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Subcritical regimes in some models of continuum percolation

Jean-Baptiste Gouéré *

Abstract

We consider some continuum percolation models. We are mainly interested in giving some sufficient conditions for absence of percolation. We give some general conditions and then focuse on two examples. The first one is a multiscale percolation model based on the Boolean model. It was introduced by Meester and Roy and subsequently studied by Menshikov, Popov and Vachkovskaia. The second one is based on the stable marriage of Poisson and Lebesgue introduced by Hoffman, Holroyd and Peres and whose percolation properties have been studied by Freire, Popov and Vachkovskaia.

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1 Introduction and statement of the main results

1.1 Introduction

In this paper, we study some continuum percolation models. We are mainly interested in giving some sufficient conditions for absence of percolation. We give some general conditions and then apply them to two examples: multifractal percolation and stable marriage of Poisson and Lebesgue. The aim of Section 1.1 is to give a quick description of this two examples. We begin by recalling the Boolean model.

The Boolean model. We center a ball of random radius at each point of a homogeneous Poisson point process of the Euclidean space $\mathbb{R}^d$, $d \geq 2$. We assume that the radii of the balls are independent copies of a given positive random variable R. We also assume that the radii are independent of the point process. We denote by λ the density of the Poisson point process. We denote by $\Sigma(\lambda)$ the union of the balls, by $S(\lambda)$ the connected component of $\Sigma(\lambda)$ that contains the origin and by $D(\lambda)$ the Euclidean diameter of $S(\lambda)$.

When R is bounded, there exists a sharp phase transition (see for example [10], Section 12.10 in [4] when $R = 1$ or the papers [17], [18], [14] and [11]): if the density λ of the point process is below a critical value $\lambda_c > 0$, then $S(\lambda)$ is almost surely bounded and its diameter $D(\lambda)$ admits exponential moments ; whereas if λ is above λ_c then $S(\lambda)$ is unbounded with positive probability.

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The case where R is unbounded was studied by Hall in [5] (see also the books [6] and [10]). Hall proved that if $E(R^{2d-1})$ is finite, then the set $S(\lambda)$ is almost surely bounded for small enough λ. If $E(R^d)$ is infinite, then such behaviour does not happen: whatever the value of the density λ, the set $\Sigma(\lambda)$ is almost surely the whole space. The latter result still holds when the underlying point process is only assumed to be an almost surely non empty and stationary point process (see [10], Proposition 7.3). In [3], we proved that the set $S(\lambda)$ is almost surely bounded for small enough λ if and only if $E(R^d)$ is finite. We also proved that, for any $s > 0$, $E(D(\lambda)^s)$ is finite for small enough λ if and only if $E(R^{d+s})$ is finite. We refer to [3] for further bibliographical information. The idea developped in [3] for the Boolean model can be developped further to investigate the following models.

A multiscale percolation model. We refer to Section 1.5 for details. We keep the objects defined in the previous paragraph. Let $(\Sigma_n(\lambda))_{n\geq 0}$ be a sequence of independent copies of $\Sigma(\lambda)$. Let $a > 1$ be a scale factor. We define a new random set $\widetilde{\Sigma}(\lambda, a)$ by:

$$\widetilde{\Sigma}(\lambda, a) = \bigcup_n a^{-n}\Sigma_n(\lambda).$$

We are interested in properties of the connected components of $\widetilde{\Sigma}(\lambda, a)$. We say that the model is subcritical if the connected components can be small (see Section 1.5 for precise statements).

This model was introduced by Meester and Roy in [10]. They considered the case where the radius $R = 1$ and the dimension $d = 2$. They proved that $\widetilde{\Sigma}(\lambda, a)$ is in a subcritical phase as soon as one of the following conditions holds:

1. the density λ is small enough ;
2. the density λ is such that $\Sigma(\lambda)$ is in the subcritical phase and a is large enough.

In [12], Menshikov, Popov and Vachkovskaia considered the case where the dimension d is arbitrary. They proved that if $\Sigma(\lambda)$ is in the subcritical phase then, for large enough a, $\widetilde{\Sigma}(\lambda, a)$ is also in a subcritical phase.

In [13], Menshikov, Popov and Vachkovskaia studied the case where R is random. Let us emphasize that they did not assume R to be bounded. They considered the following condition:

$$P(D(\lambda) \geq r)r^d \to 0 \text{ as } r \to \infty. \tag{1}$$

(Let us recall that $D(\lambda)$ denotes the Euclidean diameter of the connected component of $\Sigma(\lambda)$ containing 0). Under some further technical conditions, they proved that, if (1) holds, then, for large enough a, $\widetilde{\Sigma}(\lambda, a)$ is in a subcritical phase. This is a generalization of the previous result. Indeed, when R is bounded, (1) holds as soon as $\Sigma(\lambda)$ is in the subcritical phase. When R is unbounded, one can make the following remarks about the conditions under which there exists $\lambda > 0$ such that (1) holds. Let $\varepsilon > 0$. Condition (1) holds as soon as $E(D(\lambda)^{d+\varepsilon})$ is finite. Therefore, by the result of [3] previously cited, there exists $\lambda > 0$ such that (1) holds as soon as $E(R^{2d+\varepsilon})$ is finite. On the other hand, if (1) holds then $E(D(\lambda)^{d-\varepsilon})$ is finite and therefore $E(R^{2d-\varepsilon})$ is finite. Therefore, the existence of $\lambda > 0$ such that (1) holds is roughly equivalent to the finitness of $E(R^{2d})$.

In this paper, we prove the following result in which $a > 1$ is fixed: $\widetilde{\Sigma}(\lambda, a)$ is in a subcritical phase for small enough λ if and only if $E(R^d \ln(R)_+)$ is finite. This is a corollary of Theorem 1.1, which is one of our main abstract results.

Stable marriage of Poisson and Lebesgue. We refer to Section 1.6 for details. The following model was introduced by Hoffman, Holroyd and Peres in [8]. Let $\alpha > 0$ be a parameter called appetite. Let χ be a homogeneous Poisson point process with density 1 on $\mathbb{R}^d$. In [8], the authors showed that there was essentially a unique way to give in a stable way to points of χ disjoint territories of $\mathbb{R}^d$ of volume at most α. We defer the definition of stability to Section 1.6. Very roughly, it means that the distances between points of χ and points of their territories are minimal.

In this paper, we are interested in percolation properties of the union $T(\alpha)$ of all the territories. Let $S(\alpha)$ denote the connected component of $T(\alpha)$ that contains the origin. In [1], Freire, Popov and Vachkovskaia proved, among other things, that $S(\alpha)$ was almost surely bounded for small enough λ. In this paper, we prove the following stronger result, in which $D(\alpha)$ denotes the Euclidean diameter of $S(\alpha)$. For small enough λ, for all $n \geq 0$, $E(D(\alpha)^n)$ is finite.

To prove this result, we first show that $T(\alpha)$ is dominated by a dependent percolation process. This was already the first step in the proof of [1]. We then apply to this dependent percolation process Theorem 1.3, which is the main abstract theorem of our paper.

1.2 Some notations

For the whole of the paper, we fix an integer $d \geq 1$. Let $|\cdot|$ be the Lebesgue measure on $\mathbb{R}^d$. We denote by $\|\cdot\|$ the Euclidean norm on $\mathbb{R}^d$, by $B(x, r)$ the open Euclidean ball centered at $x \in \mathbb{R}^d$ with radius $r \geq 0$ and by $\overline{B}(x, r)$ the closed Euclidean ball centered at $x \in \mathbb{R}^d$ with radius $r \geq 0$.

When a point process ξ on $\mathbb{R}^d \times]0, +\infty[$ is given we define the following objects. We let

$$\Sigma = \bigcup_{(c,r)\in\xi} B(c, r).$$

(When we write $(c, r) \in \xi$ we implicitly assume that c belongs to $\mathbb{R}^d$ and that r belongs to $]0, +\infty[$.) We denote by S the connected component of Σ which contains 0. (We let $S = \emptyset$ if 0 does not belong to Σ.) We define a random variable M as follows:

$$M = \sup_{x\in S} \|x\|. \tag{2}$$

(We let $M = 0$ if S is empty.) We say that percolation occurs if S is unbounded:

$$\{\text{percolation}\} = \{S \text{ is unbounded}\}.$$

1.3 Boolean model induced by Poisson point processes

Let $\lambda > 0$ and let μ be a locally finite measure on $]0, +\infty[$. Let ξ be a Poisson point process on $\mathbb{R}^d \times]0, +\infty[$ whose intensity measure is the product of $\lambda|\cdot|$ and μ. We denote

by $P_{\lambda,\mu}$, $E_{\lambda,\mu}$ the associated probability measure and expectation, respectively. As distinct points of ξ have distinct coordinates on $\mathbb{R}^d$, we can write:

$$\xi = \{(c, r(c)), c \in \chi\}$$

where χ denotes the projection of ξ on $\mathbb{R}^d$. If the measure μ is a probability measure then χ is a Poisson point process on $\mathbb{R}^d$ whose intensity is $\lambda|\cdot|$. Moreover, under this assumption, if we condition on χ then the $r(c), c \in \xi$ are i.i.d. with common distribution μ. (We shall not use this result.) We refer to [9, 15, 16] for background on point processes and to [6, 10] for Boolean models.

