n) = o(n^{1/4})$. However, if $k'_n/2n \rightarrow v, r_n/2n \rightarrow u$, taking again an extraction if necessary, then we have $\bar{\epsilon}_s < \bar{\epsilon}_u \leq \bar{\epsilon}_v$ because of the ancestral relations $C_{i_n}^n \leq C_{r_n}^n \leq C_{k'_n}^n$, and the fact

$$\bar{Z}_s = \lim_{n \rightarrow \infty} (9/8n)^{1/4} L_{i_n}^n = l < l + (9/8)^{1/4} \eta \leq \lim_{n \rightarrow \infty} (9/8n)^{1/4} L_{r_n}^n = \bar{Z}_u.$$

Now the statements $d_n(a'_n, x_n) = o(n^{1/4})$ and $\bar{\epsilon}_s < \bar{\epsilon}_v$ together can only hold with zero probability by Proposition 2.

It remains to rule out the possibility that $s = 0$, i.e. that γ_n lies at distance $o(n^{1/4})$ from x_* . To see that this is not possible, argue as in the beginning of the proof and take x_n, y_n respectively in the two disjoint connected components of $\mathcal{S}_n \setminus \gamma_n$, and with labels $\mathbf{l}(x_n) \wedge \mathbf{l}(y_n) \geq n^{1/4}\varepsilon/2$. By symmetry, assume that $x_n = x(i_n)$ and $y_n = x(j_n)$ with $i_n < j_n$. Now take the least integer $k_n \in [i_n, j_n]$ such that $x(k)$ belongs to γ_n . Such a k has to exist because any path from x_n to y_n in \mathcal{S}_n must pass through γ_n . Then $L_{k_n}^n = \mathbf{l}(x(k_n)) = o(n^{1/4})$. Up to extraction, assume $i_n/2n \rightarrow s, k_n/2n \rightarrow u, j_n/2n \rightarrow t$. Then $\bar{Z}_u = 0 < \bar{Z}_s \wedge \bar{Z}_t$, so that $s < u < t$, and this contradicts the fact that \bar{Z} is strictly positive on $(0, 1)$, which is a consequence of [12, Proposition 2.5]. \square

We claim that this lemma is enough to obtain 1-regularity of the convergence, and hence to conclude by Theorem 2 that the limit (S, d) is a sphere. First choose $\varepsilon < \text{diam } S/3$ to avoid trivialities. Let γ_n be a loop in \mathcal{S}_n with diameter $\leq n^{1/4}\delta$. The boundary of the union of the closures of faces of \mathbf{q}_n that are hit by γ_n is made of pairwise disjoint simple loops of edges of \mathcal{S}_n . If x, y are elements of this union of faces, and since a face of \mathcal{S}_n has diameter less than 3, there exist points x', y' of γ_n at distance at most 3 from x, y respectively, so that the diameters of these loops all are $\leq n^{1/4}\delta + 6$. By the Jordan Curve Theorem, each of these loops splits \mathcal{S}_n into two simply connected components, one of which has diameter $\leq n^{1/4}\varepsilon$, and one of which contains γ_n entirely. It suffices to justify that these two properties (being of diameter $\leq n^{1/4}\varepsilon$ and containing γ_n) hold simultaneously for some loop in the family to conclude that γ_n is homotopic to 0 in its ε -neighborhood. So assume the contrary: the component not containing γ_n of every loop is of diameter $\leq n^{1/4}\varepsilon$. By definition, any point in the complement of the union of these components is at distance at most 3 from some point of γ_n . Take x, y such that $d_n(x, y) = \text{diam}(\mathcal{S}_n)$. Then there exist points x', y' in γ_n at distance at most $n^{1/4}\varepsilon + 3$ respectively from x, y , and we conclude that $d_n(x', y') \geq \text{diam}(\mathcal{S}_n) - 6 - 2n^{1/4}\varepsilon > n^{1/4}\delta$ for n large enough by our choice of ε , a contradiction.

References

- [1] D. J. Aldous. The continuum random tree. III. *Ann. Probab.*, 21(1):248–289, 1993.
- [2] J. Ambjørn, B. Durhuus, and T. Jonsson. *Quantum geometry. A statistical field theory approach*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 1997.
- [3] E. G. Begle. Regular convergence. *Duke Math. J.*, 11:441–450, 1944.
- [4] D. Burago, Y. Burago, and S. Ivanov. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.
- [5] P. Chassaing and G. Schaeffer. Random planar lattices and integrated superBrownian excursion. *Probab. Theory Related Fields*, 128(2):161–212, 2004.
- [6] S. N. Evans, J. Pitman, and A. Winter. Rayleigh processes, real trees, and root growth with re-grafting. *Probab. Theory Related Fields*, 134(1):81–126, 2006.
- [7] A. Greven, P. Pfaffelhuber, and A. Winter. Convergence in distribution of random metric measure spaces (Λ -coalescent measure trees). 2006.
- [8] J.-F. Le Gall. *Spatial branching processes, random snakes and partial differential equations*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1999.
- [9] J.-F. Le Gall. Conditional limit theorem for tree-indexed random walks. *Stoch. Proc. Appl.*, 116:539–567, 2006. To appear.

- [10] J.-F. Le Gall. The topological structure of scaling limits of large planar maps. *Invent. Math.*, 169:621–670, 2007.
- [11] J.-F. Le Gall and F. Paulin. Scaling limits of bipartite planar maps are homeomorphic to the 2-sphere. 2006.
- [12] J.-F. Le Gall and M. Weill. Conditioned Brownian trees. *Ann. Inst. Henri Poincaré*, 2006. To appear.
- [13] J.-F. Marckert and A. Mokkadem. Limit of normalized random quadrangulations: the Brownian map. *Ann. Probab.*, 34(6):2144–2202, 2006.
- [14] G. Miermont. Topology of scaling limits of random quadrangulations of fixed genus. *In preparation*, 2007.
- [15] G. Schaeffer. *Conjugaison d'arbres et cartes combinatoires aléatoires*. PhD thesis, Université Bordeaux I, 1998.