We prove the following results:

Theorem 1.1 *Assume $d \geq 2$. There exists $\lambda_0 > 0$ such that $P_{\lambda,\mu}(\text{percolation}) = 0$ for all $\lambda \in]0, \lambda_0[$ if and only if the following assertions hold:*

A1. The supremum $\sup_{r>0} r^d \mu([r, +\infty[)$ is finite.

A2. The integral $\int_{[1,+\infty[} \beta^d \mu(d\beta)$ is finite.

If $d = 1$, then Assumptions A1. and A2. together are sufficient conditions ; Assumption A2. is a necessary condition.

Remarks.

1. For all $\rho > 1$, Assumption A1 is equivalent to the following one:

$$\sup_{r>0} r^d \mu([r, \rho r]) < \infty$$

(see Lemma 3.2). Notice that the probability of 0 belonging to a ball of the process with radius in $[r, \rho r]$ is given by

$$1 - \exp\left(-\lambda \int_{[r,\rho r]} \mu(d\beta)|B(0, \beta)|\right).$$

Assumption A1 therefore means that those probabilities are bounded away from 1.

2. If μ is a finite measure, then Assumptions A1. and A2. together are equivalent to the finitness of the integral $\int_{]0,+\infty[} \beta^d \mu(d\beta)$.

Theorem 1.2 *Let $s > 0$ be a positive real. Assume $d \geq 2$. There exists $\lambda_0 > 0$ such that $E_{\lambda,\mu}(M^s)$ is finite for all $\lambda \in]0, \lambda_0[$ if and only if the following assertions hold:*

A1. The supremum $\sup_{r>0} r^d \mu([r, +\infty[)$ is finite.

A3. The integral $\int_{[1,+\infty[} \beta^{d+s} \mu(d\beta)$ is finite.

If $d = 1$, then Assumptions A1. and A3. together are sufficient conditions ; Assumption A3. is a necessary condition.

Remark. If μ is a finite measure, then Assumptions A1. and A3. together are equivalent to the finitess of the integral $\int_{]0,+\infty[} \beta^{d+s}\mu(d\beta)$.

Theorems 1.1 and 1.2 are essentially consequences of Theorem 1.3 stated in the next subsection. Theorems 1.1 and 1.2 are generalizations of the main results of [3] in which μ is assumed to be a finite measure.

The proofs are given in Section 3.

1.4 Boolean model induced by more general point processes

Let ξ be a point process on $\mathbb{R}^d \times]0,+\infty[$. We assume that the law of ξ is invariant under the action of the translations of $\mathbb{R}^d$: for all $t \in \mathbb{R}^d$, the point processes $\{x-(t,0), x \in \xi\}$ and ξ have the same law. We also assume that the intensity measure of ξ is locally finite. Therefore, the intensity measure of ξ is the product of the Lebesgue measure on $\mathbb{R}^d$ by a locally finite measure on $]0,+\infty[$ that we denote by μ.

The main result of this paper is the following theorem.

Theorem 1.3 *Let $C > 0$. There exists $D > 0$, that depends only on d and C, such that the following hold.*

If the following properties are fulfilled:

B0. for all $r > 0$ and all $x \in \mathbb{R}^d \setminus B(0,Cr)$ the point processes

$$\xi \cap B(0,r)\times]0,r] \text{ and } \xi \cap B(x,r)\times]0,r]$$

are independent;

B1. $\sup_{r>0} r^d \mu([r,+\infty[) \leq D$*;*

B2. the integral $\displaystyle\int_{[1,+\infty[} \beta^d \mu(d\beta)$ *is finite,*

then the set S is almost surely bounded. Let s be a positive real. If moreover

B3. $\displaystyle\int_{[1,+\infty[} \beta^{d+s}\mu(d\beta) < \infty$

then $E(M^s)$ is finite.

Remarks.

1. The independence assumption B0 is fulfilled if ξ is a Poisson point process and $C \geq 2$.

2. We give a strenghtened version of Theorem 1.3 in Section 2.3 (see Theorems 2.7, 2.8 and 2.9). In those theorems, the independence assumption is weakened and the conclusions are strenghtened.

The proof is given in Section 2. We give some ideas of the proof in Section 2.2 after the statement of key Proposition 2.1.

1.5 Multiscale percolation model

Let $\lambda > 0$ and ν be a probability measure on $]0, +\infty[$. We make the following assumption:

$$\int_{]0,+\infty[} r^d \nu(dr) < \infty. \tag{3}$$

Let $(\xi_n)_{n\geq 0}$ be a sequence of independent Poisson point processes on $\mathbb{R}^d \times]0, +\infty[$ whose intensity is the product of $\lambda|\cdot|$ by ν. Let $a > 1$. We define a new point process by:

$$\xi = \bigcup_{n\geq 0} a^{-n}\xi_n.$$

Lemma 1.4 *The point process ξ is a Poisson point process whose intensity is the product of $\lambda|\cdot|$ by the locally finite measure μ on $]0, +\infty[$ defined by:*

$$\mu(B) = \sum_{n\geq 0} a^{nd}\nu(a^n B). \tag{4}$$

As in Section 1.2 we associate with ξ two random sets Σ and S. We denote by $P^a_{\lambda,\nu}$ the associated probability measure. We also denote by Σ_n the random sets associated with the processes $a^{-n}\xi_n$.

Remarks.

1. For all integer $n \geq 1$, Σ_n is an independent copy of $a^{-n}\Sigma_0$.

2. If (3) is not fulfiled then, for all $\lambda > 0$, percolation occurs with positive probability in Σ_0 (by Theorem 1.1) and then in Σ. Actually, by Lemma 3.1, if (3) is not fulfiled then, for all $\lambda > 0$, $\Sigma_0 = \mathbb{R}^d$ almost surely. Therefore, Assumption (3) is not a restriction.

3. One can easily check that 0 belongs almost surely to Σ. Therefore, the Lebesgue measure of the complement of Σ is almost surely 0. We will nevertheless see that the connected components of Σ can be bounded.

This model was introduced by Meester and Roy in a two dimensional setting in [10]. Let us denote by δ_1 the Dirac mass at 1. Let us say that the event {left-right crossing} occurs if $[0, 1]^2 \setminus \Sigma$ contains a connected component which intersects the left and the right sides of $[0, 1]^2$. Let us denote by λ_c the critical density for the Boolean model when all radii equal 1. (Thus, if $\lambda < \lambda_c$, the connected components of the Σ_n are almost surely bounded ; whereas if $\lambda > \lambda_c$, this is not the case.) Meester and Roy proved the following result, in which the radii of the unscaled process Σ_0 equal 1.

Theorem 1.5 ([10]) *Assume $d = 2$.*

1. *Let $a > 1$. If $\lambda > 0$ is small enough, then P^a_{λ,δ_1}(left-right-crossing) is positive.*

2. *Let $\lambda < \lambda_c$. If a is large enough, then P^a_{λ,δ_1}(left-right-crossing) is positive.*

In [12], Menshikov, Popov and Vachkovskaia considered the case where the dimension d is arbitrary and the radii of the unscaled process Σ_0 equal 1. They proved the following result.

Theorem 1.6 ([12]) *Assume $d \geq 2$. If $\lambda < \lambda_c$ then, for all a large enough,*

$$P^a_{\lambda,\delta_1}(S \text{ is bounded}) = 1.$$

The ideas of their proof are the following. (Those ideas are used in their paper through a discretization of space ; we describe them in a slightly more geometric way.) Assume that C is a connected component of $\Sigma_n \cup \Sigma_{n+1}$ whose diameter is a least αa^{-n} for a given $\alpha > 0$. Then, C is included is the union of the following kind of sets:

1. connected components of Σ_{n+1} whose diameter is at least αa^{-n} ;
2. balls of Σ_n enlarged by the factor $1+\alpha$ (same centers but the radii are $(1+\alpha)a^{-n}$ instead of a^{-n}).

Then, they show that the union of all those sets is stochastically dominated by a Boolean model where all radii equals $(1+\alpha)a^{-n}$ ($1+\alpha$ times those of Σ_n) and where the density of the set of centers is $(1+\alpha')a^{nd}$ for a suitable $\alpha' > 0$ ($1+\alpha'$ times the corresponding density for Σ_n). The proof of this fact relies partly on the exponential decay of the size of the components in the subcritical phase. In some sense, one can therefore control percolation in the union of two models by percolation in one model. Iterating the argument with some care in the constants α and α', one sees that one can control percolation in the multiscale model by percolation in a subcritical model. This yields the result.

In [13] the same authors considered the case where the radii are random and unbounded. Let us define Θ by:

$$\Theta = \{\lambda > 0 : P_{\lambda,\nu}(D_0 > n)n^d \to 0 \text{ as } n \to \infty\}$$

where D_0 denotes here the diameter of the connected component of Σ_0 containing 0. Let $\widetilde{\lambda}_c$ denote the supremum of Θ. They proved the following generalization of Theorem 1.6.

Theorem 1.7 ([13]) *Assume $d \geq 2$ and the following:*

1. *the set Θ is non empty (and thus $\widetilde{\lambda}_c$ is positive) ;*
2. *the measure ν satisfies:*

$$\lim_{a\to\infty} \sup_{r\geq 1/2} \frac{a^d \nu([ar, +\infty[)}{\nu([r, +\infty[)} = 0$$

with the convention $0/0 = 0$.

Then, for all $\lambda < \widetilde{\lambda}_c$, for all large enough a, $P^a_{\lambda,\nu}(S \text{ is bounded}) = 1$.

Remarks.

1. When $R = 1$, $P_{\lambda,\delta_1}(D_0 > n)$ decays exponentialy as soon as $\lambda < \lambda_c$. Therefore $\widetilde{\lambda_c} = \lambda_c$ in that case. Theorem 1.7 is thus a generalization of Theorem 1.6.

2. The assumption $\lambda < \widetilde{\lambda_c}$ is used where, in the proof of Theorem 1.6, the exponential decay of the size of the connected components in the subcritical phase were used.

3. As explained in the introduction (see Section 1.1), the first assumption means roughly that the integral $\int r^{2d}\nu(dr)$ is finite.

By Theorem 1.1, we easily get the following result.

Theorem 1.8 *There exists $\lambda_0 > 0$ such that $P^a_{\lambda,\nu}(S$ is bounded$) = 1$ for all $\lambda \in]0, \lambda_0[$ if and only if the integral*

$$\int_{[1,+\infty[} \beta^d \ln(\beta)\nu(d\beta)$$

is finite.

The proof is given in Section 4.

Remark. We can get a similar result about the finiteness of moments of the diameter of S by Theorem 1.2.

1.6 Stable marriage of Poisson and Lebesgue

The following model was introduced in [8] by Hoffman, Holroyd and Peres. Let χ be a locally finite subset of $\mathbb{R}^d$. We call the elements of $\mathbb{R}^d$ sites and the elements of χ centers. Let $\alpha \in]0, \infty[$ be a parameter, called the appetite. An allocation of $\mathbb{R}^d$ to χ with appetite α is a measurable function

$$\psi : \mathbb{R}^d \to \chi \cup \{\infty, \Delta\}$$

such that $|\psi^{-1}(\Delta)| = 0$, and $|\psi^{-1}(a)| \leq \alpha$ for all $a \in \chi$. We call $\psi^{-1}(a)$ the territory of the center a. A center $a \in \chi$ is sated if $|\psi^{-1}(a)| = \alpha$ and unsated otherwise. A site $x \in \mathbb{R}^d$ is claimed if $\psi(x) \in \chi$, and unclaimed if $\psi(x) = \infty$. The allocation is undefined at x if $\psi(x) = \Delta$.

The following definition, given in [8], is an adaptation of that introduced by Gale and Shapley [2]. Let a be a center and let x be a site with $\psi(x) \notin \{a, \Delta\}$. We say that x desires a if

$$\|x - a\| < \|x - \psi(x)\| \text{ or } x \text{ is unclaimed.}$$

We say that a covets x if

$$\|x - a\| < \|x' - a\| \text{ for some } x' \in \psi^{-1}(a), \text{ or } a \text{ is unsated.}$$

We say that a site-center pair (x, a) is unstable for the allocation ψ if x desires a and a covets x. An allocation is stable if there are no unstable pairs.

We now assume that χ is a translation invariant Poisson point process on $\mathbb{R}^d$. We assume that its intensity measure is the Lebesgue measure. (We can see by scaling arguments that there is no loss of generality in this assumption.) In [8] it was proved among other things, that for any such process there exists a.s. a $|\cdot|$-a.e. unique stable allocation ψ from $\mathbb{R}^d$ to χ. Furthermore we have the following phase transition phenomenon.

1. If $\alpha < 1$ (subcritical) then a.s. all centers are sated but there is an infinite volume of unclaimed sites.

2. If $\alpha = 1$ (critical) then a.s. all centers are sated and $|\cdot|$-a.a. sites are claimed.

3. If $\alpha > 1$ (supercritical) then a.s. not all centers are sated but $|\cdot|$-a.a. sites are claimed.

Let $\mathcal{C}$ be the closure of the union of all territories:

$$\mathcal{C} = \overline{\psi^{-1}(\chi)}.$$

In [1], Freire, Popov and Vachkovskaia proved, among other things, the following result:

Theorem 1.9 ([1]) *If α is small enough, then a.s. there is no percolation in $\mathcal{C}$.*

Let D be the diameter of the connected component of $\mathcal{C}$ that contains the origin. In this paper we give the following consequence of Theorem 1.3:

Theorem 1.10 *If α is small enough, then for all $s > 0$, $E(D^s)$ is finite.*

Remark. We must admit that we have not checked out the measurability of D. Actually, we prove that, for small enough α, D^s is bounded above by an integrable random variable.

In order to prove Theorem 1.9, we first define a process that dominates the previous one. This relies on an idea that appeared in [7] (see the proof of Proposition 11(ii)) and that is used with the same purpose as ours in [1] (see Lemma 2.1). For all $a \in \chi$ we define $R(a, \chi)$ by:

$$R(a, \chi) = \inf\{r \geq 0 : \alpha \mathrm{card}(\chi \cap \overline{B}(a, 2r)) \leq |B(a, r)|\}.$$

We let $R(a, \chi) = \infty$ if there is no such r. We assume henceforth that α is strictly smaller than 2^{-d}. This ensures that, almost surely, all the $R(a, \chi)$ are finite (see Lemma 5.1 for a stronger statement). We can also check that all the $R(a, \chi)$ are positive. We then define a point process ξ on $\mathbb{R}^d \times]0, +\infty[$ by:

$$\xi = \{(a, 2R(a, \chi)), a \in \chi\}.$$

As in Section 1.2, we associate with this process a random set Σ. We have:

Lemma 1.11 *For all $\alpha \in]0, 2^{-d}[$, the set $\mathcal{C}$ is almost surely contained in the set Σ.*

It is therefore sufficient to study the percolation properties of Σ. Theorem 1.10 follows from an application of Theorem 1.3 to the process ξ. A full proof is given in Section 5.

2 Proof of Theorem 1.3 (models induced by general processes)

2.1 Some further notations

For the whole of the section, we fix a point process ξ. We assume that ξ satisfies the properties given above Theorem 1.3.

For all $\alpha \geq 0, \beta > 0$ we define a random set $\Sigma(\alpha, \beta)$ by:

$$\Sigma(\alpha, \beta) = \bigcup_{(c,r)\in\xi \,:\, r\in[\alpha,\beta]} B(c, r).$$

Notice that this set is empty if β is strictly smaller than α. If x belongs to $\mathbb{R}^d$, we say that $G(x, \alpha, \beta)$ occurs if the connected component of

$$\Sigma(\alpha, \beta) \cup B(x, \beta)$$

containing x is not contained in $B(x, 2\beta)$. In other words, $G(x, \alpha, \beta)$ occurs if one can go from $B(x, \beta)$ to the complement of $B(x, 2\beta)$ using balls of the percolation process whose radii belong to $[\alpha, \beta]$. By stationarity of ξ, the probability of $G(x, \alpha, \beta)$ does not depend on x. We denote it by $\pi(\alpha, \beta)$:

$$\pi(\alpha, \beta) = P(G(0, \alpha, \beta)).$$

Similarly, for all $\beta > 0$, we say that $\widetilde{G}(\beta)$ occurs if the connected component of

$$\Sigma \cup B(0, \beta)$$

containing 0 is not contained in $B(0, 2\beta)$. We denote its probability by $\widetilde{\pi}(\beta)$:

$$\widetilde{\pi}(\beta) = P(\widetilde{G}(\beta)).$$

In order to state some relations between percolation and the various events we have already introduced, we shall need the following two events. For all $\beta > 0$ and $\rho > 1$ we define $\widetilde{H}(\beta)$ and $H(\rho, \beta)$ by:

$$\widetilde{H}(\beta) = \{\exists (c, r) \in \xi : B(c, r) \cap B(0, 2\beta) \neq \emptyset \text{ and } r > \beta\}$$

and:

$$H(\rho, \beta) = \{\exists (c, r) \in \xi : c \in B(0, 3\rho\beta) \text{ and } r \in [\beta, \rho\beta]\}.$$

We will give a strenghtened version of Theorem 1.3 in which we loose the independance assumption. To state this result, we shall need the following definition, in which ρ is strictly larger that 1 and α, β are as above.

$$I(\rho, \alpha, \beta) = \sup_{x\in\mathbb{R}^d\setminus B(0,\rho\beta)} \Big[P\big(G(0, \alpha, \beta) \cap G(x, \alpha, \beta)\big) - P\big(G(0, \alpha, \beta)\big)P\big(G(x, \alpha, \beta)\big)\Big].$$

Note that, under Assumption B0 of Theorem 1.3, $I(\rho, \alpha, \beta) = 0$ for large enough ρ (see the beginning of the proof of Theorem 1.3). We also let:

$$I^+(\rho, \alpha, \beta) = \max\big(I(\rho, \alpha, \beta), 0\big).$$

2.2 Proof of key inequalities

Let us recall that μ is defined above the statement of Theorem 1.3 and that M is defined in Section 1.2. The aim of this subsection is to prove the following result.

Proposition 2.1 *Let $\rho \geq 2$. There exists a constant $\widetilde{D} > 0$, that depends only on the dimension d and on ρ, such that the following assertion holds for all $\alpha \geq 0$ and all $\beta > 0$:*

$$\pi(\alpha, \rho\beta) \leq \widetilde{D}\pi(\alpha, \beta)^2 + \widetilde{D} \int_{[\beta, \rho\beta]} r^d \mu(dr) + \widetilde{D} I^+(\rho, \alpha, \beta). \tag{5}$$

Moreover, for all $\beta > 0$, we have:

$$\pi(0, \beta) = \lim_{\alpha \to 0} \pi(\alpha, \beta), \tag{6}$$

and

$$P(M > 2\beta) \leq \widetilde{\pi}(\beta) \leq \pi(0, \beta) + \widetilde{D} \int_{[\beta, +\infty[} r^d \mu(dr). \tag{7}$$

Remark. With (5), we relate percolation probabilities at different scales. Our strategy is therefore related to multiscale strategies developped for example in [12] and [13] (which use some stochastic domination properties) or in [1] (from which our approach is closer).

Ideas of the proof of the first part of Theorem 1.3 using Proposition 2.1. The aim is to prove that $P(M \geq \beta)$ tends to 0 when β tends to infinity. By (7) we get, under Assumption B2, that it is sufficient to prove that $\pi(0, \beta)$ tends to 0. By (6) we get that it is sufficient to prove that $\pi(\alpha, \beta)$ tends to 0 uniformly in α. But by (5), $\pi(\alpha, \rho\beta)$ is bounded above by $\widetilde{D}\pi(\alpha, \beta)^2$ up to error terms which satisfy the following properties:

1. They are bounded above, by Assumption B1;
2. They tend to 0 when β tends to infinity, by Assumptions B0 and B2.

As $\pi(\alpha, \beta) = 0$ for small enough β (this is why the parameter α has been introduced) and as the bound given by Assumption B1 is small enough, we first deduce that $\pi(\alpha, \beta)$ remains small far all values of β and then that $\pi(\alpha, \beta)$ tends to 0 as β tends to infinity.

The key lemma in the proof of Proposition 2.1 is the following one.

Lemma 2.2 *Let $\rho \geq 2$. There exists a positive constant D_1 that depends only on the dimension d and on ρ such that, for all $\alpha \geq 0$ and all $\beta > 0$, the following holds:*

$$\pi(\alpha, \rho\beta) \leq D_1 \pi(\alpha, \beta)^2 + D_1 I(\rho, \alpha, \beta) + P(H(\rho, \beta)).$$

Proof.
• For all $r \geq 0$ we denote by S_r the Euclidean sphere centered at the origin with radius r:

$$S_r = \{x \in \mathbb{R}^d : \|x\| = r\}.$$

We fix K and L, two finite subsets of $\mathbb{R}^d$ such that the following properties hold:

$$K \subset S_\rho \subset K + B(0,1) \text{ and } L \subset S_{2\rho} \subset L + B(0,1).$$

We define D_1 as the product of the cardinalities of the sets K and L.
• Let $\alpha \geq 0$ and $\beta > 0$. In this step, we prove the following inclusion:

$$G(0,\alpha,\rho\beta) \setminus H(\rho,\beta) \subset \left(\bigcup_{k\in K} G(\beta k,\alpha,\beta) \right) \cap \left(\bigcup_{l\in L} G(\beta l,\alpha,\beta) \right). \tag{8}$$

We assume that the event $G(0,\alpha,\rho\beta)$ occurs but that the event $H(\rho,\beta)$ does not occur. As $G(0,\alpha,\rho\beta)$ occurs, one can go from $S_{\rho\beta}$ to $S_{2\rho\beta}$ using only balls of the percolation process whose radii belong to $[\alpha,\rho\beta]$. One can furthermore assume that the center of each such ball belongs to $B(0,\rho 3\beta)$.

One of these balls touches $S_{\rho\beta}$. This ball then touches $B(\beta k,\beta)$ for some $k \in K$. We then see that one can go from $B(\beta k,\beta)$ to the complement of $B(\beta k, 2\beta)$ using only balls whose radii belong to $[\alpha,\rho\beta]$ and whose centers belong to $B(0,3\rho\beta)$.

But, as $H(\rho,\beta)$ does not occur, the radius of each such ball $B(c,r)$ is less than β. Therefore, $G(\beta k,\alpha,\beta)$ occurs. We have proved that the event $\cup_{k\in K} G(\beta k,\alpha,\beta)$ occurs. We can prove in a similar way that the event $\cup_{l\in L} G(\beta l,\alpha,\beta)$ occurs. Therefore the inclusion (8) is proved.
• We then get:

$$\pi(\alpha,\rho\beta) \leq P(H(\rho,\beta)) + \sum_{k\in K, l\in L} P(G(\beta k,\alpha,\beta) \cap G(\beta l,\alpha,\beta)).$$

For all $k \in K$ and all $l \in L$, we have $\|\beta k - \beta l\| \geq \beta\rho$. By stationarity and by definition of $I(\rho,\alpha,\beta)$ and of D_1, we then get:

$$\pi(\alpha,\rho\beta) \leq P(H(\rho,\beta)) + D_1\big(\pi(\alpha,\beta)^2 + I(\rho,\alpha,\beta)\big).$$

This ends the proof. □

Lemma 2.3 *For all $\beta > 0$, the following holds:*

$$\pi(0,\beta) = \lim_{\alpha\to 0} \pi(\alpha,\beta).$$

Proof. Let $\beta > 0$. As $\alpha \mapsto \Sigma(\alpha,\beta)$ is non-increasing, $\alpha \mapsto G(0,\alpha,\beta)$ is non-increasing. Consequently,

$$\lim_{\alpha\to 0} \pi(\alpha,\beta) = P\left(\bigcup_{\alpha>0} G(0,\alpha,\beta) \right).$$

Therefore, it is sufficient to prove the following equality:

$$\bigcup_{\alpha>0} G(0,\alpha,\beta) = G(0,0,\beta).$$

If the event $G(0,0,\beta)$ occurs, then one can go from $B(0,\beta)$ to the complement of $B(0,2\beta)$ using balls of the percolation process whose radii belongs to $]0,\beta]$. By a compactness argument, we get the existence of a real $\alpha > 0$ such that one can go from $B(0,\beta)$ to the complement of $B(0,2\beta)$ using balls of the percolation process whose radii belongs to $[\alpha,\beta]$. In other words, $G(0,\alpha,\beta)$ occurs. This proves one of the required inclusions. The other inclusion is straightforward. □

Lemma 2.4 *For all $\beta > 0$, the following inclusion holds:*

$$\{M > 2\beta\} \subset \widetilde{G}(\beta) \subset G(0,0,\beta) \cup \widetilde{H}(\beta).$$

Proof. Let $\beta > 0$. If $G(0,0,\beta)$ does not occur, then one can not go from $B(0,\beta)$ to the complement of $B(0,2\beta)$ using balls of the percolation process whose radii belongs to $]0,\beta]$. If moreover $\widetilde{H}(\beta)$ does not occur, then balls of the percolation process whose radii do not belong to $]0,\beta]$ will not help to connect $B(0,\beta)$ to the complement of $B(0,2\beta)$. Therefore $\widetilde{G}(\beta)$ does not occur. This proves one inclusion. The other one is straightforward. □

Lemma 2.5 *There exists a positive constant D_2, that depends only on the dimension d, such that for all $\beta > 0$, the following inequality holds:*

$$P(\widetilde{H}(\beta)) \leq D_2 \int_{[\beta,+\infty[} r^d \mu(dr).$$

Proof. We have:

$$\widetilde{H}(\beta) = \{\xi \cap V(\beta) \neq \emptyset\}$$

where

$$V(\beta) = \{(c,r) \in \mathbb{R}^d \times]0,+\infty[: B(c,r) \cap B(0,2\beta) \neq \emptyset \text{ and } r > \beta\}.$$

We therefore have:

$$\begin{aligned} P(\widetilde{H}(\beta)) &= P(\xi \cap V(\beta) \neq \emptyset) \\ &\leq E(\mathrm{card}(\xi \cap V(\beta))) \\ &= \int_{\mathbb{R}^d} dc \int_{]0,+\infty[} \mu(dr) 1_{V(\beta)}(c,r). \end{aligned}$$

As

$$V(\beta) = \{(c,r) \in \mathbb{R}^d \times]0,+\infty[: \|c\| < r + 2\beta \text{ and } r > \beta\}$$

we get:

$$\begin{aligned} P(\widetilde{H}(\beta)) &\leq \int_{]\beta,+\infty[} |(B(0,r+2\beta)| \mu(dr) \\ &\leq \int_{]\beta,+\infty[} |B(0,3r)| \mu(dr). \end{aligned}$$

The inequality stated in the lemma is therefore fulfilled with $D_2 = |B(0,3)|$. □

Lemma 2.6 *Let $\rho \geq 2$. There exists a positive constant D_3, that depends only on the dimension d and on ρ, such that for all $\beta > 0$, the following inequality holds:*

$$P(H(\rho,\beta)) \leq D_3 \int_{[\beta,\rho\beta]} r^d \mu(dr).$$

Proof. We have:

$$\begin{aligned} P(H(\rho,\beta)) &\leq E\left(\mathrm{card}\big(\{(c,r)\in\xi : c\in B(0,3\rho\beta) \text{ and } r\in[\beta,\rho\beta]\}\big)\right) \\ &= |B(0,3\rho\beta)|\mu([\beta,\rho\beta]) \\ &= |B(0,3\rho)|\beta^d\mu([\beta,\rho\beta]). \end{aligned}$$

The inequality stated in the lemma is therefore fulfilled with $D_3 = |B(0,3\rho)|$. □

Proof of Proposition 2.1. This is a consequence of Lemmas 2.4, 2.3, 2.2, 2.5 and 2.6. □

2.3 Proof of Theorem 1.3

We first state three theorems which, together, give a strenghtened version of Theorem 1.3. Notice that the conclusion of each of the first two theorems is one of the assumptions of the following one.

Theorem 2.7 *Let $\rho \geq 2$ and $D' > 0$. There exists $D > 0$, that depends only on d, ρ and D', such that the following holds. Assume the existence of:*

1. *a sequence $(\alpha_n)_n$ of non-negative real numbers which converges to 0*
2. *and a bounded sequence $(\beta_n)_n$ of positive real numbers*

such that, for all $n \in \mathbb{N}$:

$$\sup_{\beta\geq\beta_n} I^+(\rho,\alpha_n,\beta) \leq D \text{ and } \sup_{\beta\geq\beta_n} \beta^d\mu([\beta,+\infty[) \leq D \tag{9}$$

$$\sup_{\beta\in[\beta_n,\rho\beta_n]} \pi(\alpha_n,\beta) \leq D. \tag{10}$$

Then, the probability $\pi(0,\beta)$ is smaller than D' for large enough β.

Remarks.

1. Note that, in the first assumption, one allows the sequence to be constant equal to 0.
2. If β belongs to $]0,\alpha_n[$ then $\Sigma(\alpha_n,\beta)$ is empty and therefore $G(0,\alpha_n,\beta)$ can not occur. The probability $\pi(\alpha_n,\beta)$ then equals 0. Therefore, (10) is always satisfied when β_n is strictly smaller than $\alpha_n\rho^{-1}$. This is the reason why we introduced the parameter α in the definition of G and π.

Theorem 2.8 *Let $\rho \geq 2$. There exists $D' > 0$, that depends only on d and ρ, such that the following holds. Assume the following:*

1. *The probability $\pi(0,\beta)$ is smaller than D' for large enough β.*
2. *$I^+(\rho,0,\beta)$ tends to 0 as β tends to infinity.*

3. *the integral* $\int_{[1,+\infty[} \beta^d \mu(d\beta)$ *is finite.*

Then, the probability $\widetilde{\pi}(\beta)$ *tends to* 0 *as* β *tends to infinity. Therefore, there is almost surely no percolation.*

Theorem 2.9 *Let* $\rho \geq 2$ *and* $s > 0$*. Assume the following:*

1. *The probability* $\widetilde{\pi}(\beta)$ *tends to* 0 *as* β *tend to infinity.*
2. $\int_{[1,+\infty[} \beta^{s-1} I^+(\rho, 0, \beta) d\beta < \infty$.
3. $\int_{[1,+\infty[} \beta^{d+s} \mu(d\beta) < \infty$.

Then, the integral

$$\int_0^{+\infty} \beta^{s-1} \widetilde{\pi}(\beta) d\beta$$

is finite. Therefore, the moment $E(M^s)$ *is finite.*

The proof of the previous theorems relies on Proposition 2.1 and on the following elementary lemma. There are three items in the lemma. Each of them corresponds to one of the previous theorems.

Lemma 2.10 *Let* f *and* g *be two measurable functions from* $]0, +\infty[$ *to* $[0, +\infty[$*. Let* $\rho > 1$*. We assume that, for all* $\beta > 0$*, the following inequality holds:*

$$f(\rho\beta) \leq f(\beta)^2 + g(\beta). \tag{11}$$

Then:

1. *Let* $\varepsilon \in]0, 1]$*. If there exists* $\beta_0 > 0$ *such that* $f(\beta) \leq \varepsilon/2$ *for all* $\beta \in [\beta_0, \rho\beta_0]$ *and* $g(\beta) \leq \varepsilon/4$ *for all* $\beta \geq \beta_0$ *then, for all* $\beta \geq \beta_0$*, we have* $f(\beta) \leq \varepsilon/2$.
2. *If, for all large enough* $\beta > 0$*, the inequality* $f(\beta) \leq 1/2$ *holds and if* $g(\beta)$ *converges to* 0 *as* β *tends to infinity then,* $f(\beta)$ *converges to* 0 *as* β *tends to infinity.*
3. *Let* $s > -1$ *be a real number. If* f *is bounded, if* $f(\beta)$ *converges to* 0 *as* β *tends to infinity and if the integral* $\int_1^{+\infty} \beta^s g(\beta) d\beta$ *is finite then, the integral* $\int_0^{+\infty} \beta^s f(\beta) d\beta$ *is finite.*

Proof.
• *Proof of Item* 1. If $\beta > 0$ is such that $f(\beta) \leq \varepsilon/2$ and $g(\beta) \leq \varepsilon/4$, then :

$$f(\rho\beta) \leq \varepsilon^2/4 + \varepsilon/4 \leq \varepsilon/2.$$

The result follows.
• *Proof of Item* 2. By (11) we get:

$$\limsup_{\beta\to\infty} f(\beta) \leq \left[\limsup_{\beta\to\infty} f(\beta)\right]^2 + \limsup_{\beta\to\infty} g(\beta).$$

By assumption,

$$\limsup_{\beta\to\infty} f(\beta) \leq 1/2 \text{ and } \limsup_{\beta\to\infty} g(\beta) = 0.$$

As f is non-negative, we get that $f(\beta)$ converges to 0 as β tends to infinity.

• *Proof of Item* 3. Let $s > -1$. By assumption, there exists a real $A \geq \rho$ such that:

$$\forall \beta \geq A\rho^{-1} : f(\beta) \leq \rho^{-s-1}/2. \tag{12}$$

For all real $r \geq A$, we get, by (11) and (12):

$$\begin{aligned}
\int_A^r f(\beta)\beta^s d\beta &\leq \int_A^r f(\beta\rho^{-1})^2\beta^s d\beta + \int_A^r g(\beta\rho^{-1})\beta^s d\beta \\
&\leq \rho^{s+1}\int_{A\rho^{-1}}^{r\rho^{-1}} f(\beta)^2\beta^s d\beta + \rho^{s+1}\int_{A\rho^{-1}}^{+\infty} g(\beta)\beta^s d\beta \\
&\leq 1/2\int_{A\rho^{-1}}^{r\rho^{-1}} f(\beta)\beta^s d\beta + \rho^{s+1}\int_{A\rho^{-1}}^{+\infty} g(\beta)\beta^s d\beta \\
&\leq 1/2\int_A^r f(\beta)\beta^s d\beta + 1/2\int_{A\rho^{-1}}^{A} f(\beta)\beta^s d\beta + \rho^{s+1}\int_{A\rho^{-1}}^{+\infty} g(\beta)\beta^s d\beta.
\end{aligned}$$

As f is bounded, the integral $\int_A^r f(\beta)\beta^s d\beta$ is finite. We therefore get:

$$\int_A^r f(\beta)\beta^s d\beta \leq \int_{A\rho^{-1}}^{A} f(\beta)\beta^s d\beta + 2\rho^{s+1}\int_{A\rho^{-1}}^{+\infty} g(\beta)\beta^s d\beta$$

and then

$$\int_A^{+\infty} f(\beta)\beta^s d\beta \leq \int_{A\rho^{-1}}^{A} f(\beta)\beta^s d\beta + 2\rho^{s+1}\int_1^{+\infty} g(\beta)\beta^s d\beta.$$

As f is bounded, the lemma follows. □

Proof of Theorems 2.7, 2.8 and 2.9. Let $\widetilde{D}$ be the positive constant given by Proposition 2.1. For all $\alpha \geq 0$ we define a function $f_\alpha :]0, +\infty[\to [0, +\infty[$ by:

$$f_\alpha(\beta) = \widetilde{D}\pi(\alpha, \beta)$$

and a function $g_\alpha :]0, +\infty[\to [0, +\infty[$ by:

$$g_\alpha(\beta) = \widetilde{D}^2 I^+(\rho, \alpha, \beta) + \widetilde{D}^2 \int_{[\beta,\rho\beta]} r^d \mu(dr).$$

By (5) we get, for all $\alpha \geq 0$ and all $\beta > 0$:

$$f_\alpha(\rho\beta) \leq f_\alpha(\beta)^2 + g_\alpha(\beta). \tag{13}$$

• *Proof of Theorem 2.7.* Let

$$\varepsilon = \min\left(\frac{1}{2}, 2D'\widetilde{D}\right) > 0$$

and

$$D = \min\left(\frac{\varepsilon}{8\rho^2\widetilde{D}^2}, \frac{\varepsilon}{2\widetilde{D}}\right) > 0.$$

Let us prove that D satisfies the required properties. Let $(\alpha_n)_n$ and $(\beta_n)_n$ be as in the statement of the theorem. Let β_* be the supremum of the bounded sequence $(\beta_n)_n$.

Let $n \in \mathbb{N}$. By (9) we get, for all $\beta \geq \beta_n$,

$$\begin{aligned} g_{\alpha_n}(\beta) &\leq \widetilde{D}^2\rho^d\beta^d\mu([\beta,+\infty[) + \widetilde{D}^2 I^+(\rho,\alpha_n,\beta) \\ &\leq 2\widetilde{D}^2\rho^d D \\ &\leq \varepsilon/4. \end{aligned}$$

By (9) we get, for all $\beta \in [\beta_n, \rho\beta_n]$,

$$f_{\alpha_n}(\beta) \leq \widetilde{D}D \leq \varepsilon/2.$$

By the first item of Lemma 2.10, we then get the inequality $f_{\alpha_n}(\beta) \leq \varepsilon/2$ for all $\beta \geq \beta_n$. Therefore, for all $\beta \geq \beta_*$, we have:

$$\pi(\alpha_n, \beta) \leq D'.$$

The theorem follows thanks to Lemma 2.3.

• *Proof of theorem 2.8.* Let

$$D' = \frac{1}{2\widetilde{D}} > 0.$$

Let us check that D' satisfies the required properties. By the first assumption of the theorem, we know that the inequality $\pi(0,\beta) \leq D'$ holds for large enough β. Therefore, we have $f_0(\beta) \leq 1/2$ for large enough β. By the second and the third assumptions, we ge that $g_0(\beta)$ converges to 0 as β tends to infinity. By the second item of Lemma 2.10, we then get that $f_0(\beta)$ also converges to 0. Therefore, $\pi(0,\beta)$ converges to 0. The theorem follows thanks to the third assumption and to (7).

• *Proof of theorem 2.9.* For all $\beta > 0$, we have $\pi(0,\beta) \leq \widetilde{\pi}(\beta)$. By the first assumption of the theorem, we then have the convergence of $\pi(0,\beta)$ to 0. Therefore, $f_0(\beta)$ converges to 0. Let us notice the following:

$$\begin{aligned} \int_1^{+\infty} d\beta\beta^{s-1}\int_{[\beta,+\infty[}\mu(dr)r^d &= \int_{[1,\infty[}\mu(dr)r^d\int_1^r d\beta\beta^{s-1} \\ &\leq \int_{[1,\infty[}\mu(dr)s^{-1}r^{d+s} \\ &< \infty \end{aligned} \tag{14}$$

by the third assumption. Using also the second assumption, we then get that the integral $\int_1^{+\infty}\beta^{s-1}g_0(\beta)d\beta$ is finite. By the third item of Lemma 2.10, we then get that the integral $\int_0^{+\infty}\beta^{s-1}f_0(\beta)d\beta$ is finite. The integral

$$\int_0^{+\infty}\beta^{s-1}\pi(0,\beta)d\beta \tag{15}$$

is therefore also finite. But by (7) we have, for all $\beta > 0$:

$$\widetilde{\pi}(\beta) \leq \pi(0,\beta) + \widetilde{D} \int_{[\beta,+\infty[} r^d \mu(dr).$$

By (14) and (15) , we thus get that the integral $\int_1^{+\infty} \beta^{s-1}\widetilde{\pi}(\beta)d\beta$ and then the integral $\int_0^{+\infty} \beta^{s-1}\widetilde{\pi}(\beta)d\beta$ is finite. The theorem follows by the first inequality of (7). □

Proof of Theorem 1.3. Let $\rho = \max(4C, 2)$. Let D' be the constant given by Theorem 2.8. Let D be the constant given by Theorem 2.7. Let us check that D satisfies the required properties.

Let $\alpha \geq 0$ and $\beta > 0$. Let us notice that, for all $x \in \mathbb{R}^d$, the event $G(x,\alpha,\beta)$ only depends on $\xi \cap B(x,3\beta)\times]0,\beta]$ Therefore, the event $G(x,\alpha,\beta)$ only depends on $\xi \cap B(x,3\beta)\times]0,3\beta]$. By Assumption B0 we then get that $G(0,\alpha,\beta)$ and $G(x,\alpha,\beta)$ are independent as soon as $\|x\| \geq 3\beta C$. By definition of ρ, we thus get:

$$I(\rho,\alpha,\beta) = 0. \tag{16}$$

Let n be a positive integer. We let $\alpha_n = n^{-1}$ and $\beta_n = \alpha_n(2\rho)^{-1}$. For all $\beta \in [\beta_n, \rho\beta_n]$, β belongs to $]0,\alpha_n$ Therefore the set $\Sigma(\alpha_n,\beta)$ is empty and consequently the event $G(0,\alpha_n,\beta)$ does not occur. As a consequence, (10) holds.

Property (9) holds because of (16) and Assumption B1 of Theorem 1.3. By Theorem 2.7, we then get that $\pi(0,\beta)$ is smaller than D' for large enough β. In other words, the first assumption of Theorem 2.8 holds. The second assumption of this theorem holds because of (16). The third one holds because of Assumption B2 of Theorem 1.3. We then get that S is almost surely bounded and that the first assumption of Theorem 2.9 holds. By (16), the second assumption of Theorem 2.9 holds. If Assumption B3 holds, we then get, by Theorem 2.9, that $E(M^s)$ is finite. □

3 Proof of Theorems 1.1 and 1.2 (models induced by Poisson processes)

We work under the assumption of Section 1.3. In particular, ξ is a Poisson point process on $\mathbb{R}^d\times]0,+\infty[$.

The following elementary lemma is stated and proven in [3] for a probability measure μ. The proof is the same for a locally finite measure.

Lemma 3.1 *Let μ be a locally finite measure on $]0,+\infty[$. If $\int_{[1,+\infty[}\beta^d\mu(d\beta)$ is infinite then, for all $\lambda > 0$, we have $P_{\lambda,\mu}$-almost surely $\Sigma = \mathbb{R}^d$. If $s > 0$ is such that $\int_{[1,+\infty[}\beta^{d+s}\mu(d\beta)$ is infinite then, for all $\lambda > 0$, $E_{\lambda,\mu}(M^s)$ is infinite.*

Proof. Let μ be a locally finite measure on $]0,+\infty[$ and $\lambda > 0$.
• We first prove that, for all $r > 0$, the following inequality holds:

$$P_{\lambda,\mu}\big(\exists c \in \chi : B(0,r) \subset B(c,r(c))\big) \geq 1 - \exp\left(-\lambda 2^{-d}|B(0,1)| \int_{[2r,+\infty[} \beta^d \mu(d\beta)\right). \tag{17}$$

Let $r > 0$. We have:

$$P_{\lambda,\mu}(\exists c \in \chi : B(0,r) \subset B(c,r(c))) = P(\xi \cap A \neq \emptyset)$$

where

$$A = \{(c,\beta) \in \xi : \beta \geq \|c\| + r\}.$$

Therefore:

$$\begin{aligned}
P_{\lambda,\mu}(\exists c \in \chi : B(0,r) \subset B(c,r(c))) &= 1 - \exp\left(-\lambda \int_{\mathbb{R}^d} \mu([\|c\| + r, +\infty[)dc\right) \\
&= 1 - \exp\left(-\lambda \int_{[r,+\infty[} |B(0,\beta - r)|\mu(d\beta)\right) \\
&\geq 1 - \exp\left(-\lambda \int_{[2r,+\infty[} [B(0,\beta - r)|\mu(d\beta)\right) \\
&\geq 1 - \exp\Big(-\lambda \int_{[2r,+\infty[} |B(0,\beta/2)|\mu(d\beta)\Big).
\end{aligned}$$

The relation (17) is proved.
• If $\int_{[1,+\infty[} \beta^d \mu(d\beta)$ is infinite then, by (17), we get, for all $r > 0$:

$$P_{\lambda,\mu}\big(\exists c \in \chi : B(0,r) \subset B(c,r(c))\big) = 1.$$

Therefore, almost surely, we have $\Sigma = \mathbb{R}^d$.
• Let $s > 0$. We assume now that $\int_{[1,+\infty[} \beta^{d+s} \mu(d\beta)$ is infinite. If $\int_{[1,+\infty[} \beta^d \mu(d\beta)$ is infinite, the desired result is a consequence of what we have proved in the previous step. We assume henceforth that $\int_{[1,+\infty[} \beta^d \mu(d\beta)$ is finite. Let C be defined by:

$$C = \lambda 2^{-d} |B(0,1)| \int_{[1,+\infty[} \beta^d \mu(d\beta).$$

This constant is finite. By (17) we get, for all $r > 1/2$, the following inequality:

$$P_{\lambda,\mu}\big(\exists c \in \chi : B(0,r) \subset B(c,r(c))\big) \geq C^{-1}(1 - \exp(-C))\lambda 2^{-d} |B(0,1)| \int_{[2r,+\infty[} \beta^d \mu(d\beta)$$

and then

$$P(M \geq r) \geq C^{-1}(1 - \exp(-C))\lambda 2^{-d} |B(0,1)| \int_{[2r,+\infty[} \beta^d \mu(d\beta). \tag{18}$$

As $\int_{[1,+\infty[} \beta^{d+s} \mu(d\beta)$ is infinite, the integral

$$\int_{1/2}^{+\infty} \left(r^{s-1} \int_{[2r,+\infty[} \beta^d \mu(d\beta) \right) dr$$

is infinite. Therefore, by (18), the integral $\int_0^{+\infty} r^{s-1} P_{\lambda,\mu}(M \geq r) dr$ is infinite. The moment $E_{\lambda,\mu}(M^s)$ is then infinite. □

Lemma 3.2 *Let μ be a locally finite measure on $]0,+\infty[$. Let $\rho > 1$. We have:*

$$\sup_{r>0} r^d \mu([r,\rho r]) \leq \sup_{r>0} r^d \mu([r,+\infty[) \leq \frac{1}{1-\rho^{-d}} \sup_{r>0} r^d \mu([r,\rho r]).$$

Proof. The first inequality is straightforward. Let us prove the other one. Let $r > 0$. We have:

$$\begin{aligned} r^d \mu([r,+\infty[) &= \sum_{n\geq 0} \rho^{-nd} (r\rho^n)^d \mu([r\rho^n, r\rho^{n+1}[) \\ &\leq \sum_{n\geq 0} \rho^{-nd} \sup_{s>0} s^d \mu([s,s\rho[) \\ &\leq \frac{1}{1-\rho^{-d}} \sup_{s>0} s^d \mu([s,s\rho[). \end{aligned}$$

The lemma follows. □

Lemma 3.3 *Assume $d \geq 2$. Let μ be a locally finite measure on $]0,+\infty[$. If*

$$\sup_{r>0} r^d \mu([r,+\infty[)$$

is infinite, then for all $\lambda > 0$, we have $P_{\lambda,\mu}(\textit{percolation}) > 0$.

Proof. Let μ be a locally finite measure on $]0,+\infty[$ and $\lambda > 0$. Let $\lambda_c > 0$ be the critical value for the classical Boolean model when all radii equal 1 (see for example [10] or Section 12.10 in [4]). In other word, when $\mu = \delta_1$, S is almost surely bounded when $\lambda < \lambda_c$ and S is unbounded with positive probability when $\lambda > \lambda_c$.

Let $p = 2$. By assumption and by Lemma 3.2, there exists $r_0 > 0$ such that:

$$\lambda r_0^d \mu([r_0, r_0\rho]) > \lambda_c.$$

We define a new Poisson point process as follows:

$$\widetilde{\xi} = \{(c, r_0) : c \in \chi \text{ such that } r(c) \in [r_0, r_0\rho]\}.$$

The intensity measure of this point process is the product of the measure $\lambda\mu([r_0, r_0\rho])|\cdot|$ by the probability measure δ_{r_0}. Let $\widetilde{\Sigma}$ be associated with $\widetilde{\xi}$ as in Section 1.2. Let us notice that $\widetilde{\Sigma}$ is a subset of Σ. It is therefore sufficient to prove that $\widetilde{\Sigma}$ is in the supercritical phase. The random set $r_0^{-1}\widetilde{\Sigma}$ is associated with the following Poisson point process

$$r_0^{-1}\widetilde{\xi} = \{(cr_0^{-1}, 1) : c \in \chi \text{ such that } r(c) \in [r_0, r_0\rho]\}$$

whose intensity measure is the product of $r_0^d \lambda\mu([r_0, r_0\rho])|\cdot|$ by the probability measure δ_1. By our choice of r_0 and by definition of λ_c we get that $r_0^{-1}\widetilde{\Sigma}$, and therefore $\widetilde{\Sigma}$, is in the supercritical phase. This ends the proof. □

Proof of Theorems 1.1 and 1.2. • *Proof of sufficient conditions.* Let $C = 2$. Let $D > 0$ be the constant given by Theorem 1.3. Assumption B0 of Theorem 1.3 is satisfied because of independence properties of Poisson point processes. Since, under $P_{\lambda,\mu}$, the intensity measure of ξ is the product of the Lebesgue measure and of the measure $\lambda\mu$, the required results follow from Theorem 1.3.

• *Proof of necessary conditions.* This is a consequence of Lemmas 3.3 and 3.1. □

4 Proof of Lemma 1.4 and Theorem 1.8 (multifractal percolation)

Proof of Lemma 1.4. Let us first notice that, for each $n \geq 0$, $a^{-n}\chi_n$ is a Poisson point process whose intensity measure is the product of $a^{nd}\lambda|\cdot|$ by the measure ν_n defined by $\nu_n(B) = \nu_n(a^n B)$. The measure μ defined by (4) is then also defined by $\mu = \sum_n \nu_n$.

It remains to check that the measure μ defined by (4) is locally finite. Let $k \in \mathbb{Z}$. It is sufficient to prove that $\mu([a^k, a^{k+1}[)$ is finite. We have:

$$\begin{aligned}\mu([a^k, a^{k+1}[) &= \sum_{n\geq 0} a^{nd}\nu([a^{k+n}, a^{k+n+1}[) \\ &\leq \int_{[a^k,+\infty[} x^d a^{-kd}\nu(dx).\end{aligned}$$

As $\int_{]0,+\infty[} x^d\nu(dx)$ is finite, the result follows. □

Proof of Theorem 1.8. Let μ be the measure defined by (4). Thanks to Theorem 1.1 is it sufficient to check the following:

1. Condition A1 holds.
2. Condition A2 holds if and only if $\int_{[1,+\infty[} \beta^d \ln(\beta)\nu(d\beta)$ is finite.

Let us notice that, for all $f :]0,+\infty[\to \mathbb{R}$ measurable and non-negative, we have:

$$\int_{]0,+\infty[} f(\beta)\mu(d\beta) = \sum_{n\geq 0} a^{nd} \int_{]0,+\infty[} f(a^{-n}\beta)\nu(d\beta).$$

Let us check the first item. Let $r > 0$. We have:

$$\begin{aligned}\int_{[r,ra]} \beta^d\mu(d\beta) &= \sum_{n\geq 0} a^{nd} \int_{]0,+\infty[} 1_{[r,ra]}(\beta a^{-n})(\beta a^{-n})^d\nu(d\beta) \\ &= \int_{]0,+\infty[} \sum_{n\geq 0} 1_{[r,ra]}(\beta a^{-n})\beta^d\nu(d\beta) \\ &\leq \int_{]0,+\infty[} 2\beta^d\nu(d\beta).\end{aligned}$$

The first item then follows from Lemma 3.2 by (3).

Let us check the second item. As above, we get:

$$\begin{aligned}\int_{[1,+\infty[} \beta^d\mu(d\beta) &= \sum_{n\geq 0} a^{nd} \int_{]0,+\infty[} 1_{[1,+\infty[}(\beta a^{-n})(\beta a^{-n})^d\nu(d\beta) \\ &= \int_{]0,+\infty[} \sum_{n\geq 0} 1_{[1,+\infty[}(\beta a^{-n})\beta^d\nu(d\beta) \\ &= \int_{[1,+\infty[} \big(\lfloor \ln(\beta)\ln(a)^{-1}\rfloor + 1\big)\beta^d\nu(d\beta).\end{aligned}$$

The second idem follows. This concludes the proof. □

5 Proof of Lemma 1.11 and Theorem 1.10 (marriage)

Let us recall the definition of ξ. We assume that χ is a Poisson point process on $\mathbb{R}^d$ whose intensity measure is the Lebesgue measure. For all $a \in \chi$ we define $R(a,\chi)$ by:

$$R(a,\chi) = \inf\left\{r \geq 0 : \alpha \mathrm{card}(\chi \cap \overline{B}(a,2r)) \leq |B(a,r)|\right\}.$$

(We let $R(a,\chi) = \infty$ if there is no such r.) Using some elementary properties of the map defined by $r \mapsto \alpha \mathrm{card}(\chi \cap \overline{B}(a,2r)) - |B(a,r)|$, we get that $R(a,\chi)$ is always positive and that:

$$R(a,\chi) = \min\left\{r \geq 0 : |B(0,r)| \in \alpha\mathbb{N} \text{ and } \alpha \mathrm{card}(\chi \cap \overline{B}(a,2r)) = |B(a,r)|\right\}.$$

(With the same convention as before if there is no such r.) Among other things, this remark enables us to easily solve some measurability issues. We define a point process ξ on $\mathbb{R}^d \times]0,+\infty]$ by:

$$\xi = \{(a, 2R(a,\chi)), a \in \chi\}.$$

Let us notice that the law of ξ is invariant under the action of the translations of $\mathbb{R}^d$ and that the intensity measure of ξ is locally finite. The intensity measure is therefore the product of the Lebesgue measure on $\mathbb{R}^d$ by a locally finite measure on $]0,+\infty]$. We denote this measure on $]0,+\infty]$ by μ. Let us notice that μ is a probability measure.

Lemma 5.1 *There exists an absolute constant $K > 0$ and a function $F :]0, 2^{-d}[\to]0,+\infty[$ that depends only on the dimension d such that:*

1. $\lim_{\alpha \to 0} F(\alpha) = +\infty$.
2. *For all $\alpha \in]0, 2^{-d}[$ and all $r > 0$, we have: $\mu(]r,+\infty]) \leq K \exp(-F(\alpha) r^d)$.*

Proof. Assume $\alpha \in]0, 2^{-d}[$. Let $r > 0$. By definition of μ and ξ we have:

$$\begin{aligned}
\mu(]r,+\infty]) &= E\left(\mathrm{card}(\xi \cap [0,1]^d \times]r,+\infty])\right) \\
&= E\left(\sum_{a \in \chi \cap [0,1]^d} 1_{2R(a,\chi) > r}\right) \\
&= E\left(\sum_{a \in \chi \cap [0,1]^d} 1_{2R(0,\chi - a) > r}\right).
\end{aligned}$$

As the Palm measure of the Poisson point process χ is the law of $\chi \cup \{0\}$ (see for example [15]), we get:

$$\mu(]r,+\infty]) = P(2R(0,\chi \cup \{0\}) > r).$$

By definition of $R(0,\chi \cup \{0\})$, we then get:

$$\begin{aligned}
\mu(]r,+\infty]) &\leq P\left(\alpha \mathrm{card}\big((\chi \cup \{0\}) \cap \overline{B}(0,r)\big) > |B(0,r/2)|\right) \\
&= P\left(\alpha(N(r)+1) > \omega_d r^d 2^{-d}\right) \\
&= P\left(N(r) > \alpha^{-1}\omega_d r^d 2^{-d} - 1\right)
\end{aligned}$$

where $N(r) = \text{card}(\chi \cap \overline{B}(0,r))$ and $\omega_d = |B(0,1)|$.

If

$$1 < \alpha^{-1}\omega_d r^d 2^{-d}(1-\sqrt{\alpha 2^d}) \tag{19}$$

we have:

$$\mu(]r,+\infty]) \leq P\left(N(r) > \omega_d r^d \sqrt{\alpha 2^d}^{-1}\right).$$

As $N(r)$ is a Poisson random variable with mean $w_d r^d$ we then get, using Chernoff's bound:

$$\mu(]r,+\infty]) \leq \exp(-w_d r^d g(\sqrt{\alpha 2^d}))$$

where $g :]0,1[\to \mathbb{R}$ is defined by:

$$g(x) = (x - 1 - \ln(x))/x.$$

The previous inequality holds as soon as (19) holds. It therefore holds as soon as $\omega_d r^d > \alpha 2^d (1-\sqrt{\alpha 2^d})^{-1}$. For all $r > 0$ we thus have:

$$\mu(]r,+\infty]) \leq \exp(h(\sqrt{\alpha 2^d}))\exp(-w_d r^d g(\sqrt{\alpha 2^d}))$$

where $h :]0,1[\to \mathbb{R}$ is defined by:

$$h(x) = x^2(1-x)^{-1}g(x).$$

As h is bounded and as $\lim_{x\to 0} g(x) = +\infty$, the lemma follows. □

We assume henceforth that α is strictly smaller than 2^{-d}. By the previous lemma, we can therefore consider that ξ is a point process on $\mathbb{R}^d \times]0,+\infty[$ and that μ is a probability measure on $]0,+\infty[$. We are therefore in the same framework as in Section 1.4. We associate with ξ a random set Σ and a random variable M.

Proof of Lemma 1.11. We work on a full event on which there exists an a.e. unique stable allocation and denote by ψ one of those allocations. Let $a \in \chi$. Let us recall that $R(a,\chi)$ is finite. To simplify notations, we write R instead of $R(a,\chi)$. To prove the lemma, it suffices to check that $\psi^{-1}(a)$ is a subset of $\overline{B}(a,R)$. We have :

$$\alpha \text{card}(\chi \cap \overline{B}(a,2R)) = |B(a,R)|.$$

Let $\varepsilon > 0$ be such that there is no point of χ in the shell $\overline{B}(a,2R+2\varepsilon) \setminus \overline{B}(a,2R)$. We then have:

$$\alpha \text{card}(\chi \cap \overline{B}(a,2R+2\varepsilon)) < |B(a,R+\varepsilon)|.$$

Therefore:

$$\left|\psi^{-1}\left(\chi \cap \overline{B}(a,2R+2\varepsilon)\right)\right| < |B(a,R+\varepsilon)|.$$

As a consequence, there exists x in $B(a,R+\varepsilon)$ such that $\psi(x)$ belongs to $\chi \cup \{\infty\}$ and does not belong to $\overline{B}(a,2R+2\varepsilon)$. If $\psi(x) \in \chi$, we have:

$$\|x - \psi(x)\| > R + \varepsilon \text{ and } \|x - a\| \leq R + \varepsilon.$$

In particular, x desires a. Otherwise, that is, if $\psi(x) = \infty$, then x also desires a. As ψ is stable, we therefore get that a does not covet x. As a consequence, $\psi^{-1}(a)$ is contained

in $\overline{B}(a, \|x-a\|)$ and therefore in $\overline{B}(a, R+\varepsilon)$. As this result holds for arbitrary small $\varepsilon > 0$, we get that $\psi^{-1}(a)$ is contained in $\overline{B}(a, R)$. The lemma follows. $\square$

Proof of Theorem 1.10. Thanks to Lemma 1.11, it suffices to check that ξ satisfies the assumptions of Theorem 1.3.

B0. We show that the assumption is fulfiled with $C = 7$. Let $r > 0$. For all $a \in \chi$ we let:

$$\widetilde{R}(a, \chi) = \inf\{s \in [0, r] : \alpha \mathrm{card}(\chi \cap \overline{B}(a, 2s)) \leq |B(a, s)|)\}.$$

(We let $\widetilde{R}(a, \chi) = r$ if there exists no such s.) Let us notice that, for all $a \in \chi$, we have $\widetilde{R}(a, \chi) = R(a, \chi)$ as soon as $R(a, \chi) < r$ or $\widetilde{R}(a, \chi) < r$ Therefore, for all $x \in \mathbb{R}^d$,

$$\xi \cap \mathbb{R}^d \times [0, r[= \widetilde{\xi} \cap \mathbb{R}^d \times [0, r[$$

where $\widetilde{\xi}$ is defined by

$$\widetilde{\xi} = \{(a, 2\widetilde{R}(a, \chi)), a \in \chi\}.$$

As a consequence, we see that $\xi \cap B(x, r) \times [0, r[$ only depends on $\chi \cap B(x, 3r)$. By the independance property of Poisson point processes, we then get that, if x belongs to $\mathbb{R}^d \setminus B(0, 6r)$, the point processes $\xi \cap B(0, r) \times [0, r[$ and $\xi \cap B(x, r) \times [0, r[$ are independent. The required result follows.

B1. By Lemma 5.1, we have:

$$\begin{aligned} \sup_{r>0} r^d \mu([r, +\infty[) &\leq \sup_{r>0} r^d \mu(]r/2, +\infty[) \\ &\leq \sup_{r>0} r^d K \exp(-F(\alpha) r^d 2^{-d}) \\ &= K 2^d F(\alpha)^{-1} \sup_{x>0} x \exp(-x). \end{aligned}$$

As $F(\alpha)$ tends to infinity when α tends to 0, Assumption B1 is fulfiled for small enough α.

B2 and B3. By Lemma 5.1, we get that $\int_{]0,+\infty[} r^{d+s} \mu(dr)$ is finite for all $s \geq 0$.

When α is small enough, we can thus use Theorem 1.3. We get that $E(M^s)$ is finite for all $s > 0$. By Lemma 1.11 we then get that $E(D^s)$ is finite for all $s > 0$. $\square$

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